JOURNAL OF APPROXIMATION THEORY **30**, 334–336 (1980)

Note

On Approximation by Linear Positive Operators

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Communicated by Oved Shisha

Received January 19, 1979

In [7, 8], Shisha and Mond gave a quantitative formulation of some well-known results of Korovkin [4]. In [6], Mond showed how a proof in [7] is easily modified to yield a more general and often better result. Here we show how the proofs of Censor [1] can be similarly modified to obtain corresponding generalizations. For simplicity we utilize the notation of [1].

THEOREM. *Let A be a positive number. Let $L_1, L_2,\ldots$ be linear positive operators on $C[a, b]$. Suppose that $\{L_n(1)\}_{n=1}^{\infty}$ is uniformly bounded in $[a, b]$. Let $f \in C^1[a, b]$ and let $\omega(f'; \cdot)$ be the modulus of continuity of f'. Then, for $n = 1, 2,\ldots$,*

$$\|L_n(f) - f\| \leqslant \|f\| \cdot \|L_n(1) - 1\| + C_n \|f'\| \mu_n + C_n \mu_n \omega(f'; A\mu_n), \quad (1)$$

where

$$C_n = A^{-1} + \|L_n(1)\|^{1/2}$$

and

$$\mu_n = \|L_n\{(t-x)^2; x\}\|^{1/2}.$$

In particular, if $L_n(1) = 1$, (1) reduces to

$$\|L_n(f) - f\| \leqslant \|f'\| \mu_n + (A^{-1} + 1)\mu_n \omega(f'; A\mu_n).$$

0021-9045/80/120334-03$02.00/0

If, in addition, $L_n\{t; x\} \equiv x$, *we obtain*

$$\|L_n(f) - f\| \leqslant (A^{-1} + 1)\,\mu_n\,\omega(f'; A\mu_n).$$

Note that, if we take $A = 1$, the theorem reduces to that of Censor [1]. The proof of the theorem is analogous to that of Theorem 5 of [1] except that, in the appropriate step of the proof, one takes $\delta = A\mu_n$ instead of $\delta = \mu_n$. A number of other results of Censor [1] can similarly be improved by this change, introducing the arbitrary constant A into the estimate for $\|L_n(f) - f\|$.

EXAMPLE. Let D be the set of all real functions with domain $[0, 1]$. For $n = 1, 2, \ldots$, let L_n be the linear positive operator with domain D, defined by

$$(L_n\phi)(x) \equiv \sum_{i=0}^{n} \phi(i/n)\binom{n}{i} x^i(1-x)^{n-i}.$$

Let f be a real function in $C^1[0, 1]$. Let n be a positive integer. Then $L_n(1) \equiv 1$, $[L_n(t)](x) \equiv x$,

$$[L_n(t^2)](x) \equiv (n-1)\,n^{-1}x^2 + n^{-1}x,\ (L_n(|t-x|^2))(x) = n^{-1}(x - x^2).$$

Taking $A = 2$, our theorem gives

$$\max_{0 \leqslant x \leqslant 1} |f(x) - L_n f(x)| \leqslant (\tfrac{1}{2} + 1)\,2^{-1}n^{-1/2}\omega(f'; 2/(2n^{1/2}))$$
$$= \tfrac{3}{4}n^{-1/2}\omega(f'; n^{-1/2}).$$

Thus, by selecting $A = 2$, our theorem yields the estimate for the rate of convergence of Bernstein polynomials of functions in $C^1[0, 1]$ given in [5, p. 21], whereas in [1–3], as good a result is not achieved.

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