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Applications of Frobenius Algebras to Representation Theory of Schur Algebras

L. Delvaux and E. Nauwelaerts

Department of Mathematics, Limburgs Universitair Centrum,

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A Schur algebra is a subalgebra of the group algebra RG associated to a partition of G, where G is a finite group and R is a commutative ring. For two classes of Schur algebras we study the relationship between indecomposable modules over the Schur algebra and over RG, but we discuss this problem in a more general context. Further we develop a character theory for Schur algebras; in particular, we express primitive central idempotents in terms of trace functions and we derive orthogonality relations for trace functions. These results are also presented in a more general context, namely for Frobenius algebras over rings. Moreover, we focus on class functions on Schur algebras.

INTRODUCTION

A Schur algebra is a subalgebra of the group algebra RG associated to a partition of G, where G is a finite group and R is a commutative ring, see Definition 1.1. Schur algebras over $\mathbb{C}$ were introduced by Schur and Wielandt [14], and were first studied by Tamaschke [13] and Roesler [11].

In Section 2 we study indecomposable modules for two important classes of Schur algebras: double coset algebras (and their generalizations, see Proposition 2.7) and fixed rings of certain automorphism groups. Double coset algebras are studied in the more general situation of Hecke algebras: if A is an R-algebra and ε a nonzero idempotent of A, then $\varepsilon A \varepsilon$ is called a Hecke in A. We investigate the relationship between indecomposable modules over $\varepsilon A \varepsilon$ and indecomposable modules over A, and we determine the primitive central idempotents of $\varepsilon A \varepsilon$ in terms of the idempotents of A, see Proposition 2.4 and Theorem 2.5. In particular, we prove that a connected ring R is a splitting ring for $\varepsilon A \varepsilon$ whenever it is a splitting ring for A.

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Furthermore, we study the relationship between indecomposable modules over an R-algebra A and indecomposable modules over the fixed ring A^H, which is associated to a group homomorphism $\sigma: H \to \mathrm{Aut}_R(A)$, see Theorem 2.15 and Corollary 2.16. These results can be applied to the situation where A is the group algebra RG and σ is derived from a group homomorphism $H \to \mathrm{Aut}(G)$, and in this case A^H is a Schur algebra. In the special case where H is a subgroup of G and σ_h, $h \in H$, is an inner automorphism, the fixed ring coincides with the centralizer of RH in RG, and in [10, 3] we obtained more relations between indecomposable modules over RH, RG, and RG^H.

In Section 3 we develop a character theory for Schur algebras. We show that Schur algebras are Frobenius algebras (under a suitable condition). Therefore we set up this problem in the more general context of Frobenius algebras over commutative rings. First we collect some generalities on Frobenius algebras, including a criterion for separability. We then study symmetric functions on Frobenius R-algebras and we show that, under certain conditions, they are generated over R by trace functions. We express primitive central idempotents of a Frobenius algebra A in terms of trace functions and we derive orthogonality relations for trace functions on A, see Theorem 3.10 and Proposition 3.12.

In the case of Schur algebras we introduce class functions and we investigate when the set of class functions coincides with the set of symmetric functions. In fact, this latter study yields an analysis of the center of a Schur algebra. To conclude we calculate the trace function of induced modules between two Schur algebras (Section 5).

Throughout this paper rings are assumed to have a unit element and modules are unitary:

1. PRELIMINARIES

Throughout R is a commutative ring. A ring is said to be connected if 0 and 1 are the only idempotent elements. We begin with some useful facts about indecomposable modules.

Let A be an R-algebra and suppose that R is connected. We first remark that a left A-module, which is finitely generated and projective over R, is a finite direct sum of indecomposable left A-modules (use rank_R). Now assume that A is finitely generated and projective as an R-module. Then there exist primitive central orthogonal nonzero idempotents $e_1, \ldots, e_q$ in A such that $1 = e_1 + \cdots + e_q$ (use rank_R). Moreover, each central nonzero idempotent of A is uniquely a sum of some e_i. If M is an indecomposable left A-module, then there is a unique i such that $e_i M \neq 0$ and we say that M lies over e_i. Further, if any two indecomposable finitely generated projective left A-modules lying over the same e_i are isomorphic as A-modules, then it is easily seen that any finitely generated projective left A-module is uniquely expressible as a finite direct sum of indecomposable left A-modules (up to isomorphism). The same remark holds if we replace projectivity over A by projectivity over R.

In addition, suppose that A is a separable R-algebra. Then a left A-module is projective as an R-module if and only if it is projective as an A-module, cf. [5, p. 48]. Furthermore, if R is semilocal, then any two indecomposable finitely generated projective left A-modules lying over the same primitive central idempotent of A are isomorphic as A-modules, see [4, Theorem 1; 10, Note 3.4]. Note also that a separable R-algebra, where R is a field, is semisimple.

We now assume that R is a splitting ring for A (or A is split separable over R); that is, $A \cong \mathrm{End}_R(M_1) \oplus \cdots \oplus \mathrm{End}_R(M_q)$ as R-algebras, $M_1, \ldots, M_q$ being finitely generated projective faithful R-modules. Recall that finitely generated projective nonzero modules over connected commutative rings are always faithful, see [5, p. 8]. Note also that the center of A is a free R-module of rank q. Obviously M_i can be viewed as a left A-module by setting $(\varphi_1, \ldots, \varphi_q).m = \varphi_i(m)$, where $m \in M_i$ and $\varphi_j \in \mathrm{End}_R(M_j)$. Since R is connected, each M_i is an indecomposable left A-module, and they are not isomorphic as such. Now assume that M_i lies over the primitive central idempotent e_i of A. If finitely generated projective R-modules are free, for example, when R is semilocal or a principal ideal domain, then M_i is, up to isomorphism, the only indecomposable finitely generated projective left A-module lying over e_i (see [9, 18]). Note also that any semisimple $\mathbb{C}$-algebra is split separable over $\mathbb{C}$.

Next, we recall some basic facts about trace functions. Let A be an R-algebra and V a left A-module which is finitely generated and projective over R. Let $\{v_1, \ldots, v_n\} \subset V$, $\{\varphi_1, \ldots, \varphi_n\} \subset \mathrm{Hom}_R(V, R)$ be an R-dual basis for V. The trace function (or character) from A to R afforded by V, notation t_V, is defined as $t_V(a) = \sum_{i=1}^{n} \varphi_i(av_i)$, for all $a \in A$. It is easily seen that t_V does not depend on the choice of the dual basis. Further, $t_V(xy) = t_V(yx)$ for all $x, y \in A$, and if R is connected, then $t_V(1) = \mathrm{rank}_R(V)1_R$; see [9, 2.5].

To conclude this subsection, let us focus on group rings. Let G be a finite group and consider the group ring RG. As R-module, RG will be freely generated by symbols $\{u_g;\ g \in G\}$. Recall that in case $|G|^{-1} \in R$, RG is separable over R. Further, suppose R is connected and $|G|^{-1} \in R$. Let m be the exponent of G and let η be a primitive mth root of unity. Then $L = R[\eta]$ is a splitting ring for LG, see [12].

We now turn to Schur algebras in RG.

1.1. DEFINITION. Let G be a finite group and let $\{E_g;\ g \in G\}$ $(g \in E_g)$ be a partition of G such that $E_g^{-1} = E_{g^{-1}}$. Denote by G_0 a set of representatives of the distinct E_g. Now let R be a commutative ring and put $s_g = \sum_{x \in E_g} u_x$ in RG. If $S = \oplus_{g \in G_0} Rs_g$ is a subalgebra of RG with unit element 1_S, then S is said to be a *Schur algebra* in RG.

1.2. *Remarks*. (1) Keep the notation of Definition 1.1 and suppose that $S = \oplus_{g \in G_0} Rs_g$ is a subalgebra of RG with unit element. Then the following statement need not hold:

$$\forall g, h \in G, \qquad E_g E_h = \bigcup_k E_k \text{ for some } k \in G. \tag{$*$}$$

However, if $\operatorname{char}(R) = 0$, then property $(*)$ follows from the ring structure of S. We also remark that property $(*)$ holds for the Schur algebras considered in Section 2.

(2) An example of a Schur algebra for which property $(*)$ does not hold is given in [1]. Namely, take $R = \mathbb{Z}_2$, $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and consider the partition $\{(0,0),(0,1),(0,2),(1,0),(2,0)\}$, $\{(1,1),(2,2),(1,2),(2,1)\}$. Here, $s_{(0,0)}$ is the unit element.

Of course, if $E_e = \{e\}$, then $s_e = 1_S$. Furthermore:

1.3. LEMMA. *Let R, G, E_g, s_g be as in Definition* 1.1.

(1) *Suppose for all $g, h \in G$ we have $E_g E_h = \bigcup_k E_k$ (some $k \in G$). Then E_e is a subgroup of G and $s_e s_g = s_g s_e = |E_e| s_g$ for all $g \in G$.*

(2) *Suppose that $S = \oplus_{g \in G_0} Rs_g$ is a subalgebra of RG with unit element 1_S. Then $|E_e|$ is invertible in R. Moreover, if $|E_g| 1_R \neq 0$ and $|E_g| 1_R$ is not a zero divisor in R for each $g \in G$, then $1_S = |E_e|^{-1} s_e$.*

Proof. (1) We shall prove that $xE_g \subset E_g$ for all $x \in E_e$. But then equality must hold, because $|xE_g| = |E_g|$. Analogously $E_g x = E_g$, and the assertions follow. Now take $y \in E_g$ and put $h = xy$. Then $E_h E_{g^{-1}} \cap E_e \neq \varnothing$, and thus by our hypothesis $E_e \subset E_h E_{g^{-1}}$. Therefore $E_h = E_g$.

(2) Write $1_S = \sum_{g \in G_0} r_g s_g$ with $r_g \in R$, and let $e \in G_0$. Then $s_t = \sum_{g \in G_0} r_g s_g s_t$ for all $t \in G$. Comparing coefficients of u_e, we obtain $1 = |E_e| r_e$ and $0 = |E_g| r_g$ for all $g \in G_0 \setminus \{e\}$. The result now follows. ∎

We also mention the following elementary fact.

1.4. LEMMA. *Let R, G be as above.*

(1) *The map θ: $RG \to RG$: $\sum_{g \in G} r_g u_g \mapsto \sum_{g \in G} r_g u_{g^{-1}}$ is an anti-isomorphism and $\theta \circ \theta = I$.*

(2) *If S is a Schur algebra in RG, then $\theta(S) = S$.*

Again, R is a commutative ring and G is a finite group. We may consider the following componentwise multiplication on RG. Let $a, a' \in RG$, $a = \sum_{g \in G} r_g u_g$, and $a' = \sum_{g \in G} r'_g u_g$ with $r_g, r'_g \in R$. Then we define $a * a' = \sum_{g \in G} r_g r'_g u_g$. Note that $RG, *$ is a commutative R-algebra with $\sum_{g \in G} u_g$ as unit element. Evidently, every Schur algebra in RG is closed under this multiplication and contains $\sum_{g \in G} u_g$. On the other hand, we have:

1.5. PROPOSITION. *Suppose R is a field.*

(1) *Let S be an R-submodule of RG. If S is closed under the multiplication $*$ and $\sum_{g \in G} u_g \in S$, then there is a partition $\{E_g;\ g \in G\}$ of G such that $S = \oplus_{g \in G_0} Rs_g$, where $s_g = \sum_{x \in E_g} u_x$ and G_0 denotes a set of representatives of the distinct E_g.*

(2) *Let S be an R-subalgebra of RG with unit element. If S satisfies the conditions in* (1) *and $\theta(S) \subset S$, then S is a Schur algebra in RG.*

Proof. (1) We consider the R-algebra $S, *$. There exist orthogonal primitive nonzero idempotents in $S, *$, say $e_1, \ldots, e_m$, such that $\sum_{g \in G} u_g = e_1 + \cdots + e_m$. Clearly, $\{u_g;\ g \in G\}$ is the set of primitive idempotents of $RG, *$ and thus we have $e_1 = u_{g_1} + \cdots + u_{g_t}$ and so on. By the above remarks we obtain a partition of G, namely $E_{g_1} = \{g_1, \ldots, g_t\}$, etc.

Next, the multiplication $*$ makes Ru_g into a left S-module. Since $\dim_R(Ru_g) = 1$, Ru_g is a simple S-module. So RG is a semisimple left S-module and thus $S, *$ is a semisimple ring. But then $S * s_g \cong Ru_g$ as S-modules $(s_{g_1} = e_1)$. Consequently $\dim_R(S * s_g) = 1$, and thus $Rs_g \subset S * s_g$ must be an equality.

(2) Let θ be as above. Clearly, θ: $RG, * \to RG, *$ is an isomorphism of R-algebras and $\theta \circ \theta = I$. Since $\theta(S) \subset S$, it follows that $\theta(s_g) = s_{g^{-1}}$ is a primitive idempotent of $S, *$. This proves our assertion. ∎

1.6. *Note.* Let S be a Schur algebra in RG with associated partition $\{E_g;\ g \in G\}$. Assume R is connected, $|G|^{-1} \in R$, and consider the idempotent $\varepsilon = |G|^{-1} \sum_{g \in G} u_g$. Clearly $\varepsilon \in S$ and $s_g \varepsilon = |E_g| \varepsilon = \varepsilon s_g$, with $s_g = \sum_{x \in E_g} u_x$. Now $S\varepsilon = R\varepsilon$ is an indecomposable left S-module, and thus ε is a primitive idempotent of S. Moreover ε is an element of the center of S. Furthermore, $t_{S\varepsilon}(s_g) = |E_g| 1_R$.

2. INDECOMPOSABLE MODULES OVER SCHUR ALGEBRAS

Throughout, R is a commutative ring and G is a finite group. Our aim is to investigate the relationship between indecomposable RG-modules and indecomposable modules over certain Schur algebras.

We begin with some results about double coset algebras. Let H be a subgroup of G. Suppose that $|H|^{-1} \in R$ and consider the idempotent $\varepsilon = |H|^{-1} \sum_{h \in H} u_h$ in RG. Then $\varepsilon RG \varepsilon$ is a Schur algebra, called a double coset algebra. Indeed, $H \times H$ acts on G as follows: $((h,k), g) \mapsto hgk^{-1}, h, k \in H, g \in G,$ and $(HgH)^{-1} = Hg^{-1}H$. Furthermore, $|HgH|$ is invertible in R and $\sum_{x \in HgH} u_x = |HgH| \varepsilon u_g \varepsilon$.

We shall discuss relations between RG and $\varepsilon RG \varepsilon$ in a more general context. Until further notice, A denotes an R-algebra and ε a nonzero idempotent of A. Note that $(\mathrm{End}_A(A\varepsilon))^{\mathrm{o}} \to \varepsilon A \varepsilon$: $\psi \mapsto \psi(\varepsilon)$ is an isomorphism of R-algebras. Further, if A is finitely generated and projective as R-module, then so is $\varepsilon A \varepsilon$. From [10, 4.1] we retain:

2.1. PROPOSITION. *Suppose that A is finitely generated and projective as R-module and suppose that $\varepsilon A \varepsilon$ is a faithful R-algebra* (*this follows whenever R is connected*). *If A is separable over R, then so is $\varepsilon A \varepsilon$.*

In particular, if $|G|$ is invertible in R, then every double coset algebra in RG is separable over R. From [2, 6.3] we may deduce:

2.2. PROPOSITION. *Let $\mathscr{P}$ be the category of all finitely generated projective left $\varepsilon A \varepsilon$-modules and let $\mathscr{C} = \mathscr{C}(A\varepsilon)$ be the category of all left A-modules which are isomorphic to A-direct summands of $(A\varepsilon)^m$ for some m. Then the functors $A\varepsilon \otimes_{\varepsilon A \varepsilon} - : \mathscr{P} \to \mathscr{C}$ and $\mathrm{Hom}_A(A\varepsilon, -)$: $\mathscr{C} \to \mathscr{P}$, denoted by F_1 resp. F_2, define an equivalence of categories between $\mathscr{P}$ and $\mathscr{C}$. Consequently, indecomposable modules in $\mathscr{P}$ correspond to indecomposable modules in $\mathscr{C}$ under F_1 and F_2.*

It is clear that $\mathrm{Hom}_A(A\varepsilon, M)$ is a right $\mathrm{End}_A(A\varepsilon)$-module, hence it is a left $\varepsilon A \varepsilon$-module ($M$ being a left A-module). Moreover, $\mathrm{Hom}_A(A\varepsilon, M) \to \varepsilon M$: $\psi \mapsto \psi(\varepsilon)$ is an isomorphism of left $\varepsilon A \varepsilon$-modules.

Further, if $0 \neq M \in \mathscr{C}(A\varepsilon)$, then M is a finitely generated projective left A-module and $\varepsilon M \neq 0$. We now focus on central idempotents.

2.3. *Remarks.* Suppose that R is connected and that A is finitely generated and projective as R-module. Let $\{e_1, \dots, e_q\}$, resp. $\{d_1, \dots, d_m\}$, be the set of primitive central nonzero idempotents of A, resp. $\varepsilon A \varepsilon$.

(1) Each nonzero εe_i is uniquely expressible as a sum of d_j's and each d_j appears in one and only one of the nonzero εe_i.

(2) Let $P \in \mathscr{P}$ be indecomposable (notation as in Proposition 2.2). Then $A\varepsilon \otimes_{\varepsilon A \varepsilon} P$ lies over e_i if and only if $\varepsilon e_i \neq 0$ and P lies over some d_j appearing in the decomposition of εe_i.

(3) We may write d_j as a sum of orthogonal primitive nonzero idempotents of $\varepsilon A \varepsilon$, say $d_j = \eta_1 + \cdots + \eta_k$ (use rank_R). It is clear that $\varepsilon A \varepsilon \eta_i$ is an indecomposable module in $\mathscr{P}$ lying over d_j.

2.4. PROPOSITION. *Let R, A, and $\{e_1, \dots, e_q\}$ be as in Remarks* 2.3. *Suppose that any two indecomposable finitely generated projective left A-modules lying over the same e_i are isomorphic as A-modules, then*:

(1) *The nonzero εe_i are precisely the distinct primitive central idempotents of $\varepsilon A \varepsilon$.*

(2) *Any two indecomposable finitely generated projective left $\varepsilon A \varepsilon$-modules lying over the same nonzero εe_i are isomorphic as $\varepsilon A \varepsilon$-modules.*

(3) *Let M be an indecomposable finitely generated projective left A-module lying over e_i. Then $\varepsilon M \neq 0$ if and only if $\varepsilon e_i \neq 0$, and this is equivalent to $M \in \mathscr{C}(A\varepsilon)$.*

Proof. The result follows readily from Proposition 2.2 and Remarks 2.3. ∎

2.5. THEOREM. *Let R, A, and $\{e_1, \dots, e_q\}$ be as in Remarks* 2.3. *Suppose $A \cong \mathrm{End}_R(M_1) \oplus \cdots \oplus \mathrm{End}_R(M_q)$ as R-algebra, M_i being finitely generated projective R-modules, and assume that M_i lies over e_i. Then*:

(1) *The nonzero εe_i are the primitive central idempotents of $\varepsilon A \varepsilon$.*

(2) *$\varepsilon M_i \neq 0$ if and only if $\varepsilon e_i \neq 0$, and this is equivalent to $M_i \in \mathscr{C}(A\varepsilon)$.*

(3) *Each nonzero εM_i is an indecomposable left $\varepsilon A \varepsilon$-module and $\varepsilon A \varepsilon \cong \bigoplus_i \mathrm{End}_R(\varepsilon M_i)$ as R-algebras, where the sum is taken over the nonzero εM_i.*

Proof. Recall that each M_i is an indecomposable left A-module under the operation $(\varphi_1, \dots, \varphi_q) \cdot m = \varphi_i(m)$, $m \in M_i$, and $\varphi_j \in \mathrm{End}_R(M_j)$.

It is easily seen that each nonzero εM_i is a finitely generated projective R-module. Further, since A is separable over R, projectivity over R is equivalent to projectivity over A. The same remark holds for $\varepsilon A \varepsilon$.

(a) Obviously $\varepsilon M_i \neq 0$ yields $\varepsilon e_i \neq 0$. Now assume $\varepsilon e_i \neq 0$. Let $\{d_1, \dots, d_m\}$ be as in Remarks 2.3; then εe_i is a sum of d_j's. Consider an indecomposable module $P \in \mathscr{P}$ which lies over some d_j, appearing in the decomposition of εe_i. We know that $Ae_i \cong \mathrm{End}_R(M_i)$, and $A\varepsilon \otimes_{\varepsilon A \varepsilon} P$ is a unitary left Ae_i-module. Therefore there is an R-module L such that $A\varepsilon \otimes_{\varepsilon A \varepsilon} P \cong L \otimes_R M_i$ as left A (or Ae_i)-modules, see, e.g., [5, p. 19]. Then $P \cong L \otimes_R \varepsilon M_i$ as left $\varepsilon A \varepsilon$-modules. Consequently, $\varepsilon M_i \neq 0$ and $d_j \varepsilon M_i \neq 0$.

(b) Assume $\varepsilon M_i \neq 0$. We observe that $\varepsilon M_i \in \mathscr{P}$. Thus $A\varepsilon \otimes_{\varepsilon A \varepsilon} \varepsilon M_i \in \mathscr{C}(A\varepsilon)$ and it is a unitary left Ae_i-module. But then there is a finitely generated projective R-module N such that $A\varepsilon \otimes_{\varepsilon A \varepsilon} \varepsilon M_i \cong N \otimes_R M_i$ as left A (or Ae_i)-modules, see, e.g., [5, p. 19 and 23]. As a consequence,

$\varepsilon M_i \cong N \otimes_R \varepsilon M_i$ as left $\varepsilon A\varepsilon$-modules. This implies that $\mathrm{rank}_R(\varepsilon M_i) = \mathrm{rank}_R(N)\mathrm{rank}_R(\varepsilon M_i)$, whence $\mathrm{rank}_R(N) = 1$. Therefore $\mathrm{End}_R(N) = RI \cong R$, see, e.g., [5, p. 32]. Since we are dealing with equivalent categories, we have $\mathrm{End}_A(N \otimes_R M_i) \cong \mathrm{End}_R(N)$ and $\mathrm{End}_A(A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon M_i) \cong \mathrm{End}_{\varepsilon A\varepsilon}(\varepsilon M_i)$ as R-algebras, see, e.g., [5, p. 17]. We conclude that $\mathrm{End}_{\varepsilon A\varepsilon}(\varepsilon M_i) = RI \cong R$. In particular, εM_i is an indecomposable left $\varepsilon A\varepsilon$-module, see [2, 6.4].

(c) Since each nonzero εM_i is indecomposable, it follows from (a) that each nonzero εe_i is a primitive central idempotent of $\varepsilon A\varepsilon$. Let $\varepsilon M_i \neq 0$. Since $\mathrm{End}_{\varepsilon A\varepsilon}(\varepsilon M_i) = RI$, we then obtain $\varepsilon A\varepsilon e_i \cong \mathrm{End}_R(\varepsilon M_i)$ as R-algebras, see [9, 1.7] (the isomorphism associates to $\varepsilon a\varepsilon e_i$ the left multiplication by $\varepsilon a\varepsilon e_i$). Now, $N \otimes_R \varepsilon M_i \cong \varepsilon M_i \cong R \otimes_R \varepsilon M_i$ as left $\varepsilon A\varepsilon$-modules, and thus $N \cong R$ (N as in (b)). Consequently, $M_i \cong A\varepsilon \otimes_{\varepsilon A\varepsilon} \varepsilon M_i \in \mathscr{C}(A\varepsilon)$, completing the proof. ∎

2.6 *Remarks*. Let M be a left A-module such that $\varepsilon M \neq 0$. If M is finitely generated projective over R, then so is εM and we have $t_{\varepsilon M}(\varepsilon x\varepsilon) = t_M(\varepsilon x\varepsilon) = t_M(x\varepsilon)$ for all $x \in A$. More details on trace functions may be found in [3, Sect. 4]. The case where ε is central is less complicated, see also [3, 4.8 and 4.9]. Finally, if A is semisimple, then $\varepsilon A\varepsilon$ is semisimple too, as is well known.

One may also apply the preceding results to the following situation.

2.7. Proposition. *Let S be a Schur algebra in RG with associated partition $\{E_g;\ g \in G\}$. Let H be a subgroup of G such that $|H|^{-1} \in R$ and consider the idempotent $\varepsilon = |H|^{-1}\sum_{h\in H} u_h$. If $\varepsilon \in S$ and $|E_g|1_R \neq 0$ for all $g \in G$, then $\varepsilon S\varepsilon$ is a Schur algebra in RG with partition $\{HE_gH;\ g \in G\}$. Moreover we have $m|H|^{-2}|HE_gH|1_R = |E_g|1_R$ with $m \in \mathbb{N}$.*

Proof. Put $s_g = \sum_{x\in E_g} u_x$ and let G_0 denote a set of representatives of the distinct E_g. Now let $g \in G_0$. Clearly, $\varepsilon s_g\varepsilon = \sum_{i=1}^{l} n_i|Hx_iH|^{-1}\overline{Hx_iH}$ with $n_i \in \mathbb{N}$, where $x_i \in E_g$ are representatives of the distinct HxH, $x \in E_g$, and $\overline{Hx_iH} = \sum_{y\in Hx_iH} u_y$. Note that $n_1 + \cdots + n_l = |E_g|$. So there is some n_i such that $n_i1_R \neq 0$, because $|E_g|1_R \neq 0$. Since $\varepsilon \in S$, we have also $\varepsilon s_g\varepsilon = \sum_{t\in G_0} r_ts_t$ with $r_t \in R$ ($r_t = m_t|H|^{-2}1_R$ with $m_t \in \mathbb{N}$).

Comparing these expressions for $\varepsilon s_g\varepsilon$, we obtain $n_i|Hx_iH|^{-1}1_R = r_g$ for $i = 1, \ldots, l$, whence also $r_t = r_g$ or $r_t = 0$. Moreover $r_g \neq 0$. Consequently, $\varepsilon s_g\varepsilon = r_g\overline{HE_gH}$ with $\overline{HE_gH} = \sum_{y\in HE_gH} u_y$. We also deduce that $\varepsilon s_g\varepsilon = r_g\sum_k s_k$ for some $k \in G_0$, and we conclude that $\overline{HE_gH} = \sum_k s_k$. Therefore $\overline{HE_gH} \in S \cap \varepsilon RG\varepsilon$, and this intersection is equal to $\varepsilon S\varepsilon$.

Next, the above discussion shows that for each $g \in G_0$, $HE_gH = \bigcup_k E_k$ for some $k \in G_0$. Using this, it is easily seen that sets of the form HE_gH coincide or are disjoint. Moreover $(HE_gH)^{-1} = HE_{g^{-1}}H$.

Finally, since $n_1 + \cdots + n_l = |E_g|$ and $n_i|Hx_iH|^{-1}1_R = r_g$, we have $|HE_gH|r_g = |E_g|1_R$, completing the proof. ∎

2.8. *Remark*. Proposition 2.7 remains valid if we replace the condition $|E_g|1_R \neq 0$ by the following condition: for any $g, h \in G$, $E_gE_h = \bigcup_l E_l$ for some $l \in G$. In this case, it follows at once from the hypotheses that $HE_gH = \bigcup_k E_k$ for some $k \in G$.

We now turn to Schur algebras which are fixed rings of automorphism groups. Let G, H be finite groups and let $\sigma: H \to \operatorname{Aut}(G)$ be a homomorphism of groups. The orbits $E_g = \{\sigma_h(g) \mid h \in H\}$, $g \in G$, form a partition of G; $E_g^{-1} = E_{g^{-1}}$ and $E_e = \{e\}$. Each σ_h extends to an R-algebra isomorphism of RG (again denoted by σ_h) as follows: $\sigma_h(\sum_g r_g u_g) = \sum_g r_g u_{\sigma_h(g)}$, with $g \in G$ and $r_g \in R$. Furthermore, $\sigma: H \to \operatorname{Aut}_R(RG)$: $h \mapsto \sigma_h$ is a homomorphism of groups.

Consider the fixed ring $RG^H = \{a \in RG \mid \forall h \in H: \sigma_h(a) = a\}$; we have:

2.9. LEMMA. *Keep the above notation, put $s_g = \sum_{x \in E_g} u_x$ in RG, and let G_0 denote a set of representatives of the distinct E_g. Then $RG^H = \bigoplus_{g \in G_0} Rs_g$, i.e., RG^H is a Schur algebra in RG.*

Proof. Clearly $s_g \in RG^H$. Conversely, let $\sum_{g \in G} r_g u_g \in RG^H$, $r_g \in R$. Then for each $h \in H$ we have $\sum_{g \in G} r_g u_g = \sum_{g \in G} r_g u_{\sigma_h(g)}$, whence $r_{\sigma_h(g)} = r_g$ (for nonzero r_g). The result follows at once. ∎

We recall a few facts about fixed rings of automorphism groups. Throughout A is an R-algebra, H a finite group, and $\sigma: H \to \operatorname{Aut}_R(A)$ a homomorphism of groups.

For any $a \in A$, denote by $O(a)$ the orbit $\{\sigma_h(a) \mid h \in H\}$ and set $s(a) = \sum_{x \in O(a)} x$. Clearly, $A^H = \{a \in A \mid \forall h \in H: \sigma_h(a) = a\}$ is an R-subalgebra of A containing 1_A. Moreover, for any $a \in A$ we have $s(a) \in A^H$ as well as $\sum_{h \in H} \sigma_h(a) \in A^H$. Further, the associated skew group ring is denoted by $A * H$. As a left A-module $A * H$ is freely generated by symbols $\{w_h \mid h \in H\}$ and multiplication is defined by $(aw_h) \cdot (bw_k) = a\sigma_h(b)w_{hk}$ for all $a, b \in A, h, k \in H$. Of course $A * H$ is also an R-algebra, where the R-module structure is inherited from A.

If $|H|^{-1} \in R$, then we may consider the idempotent $e_H = |H|^{-1}\sum_{h \in H} w_h$ in $A * H$. From [7, Lemma 2.1] we retain:

2.10. LEMMA. *Assume $|H|^{-1} \in R$. Then $e_H(A * H)e_H = A^He_H$, and A^He_H is isomorphic to A^H as R-algebra.*

Proof. Set $\varepsilon = e_H$, and observe that $(aw_e)v = av$ for all $a \in A$, $v \in A * H$.

For $a \in A$ and $k \in H$ we have $\varepsilon(aw_k) = |H|^{-1}\sum_{h \in H} \sigma_h(a)w_{hk}$. But $w_t\varepsilon = \varepsilon$. Therefore $\varepsilon(aw_k)\varepsilon = |H|^{-1}\sum_{h \in H} \sigma_h(a)\varepsilon$, and this shows that

$\varepsilon(A * H)\varepsilon \subset A^H\varepsilon$. On the other hand, $a = |H|^{-1}\sum_{h \in H} \sigma_h(a)$ for all $a \in A^H$, and the equality follows.

Using the expressions given above, it is easily verified that $A^H \to A^H\varepsilon$: $a \mapsto a\varepsilon$ is an isomorphism of R-algebras. ∎

We may use the preceding lemma to prove:

2.11. PROPOSITION. *Let A, H, σ be as before and assume $|H|^{-1} \in R$.*

(1) *If A is finitely generated and projective as R-module, then so is A^H.*

(2) *Suppose that A is finitely generated projective and faithful as R-module. If A is separable over R, then so is A^H.*

Proof. (1) Let $\{a_1, \dots, a_n\} \subset A, \{\varphi_1, \dots, \varphi_n\} \subset \mathrm{Hom}_R(A, R)$ be a dual basis for A. Then it is easily checked that $\{|H|^{-1}\sum_{h \in H} \sigma_h(a_i)\}, \{\varphi_i|_{A^H}\}$ is a dual basis for A^H.

(2) Let $\sum_{i=1}^m x_i \otimes y_i \in A \otimes_R A^0$ be a separability idempotent for A. Then it is easily verified that $|H|^{-1}\sum_{h \in H} \sum_{i=1}^m (\sigma_h(x_i)w_h \otimes y_i w_{h^{-1}})$ is a separability idempotent for $A * H$. So $A * H$ is separable over R. Moreover, $A * H$ is finitely generated projective as R-module. We now apply Lemma 2.10 and Proposition 2.1. ∎

Let us return to the case where $A = RG$ and H acts on G. Then $A * H$ is isomorphic to $R(G \times_\sigma H)$ as R-algebra, where $G \times_\sigma H$ is the semidirect product of G and H (i.e., $(g_1, h_1) \cdot (g_2, h_2) = (g_1\sigma_{h_1}(g_2), h_1h_2)$ for $g_i \in G, h_i \in H$). The isomorphism maps $u_g w_h \in A * H$ onto (g, h) for any $g \in G, h \in H$.

In case $|H|^{-1} \in R$, the algebra RG^H is isomorphic to a double coset algebra in $R(G \times_\sigma H)$, see Lemma 2.10. Furthermore we have:

2.12. PROPOSITION. (1) *If $|H|$ and $|G|$ are invertible in R, then RG^H is separable over R.*

(2) *Suppose R is connected, and $|H|$ and $|G|$ are invertible in R. If R is a splitting ring for $R(G \times_\sigma H)$, then R is a splitting ring for RG^H.*

In particular, let m be the exponent of $G \times_\sigma H$ and η a primitive mth root of unity. Then $L = R[\eta]$ is a splitting ring for LG^H.

Proof. (1) We know that $|G|^{-1} \in R$ implies that RG is separable over R, and we may apply Proposition 2.11(2).

(2) The first statement follows from Lemma 2.10 and Proposition 2.5(3). The second part follows from [12]. ∎

Next we deal with indecomposable modules. Connections between RG^H-modules and $R(G \times_\sigma H)$-modules are given by the theory of double coset algebras, developed in the first part of this section. We now investigate the relationship between indecomposable RG^H-modules and indecomposable RG-modules. We return to the general situation where A is an R-algebra, H a finite group, and $\sigma\colon H \to \mathrm{Aut}_R(A)$ a homomorphism of groups. We require the following definition.

2.13. DEFINITION. Let M be a left A-module and let $h \in H$. We obtain a left A-module ${}^hM$ as follows: consider the underlying abelian group of M and let A act on it by setting $a \circ m = \sigma_h^{-1}(a)m$ for all $a \in A$, $m \in M$.

Observe that the induced R-module structure on ${}^hM$ coincides with that on M.

2.14. *Remarks*. (1) Let M, N be left A-modules and let $h, k \in H$. Then ${}^k({}^hM) = {}^{kh}M$ as A-modules, and $\mathrm{Hom}_A({}^hM, {}^hN) = \mathrm{Hom}_A(M, N)$.

(2) Let M be a left A-module which is finitely generated and projective over R. For the trace functions we get $t_{{}^hM}(a) = t_M(\sigma_h^{-1}(a))$ for all $a \in A$, $h \in H$.

(3) If M is an indecomposable, resp. a finitely generated projective, left A-module, then so is ${}^hM$ for all $h \in H$.

(4) Suppose that R is connected and that A is finitely generated and projective as R-module. Let $\{e_1, \ldots, e_q\}$, resp. $\{d_1, \ldots, d_m\}$, be the set of primitive central nonzero idempotents of A, resp. A^H (use rank_R). Then H acts on $\{e_1, \ldots, e_q\}$ by σ. Again, let $s(e_i)$ denote the sum of the idempotents in the orbit of e_i. Each $s(e_i)$ is uniquely expressible as a sum of d_j's, and each d_j appears in one and only one of the $s(e_i)$. Note also that d_j appears in $s(e_i)$ if and only if $d_je_i \neq 0$.

(5) Let R, A, e_i, d_j be as in (4), and let M be an indecomposable left A-module lying over e_i. We observe that ${}^hM$ lies over $\sigma_h(e_i)$, $h \in H$.

Further, it is clear that $d_je_i = 0$ implies $d_jM = 0$. Moreover, if M is finitely generated projective over A and if any two indecomposable finitely generated projective left A-modules lying over the same primitive central idempotent are isomorphic as A-modules, then the converse is true. Indeed, suppose $d_jM = 0$ and write $e_i = \eta_1 + \cdots + \eta_t$, η_k being primitive idempotents of A. Then $d_jA\eta_k = 0$ for $k = 1, \ldots, t$, whence $d_je_i = 0$.

Note also that $M|_{A^H}$ is the direct sum of the nonzero d_jM.

2.15. THEOREM. *Suppose that R is connected and that A is finitely generated and projective as R-module. Let P be an indecomposable left A^H-module, and let e be a primitive central idempotent of A such that $e(A \otimes_{A^H} P) \neq 0$. Set $W = e(A \otimes_{A^H} P)$ and $F = \{h \in H \mid \sigma_h(e) = e\}$. Then*

(1) *$A \otimes_{A^H} P \cong \bigoplus_{i=1}^r {}^{h_i}W$ as left A-modules, where $\{h_1, \ldots, h_r\}$ is a set of left coset representatives of F in H.*

Moreover, $F = \{h \in H \mid {}^hW \cong W$ as A-modules$\}$.

(2) *If P is finitely generated and projective over A^H, then we may write $A \otimes_{A^H} P = M_1 \oplus \cdots \oplus M_s$ where each M_i is an indecomposable left A-module. In this case W is the direct sum of all M_i lying over e.*

Proof. (1) Let $\{e = e_1, \ldots, e_t\}$ be the set of all primitive central idempotents of A for which $e_j(A \otimes_{A^H} P) \neq 0$, and set $W_j = e_j(A \otimes_{A^H} P)$. Then $A \otimes_{A^H} P = W_1 \oplus \cdots \oplus W_t$ $(W = W_1)$.

Further, let d denote the primitive central idempotent of A^H for which $dP \neq 0$. Then $e_j(A \otimes_{A^H} P) \neq 0$ implies $e_j d \neq 0$. By Remark 2.14(4), it follows that $e_1, \ldots, e_t$ belong to the same orbit (of the action of H).

Now let $h \in H$. We observe that $A \otimes_{A^H} P \to {}^h(A \otimes_{A^H} P)$: $\sum_i a_i \otimes p_i \mapsto \sum_i \sigma_h^{-1}(a_i) \otimes p_i$ is an isomorphism of left A-modules. Thus $\sigma_h(e)(A \otimes_{A^H} P) \cong \sigma_h(e) \circ {}^h(A \otimes_{A^H} P) = {}^hW \neq 0$ as A-modules. This yields $\sigma_h(e) = e_j$ for some $j \in \{1, \ldots, t\}$.

Moreover we obtain $W_j \cong {}^hW$. Furthermore, if $\sigma_h(e) = e$, then $W \cong {}^hW$. The converse follows from the fact that $eW = W$ and $\sigma_h(e) \circ {}^hW = {}^hW$.

(2) It is clear that $A \otimes_{A^H} P$ is nonzero, finitely generated, and projective over A, hence also over R, and use rank_R. ∎

From the proof it follows that $e_j(A \otimes_{A^H} P) \neq 0$ if and only if $A \otimes_{A^H} P \neq 0$ and $e_j d \neq 0$.

As an immediate consequence of Theorem 2.15, we obtain:

2.16. Corollary. *Keep the hypotheses and notation of Theorem* 2.15(2), *and suppose that any two indecomposable finitely generated projective left A-modules lying over the same primitive central idempotent are isomorphic as A-modules.*

Then $A \otimes_{A^H} P \cong \bigoplus_{i=1}^{r} ({}^{h_i}M)^k$ as A-modules, where M is an indecomposable finitely generated projective left A-module lying over e and $k \in \mathbb{N}$. Moreover $\sigma_h(e) = e$ if and only if ${}^hM \cong M$.

Let $\operatorname{Inn}(A)$ denote the group of inner automorphisms of A. As a special case we now obtain:

2.17. Corollary. *Suppose $\sigma(H) \subset \operatorname{Inn}(A)$. Then we have $A \otimes_{A^H} P = W$ in Theorem* 2.15, *and we have $A \otimes_{A^H} P \cong M^k$ in Corollary* 2.16.

Note. Suppose that $\sigma(H) \subset \operatorname{Inn}(A)$. Let U denote the group of invertible elements of A and consider $j: U \to \operatorname{Inn}(A)$: $u \mapsto j_u$ with $j_u(a) = uau^{-1}$ for all $a \in A$. Take the subgroup $L = j^{-1}(\sigma(H))$ of U and restrict j to L. Then $A^H = A^L$ and A^L is the centralizer in A of the R-subalgebra generated by L.

To conclude, let us return to Schur algebras. So let G, H be finite groups, let σ: $H \to \operatorname{Aut}(G)$ be a homomorphism of groups, and suppose that $\sigma(H) \subset \operatorname{Inn}(G)$. Consider i: $G \to \operatorname{Inn}(G)$: $g \mapsto i_g$ with $i_g(x) = gxg^{-1}$ for all $x \in G$. In this case, we take the subgroup $K = i^{-1}(\sigma(H))$ of G and we restrict i to K. Extending to automorphisms of RG, we get $RG^H = RG^K$. Now, for any subgroup K of G and homomorphism $i\colon K \to \mathrm{Inn}(G)$, we see that RG^K is the centralizer of RK in RG. Further results on modules over centralizers can be found in [10, 3].

3. SYMMETRIC FUNCTIONS ON FROBENIUS ALGEBRAS

Throughout, R is a commutative ring and A is a faithful R-algebra which is a finitely generated free R-module. Recall that $A^* = \mathrm{Hom}_R(A, R)$ is a left A-module under the operation $(a \,.\, f)(x) = f(xa)$ for $a, x \in A$, $f \in A^*$.

3.1. *Remarks*. An R-bilinear form on A is called associative if $b(xy, v) = b(x, yv)$ for all $x, y, v \in A$. As is well known, there is a one-to-one correspondence between associative R-bilinear forms $b\colon A \times A \to R$ and (left) A-linear maps $\beta\colon A \to A^*$, given by $b(x, y) = \beta(y)(x)$, $x, y \in A$.

On the other hand, an A-linear map $\beta\colon A \to A^*$ is completely determined by $\beta(1) = \tau$, and the above correspondence yields $b(x, y) = \tau(xy)$, $x, y \in A$.

3.2. LEMMA. *Let b be an associative R-bilinear form on A, let β: $A \to A^*$ be the corresponding left A-linear map, and $\tau = \beta(1)$. The following statements are equivalent*:

(1) *There are R-bases $\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}$ in A such that $b(a_i, b_j)$ form an invertible matrix.*

(2) *For each R-basis $\{a_1, \dots, a_n\}$ of A there exists an R-basis $\{b_1, \dots, b_n\}$ of A with $b(a_i, b_j) = \delta_{ij}$.*

(3) *β is an isomorphism.*

(4) *For every $f \in A^*$ there is a unique $a \in A$ such that $f = a \,.\, \tau$.*

Proof. This is straightforward; see also [3, Lemma 3.1]. ∎

A bilinear form satisfying property (2) is said to be nonsingular, and $\{a_k\}, \{b_k\}$ in (2) are called dual bases with respect to b. The R-algebra A is called a *Frobenius algebra* if there exists a nonsingular associative R-bilinear form on A.

Now let τ, f, a be as in Lemma 3.2(4); then it is easily seen that f is also a free generator of A^* viewed as a left A-module if and only if a is invertible in A. Furthermore, we have:

3.3. LEMMA. *Let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}$, and let $\beta\colon A \to A^*$ be the corresponding left A-linear map. Then $\beta^{-1}\colon A^* \to A$ is given by $\beta^{-1}(f) = \sum_{i=1}^n f(a_i) b_i$.*

Proof. We have $\beta(\sum_i f(a_i) b_i)(a_j) = b(a_j, \sum_i f(a_i) b_i) = f(a_j)$. ∎

Recall that $f \in A^*$ is said to be symmetric if $f(xy) = f(yx)$ for all $x, y \in A$. The set of all symmetric functions $f \in A^*$ will be denoted by $\operatorname{Sym}(A, R)$. The A-module structure on A^* makes $\operatorname{Sym}(A, R)$ into a $Z(A)$-module, where $Z(A)$ denotes the center of A.

Furthermore, we say that A is a *symmetric Frobenius algebra* if there exists a nonsingular associative R-bilinear form on A which is symmetric.

3.4. LEMMA. *Let b be a nonsingular symmetric associative R-bilinear form on A, and let β: $A \to A^*$ be the corresponding left A-linear map. Then β induces an isomorphism of $Z(A)$-modules between $Z(A)$ and* $\operatorname{Sym}(A, R)$.

Proof. Let $\tau = \beta(1)$; τ is symmetric. Obviously, if $a \in Z(A)$, then $\beta(a) = a \,.\, \tau$ is symmetric. Now let $f \in \operatorname{Sym}(A, R)$, hence $f = a \,.\, \tau$ for some $a \in A$. From $f(yx) = f(xy)$ it follows that $\tau(yxa) = \tau(xya) = \tau(yax)$, for all $x, y \in A$. Therefore $xa \,.\, \tau = ax \,.\, \tau$, whence $xa = ax$, for all $x \in A$. ∎

3.5. EXAMPLES. (1) Let G be a finite group and consider the twisted group ring $R *_\alpha G$ with R-basis $\{u_g \mid g \in G\}$. Consider the R-linear map τ: $R *_\alpha G \to R$: $\sum_{g \in G} r_g u_g \mapsto r_e$. It is clear that τ defines a symmetric associative R-bilinear form on $R *_\alpha G$ with dual bases $\{u_g \mid g \in G\}$, $\{\alpha(g, g^{-1})^{-1} u_{g^{-1}} \mid g \in G\}$ (α is 2-cocycle).

(2) Let G be a finite group, let $\{E_g;\ g \in G\}$ be a partition of G such that $E_g^{-1} = E_{g^{-1}}$, and let G_0 denote a set of representatives of the distinct E_g. Put $s_g = \sum_{x \in E_g} u_x$ in RG and suppose that $S = \bigoplus_{g \in G_0} Rs_g$ is a Schur algebra in RG.

Now consider τ: $S \to R$: $\sum_{g \in G_0} r_g s_g \to r_e$. If each $|E_g|$ is invertible in R, then τ defines a symmetric associative R-bilinear form on S with dual bases $\{s_g \mid g \in G_0\}, \{|E_g|^{-1} s_{g^{-1}} | g \in G_0\}$.

(3) For some other examples we refer to [3, 3.8].

Now let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}$. Consider the $Z(A)$-linear map ζ: $A \to A$: $x \mapsto \sum_{i=1}^n b_i x a_i$. We have:

3.6. PROPOSITION. (1) *$\zeta(x)$ is independent of the choice of the dual bases and $\zeta(A)$ is independent of the choice of the nonsingular associative bilinear form.*

(2) *$\zeta(A)$ is an ideal of the center $Z(A)$ of A.*

(3) *A is a separable R-algebra if and only if $1 \in \zeta(A)$.*

(4) *If b is symmetric, then $\zeta(xy) = \zeta(yx)$ for all $x, y \in A$.*

Proof. Parts (1) and (2) follow from [3, Proposition 3.6] and (3) follows from [3, Propositions 3.7 and 3.10]. As for (4), $\{b_i\}, \{a_i\}$ are dual bases with respect to b, i.e., $b(b_j, a_i) = \delta_{ij}$, because b is symmetric. Then by (1),

$\zeta(xy) = \sum_{i=1}^{n} a_i xy b_i$. Now for each $x \in A$, we have

$$a_i x = \sum_{j=1}^{n} r_{ji} a_j \quad \text{implies} \quad x b_i = \sum_{j=1}^{n} r_{ij} b_j, \qquad r_{ji} \in R.$$

Using these relations, we obtain $\zeta(xy) = \zeta(yx)$. ∎

Keep the above notation and put $z = z_b = \sum_{i=1}^{n} a_i b_i$.

It is easily verified (see [3, 3.4]) that $b(x, z) = t_A(x)$ for all $x \in A$, where t_A denotes the trace function from A to R afforded by A viewed as left A-module. In other words, $t_A = z \,.\, \tau$ where $\tau \in A^*$ is associated to b as in Remarks 3.1.

As a consequence, we obtain that z is independent of the choice of the dual bases for b. Moreover, if b is symmetric, then z is central and $z = \sum_{i=1}^{n} b_i a_i = \zeta(1)$. Furthermore, if b' is another nonsingular associative R-bilinear form on A, then we can find an invertible element $u \in A$ such that $z_b = z_{b'} u$.

The invertibility of z has been investigated in [3]. In particular, if R is a field of characteristic zero and A is separable over R, then z is invertible in A. Here we give an additional result on invertibility. We shall need the $Z(A)$-module $\ker \zeta$. Clearly $\ker \zeta$ is independent of the choice of the dual bases for b and, in case b is symmetric, $\ker \zeta$ is also independent of the choice of the nonsingular symmetric form.

3.7. PROPOSITION. *Keep the above notation and assumptions and suppose that b is symmetric. Then the following statements are equivalent*:

(1) *z is invertible in A.*

(2) *A is separable over R and $A = \ker \zeta \oplus Z(A)$.*

Proof. Note that $\zeta(c) = zc$ for all $c \in Z(A)$.

(1) ⇒ (2). Clearly $\zeta(z^{-1}) = 1$, hence $\zeta(A) = Z(A)$ and A is separable over R, see Proposition 3.6. For each $x \in A$, we write $x = (x - \zeta(z^{-1}x)) + \zeta(z^{-1}x)$, and then it is easily checked that $A = \ker \zeta \oplus \zeta(A)$.

(2) ⇒ (1). By the separability, we have $1 = \zeta(x)$ for some $x \in A$. There exist elements $y_1 \in \ker \zeta$, $y_2 \in Z(A)$ such that $x = y_1 + y_2$. Thus $1 = \zeta(y_2) = zy_2$. ∎

Next we show that, under certain conditions, symmetric functions are determined by their values on the center. Again let A be a Frobenius algebra, let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_i\}$, $\{b_i\}$, and let ζ, z be as before.

3.8. PROPOSITION. *Assume that b is symmetric and that z is invertible in A. Given $f \in A^*$, the following conditions are equivalent*:

(1) $f \in \mathrm{Sym}(A, R)$.

(2) $f(x) = f(\zeta(z^{-1}x))$ *for all* $x \in A$.

(3) $\ker \zeta \subset \ker f$.

Proof. (1) $\Rightarrow$ (2). We have $f(\zeta(z^{-1}x)) = f(\sum_i b_i z^{-1} x a_i) = f(\sum_i a_i b_i z^{-1} x) = f(x)$.

(2) $\Rightarrow$ (3). Note that $\zeta(z^{-1}x) = z^{-1}\zeta(x)$.

(3) $\Rightarrow$ (1). For all $x, y \in A$, we have $\zeta(xy) = \zeta(yx)$, hence $xy - yx \in \ker \zeta \subset \ker f$. ∎

3.9. PROPOSITION. *Let b, ζ, z be as before and suppose that b is symmetric. Then $\bigcap_f \ker f \subset \ker \zeta$ where f ranges over all elements of $\mathrm{Sym}(A, R)$. If z is invertible in A, then we get an equality.*

Proof. Let $\tau \in A^*$ be associated to b as in Remarks 3.1. Let $x \in A$ be such that $f(x) = 0$ for all $f \in \mathrm{Sym}(A, R)$. Then by Lemma 3.4, $\tau(xc) = 0$ for all $c \in Z(A)$. For each $y \in A$, we now have $\tau(y\zeta(x)) = \tau(\sum_i y b_i x a_i) = \tau(\sum_i a_i y b_i x) = \tau(\zeta(y)x) = 0$ using Proposition 3.6. Thus $\zeta(x) . \tau = 0$, whence $\zeta(x) = 0$.

In case z is invertible, we may apply Proposition 3.8 and we obtain an equality. ∎

We now show that, under certain conditions, $\mathrm{Sym}(A, R)$ has an R-basis consisting of characters and we derive orthogonality relations for characters. Again let A be a Frobenius R-algebra, let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}$, and put $z = \sum_{i=1}^n a_i b_i$. Moreover we assume that b is symmetric, although some results can be proved without this assumption. Further, suppose that R is connected and let $\{e_1, \dots, e_q\}$ be the set of primitive central nonzero idempotents of A. Let now $M_1, \dots, M_q$ be nonzero left A-modules which are finitely generated and projective over R, and assume that $e_k M_i = 0$ for $k \neq i$. Note that an indecomposable A-module P lies over exactly one e_i. Finally, we let rank stand for rank_R, and we recall that t_{M_i} denotes the trace function from A to R afforded by M_i.

3.10. THEOREM. *Keep the above hypotheses and notation.*

(1) *If R is a splitting ring for the center, that is, $Z(A) = Re_1 \oplus \cdots \oplus Re_q$, then*

$$\mathrm{rank}(M_j) e_j = b(e_j, e_j) \sum_{i=1}^n t_{M_j}(a_i) b_i$$

$$t_{M_j}(z) e_j = \mathrm{rank}(Ae_j) \sum_{i=1}^n t_{M_j}(a_i) b_i.$$

(2) *For* $j \neq k$ *we have* $\sum_{i=1}^{n} t_{M_j}(a_i) t_{M_k}(b_i) = 0$.

(3) *Let* L_j *be any nonzero left* A*-module which is finitely generated projective over* R *and has the property that* $e_k L_j = 0$ *for* $k \neq j$ (*special case*: $L_j = M_j$). *If* R *is a splitting ring for* $Z(A)$, *then*

$$b(e_j, e_j) \sum_{i=1}^{n} t_{M_j}(a_i) t_{L_j}(b_i) = \operatorname{rank}(M_j) \operatorname{rank}(L_j) 1_R.$$

(4) *With assumptions as in* (3) *we have*

$$\operatorname{rank}(M_j) t_{L_j} = \operatorname{rank}(L_j) t_{M_j}.$$

(5) *If* $\operatorname{rank}(M_i) 1_R \neq 0$ *and* $\operatorname{rank}(M_i) 1_R$ *is not a zero divisor in* R *for* $i = 1, \ldots, q$, *then* $t_{M_1}, \ldots, t_{M_q}$ *are linearly independent over* R.

(6) *If* R *is a splitting ring for* $Z(A)$ *and* $\operatorname{rank}(M_i) 1_R$ *is invertible in* R *for* $i = 1, \ldots, q$, *then* $t_{M_1}, \ldots, t_{M_q}$ *form an* R*-basis of* $\operatorname{Sym}(A, R)$.

(7) *We have*

$$z e_j = \sum_{i=1}^{n} t_{Ae_j}(a_i) b_i.$$

(8) *If* R *is a splitting ring for* $Z(A)$ *and* z *is invertible in* A, *then* $t_{Ae_1}, \ldots, t_{Ae_q}$ *form an* R*-basis of* $\operatorname{Sym}(A, R)$ (Ae_i *viewed as left* A*-module*).

Proof. Let $\tau \in A^*$ be associated to b as in Remarks 3.1. For each t_{M_j} there is a unique $c_j \in A$ such that $t_{M_j} = c_j . \tau$. By Lemma 3.3, $c_j = \sum_{i=1}^{n} t_{M_j}(a_i) b_i$.

Further, it is easily seen that $e_k . t_{M_j} = 0$ for $k \neq j$. Consequently $(e_k c_j) . \tau = 0$, whence $e_k c_j = 0$ for $k \neq j$. Therefore $c_j \in Ae_j$.

(1) Since b is symmetric, $c_j \in Z(A)$, see Lemma 3.4. Thus $c_j = r_j e_j$ with $r_j \in R$. We now have $t_{M_j}(1) = \tau(c_j) = r_j \tau(e_j)$ and $t_{M_j}(1) = \operatorname{rank}(M_j) 1_R$. Then $\operatorname{rank}(M_j) e_j = \tau(e_j) c_j$ and we obtain the first formula.

Further, we know that $t_A = z . \tau$. Using the fact that $t_A = \sum_{i=1}^{q} t_{Ae_i}$ on A, it is easily seen that $t_{Ae_j} = e_j . t_A$ (we view A and Ae_i as left A-modules). We thus obtain $t_{Ae_j} = (e_j z) . \tau$. Since b is symmetric, z is central and thus $z = \sum_{i=1}^{q} \lambda_i e_i$ with $\lambda_i \in R$. Therefore $t_{Ae_j} = (\lambda_j e_j) . \tau$. As a consequence, we have $\operatorname{rank}(Ae_j) 1_R = \lambda_j \tau(e_j)$. On the other hand, $t_{M_j}(z) = \operatorname{rank}(M_j) \lambda_j$. We now have $t_{M_j}(z) e_j = \operatorname{rank}(M_j) \lambda_j e_j = \lambda_j \tau(e_j) c_j = \operatorname{rank}(Ae_j) c_j$ and this gives the second formula.

(2) Apply t_{M_k}, $k \neq j$, to the expression $c_j = \sum_{i=1}^{n} t_{M_j}(a_i) b_i$.

(3) Apply t_{L_j} to the first formula in (1).

(4) There is a unique $c'_j \in A$ such that $t_{L_j} = c'_j . \tau$, and $c'_j = r'_j e_j$ with $r'_j \in R$. Moreover, $\operatorname{rank}(L_j) 1_R = r'_j \tau(e_j)$.

Let c_j, r_j be as above. Then we have $c_j \,.\, t_{L_j} = c'_j \,.\, t_{M_j}$ and thus $r_j t_{L_j} = r'_j t_{M_j}$. Multiplying by $\tau(e_j)$, we obtain the formula in (4).

(5) Suppose that $\sum_{i=1}^{q} \mu_i t_{M_i} = 0$ with $\mu_i \in R$. Then $\sum_i \mu_i t_{M_i}(e_k) = 0$ for $k = 1, \ldots, q$. We get $\operatorname{rank}(M_k)\mu_k = 0$, whence $\mu_k = 0$ for $k = 1, \ldots, q$.

(6) As before, we have $t_{M_j} = (r_j e_j) \,.\, \tau$ with $r_j \in R$. The invertibility of $\operatorname{rank}(M_j)$ in R implies the invertibility of r_j in R, because $\operatorname{rank}(M_j)1_R = r_j \tau(e_j)$. Now, $e_1, \ldots, e_q$ form an R-basis of $Z(A)$, and thus also $r_1 e_1, \ldots, r_q e_q$. By Lemma 3.4, it follows that $t_{M_1}, \ldots, t_{M_q}$ form an R-basis of $\operatorname{Sym}(A, R)$.

(7) As in the proof of (1), $t_{Ae_j} = (ze_j) \,.\, \tau$. The assertion follows from Lemma 3.3.

(8) We have $z = \sum_{i=1}^{q} \lambda_i e_i$ with $\lambda_i \in R$ and $t_{Ae_j} = (\lambda_j e_j) \,.\, \tau$. Since z is invertible in A, each λ_i is invertible in R. We now proceed as in (6) in order to show that $t_{Ae_1}, \ldots, t_{Ae_q}$ form an R-basis of $\operatorname{Sym}(A, R)$. ∎

3.11. *Remarks.* Keep the hypotheses and notation of Theorem 3.10 and assume that R is a splitting ring for $Z(A)$.

(1) From the proof of Theorem 3.10 we retain that $\operatorname{rank}(M_j)1_R = r_j b(e_j, e_j)$ with $r_j \in R$. Further, $z = \sum_{i=1}^{q} \lambda_i e_i$ with $\lambda_i \in R$ and $t_{Ae_j} = (\lambda_j e_j) \,.\, \tau$, in particular $\operatorname{rank}(Ae_j)1_R = \lambda_j b(e_j, e_j)$.

(2) If $b(e_i, e_i)$ is invertible in R for $i = 1, \ldots, q$, then $b\colon Z(A) \times Z(A) \to R$ is nonsingular. The converse also holds.

(3) For all $x \in A$ we have $t_{M_j}(x)ze_j = \operatorname{rank}(M_j)\zeta(x)e_j = t_{M_j}(\zeta(x))e_j$. Indeed, we may write $z = \sum_{i=1}^{q} \lambda_i e_i$ and $\zeta(x) = \sum_{i=1}^{q} \mu_i e_i$. Clearly, $t_{M_j}(\zeta(x)) = \operatorname{rank}(M_j)\mu_j$. On the other hand, $t_{M_j}(\zeta(x)) = t_{M_j}(\sum_{i=1}^{n} b_i x a_i) = t_{M_j}(zx) = \lambda_j t_{M_j}(x)$.

As before, let b be a nonsingular symmetric associative R-bilinear form on A with dual bases $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}$, and put $z = \sum_{i=1}^{n} a_i b_i$. Suppose that R is connected and let $\{e_1, \ldots, e_q\}$ be the set of primitive central nonzero idempotents of A. We now assume that $A \cong \operatorname{End}_R(P_1) \oplus \cdots \oplus \operatorname{End}_R(P_q)$ as R-algebras, $P_1, \ldots, P_q$ being finitely generated projective R-modules.

Observe that $Z(A) = Re_1 \oplus \cdots \oplus Re_q$. We recall that each P_i is an indecomposable left A-module under the operation $(\varphi_1, \ldots, \varphi_q) \,.\, p = \varphi_i(p)$, $p \in P_i$, and $\varphi_j \in \operatorname{End}_R(P_j)$, and we may assume that P_i lies over e_i. All indecomposable left A-modules and their characters are described in [9, 1.8 and 2.2].

Further, from the proof of Theorem 3.13 in [3] we retain that $t_{Ae_j} = \operatorname{rank}(P_j) t_{P_j}$ on A, in particular $\operatorname{rank}(Ae_j) = (\operatorname{rank} P_j)^2$.

Clearly we may apply Theorem 3.10 to t_{P_i}. Moreover the following holds true.

3.12. PROPOSITION. *Keep the above hypotheses and notation. Then*

(1) *We have*

$$ze_j = \sum_{i=1}^{n} \operatorname{rank}(P_j) t_{P_j}(a_i) b_i$$

$$t_{P_j}(z) = \sum_{i=1}^{n} \operatorname{rank}(P_j) t_{P_j}(a_i) t_{P_j}(b_i).$$

(2) *z is invertible in A if and only if all* $\operatorname{rank}(P_j)1_R$ *are invertible in R. Moreover,* $\operatorname{rank}(P_j)1_R$ *is invertible in R if and only if $t_{P_j}(z)$ is invertible in R.*

Proof. (1) We have $t_{Ae_j} = \operatorname{rank}(P_j)t_{P_j}$. The first formula now follows from Theorem 3.10(7). Applying t_{P_j}, we obtain the second formula.

(2) Let $\tau \in A^*$ be associated to b. There is a unique $c_j \in A$ such that $t_{P_j} = c_j . \tau$ and $c_j \in Ae_j$. Then $t_{Ae_j} = \operatorname{rank}(P_j)t_{P_j} = \operatorname{rank}(P_j)c_j . \tau$. On the other hand, we know that $t_{Ae_j} = (ze_j) . \tau$, see Theorem 3.10. Therefore $ze_j = \operatorname{rank}(P_j)c_j$ and thus $z = (\sum_j \operatorname{rank}(P_j)e_j)(\sum_j c_j)$. So the invertibility of z implies that all $\operatorname{rank}(P_j)$ are invertible in R. To prove the converse, we write $z = \sum_{i=1}^{q} \lambda_i e_i$ with $\lambda_i \in R$ and we observe that $(\operatorname{rank} P_j)^2 1_R = \operatorname{rank}(Ae_j)1_R = \lambda_j b(e_j, e_j)$, see Remarks 3.11.

The last statement follows from $t_{P_j}(z) = \operatorname{rank}(P_j)\lambda_j$ and the preceding formula. ∎

3.13. *Remarks*. (1) We do not need the fact that b is symmetric in the proofs of Theorem 3.10(2)–(5)–(7) and Proposition 3.12(1), and in the proof of the implication: z invertible $\Rightarrow$ $\operatorname{rank}(P_j)1_R$ invertible.

(2) We recover the special case considered in [2, 9.17].

3.14. EXAMPLES. (1) As in Example 3.5(1), let $A = R *_\alpha G$ with bilinear form associated to τ: $A \to R$: $\sum_{g \in G} r_g u_g \mapsto r_e$. In this case $\{u_g\}, \{\alpha(g, g^{-1})^{-1} u_{g^{-1}}\}$, $g \in G$, are dual bases and $z = |G| u_e = |G| \alpha(e, e) 1_A$.

If R is connected, $|G|$ is invertible in R, and $R *_\alpha G \cong \operatorname{End}_R(P_1) \oplus \cdots \oplus \operatorname{End}_R(P_q)$ with P_i as above (see, for example, [8]), then we may apply Theorem 3.10 and Proposition 3.12. Now all $\operatorname{rank}(P_j)$ are invertible in R and $t_{P_j}(z) = \operatorname{rank}(P_j)|G|\alpha(e, e)$.

(2) Recall that a Schur algebra in RG (with associated partition $\{E_g;\ g \in G\}$) is a symmetric Frobenius R-algebra, whenever $|E_g|$ is invertible in R for all $g \in G$, cf. Example 3.5.

3.15. *Note.* Let b be a nonsingular associative R-bilinear form on A with dual bases $\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}$, and let $\beta\colon A \to A^*$ be associated to b as in Remarks 3.1.

(1) Since β is bijective, β induces a ring structure on A^*. Explicitly, let $\varphi, \psi \in A^*$; $\varphi = \beta(s)$, $\psi = \beta(t)$. Then $\varphi \times \psi = \beta(st)$.

Now let $A = R *_\alpha G$ with bilinear form associated to $\tau\colon A \to R$: $\sum_{g \in G} r_g u_g \mapsto r_e$, as in Example 3.5(1). By Lemma 3.3, we have

$$st = \sum_{k \in G} \varphi \times \psi(u_{k^{-1}}) \alpha(k, k^{-1})^{-1} u_k.$$

On the other hand,

$$st = \sum_{g \in G} \sum_{h \in G} \varphi(u_{g^{-1}}) \psi(u_{h^{-1}}) \alpha\left(g, g^{-1}\right)^{-1} \alpha(h, h^{-1})^{-1} \alpha(g, h) u_{gh}.$$

But

$$\begin{aligned} \alpha(h, h^{-1})^{-1} & \alpha\left(g, g^{-1}\right)^{-1} \alpha(g, h) \alpha\left(gh, (gh)^{-1}\right) \\ &= \alpha(h, h^{-1})^{-1} \alpha\left(h, (gh)^{-1}\right) \\ &= \alpha(e, e) \alpha\left(h^{-1}, g^{-1}\right)^{-1}. \end{aligned}$$

Consequently,

$$st = \sum_{k \in G} \sum_{g \in G} \varphi(u_{g^{-1}}) \psi(u_{k^{-1}g}) \alpha(e, e) \alpha\left(k^{-1}g, g^{-1}\right)^{-1} \alpha(k, k^{-1})^{-1} u_k.$$

So we obtain

$$\varphi \times \psi(u_k) = \sum_{g \in G} \varphi(u_{g^{-1}}) \psi(u_{kg}) \alpha(e, e) \alpha\left(kg, g^{-1}\right)^{-1}.$$

(2) The map β also induces an R-bilinear form b^* on A^*. Explicitly, let $\varphi, \psi \in A^*$; $\varphi = \beta(s)$, $\psi = \beta(t)$. Then $b^*(\varphi, \psi) = b(s, t)$. Now let b be symmetric. Then we may write $s = \sum_i \varphi(b_i) a_i$ and $t = \sum_j \psi(a_j) b_j$. Consequently, $b^*(\varphi, \psi) = \sum_{i=1}^n \varphi(b_i) \psi(a_i)$. The formulas in Theorem 3.10(2)–(3) and in Proposition 3.12(1) can be rewritten using the R-bilinear form b^*.

(3) We also have the following multiplication on A^*. For $\varphi, \psi \in A^*$, define $\varphi * \psi(a_i) = \varphi(a_i)\psi(a_i)$ and extend by linearity. On the other hand, we may consider the following componentwise multiplication on A. Let $s, t \in A$, write $s = \sum_{i=1}^n r_i b_i$, $t = \sum_{i=1}^n r_i' b_i$ with $r_i, r_i' \in R$, and set $s * t = \sum_{i=1}^n r_i r_i' b_i$. Then $\beta(s * t) = \beta(s) * \beta(t)$, as is easily checked.

4. CLASS FUNCTIONS ON SCHUR ALGEBRAS

Throughout this section, R is a commutative ring, G is a finite group, and $\{E_g;\ g \in G\}$ is a partition of G such that $E_g^{-1} = E_{g^{-1}}$ and $|E_g|$ is invertible in R. Put $s_g = \sum_{x \in E_g} u_x$ in RG, $\hat{s}_g = |E_g|^{-1} s_g$, and let G_0 denote a set of representatives of the distinct E_g. We assume that $S = \bigoplus_{g \in G_0} R s_g$ is a subalgebra with unit element, i.e., S is a Schur algebra in RG. Note that $\hat{s}_e = 1_S$, see Lemma 1.3.

Recall that τ: $S \to R$: $\sum_{g \in G_0} r_g s_g \mapsto r_e$ defines a symmetric associative R-bilinear form b on S with dual bases $\{\hat{s}_g\}, \{s_{g^{-1}}\}$. As in Section 3, let $z = \sum_{g \in G_0} \hat{s}_g s_{g^{-1}}$ and ζ: $S \to Z(S)$: $s \mapsto \sum_{g \in G_0} \hat{s}_g s s_{g^{-1}}$. Again, $Z(S)$ denotes the center of S.

4.1. Definition. We define an equivalence relation on G as follows: $g \sim h$ if and only if $f(\hat{s}_g) = f(\hat{s}_h)$ for all $f \in \operatorname{Sym}(S, R)$. In this case we say that g and h are *S-conjugated* (see also Note 4.12).

4.2. Proposition. *Let $g, h \in G$. If $g \sim h$, then $\zeta(\hat{s}_g) = \zeta(\hat{s}_h)$. In case z is invertible in S, the converse holds true.*

Proof. The result follows from Proposition 3.9. ∎

4.3. *Remark.* Suppose R is connected and $S \cong \operatorname{End}_R(P_1) \oplus \cdots \oplus \operatorname{End}_R(P_q)$ as R-algebras, $P_1, \dots, P_q$ being finitely generated projective R-modules, and suppose that z is invertible in S. Then $g \sim h$ if and only if $t_{P_i}(\hat{s}_g) = t_{P_i}(\hat{s}_h)$ for $i = 1, \dots, q$, see Theorem 3.10(6) and Proposition 3.12(2).

4.4. Lemma. *Let $g, h \in G$. If $g \sim h$, then $g^{-1} \sim h^{-1}$.*

Proof. Let $f \in \operatorname{Sym}(S, R)$. Take the map θ: $RG \to RG$: $\sum_{g \in G} r_g u_g \mapsto \sum_{g \in G} r_g u_{g^{-1}}$ and consider the restriction to S. By Lemma 1.4, $f \circ \theta \in \operatorname{Sym}(S, R)$. Since $g \sim h$, we have $(f \circ \theta)(\hat{s}_g) = (f \circ \theta)(\hat{s}_h)$. The statement follows at once. ∎

For the remainder of this section, we fix the following notation. For $g \in G$, set $K_g = \{h \in G \mid g \sim h\}$. Obviously $\{K_g;\ g \in G\}$ is a partition of G and by Lemma 4.4, $K_{g^{-1}} = K_g^{-1}$. Put $v_g = \sum_{x \in K_g} u_x$ and let G_1 denote a set of representatives of the distinct K_g.

We observe that $K_g = E_g \cup \cdots \cup E_t$, in particular $v_g \in S$. Furthermore, $K_e = E_e$. Indeed, $\tau(\hat{s}_e) = |E_e|^{-1} 1_R$ and $\tau(\hat{s}_k) = 0$ for $k \notin E_e$.

4.5. Definition. Let $f \in S^*$. We say that f is a *class function* on S if $g \sim h$ in G implies that $f(\hat{s}_g) = f(\hat{s}_h)$. The set of all class functions forms an R-submodule of S^*, denoted by $Cf(S, R)$. Clearly $\operatorname{Sym}(S, R) \subset Cf(S, R)$.

4.6. PROPOSITION. (1) $Z(S) \subset \oplus_{g \in G_1} Rv_g$.

(2) $Z(S) = \oplus_{g \in G_1} Rv_g$ *if and only if* $\mathrm{Sym}(S, R) = Cf(S, R)$.

Proof. Consider the left S-linear map $\beta: S \to S^*$ associated to τ as in Remarks 3.1. We know that β is bijective and $\beta(Z(S)) = \mathrm{Sym}(S, R)$, by Lemma 3.4. It suffices to show that $\beta(\oplus Rv_g) = Cf(S, R)$. We have $\beta(v_g)(\hat{s}_k) = \tau(\hat{s}_k v_g) = 1$ for $k \in K_{g^{-1}}$ and $\tau(\hat{s}_k v_g) = 0$ for $k \notin K_{g^{-1}}$. Hence $\beta(\oplus Rv_g) \subset Cf(S, R)$. For the reverse inclusion, use Lemma 3.3. ∎

At the end of this section we give an example to show that the inclusion in Proposition 4.6(1) need not to be an equality. Our next objective is to analyze the equality $Z(S) = \oplus Rv_g$. We begin with a few remarks.

4.7. *Remarks.* (1) If $s_g \in Z(S)$, then $K_g = E_g$ by Proposition 4.6(1).

(2) It is easily verified that $\zeta(v_g) = |K_g| \zeta(\hat{s}_g)$. In particular, if $v_g \in Z(S)$, then $zv_g = |K_g| \zeta(\hat{s}_g)$.

(3) If $v_g \in Z(S)$ and z is invertible in S, then $|K_g|$ is invertible in R. Indeed, $v_g = |K_g| \zeta(\hat{s}_g) z^{-1} = |K_g| \Sigma_{k \in G_1} r_k v_k$ with $r_k \in R$, whence $1 = |K_g| r_g$.

4.8. PROPOSITION. *Suppose that z is invertible in S. Then $Z(S) = \oplus_{g \in G_1} Rv_g$ if and only if the distinct $\zeta(\hat{s}_k)$ are linearly independent over R.*

Proof. By Proposition 4.2, $\zeta(\hat{s}_g)$, $g \in G_1$, are all distinct $\zeta(\hat{s}_t)$. Suppose that $\zeta(\hat{s}_g)$, $g \in G_1$, are linearly independent over R. Let $f \in Cf(S, R)$. It suffices to show that f is symmetric, see Proposition 4.6. Let $x \in S$ be such that $\zeta(x) = 0$ and write $x = \Sigma_{k \in G_0} r_k \hat{s}_k, r_k \in R$. So $0 = \zeta(x) = \Sigma_{g \in G_1}(\Sigma_{k \in J(g)} r_k) \zeta(\hat{s}_g)$ with $J(g) = G_0 \cap K_g$, whence $\Sigma_{k \in J(g)} r_k = 0$. It follows that $f(x) = 0$ and thus f is symmetric, see Proposition 3.8.

For the converse, use Remarks 4.7(2) and (3). ∎

As in Section 1, we may consider the following componentwise multiplication on RG. Let $a, a' \in RG$, $a = \Sigma_{g \in G} r_g u_g$, and $a' = \Sigma_{g \in G} r'_g u_g$ with $r_g, r'_g \in R$. Then we define $a * a' = \Sigma_{g \in G} r_g r'_g u_g$. Of course S is closed under this multiplication.

4.9. PROPOSITION. *Suppose that R is a domain. If $Z(S)$ is closed under the above componentwise multiplication, then $Z(S) = \oplus_{g \in G_1} Rv_g$.*

Proof. (1) We first assume that R is a field. Note that $\Sigma_{g \in G} u_g = \Sigma_{g \in G_0} s_g \in Z(RG) \cap S$, hence $\Sigma_{g \in G} u_g \in Z(S)$. Then by Proposition 1.5, there is a partition $\{F_k;\ k \in G\}$ of G such that $Z(S) = \oplus_{w_k} Rw_k$ with $w_k = \Sigma_{x \in F_k} u_x$. Since $Z(S) \subset \oplus_{g \in G_1} Rv_g$, each w_k is a sum of certain v_g. Fix w_k; say $w_k = v_{g_1} + \cdots + v_{g_m}$, $g_i \in G_1$. We now prove that $m = 1$.

Let $f \in \mathrm{Sym}(S, R)$. By Lemmas 3.3 and 3.4, $c = \Sigma_{g \in G_0} f(\hat{s}_{g^{-1}}) s_g \in Z(S)$, and $c = \Sigma_{g \in G_1} f(\hat{s}_{g^{-1}}) v_g$. But $c * w_k = rw_k$ for some $r \in R$. Therefore

$f(\hat{s}_{g_1^{-1}}) = \cdots = f(\hat{s}_{g_m^{-1}}) = r$. From this it follows that $g_1 \sim g_i$, $i = 1, \ldots, m$. Consequently, $m = 1$ and $w_k = v_{g_1}$. Then, using $\sum_{w_k} w_k = \sum_{g \in G_1} v_g$, we obtain $Z(S) = \bigoplus_{g \in G_1} Rv_g$.

(2) Now let R be a domain with field of quotients L. Consider the Schur algebra $\bar{S} = \bigoplus_{g \in G_0} Ls_g$ in LS. We observe that $Z(S) = Z(\bar{S}) \cap S$. Then it is easily verified that $Z(\bar{S})$ is closed under componentwise multiplication in LG. Further, $g, h \in G$ are $\bar{S}$-conjugated if and only if they are S-conjugated. In order to prove this, one needs the following remarks. A map $f \in \mathrm{Sym}(S, R)$ can be extended to a map $\bar{f} \in \mathrm{Sym}(\bar{S}, L)$ by setting $\bar{f}(\sum_{g \in G_0} l_g s_g) = \sum_{g \in G_0} l_g f(s_g), l_g \in L$. On the other hand, let $\varphi \in \mathrm{Sym}(\bar{S}, L)$. Then there exists $r \in R$ such that $r\varphi(s_g) \in R$ for all $g \in G_0$, and $r\varphi|_S \in \mathrm{Sym}(S, R)$. The above discussion yields the equality $Z(\bar{S}) = \bigoplus_{g \in G_1} Lv_g$. Consequently, $v_g \in Z(\bar{S}) \cap S = Z(S)$, as desired. ∎

4.10. *Remark*. To the above defined componentwise multiplication on RG there corresponds a multiplication on $(RG)^*$; see Note 3.15(3). Namely, let $\varphi, \psi \in (RG)^*$. Then $\varphi * \psi(u_{g^{-1}}) = \varphi(u_{g^{-1}})\psi(u_{g^{-1}})$, or equivalently, $\varphi * \psi(u_g) = \varphi(u_g)\psi(u_g)$ for all $g \in G$.

In the case where $Z(S) = \oplus Rv_g$ we can derive the second orthogonality relations.

4.11. PROPOSITION. *Suppose R is connected and $S \cong \mathrm{End}_R(P_1) \oplus \cdots \oplus \mathrm{End}_R(P_q)$ as R-algebras, $P_1, \ldots, P_q$ being finitely generated projective R-modules, and suppose that z is invertible in S. If $Z(S) = \bigoplus_{g \in G_1} Rv_g$, then for $g, h \in G_1$ we have*

$$\sum_{i=1}^{q} |K_h| \mathrm{rank}(P_i) t_{P_i}(z)^{-1} t_{P_i}(\hat{s}_g) t_{P_i}(\hat{s}_{h^{-1}}) = \delta_{gh}.$$

Proof. Note that $|G_1| = q$. By Theorem 3.10 and Proposition 3.12,

$$\sum_{g \in G_0} \mathrm{rank}(P_i) t_{P_i}(z)^{-1} t_{P_i}(\hat{s}_g) t_{P_j}(s_{g^{-1}}) = \delta_{ij}.$$

This gives

$$\sum_{g \in G_1} \mathrm{rank}(P_i) t_{P_i}(z)^{-1} t_{P_i}(\hat{s}_g) t_{P_j}(v_{g^{-1}}) = \delta_{ij}$$

and $t_{P_j}(v_{g^{-1}}) = |K_g| t_{P_j}(\hat{s}_{g^{-1}})$. We can write this relation as $AB = I$; A, B being $q \times q$ matrices. Then $BA = I$, which implies the desired formula. ∎

4.12. *Note*. We discuss the case where $S = RG$. Here, $g, h \in G$ are RG-conjugated if and only if $h = tgt^{-1}$ for some $t \in G$. Indeed, suppose that $f(u_g) = f(u_h)$ for all $f \in \mathrm{Sym}(RG, R)$. In other words, $\tau(u_g c) =$

$\tau(u_h c)$ for all $c \in Z(RG)$, see Lemma 3.4. Let s denote the sum in RG of all distinct conjugates $kg^{-1}k^{-1}$, $k \in G$. Clearly, $s \in Z(RG)$ and $\tau(u_g s) = 1$. Consequently, $\tau(u_h s) = 1$, whence $tg^{-1}t^{-1} = h^{-1}$ for some $t \in G$. The converse is obvious.

With notation as before, we have $v_g \in Z(RG)$ and $Z(RG) = \oplus_{g \in G_1} Rv_g$. Moreover, $\zeta(u_g) = |C_G(g)| v_g$.

Let us now focus on the case where S is a double coset algebra. So let H be a subgroup of G with $|H|^{-1} \in R$, put $\varepsilon = |H|^{-1} \sum_{h \in H} u_h$, and consider $S = \varepsilon RG \varepsilon$, see also Section 2.

Let $Z(S)$ and τ be as before, and put $\hat{s}_g = |HgH|^{-1} \sum_{x \in HgH} u_x$, for $g \in G$.

For RG-conjugacy we now set $C_k = \{tkt^{-1} \mid t \in G\}$ and $w_k = \sum_{x \in C_k} u_x$, with $k \in G$.

4.13. PROPOSITION. *Consider $S = \varepsilon RG \varepsilon$ and let $g_1, g_2 \in G$.*

(1) *If g_1 and g_2 are S-conjugated, then*

$$|Hg_1 H|^{-1} |Hg_1 H \cap C_k| 1_R = |Hg_2 H|^{-1} |Hg_2 H \cap C_k| 1_R$$

for any RG-conjugacy class C_k.

(2) *If R is connected and R is a splitting ring for RG, then the converse of* (1) *holds.*

Proof. Note that g_1 and g_2 are S-conjugated if and only if $\tau(\hat{s}_{g_1} c) = \tau(\hat{s}_{g_2} c)$ for all $c \in Z(S)$, see Lemma 3.4.

(1) Clearly $\varepsilon w_k \in Z(S)$. Further, $\tau(\hat{s}_g \varepsilon w_k) = \tau(\hat{s}_g w_k) = |HgH|^{-1} |HgH \cap C_{k^{-1}}| 1_R$. The assertion now follows.

(2) It suffices to show that εw_k, $k \in G$, generate $Z(S)$ as R-module. Let $\{e_1, \dots, e_q\}$ be the set of primitive central nonzero idempotents of RG, and let $\varepsilon e_i \neq 0$ for $i = 1, \dots, m$. Take $a \in Z(S)$. By Theorem 2.5(1)–(3), we have $a = \sum_{i=1}^{m} r_i \varepsilon e_i$ with $r_i \in R$. Moreover, $e_i = \sum r'_k w_k$ with $r'_k \in R$. ∎

We conclude this section with a concrete example of the above situation, based on [15]. This example shows that the inclusion in Proposition 4.6(1) need not to be an equality.

EXAMPLE. Consider in $GL_3(\mathbb{Z}_3)$, the matrices

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $G = \langle a, b, d \rangle$. We have the relations $a^3 = b^3 = I$, $d^2 = I$, $da = a^2 d$, and $db = b^2 d$. Further, $c = bab^{-1}a^{-1}$, $c^3 = I$, and c commutes with a, b, d. So each element of G can be expressed as $a^i b^j d^k c^l$ with $i, j, l = 0, 1, 2$ and $k = 0, 1$.

Now let $H = \langle d \rangle$, put $\varepsilon = \frac{1}{2}(u_I + u_d)$ in $\mathbb{C}G$, and consider $S = \varepsilon(\mathbb{C}G)\varepsilon$. Using Proposition 4.13 and the RG-conjugacy classes C_a and C_d, it is easy to check that the S-conjugacy class K_a of a is equal to HaH. However, $\underline{HaH} = \sum_{x \in HaH} u_x$ does not commute with $\underline{HbH}$, and thus $\underline{HaH} \notin Z(S)$.

5. TRACE FUNCTIONS OF INDUCED MODULES

Throughout this section, R is a commutative ring, G is a finite group, and H is a subgroup of G. Let S be a Schur algebra in RG with associated partition $\{E_g;\, g \in G\}$ and let B be a Schur algebra in RH with partition $\{F_h;\, h \in H\}$. Further, let G_0, resp. H_0, denote a set of representatives of the distinct E_g, resp. F_h. Put $s_g = \sum_{x \in E_g} u_x$ and $b_h = \sum_{x \in F_h} u_x$.

5.1. DEFINITION. The Schur algebra B is called a Schur subalgebra of S if for each $h \in H$ we have $F_h = \cup E_g$, for some $g \in G$.

For the remainder of this section, we assume that B is a Schur subalgebra of S. We also assume $|E_g|^{-1} \in R$ and $|F_h|^{-1} \in R$ for all $g \in G$, $h \in H$. We set $\hat{s}_g = |E_g|^{-1} s_g$, analogously $\hat{b}_h$.

5.2. DEFINITION. Let $f \in \operatorname{Hom}_R(B, R)$. We define $\tilde{f} \in \operatorname{Hom}_R(S, R)$ as follows: $\tilde{f}(\hat{s}_g) = 0$ if $g \notin H$ and $\tilde{f}(\hat{s}_g) = f(\hat{b}_g)$ if $g \in H$, and extend by linearity.

We observe that $\tilde{f}|_B = f$.

Under certain conditions, we shall derive a formula for the trace function of an induced module. We set $z_S = \sum_{g \in G_0} \hat{s}_g s_{g^{-1}}$ and $z_B = \sum_{h \in H_0} \hat{b}_h b_{h^{-1}}$.

5.3. PROPOSITION. *Assume that $F_e = E_e$ and that z_S is invertible in S. Suppose R is connected and finitely generated projective R-modules are free. Further, suppose $S \cong \oplus_{j=1}^{q} \operatorname{End}_R(M_j)$ and $B \cong \oplus_{i=1}^{p} \operatorname{End}_R(N_i)$ as R-algebras, where M_j, N_i are finitely generated projective R-modules. Set $N_i^S = S \otimes_B N_i$. Then*

$$t_{N_i}(z_B) t_{N_i^S} = \operatorname{rank}_R(N_i)\left(\tilde{t}_{N_i} \circ \zeta\right) \qquad \text{on } S,$$

where $\zeta\colon S \to Z(S)\colon x \mapsto \sum_{g \in G_0} \hat{s}_g x s_{g^{-1}}$.

Proof. Recall that N_i is an indecomposable left B-module (similar remark for M_j). Combining Remark 3.11(3) and Proposition 3.12, we have for any $x \in S$

$$\zeta(x) = \sum_{j=1}^{q} t_{M_j}(x)\left(\sum_{g \in G_0} t_{M_j}(s_{g^{-1}})\hat{s}_g\right).$$

Applying $\tilde{t}_{N_i}$ to this expression yields

$$\begin{aligned}\tilde{t}_{N_i}(\zeta(x)) &= \sum_{j=1}^{q} t_{M_j}(x)\left(\sum_{g \in G_0 \cap H} t_{M_j}(s_{g^{-1}})t_{N_i}\left(\hat{b}_g\right)\right)\\ &= \sum_{j=1}^{q} t_{M_j}(x)\left(\sum_{g \in H_0} t_{M_j}(b_{g^{-1}})t_{N_i}\left(\hat{b}_g\right)\right).\end{aligned}$$

By the hypothesis on R, we have $M_j|_B \cong \oplus_k N_k^{c_{kj}}$ as left B-modules, where $c_{kj} \in \mathbb{N}$. Thus $t_{M_j} = \sum_k c_{kj} t_{N_k}$ on B. Using the orthogonality relations, Theorem 3.10(2) and Proposition 3.12(1), we then obtain

$$\operatorname{rank}_R(N_i)\tilde{t}_{N_i}(\zeta(x)) = \sum_{j=1}^{q} t_{M_j}(x) c_{ij} t_{N_i}(z_B).$$

By the hypothesis on R, we can apply a version of Frobenius reciprocity, see [3, 1.2]. This gives $t_{N_i^s} = \sum_{j=1}^{q} c_{ij} t_{M_j}$, which completes the proof. ∎

To conclude, let $S = RG$ and $B = RH$. In this case we have $z_S = |G|u_e$ and $z_B = |H|u_e$. With hypotheses and notation as in Proposition 5.3 (in particular $|G|^{-1} \in R$), we now obtain

$$|H| t_{N_i^s}(u_x) = \tilde{t}_{N_i}\left(\sum_{g \in G} u_{gxg^{-1}}\right), \qquad \text{for } x \in G$$

(use also Proposition 3.12(2)). Of course, this formula can be proved without any assumption (N being a left RH-module, which is finitely generated and projective over R).

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