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Asymptotic behavior of solutions of third-order nonlinear dynamic equations on time scales☆

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ABSTRACT

In this paper, we will study asymptotic behavior of solutions to third-order nonlinear dynamic equations on time scales of the form

$$\left(\frac{1}{a_2(t)}\left(\left(\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}+q(t)f(x(t))=0.$$

By using the Riccati technique and integral averaging technique, two different types of criteria are established, one of which extends some existing results and the other is new. Two examples of dynamic equations on different time scales are given to show the applications of the obtained results.

1. Introduction

Recently, there has been much research activity concerning the oscillation and nonoscillation of solutions of some ordinary differential equations and dynamic equations on time scales, we refer the reader to [2,3,8,9,12] and [1,4–7,10,11, 13], respectively, and the references therein. Especially in 2005, Erbe, Peterson and Saker [7] studied asymptotic behavior of solutions of the following third-order nonlinear dynamic equation (1.1) for $\alpha_1 = \alpha_2 = 1$. Following this trend, in this paper, we will study the asymptotic behavior of solutions to more general third-order nonlinear dynamic equations of the form

$$\left(\frac{1}{a_2(t)}\left(\left(\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}+q(t)f(x(t))=0,\quad t\in\mathbb{T},$$

or for short,

$$L_3x(t)+q(t)f(x(t))=0,\quad t\in\mathbb{T},\tag{1.1}$$

where $\mathbb{T}$ is a time scale, $L_0x(t)=x(t)$, $L_3x(t)=(L_2x(t))^{\Delta}$, and

$$L_kx(t)=\frac{1}{a_k(t)}\left((L_{k-1}x(t))^{\Delta}\right)^{\alpha_k},\quad k=1,2.$$

In Eq. (1.1), we assume that the following conditions are satisfied:

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(A1) The functions $a_i(t)$ $(i = 1, 2)$ and $q(t)$ are positive, real-valued, rd-continuous functions defined on the time scale interval $[a, b]$ (throughout $a, b \in \mathbb{T}$ with $a < b$);
(A2) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $xf(x) > 0$ $(x \neq 0)$, and for all $k > 0$, $\exists M = M_k > 0$, $\frac{f(x)}{x} \geq M$, $|x| \geq k$;
(A3) α_i is a quotient of odd positive integers, $i = 1, 2$.

Since we study the asymptotic behavior of solutions to Eq. (1.1), we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)$. By a solution of Eq. (1.1) we mean a nontrivial real-valued function $x(t)$ satisfying (1.1) for $t \geq t_0$. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. Our attention is restricted to those solutions of Eq. (1.1) which exist on some half-time $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > T\} > 0$ for any $T \geq t_x$.

In this paper, we will use the Riccati transformation technique and integral averaging technique to give some sufficient conditions in terms of the coefficients and the graininess function which guarantee that every solution of (1.1) is oscillatory on $[t_0, \infty)$ or converges as $t \to \infty$. Two different types of criteria are established, one of which extends the results in [7] and the other is new. Two examples of dynamic equations on different time scales are given to show the applications of the obtained results.

To present our work better, we will organize this paper as follows. After this introduction, in Section 2, we state the main results and give two examples. Some technical lemmas and the proofs of the main results are given in Section 3. We assume the reader is familiar with the basic knowledge on dynamic equations on time scales. Those who are not may read the Appendix at the end of this paper for a brief summary on the concepts and results on time scales related to our work, and are referred to [4] for further details.

2. Main results

In this section, we state the main results which guarantee that every solution of (1.1) oscillates on $[t_0, \infty)$ or converges as $t \to \infty$ and give two examples to illustrate the significance of the results.

Theorem 2.1. *Assume* (A1)–(A3) *and* $\alpha_1\alpha_2 = 1$ *hold, and*

$$\int_{t_0}^{+\infty} [a_i(s)]^{\frac{1}{\alpha_i}} \Delta s = \infty, \quad i = 1, 2. \tag{2.1}$$

Furthermore, assume that there exists a positive function $r(t)$ *such that* $r^{\Delta}(t)$ *is rd-continuous on* $[t_0, \infty)$, *and that for all* $M > 0$ *and sufficiently large* t_1, t_2 *with* $t_2 > t_1$, *we have*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[Mr(s)q(s) - \frac{(r^{\Delta}(s))^2}{4Q(s)} \right] \Delta s = \infty, \tag{2.2}$$

where $Q(t) = r(t)\,[a_1(t)\delta(t, t_1)]^{\frac{1}{\alpha_1}}$, $\delta(t, t_1) = \int_{t_1}^{t} [a_2(s)]^{\frac{1}{\alpha_2}} \Delta s$. *Then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

Taking $r(t) = 1$ and $r(t) = t$ in Theorem 2.1 respectively, we shall have the following two corollaries.

Corollary 2.1. *Assume* (A1)–(A3) , (2.1) *and* $\alpha_1\alpha_2 = 1$ *hold, and*

$$\int_{t_0}^{\infty} q(s)\Delta s = \infty. \tag{2.3}$$

Then every solution $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Corollary 2.2. *Assume* (A1)–(A3) *and* (2.1) *hold,* $\alpha_1\alpha_2 = 1$. *If for all* $M > 0$ *and sufficiently large* t_1, t_2 *with* $t_2 > t_1$,

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[Msq(s) - \frac{1}{4s} \left(a_1(s)\delta(s, t_1)\right)^{-\frac{1}{\alpha_1}} \right] \Delta s = \infty,$$

where $\delta(t, t_1) = \int_{t_1}^{t} [a_2(s)]^{\frac{1}{\alpha_2}} \Delta s$, *then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

The first example illustrates the application of Corollary 2.2.

Example 2.1. Consider the equation

$$\left(\left(\left(\frac{1}{a_1(t)} \left(x^{\Delta}(t)\right)^{\frac{1}{k}} \right)^{\Delta} \right)^{k} \right)^{\Delta} + q(t)|x(t)|^{\gamma-1}x(t) = 0, \tag{2.4}$$

where $t \in \mathbb{T} = q_0^{\mathbb{N}}$, $q_0 > 1$, k is any positive odd, $a_1(t), q(t) \in C_{rd}(\mathbb{T}, \mathbb{R})$ with $a_1(t) \geq 1$ and $q(t) \geq \frac{1}{t}$, $\gamma \geq 1$ is a constant.

Taking $\alpha_1 = \frac{1}{k}$, $\alpha_2 = k$, $a_2(t) = 1$ and $f(x) = |x(t)|^{\gamma-1}x(t)$, then for sufficiently large t_1,

$$\delta(t, t_1) = \int_{t_1}^{t} [a_2(s)]^{\frac{1}{\alpha_2}} \Delta s = \int_{t_1}^{t} \Delta s = t - t_1,$$

$$(a_1(t)\delta(t, t_1))^{-\frac{1}{\alpha_1}} = (a_1(t)(t - t_1))^{-k} \le \frac{1}{(t - t_1)^k},$$

and for all $M > 0$ and sufficiently large t_1, t_2 with $t_2 > t_1$, we have

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[Msq(s) - \frac{1}{4s} (a_1(s)\delta(s, t_1))^{-\frac{1}{\alpha_1}} \right] \Delta s \ge \limsup_{t\to\infty} \int_{t_2}^{t} \left[M - \frac{1}{4s(s - t_1)^k} \right] \Delta s = \infty.$$

We see that all conditions of Corollary 2.2 are satisfied and then every solution $x(t)$ of (2.4) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

By using the functions of the form $(t - s)^m$, we have the following theorem.

Theorem 2.2. *Assume* (A1)–(A3) *and* (2.1) *hold,* $\alpha_1\alpha_2 = 1$. *If there exist* $m \ge 1$ *and a positive function* $r(t)$ *such that* $r^{\Delta}(t)$ *is rd-continuous on* $[t_0, \infty)$, *and that for all* $M > 0$ *and sufficiently large* t_1, t_2 *with* $t_2 > t_1$,

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_2}^{t} (t - s)^m \left[Mr(s)q(s) - \frac{(r^{\Delta}(s))^2}{4Q(s)} \right] \Delta s = \infty, \tag{2.5}$$

where $Q(t) = r(t) [a_1(t)\delta(t, t_1)]^{\frac{1}{\alpha_1}}$, $\delta(t, t_1) = \int_{t_1}^{t} [a_2(s)]^{\frac{1}{\alpha_2}} \Delta s$, *then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

Taking $r(t) = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Assume* (A1)–(A3) *and* (2.1) *hold,* $\alpha_1\alpha_2 = 1$. *If there exists* $m \ge 1$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t - s)^m q(s) \Delta s = \infty, \tag{2.6}$$

then every solution $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

Remark 2.1. (2.6) can be considered as the extension of Kamenev-type oscillation criteria for second-order differential equations (see [6,8]). When $\mathbb{T} = \mathbb{R}^+ := [0, \infty)$, (2.6) becomes

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t - s)^m q(s) \mathrm{d}s = \infty;$$

when $\mathbb{T} = \mathbb{N}_0$, (2.6) becomes

$$\lim_{n\to\infty} \frac{1}{n^m} \sum_{k=0}^{n-1} (n - k)^m q(k) = \infty;$$

when $\mathbb{T} = p^{\mathbb{N}_0}$, where $p > 1$ is a constant, (2.6) becomes

$$\lim_{n\to\infty} \frac{1}{p^{mn}} \sum_{k=0}^{n-1} p^k (p^n - p^k)^m q(p^k) = \infty.$$

Remark 2.2. In Theorems 2.1 and 2.2, $Q(t_1) = \delta(t_1, t_1) = 0$, so we replace "for sufficiently large t_1" in Theorems 1 and 2 of [7] with "for sufficiently large t_1, t_2 with $t_2 > t_1$". When $\alpha_1 = \alpha_2 = 1$, Theorems 2.1 and 2.2 reduce to Theorems 1 and 2 of [7].

When $f'(x) \ge C > 0$ for some constant C, we present the following two different types of theorems from Theorems 2.1 and 2.2.

Theorem 2.3. *Assume* (A1)–(A3) *and* (2.1) *hold,* $\alpha_1\alpha_2 = 1$, *and* $f'(x) \ge C > 0$ *for some constant* C. *If there exists a function* $r(t) > 0$ *such that* $r^{\Delta}(t)$ *is rd-continuous on* $[t_0, \infty)$ *and for all sufficiently large* t_1, t_2 *with* $t_2 > t_1$,

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[r(s)q(s) - \frac{(r^{\Delta}(s))^2}{4CQ(s)} \right] \Delta s = \infty, \tag{2.7}$$

where $Q(t) = r(t)[a_1(t)\delta(t,t_1)]^{\frac{1}{\alpha_1}}$, $\delta(t,t_1) = \int_{t_1}^{t}[a_2(s)]^{\frac{1}{\alpha_2}}\Delta s$, *then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

Taking $r(t) = t$ in Theorem 2.3, we have the following corollary.

Corollary 2.4. *Assume* (A1)–(A3) *and* (2.1) *hold,* $\alpha_1\alpha_2 = 1$ *and* $f'(x) \geq C > 0$ *for some constant* C*. If for sufficiently large* t_1, t_2 *with* $t_2 > t_1$*,*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[sq(s) - \frac{1}{4Cs}\left(a_1(s)\delta(s,t_1)\right)^{-\frac{1}{\alpha_1}} \right] \Delta s = \infty,$$

where $\delta(t,t_1) = \int_{t_1}^{t}[a_2(s)]^{\frac{1}{\alpha_2}}\Delta s$*, then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

By using the functions of the form $(t-s)^m$, we also have the following theorem.

Theorem 2.4. *Assume* (A1)–(A3) *and* (2.1) *hold,* $\alpha_1\alpha_2 = 1$ *and* $f'(x) \geq C > 0$ *for some constant* C*. If there exist* $m \geq 1$ *and a function* $r(t) > 0$ *such that* $r^{\Delta}(t)$ *is rd-continuous on* $[t_0,\infty)$ *and for all sufficiently large* t_1, t_2 *with* $t_2 > t_1$*,*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_2}^{t} (t-s)^m \left[r(s)q(s) - \frac{(r^{\Delta}(s))^2}{4CQ(s)} \right] \Delta s = \infty, \tag{2.8}$$

where $Q(t) = r(t)[a_1(t)\delta(t,t_1)]^{\frac{1}{\alpha_1}}$, $\delta(t,t_1) = \int_{t_1}^{t}[a_2(s)]^{\frac{1}{\alpha_2}}\Delta s$, *then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists (finite).*

The next example illustrates the application of Corollary 2.4.

Example 2.2. Consider the following equation

$$\left(\left(\left(\frac{1}{a_1(t)}\left(x^{\Delta}(t)\right)^{\frac{1}{k}}\right)^{\Delta}\right)^{k}\right)^{\Delta} + q(t)x(t)\left(1 + x^6(t)\right) = 0, \tag{2.9}$$

where $t \in \mathbb{T}$, $\mathbb{T} = \mathbb{R}$ or $h\mathbb{N}$ with $h > 0$, k is any positive odd, $a_1(t), q(t) \in C_{rd}(\mathbb{T},\mathbb{R})$ with $a_1(t) \geq 1$ and $q(t) \geq \frac{1}{t}$.

Taking $\alpha_1 = \frac{1}{k}$, $\alpha_2 = k$, $a_2(t) = 1$, $f(x) = x(1+x^6)$, then $f'(x) = 1 + 7x^6 \geq C = 1$, and for sufficiently large $t_1 < t_2$,

$$\delta(t,t_1) = \int_{t_1}^{t}[a_2(s)]^{\frac{1}{\alpha_2}}\Delta s = \int_{t_1}^{t}\Delta s = t - t_1,$$

$$(a_1(t)\delta(t,t_1))^{-\frac{1}{\alpha_1}} \leq (t-t_1)^{-k},$$

and

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[sq(s) - \frac{1}{4Cs}\left(a_1(s)\delta(s,t_1)\right)^{-\frac{1}{\alpha_1}} \right] \Delta s \geq \limsup_{t\to\infty} \int_{t_2}^{t} \left[1 - \frac{1}{4s(s-t_1)^k} \right] \Delta s = \infty.$$

We see that all conditions of Corollary 2.4 are satisfied and then every solution $x(t)$ of (2.9) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Remark 2.3. The conditions imposed on function $f(x)$ in Theorems 2.3 and 2.4 are stronger than those in Theorems 2.1 and 2.2. However, when $f'(x) \geq C > 0$, the restriction "for all $M > 0$" is dropped, and the corresponding conditions (2.7) and (2.8) seem to be simpler and weaker than (2.2) and (2.5), respectively.

3. Basic lemmas and proofs

First, we state and prove some lemmas which we will need in the proofs of our main results.

Lemma 3.1. *Assume* (A1)–(A3) *and* (2.1) *hold,* $x(t)$ *is an eventually positive solution of* (1.1)*. Then there exists a* $t_1 \in [t_0,\infty)$ *such that either:*

(1) $x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) > 0$, $t \in [t_1,\infty)$;

or

(2) $x(t) > 0$, $L_1x(t) < 0$, $L_2x(t) > 0$, $t \in [t_1,\infty)$.

Proof. Let $x(t)$ be an eventually positive solution of (1.1), then there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$ for $t \in [t_1, \infty)$. Since $f(x(t)) > 0$ for $t \in [t_1, \infty)$, from (1.1) we have

$$L_3x(t) = -q(t)f(x(t)) < 0, \quad t \in [t_1, \infty),$$

which implies that $L_2x(t)$ is strictly decreasing on $[t_1, \infty)$.

We claim that $L_2x(t) > 0$. Otherwise, there exists a $t_2 \in [t_1, \infty)$ such that

$$L_2x(t) \le L_2x(t_2) < 0, \quad t \in [t_2, \infty),$$

that is,

$$\frac{1}{a_2(t)}\left((L_1x(t))^{\Delta}\right)^{\alpha_2} \le L_2x(t_2) < 0, \quad t \in [t_2, \infty).$$

Hence we have

$$(L_1x(t))^{\Delta} \le (a_2(t)L_2x(t_2))^{\frac{1}{\alpha_2}}, \quad t \in [t_2, \infty), \tag{3.1}$$

which implies that $L_1x(t)$ is strictly decreasing on $[t_2, \infty)$. Integrating (3.1) from t_2 to t $(\ge t_2)$, we obtain

$$L_1x(t) \le L_1x(t_2) + (L_2x(t_2))^{\frac{1}{\alpha_2}} \int_{t_2}^{t} [a_2(s)]^{\frac{1}{\alpha_2}} \Delta s.$$

Letting $t \to \infty$, from (2.1) we have $L_1x(t) \to -\infty$. Thus, there exists $t_3 \in [t_2, \infty)$ such that

$$L_1x(t) \le L_1x(t_3) < 0, \quad t \in [t_3, \infty),$$

that is,

$$\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1} \le L_1x(t_3) < 0, \quad t \in [t_3, \infty).$$

It follows that

$$x^{\Delta}(t) \le (a_1(t)L_1x(t_3))^{\frac{1}{\alpha_1}}, \quad t \in [t_3, \infty).$$

Integrating from t_3 to $t(\ge t_3)$, we have

$$x(t) \le x(t_3) + (L_1x(t_3))^{\frac{1}{\alpha_1}} \int_{t_3}^{t} [a_1(s)]^{\frac{1}{\alpha_1}} \Delta s.$$

Letting $t \to \infty$, from (2.1) we have $x(t) \to -\infty$, which is a contradiction with the fact that $x(t) > 0$. Hence $L_2x(t) > 0$, $t \in [t_1, \infty)$. This implies that $L_1x(t)$ is strictly increasing on $[t_1, \infty)$. It follows that either $L_1x(t) > 0$ or $L_1x(t) < 0$ and the proof is complete. □

Lemma 3.2. *Assume* (A1)–(A3) *and* (2.3) *hold. If* $x(t)$ *is a solution of* (1.1) *that satisfies Case* (2) *in Lemma* 3.1, *then*

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Let $x(t)$ be a solution of (1.1) satisfying Case (2) in Lemma 3.1, that is, there exists $t_1 \in [t_0, \infty)$ such that

$$x(t) > 0, \qquad L_1x(t) < 0, \qquad L_2x(t) > 0, \quad t \ge t_1.$$

Then from $L_1x(t) < 0$, we see that $\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1} < 0$ for $t \ge t_1$. So, $x^{\Delta}(t) < 0$ for $t \ge t_1$ and $\lim_{t\to\infty} x(t) = b \ge 0$.

We claim that $b = 0$. Assume not, $b > 0$, then $x(t) \ge b$ for $t \ge t_1$. With $k = b$, then from (A2) there exists $M = M_b > 0$ such that

$$L_3x(t) = -q(t)f(x(t)) < -Mq(t)x(t) \le -Mbq(t), \quad t \ge t_1.$$

Letting $u(t) := L_2x(t) > 0$, $t \ge t_1$, then

$$u^{\Delta}(t) = L_3x(t) < -Mbq(t), \quad t \ge t_1.$$

Integrating from t_1 to t $(\ge t_1)$, we have

$$u(t) \le u(t_1) - bM \int_{t_1}^{t} q(s)\Delta s.$$

From (2.3), there exists a sufficiently large $t_2 \in [t_1, \infty)$ such that for all $t \in [t_2, \infty)$,

$$u(t) < 0,$$

which is a contradiction with the fact that $u(t) > 0$, $t \ge t_1$.

Therefore, $b = 0$, that is, $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Lemma 3.3. *Assume* (A1)–(A3) *hold. If $x(t)$ is a solution of* (1.1) *satisfying Case* (1) *of Lemma* 3.1*, then there exists $t_1 \in [t_0, \infty)$ such that*

$$L_1x(t) \geq \delta(t, t_1)\,(L_2x(t))^{\frac{1}{\alpha_2}} \quad \text{or} \quad x^{\Delta}(t) \geq (a_1(t)\delta(t, t_1))^{\frac{1}{\alpha_1}}\,(L_2x(t))^{\frac{1}{\alpha_1\alpha_2}},$$

and $\frac{L_1x(t)}{\delta(t,t_1)}$ is decreasing on (t_1, ∞), where $\delta(t, t_1) = \int_{t_1}^{t}[a_2(s)]^{\frac{1}{\alpha_2}}\Delta s$.

Proof. Let $x(t)$ be a solution of (1.1) satisfying Case (1) in Lemma 3.1, that is, there exists $t_1 \in [t_0, \infty)$ such that

$$x(t) > 0, \qquad L_1x(t) > 0, \qquad L_2x(t) > 0, \quad t \geq t_1.$$

Then from (1.1) we have $L_3x(t) < 0$ for $t \in [t_1, \infty)$, so $L_2x(t)$ is strictly decreasing on $[t_1, \infty)$. From $L_2x(t) = \frac{1}{a_2(t)}((L_1x(t))^{\Delta})^{\alpha_2}$, we obtain

$$(L_1x(t))^{\Delta} = (a_2(t)L_2x(t))^{\frac{1}{\alpha_2}}.$$

Then for $t \geq t_1$, we have

$$\begin{aligned}\int_{t_1}^{t}(L_1x(s))^{\Delta}\Delta s = L_1x(t) - L_1x(t_1) &= \int_{t_1}^{t}(a_2(s)L_2x(s))^{\frac{1}{\alpha_2}}\Delta s\\ &\geq (L_2x(t))^{\frac{1}{\alpha_2}}\int_{t_1}^{t}[a_2(s)]^{\frac{1}{\alpha_2}}\Delta s.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1} &= L_1x(t) \geq L_1x(t_1) + \delta(t, t_1)(L_2x(t))^{\frac{1}{\alpha_2}}\\ &\geq \delta(t, t_1)(L_2x(t))^{\frac{1}{\alpha_2}}, \quad t \geq t_1,\end{aligned} \tag{3.2}$$

that is,

$$x^{\Delta}(t) \geq (a_1(t)\delta(t, t_1))^{\frac{1}{\alpha_1}}\,(L_2x(t))^{\frac{1}{\alpha_1\alpha_2}}, \quad t \geq t_1.$$

We claim that $\frac{L_1x(t)}{\delta(t,t_1)}$ is decreasing on (t_1, ∞). In fact for $t > t_1$, from (3.2) we obtain

$$\begin{aligned}\left[\frac{L_1x(t)}{\delta(t, t_1)}\right]^{\Delta} &= \frac{(L_1x(t))^{\Delta}\delta(t, t_1) - L_1x(t)(\delta(t, t_1))^{\Delta}}{\delta(t, t_1)\delta(\sigma(t), t_1)}\\ &= \frac{(a_2(t))^{\frac{1}{\alpha_2}}(L_2x(t))^{\frac{1}{\alpha_2}}\delta(t, t_1) - L_1x(t)(a_2(t))^{\frac{1}{\alpha_2}}}{\delta(t, t_1)\delta(\sigma(t), t_1)}\\ &\leq \frac{(a_2(t))^{\frac{1}{\alpha_2}}L_1x(t) - L_1x(t)(a_2(t))^{\frac{1}{\alpha_2}}}{\delta(t, t_1)\delta(\sigma(t), t_1)} = 0.\end{aligned}$$

Hence, $\frac{L_1x(t)}{\delta(t,t_1)}$ is decreasing on (t_1, ∞). The proof is complete. □

Now, we are in a position to prove our main results.

Proof of Theorem 2.1. Let $x(t)$ be a nonoscillatory solution of (1.1). We only consider the case when $x(t)$ is eventually positive, since the case when $x(t)$ is eventually negative is similar. Since (2.1) holds, by Lemma 3.1 we see that $x(t)$ satisfies either Case (1) or Case (2).

We claim that Case (1) of Lemma 3.1 is not true. Assume not, then there exists $t_1 \in [t_0, \infty)$, such that $x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) > 0$ for $t \geq t_1$. Define the "Riccati"-type function

$$w(t) = \frac{r(t)L_2x(t)}{x(t)}, \quad t \in [t_1, \infty), \tag{3.3}$$

then $w(t) > 0$, $t \in [t_1, \infty)$. From (1.1) we obtain

$$\begin{aligned}w^{\Delta}(t) &= \left(\frac{r(t)}{x(t)}\right)^{\Delta}L_2x(\sigma(t)) + \frac{r(t)}{x(t)}(L_2x(t))^{\Delta}\\ &= \frac{r^{\Delta}(t)x(t) - r(t)x^{\Delta}(t)}{x(t)x(\sigma(t))}L_2x(\sigma(t)) - \frac{r(t)q(t)f(x(t))}{x(t)}.\end{aligned}$$

With $k = x(t_1) > 0$, from (A2) there exists $M = M_k > 0$ such that

$$\frac{f(x)}{x} \geq M, \quad |x| \geq k.$$

Noting that $\alpha_1\alpha_2 = 1$, from Lemma 3.3 we have

$$\begin{aligned} w^{\Delta}(t) &\leq -Mr(t)q(t) + w(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} - \frac{r(t)x^{\Delta}(t)}{x(t)x(\sigma(t))}L_2x(\sigma(t)) \\ &\leq -Mr(t)q(t) + w(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} - \frac{Q(t)L_2x(t)}{x(t)x(\sigma(t))}L_2x(\sigma(t)). \end{aligned}$$

Since $L_1x(t) > 0$, we have $x^{\Delta}(t) > 0$, that is, $x(t)$ is increasing. So $x(t) \leq x(\sigma(t))$ for $t \geq t_1$. Since $(L_2x(t))^{\Delta} = -q(t)f(x(t)) < 0$, we see that $L_2x(t)$ is decreasing and so $L_2x(t) \geq L_2x(\sigma(t))$ for $t \geq t_1$. It follows that

$$\begin{aligned} w^{\Delta}(t) &\leq -Mr(t)q(t) + w(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} - \frac{Q(t)}{x^2(\sigma(t))}L_2^2x(\sigma(t)) \\ &= -Mr(t)q(t) + w(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} - \frac{Q(t)}{r^2(\sigma(t))}w^2(\sigma(t)) \\ &= -Mr(t)q(t) - \left[\frac{w(\sigma(t))\sqrt{Q(t)}}{r(\sigma(t))} - \frac{r^{\Delta}(t)}{2\sqrt{Q(t)}}\right]^2 + \frac{(r^{\Delta}(t))^2}{4Q(t)} \\ &\leq -Mr(t)q(t) + \frac{(r^{\Delta}(t))^2}{4Q(t)}, \end{aligned}$$

that is,

$$w^{\Delta}(t) \leq -\left(Mr(t)q(t) - \frac{(r^{\Delta}(t))^2}{4Q(t)}\right). \tag{3.4}$$

Integrating (3.4) from t_2 to $t(\geq t_2)$, we have

$$-w(t_2) \leq w(t) - w(t_2) \leq -\int_{t_2}^{t}\left[Mr(s)q(s) - \frac{(r^{\Delta}(s))^2}{4Q(s)}\right]\Delta s,$$

that is,

$$\int_{t_2}^{t}\left[Mr(s)q(s) - \frac{(r^{\Delta}(s))^2}{4Q(s)}\right]\Delta s \leq w(t_1),$$

which is a contradiction with (2.2). Hence, Case (1) of Lemma 3.1 is not true. If Case (2) of Lemma 3.1 holds, then clearly $\lim_{t\to\infty} x(t)$ exists (finite). The proof is complete. □

Proof of Corollary 2.1. Taking $r(t) = 1$ in Theorem 2.1, by the proof of Theorem 2.1 we have that every solution $x(t)$ of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite). For the last case, by Lemma 3.2 we obtain $\lim_{t\to\infty} x(t) = 0$. □

Proof of Theorem 2.2. Proceeding as in the proof of Theorem 2.1, we assume that (1.1) has a nonoscillatory solution, say $x(t) > 0$ for all $t \geq t_1$ where t_1 is chosen so large that Lemmas 3.1 and 3.3 hold. By Lemma 3.1 there are two possible cases. First, if Case (1) holds, then by defining again $w(t)$ by (3.3) as in the proof of Theorem 2.1 we have that $w(t) > 0$ and (3.4) holds.

Multiplying (3.4) by $(t-s)^m$ (with t replaced by s) and then integrating from t_2 $(> t_1)$ to t $(\geq t_2)$, we have

$$\int_{t_2}^{t}(t-s)^m\left[Mr(s)q(s) - \frac{(r^{\Delta}(s))^2}{4Q(s)}\right]\Delta s \leq -\int_{t_2}^{t}(t-s)^m w^{\Delta}(s)\Delta s. \tag{3.5}$$

An integration by parts of the right-hand side leads to

$$\begin{aligned} \int_{t_2}^{t}(t-s)^m w^{\Delta}(s)\Delta s &= (t-s)^m w(s)\Big|_{s=t_2}^{s=t} - \int_{t_2}^{t} h(t,s)w(\sigma(s))\Delta s \\ &= -(t-t_2)^m w(t_2) - \int_{t_2}^{t} h(t,s)w(\sigma(s))\Delta s, \end{aligned}$$

where $h(t,s) := ((t-s)^m)^{\Delta_s}$. Since

$$h(t,s) = \begin{cases} -m(t-s)^{m-1}, & \text{if } \mu(s)=0,\\ \dfrac{(t-\sigma(s))^m-(t-s)^m}{\mu(s)}, & \text{if } \mu(s)>0, \end{cases}$$

and when $m \ge 1$, $h(t,s) \le 0$ for $t \ge \sigma(s)$, it follows that

$$\int_{t_2}^{t}(t-s)^m w^{\Delta}(s)\Delta s \ge -(t-t_2)^m w(t_2).$$

From (3.5) we have

$$\int_{t_2}^{t}(t-s)^m\left[Mr(s)q(s)-\frac{(r^{\Delta}(s))^2}{4Q(s)}\right]\Delta s \le (t-t_2)^m w(t_2),$$

or

$$\frac{1}{t^m}\int_{t_2}^{t}(t-s)^m\left[Mr(s)q(s)-\frac{(r^{\Delta}(s))^2}{4Q(s)}\right]\Delta s \le \left(\frac{t-t_2}{t}\right)^m w(t_2) \le w(t_2),$$

a contradiction with (2.5). Thus, Case (1) in Lemma 3.1 is not true.

If Case (2) in Lemma 3.1 holds, then as before, $\lim_{t\to\infty} x(t)$ exists (finite). The proof is complete. □

Proof of Theorem 2.3. Proceeding as in the proof of Theorem 2.1, we assume that (1.1) has a nonoscillatory solution, say $x(t) > 0$ for all $t \ge t_1$ where t_1 is chosen so large that Lemmas 3.1 and 3.3 hold. By Lemma 3.1 there are two possible cases. We will claim that Case (1) is not true.

Otherwise, if Case (1) holds, then there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) > 0$ for $t \ge t_1$. Define the "Riccati"-type function

$$v(t) = -\frac{r(t)L_2x(t)}{f(x(t))}, \quad t\in[t_1,\infty),$$

then $v(t) < 0$ for $t \in [t_1,\infty)$. From (1.1), Lemma 3.3 with $\alpha_1\alpha_2 = 1$ and [4, Theorem 1.90], we have

$$\begin{aligned} v^{\Delta}(t) &= -\left[\frac{r(t)}{f(x(t))}\right]^{\Delta} L_2x(\sigma(t)) - \frac{r(t)}{f(x(t))}(L_2x(t))^{\Delta}\\ &= \frac{r(t)}{f(x(t))}q(t)f(x(t)) - L_2x(\sigma(t))\frac{r^{\Delta}(t)f(x(t)) - r(t)(f\circ x)^{\Delta}(t)}{f(x(t))f(x(\sigma(t)))}\\ &= r(t)q(t) - L_2x(\sigma(t))\frac{r^{\Delta}(t)f(x(t)) - r(t)\{\int_0^1 f'(x(t)+h\mu(t)x^{\Delta}(t))\mathrm{d}h\}x^{\Delta}(t)}{f(x(t))f(x(\sigma(t)))}\\ &= r(t)q(t) - L_2x(\sigma(t))\frac{r^{\Delta}(t)}{f(x(\sigma(t)))} + L_2x(\sigma(t))\frac{r(t)\{\int_0^1 f'(x(t)+h\mu(t)x^{\Delta}(t))\mathrm{d}h\}x^{\Delta}(t)}{f(x(t))f(x(\sigma(t)))}\\ &\ge r(t)q(t) + v(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} + L_2x(\sigma(t))\frac{Cr(t)x^{\Delta}(t)}{f(x(t))f(x(\sigma(t)))}\\ &\ge r(t)q(t) + v(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} + L_2x(\sigma(t))\frac{CQ(t)L_2x(t)}{f(x(t))f(x(\sigma(t)))}. \end{aligned} \tag{3.6}$$

As in the proof of Theorem 2.1, we see that $x(t)$ is increasing and $L_2x(t)$ is decreasing on $[t_1,\infty)$. Also, $f'(x) > 0$ implies that $f(x)$ is increasing. It follows from (3.6) that

$$\begin{aligned} v^{\Delta}(t) &\ge r(t)q(t) + v(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} + L_2x(\sigma(t))\frac{CQ(t)L_2x(\sigma(t))}{(f(x(\sigma(t))))^2}\\ &\ge r(t)q(t) + v(\sigma(t))\frac{r^{\Delta}(t)}{r(\sigma(t))} + v^2(\sigma(t))\frac{CQ(t)}{r^2(\sigma(t))}\\ &= r(t)q(t) + \left[\frac{v(\sigma(t))\sqrt{Q(t)C}}{r(\sigma(t))} + \frac{r^{\Delta}(t)}{2\sqrt{Q(t)C}}\right]^2 - \frac{(r^{\Delta}(t))^2}{4CQ(t)}\\ &\ge r(t)q(t) - \frac{(r^{\Delta}(t))^2}{4CQ(t)}, \end{aligned}$$

that is

$$v^{\Delta}(t) \geq r(t)q(t) - \frac{(r^{\Delta}(t))^2}{4CQ(t)}. \tag{3.7}$$

Integrating (3.7) from t_2 ($> t_1$) to t ($\geq t_2$) we have

$$\int_{t_2}^{t} \left[r(s)q(s) - \frac{(r^{\Delta}(s))^2}{4CQ(s)} \right] \Delta s \leq v(t) - v(t_2) \leq -v(t_2),$$

a contradiction with (2.7). Thus, Case (1) in Lemma 3.1 is not true.

If Case (2) in Lemma 3.1 holds, then as before, $\lim_{t\to\infty} x(t)$ exists (finite). The proof is complete. □

Using (3.7), similarly to the proof of Theorem 2.2 we can prove Theorem 2.4 and hence omit its proof.

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Appendix. Preliminaries on time scales

In this section, we present the basic concepts and results on time scales related to our work, which are contained in [4].

Definition A.1. A time scale is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers with the topology and ordering inherited from $\mathbb{R}$. Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$ the forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, the backward jump operator by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, and the graininess function by $\mu(t) := \sigma(t) - t$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered; otherwise, it is left-dense. The set $\mathbb{T}^{\kappa}$ is defined as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition A.2. For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, we define the delta-derivative $f^{\Delta}(t)$ of $f(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some δ) such that

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s| \qquad \text{for all } s \in U.$$

We say that f is delta-differentiable (or in short: differentiable) on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

It is easily seen that if f is continuous at $t \in \mathbb{T}$ and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Moreover, if t is right-dense then f is differential at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

In addition, if $f^{\Delta} \geq 0$, then f is nondecreasing. A useful formula is

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t), \quad \text{where } f^{\sigma}(t) := f(\sigma(t)).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g g^{\sigma} \neq 0$) of two differentiable functions f and g:

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma},$$

$$\left(\frac{f}{g} \right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}.$$

Definition A.3. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, f is called right-dense continuous (rd-continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of f provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{k}$. By the antiderivative, the Cauchy integral of f is defined as $\int_a^b f(s)\Delta s = F(b) - F(a)$, and $\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s$.

Let $C_{rd}(\mathbb{T}, \mathbb{R})$ denote the set of all rd-continuous functions mapping $\mathbb{T}$ to $\mathbb{R}$. It is shown in [4] that every rd-continuous function has an antiderivative.

An integration by parts formula is

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]|_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t.$$

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