Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Stochastic Processes and their Applications 117 (2007) 677–707
stochastic processes and their applications
www.elsevier.com/locate/spa

Tempering stable processes

Jan Rosiński*

Department of Mathematics, University of Tennessee, Knoxville, TN 39996, United States

Received 17 August 2004; received in revised form 30 September 2006; accepted 6 October 2006
Available online 3 November 2006

Abstract

A tempered stable Lévy process combines both the α-stable and Gaussian trends. In a short time frame it is close to an α-stable process while in a long time frame it approximates a Brownian motion. In this paper we consider a general and robust class of multivariate tempered stable distributions and establish their identifiable parametrization. We prove short and long time behavior of tempered stable Lévy processes and investigate their absolute continuity with respect to the underlying α-stable processes. We find probabilistic representations of tempered stable processes which specifically show how such processes are obtained by cutting (tempering) jumps of stable processes. These representations exhibit α-stable and Gaussian tendencies in tempered stable processes and thus give probabilistic intuition for their study. Such representations can also be used for simulation. We also develop the corresponding representations for Ornstein–Uhlenbeck-type processes.

MSC: primary 60E07; 60G52; secondary 60E10; 60G51

Keywords: Tempered stable distributions and processes; Stable processes; Lévy processes; Ornstein–Uhlenbeck-type processes; Shot noise representations

1. Introduction

Tempered stable processes were introduced in statistical physics to model turbulence and are known in physics literature as the *truncated Lévy flight model* [14,13,15]. They were also introduced in mathematical finance to model stochastic volatility (the *CGMY model* in [8,9]), (the *Ornstein–Uhlenbeck-based model* in [3,4]); option pricing based on such processes was

* Tel.: +1 865 9742461; fax: +1 865 9746576.
E-mail address: rosinski@math.utk.edu.

0304-4149/$ - see front matter
doi:10.1016/j.spa.2006.10.003

considered in [6], just to mention a few. Furthermore, these processes play an important role in the construction of certain Poisson–Dirichlet laws studied in [16]. The importance of tempered stable processes comes from the fact that they combine both the α-stable and Gaussian trends.

In this paper we introduce and study a more general and robust class of tempered stable distributions. It contains previously investigated tempered stable distributions as a special subclass. We show that tempered stable distributions admit parametrization similarly to stable distributions. Namely, a multivariate tempered stable distribution is characterized by an index $\alpha \in (0, 2)$, a spectral measure R, and a shift b (Theorems 2.3 and 2.9, Definition 2.11). Moreover, this parametrization is identifiable in the subclass of proper tempered stable distributions. Unlike stable distributions, tempered ones may have all moments finite, including exponential moments of some order (Proposition 2.7). In Section 3 we give a rigorous proof of the statement that a tempered stable Lévy process in a short time looks like a stable process while in a large time scale it looks like a Brownian motion (Theorem 3.1). In Section 4 we show that a large class of tempered stable Lévy processes can be obtained from stable processes by a change of measure on the probability space (Theorem 4.1). The heart of this paper is in Section 5. We consider the question of how does the tempering occur, that changes a sample path of a stable process into a sample path of a tempered one? This question with a view toward simulation was posed to the author by Ole E. Barndorff-Nielsen and Neil Shephard with the condition that in a possible answer the tempering procedure should be continuous pathwise with respect to the parameters, and thus it should not include any removal of jumps from a sample path of a stable process. We answer this question in Theorem 5.3 obtaining shot noise representations of tempered stable Lévy processes. Our representations exhibit stable and Gaussian trends in tempered stable processes and give probabilistic insight into Theorems 3.1 and 4.1. They can also be used for computer simulation. In Section 6 we give shot noise representations for Ornstein–Uhlenbeck-type processes with tempered stable one-dimensional marginal distributions. A special case of such representations has already been used in [3,4]; cf. [18].

Before going to formal definitions let us sketch some ideas leading to tempered infinitely divisible distributions and processes, in general. The first one is an old idea of tilting density functions. Let f be a probability density function on $\mathbb{R}_+$ whose Laplace transform is $L(\lambda) = \int_0^\infty e^{-\lambda x} f(x)dx$. For every $\theta > 0$ define a tilted density f_θ by

$$f_\theta(x) = \frac{1}{L(\theta)} e^{-\theta x} f(x) = \exp\{-\theta x + \ell(\theta) + k(x)\}, \tag{1.1}$$

where $f(x) = \exp\{k(x)\}$ and $L(\theta) = \exp\{-\ell(\theta)\}$. We see that $\{f_\theta\}_\theta$ forms a one-parameter exponential family of distributions with the natural statistic $T(x) = -x$. The Laplace transform L_θ of f_θ is given by

$$L_\theta(\lambda) = \exp\{-(\ell(\lambda + \theta) - \ell(\theta))\}. \tag{1.2}$$

Assume additionally that f is infinitely divisible, so that we have

$$\ell(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) M(dx) + \lambda b,$$

where M is a Lévy measure on $\mathbb{R}_+$ and $b \geq 0$. From (1.2) we get

$$L_\theta(\lambda) = \exp\left[\int_0^\infty (e^{-\lambda x} - 1) e^{-\theta x} M(dx) - \lambda b\right].$$

Therefore, tilting an infinitely divisible density $f \mapsto f_\theta$ leads to the tilting of the corresponding Lévy measure $M \mapsto M_\theta$, where $M_\theta(\mathrm{d}x) = \mathrm{e}^{-\theta x} M(\mathrm{d}x)$.

The procedure of tilting is also related to the so-called Esscher transform. Namely, let $\{X(t) : t \geq 0\}$ be the canonical process adapted to a natural filtration $\{\mathcal{F}_t : t \geq 0\}$ (see Section 4). Suppose that under probability P, X is a Lévy process with non-decreasing trajectories such that $X(1)$ has the density f (i.e., X is a subordinator). For $\theta > 0$ define a probability measure P_θ by

$$\frac{\mathrm{d}P_\theta}{\mathrm{d}P}_{|\mathcal{F}_t} = \exp\{-\theta X(t) + \ell(\theta)t\}.$$

Then, under P_θ, X is a Lévy process such that $X(1)$ has the density f_θ; see [22, Example 33.15]. Thus Esscher's transform can be viewed as tilting (1.1) but on the level of stochastic processes.

Taking products of convolution powers $f_{\theta_i}^{*r_i}$ of f_{θ_i} with $r_i, \theta_i > 0$ (f is infinitely divisible), and then their limits, we obtain distributions having the Laplace transform of the form

$$\exp\left[\int_0^\infty (\mathrm{e}^{-\lambda x} - 1) q(x) M(\mathrm{d}x) - \lambda b\right],$$

where q is a completely monotone function with $q(\infty) = 0$. Such operation on Lévy measures M, and their multidimensional generalizations, will be called *tempering* (or tilting when $q(x) = \mathrm{e}^{-\theta x}$, $x > 0$). In this work we concentrate on tempered stable distributions obtained by tempering stable Lévy measures. We show that tempered stable distributions and related processes constitute rich classes with nice structural and analytical properties. They can be used as an attractive alternative to stable distributions and processes in modeling and theoretical considerations.

A preliminary version of these results was announced at the Second MaPhySto Conference on Lévy Processes: Theory and Applications, Aarhus 2002, and is available as an extended abstract in the Mini-proceedings [20].

2. Tempered stable distributions

In this section we will give the parametrization, the basic properties, and the canonical form of characteristic functions of multivariate tempered stable distributions. It is well known that the Lévy measure M_0 of an α-stable distribution on $\mathbb{R}^d$ in polar coordinates is of the form

$$M_0(\mathrm{d}r, \mathrm{d}u) = r^{-\alpha-1}\mathrm{d}r\sigma(\mathrm{d}u), \tag{2.1}$$

where $\alpha \in (0, 2)$ and σ is a finite measure on S^{d-1}. A tempered α-stable distribution is obtained by tempering the radial component of M_0 as follows.

Definition 2.1. A probability measure μ on $\mathbb{R}^d$ is called *tempered α-stable* (abbreviated as $T\alpha S$) if is infinitely divisible without Gaussian part and has Lévy measure M that can be written in polar coordinates as

$$M(\mathrm{d}r, \mathrm{d}u) = r^{-\alpha-1} q(r, u)\mathrm{d}r\sigma(\mathrm{d}u), \tag{2.2}$$

where α and σ are as above, and $q : (0, \infty) \times S^{d-1} \mapsto (0, \infty)$ is a Borel function such that $q(\cdot, u)$ is completely monotone with $q(\infty, u) = 0$ for each $u \in S^{d-1}$. μ is called a *proper $T\alpha S$* distribution if, in addition to the above, $q(0+, u) = 1$ for each $u \in S^{d-1}$.

The complete monotonicity of $q(\cdot, u)$ means that $(-1)^n \frac{\partial^n}{\partial r^n} q(r, u) > 0$ for all $r > 0$, $u \in S^{d-1}$, and $n = 0, 1, 2, \ldots$. In particular, $q(\cdot, u)$ is strictly decreasing and convex.

Remark 2.2. (a) The class of $T\alpha S$ distributions contains β-stable distributions with $\beta > \alpha$. Indeed, one takes $q(r, u) = r^{\alpha-\beta}$ in (2.2). However, proper $T\alpha S$ distributions do not contain any stable distributions.
(b) $T\alpha S$ distributions are self-decomposable; this follows from [22, Theorem 15.10]. Moreover, they constitute a proper subclass of the Thorin class of extended generalized gamma convolutions (see [5, p. 105] and [2]). The Thorin class is obtained when $\alpha = 0$ in (2.2).

The "tempering" function q in (2.2) can be represented as

$$q(r, u) = \int_0^\infty \mathrm{e}^{-rs} Q(\mathrm{d}s|u) \tag{2.3}$$

where $\{Q(\cdot|u)\}_{u \in S^{d-1}}$ is a measurable family of Borel measures on $(0, \infty)$. $Q(\cdot|u)$ are probability measures in the case of proper $T\alpha S$ distributions. Define a measure Q on $\mathbb{R}^d$ by

$$Q(A) := \int_{S^{d-1}} \int_0^\infty I_A(ru) Q(\mathrm{d}r|u)\sigma(\mathrm{d}u), \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.4}$$

Then $Q(\{0\}) = 0$. We also define a measure R by

$$R(A) = \int_{\mathbb{R}^d} I_A\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha Q(\mathrm{d}x), \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.5}$$

This yields the change of variable formula

$$\int_{\mathbb{R}^d} F(x) R(\mathrm{d}x) = \int_{\mathbb{R}^d} F\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha Q(\mathrm{d}x)$$

for any Borel function F in the sense that when one of the sides exists then the other exists and they are equal. Clearly $R(\{0\}) = 0$ and Q can be obtained from R by the same transformation

$$Q(A) = \int_{\mathbb{R}^d} I_A\left(\frac{x}{\|x\|^2}\right) \|x\|^\alpha R(\mathrm{d}x). \tag{2.6}$$

As it turns out, distributional properties of a $T\alpha S$ measure are described most conveniently by its measure R. The following result is fundamental for the rest of the development.

Theorem 2.3. *The Lévy measure M of a $T\alpha S$ distribution can be written in the form*

$$M(A) = \int_{\mathbb{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} \mathrm{e}^{-t} \mathrm{d}t\, R(\mathrm{d}x), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{2.7}$$

where R is a unique measure on $\mathbb{R}^d$ such that

$$R(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R(\mathrm{d}x) < \infty. \tag{2.8}$$

If M is as in (2.2) then R is given by (2.5). Conversely, if R is a measure satisfying (2.8) then (2.7) defines the Lévy measure of a $T\alpha S$ distribution. M corresponds to a proper $T\alpha S$ distribution if *and only if*

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(\mathrm{d}x) < \infty. \tag{2.9}$$

Proof. First we will show that (2.7) holds when R is given by (2.5). We have

$$\begin{aligned}
M(A) &= \int_{S^{d-1}} \int_0^\infty I_A(ru) r^{-\alpha-1} q(r,u)\,\mathrm{d}r\sigma(\mathrm{d}u) \\
&= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A(ru) r^{-\alpha-1} \mathrm{e}^{-rs} \mathrm{d}r \right) Q(\mathrm{d}s|u)\sigma(\mathrm{d}u) \\
&= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A(ts^{-1}u) t^{-\alpha-1} \mathrm{e}^{-t} \mathrm{d}t \right) s^\alpha Q(\mathrm{d}s|u)\sigma(\mathrm{d}u) \\
&= \int_0^\infty \left(\int_{S^{d-1}} \int_0^\infty I_A(ts^{-1}u) s^\alpha Q(\mathrm{d}s|u)\sigma(\mathrm{d}u) \right) t^{-\alpha-1} \mathrm{e}^{-t}\,\mathrm{d}t \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} I_A\left(t\frac{x}{\|x\|^2} \right) \|x\|^\alpha Q(\mathrm{d}x) \right) t^{-\alpha-1} \mathrm{e}^{-t}\,\mathrm{d}t \\
&= \int_0^\infty \int_{\mathbb{R}^d} I_A(ty) R(\mathrm{d}y) t^{-\alpha-1} \mathrm{e}^{-t}\,\mathrm{d}t.
\end{aligned}$$

To prove (2.8) we use (2.7) as follows:

$$\begin{aligned}
\infty &> \int_{\|x\|\le 1} \|x\|^2 M(\mathrm{d}x) = \int_{\mathbb{R}^d} \|x\|^2 \int_0^{\|x\|^{-1}} t^{1-\alpha} \mathrm{e}^{-t} \mathrm{d}t\, R(\mathrm{d}x) \\
&\ge \int_{\|x\|\le 1} \|x\|^2 \int_0^1 t^{1-\alpha} \mathrm{e}^{-t}\,\mathrm{d}t\, R(\mathrm{d}x) + \int_{\|x\|>1} \|x\|^2 \int_0^{\|x\|^{-1}} t^{1-\alpha} \mathrm{e}^{-1}\,\mathrm{d}t\, R(\mathrm{d}x) \\
&\ge \mathrm{e}^{-1}(2-\alpha)^{-1} \int_{\|x\|\le 1} \|x\|^2 R(\mathrm{d}x) + \mathrm{e}^{-1}(2-\alpha)^{-1} \int_{\|x\|>1} \|x\|^\alpha R(\mathrm{d}x).
\end{aligned}$$

To prove the uniqueness of R, we suppose that two measures R_1 and R_2 satisfy (2.7) when substituted in place of R. Such measures must satisfy (2.8), as demonstrated above. Define measures Q_i by (2.6), $Q_i(\{0\}) = 0, i = 1, 2$. Consider representations of Q_i in polar coordinates in the form

$$Q_i(\mathrm{d}r, \mathrm{d}u) = Q_i(\mathrm{d}r|u)\sigma(\mathrm{d}u) \quad i = 1, 2,$$

where σ is a probability measure on S^{d-1} and $\{Q_i(\cdot|u)\}_{u\in S^{d-1}}$ are measurable families of Borel measures on $(0,\infty)$. Since

$$\begin{aligned}
\infty &> \int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R_i(\mathrm{d}x) = \int_{\mathbb{R}^d} (\|x\|^{-2} \wedge \|x\|^{-\alpha}) \|x\|^\alpha Q_i(\mathrm{d}x) \\
&= \int_{S^{d-1}} \int_0^\infty (s^{-2+\alpha} \wedge 1) Q_i(\mathrm{d}s|u)\sigma(\mathrm{d}u),
\end{aligned}$$

$\int_0^\infty (s^{-2+\alpha} \wedge 1) Q_i(\mathrm{d}s|u) < \infty$ for σ-almost every u. Hence, for σ-almost every u and all $r > 0$, the Laplace transform

$$q_i(r,u) = \int_0^\infty \mathrm{e}^{-rs} Q_i(\mathrm{d}s|u)$$

is finite and thus it defines completely monotone functions $q_i(\cdot|u) : (0, \infty) \to [0, \infty)$. Since the R_i satisfy (2.7), the computations at the beginning of this proof show that

$$\int_{S^{d-1}} \int_0^\infty I_A(ru) r^{-\alpha-1} q_i(r, u) \,\mathrm{d}r \sigma(\mathrm{d}u) = M(A), \quad A \in \mathcal{B}(\mathbb{R}^d),\ i = 1, 2.$$

From the uniqueness of the representation in polar coordinates of M and the continuity of $q_i(\cdot, u)$ we infer that for σ-almost all u, $q_1(r, u) = q_2(r, u)$, $r > 0$. Thus $Q_1(\cdot|u) = Q_2(\cdot|u)$, which yields $Q_1 = Q_2$ and then (2.5) gives $R_1 = R_2$.

In the converse part, we will first prove that M is a Lévy measure when R satisfies (2.8). Indeed,

$$\begin{aligned}\int_{\|x\|\le 1} \|x\|^2 M(\mathrm{d}x) &= \int_{\mathbb{R}^d} \|x\|^2 \int_0^{\|x\|^{-1}} t^{1-\alpha} \mathrm{e}^{-t} \,\mathrm{d}t R(\mathrm{d}x) \\ &\le \Gamma(2-\alpha) \int_{\|x\|\le 1} \|x\|^2 R(\mathrm{d}x) + \frac{1}{2-\alpha} \int_{\|x\|>1} \|x\|^\alpha R(\mathrm{d}x) < \infty\end{aligned}$$

and

$$\begin{aligned}\int_{\|x\|>1} M(\mathrm{d}x) &= \int_{\mathbb{R}^d} \int_{\|x\|^{-1}}^\infty t^{-\alpha-1} \mathrm{e}^{-t} \mathrm{d}t R(\mathrm{d}x) \\ &\le C \int_{\|x\|\le 1} \int_{\|x\|^{-1}}^\infty t^{-3} \,\mathrm{d}t R(\mathrm{d}x) + \int_{\|x\|>1} \int_{\|x\|^{-1}}^\infty t^{-\alpha-1} \,\mathrm{d}t R(\mathrm{d}x) \\ &= 2^{-1} C \int_{\|x\|\le 1} \|x\|^2 R(\mathrm{d}x) + \alpha^{-1} \int_{\|x\|>1} \|x\|^\alpha R(\mathrm{d}x) < \infty,\end{aligned}$$

where $C := \sup_{t\ge 1} t^{2-\alpha} \mathrm{e}^{-t}$. The fact that M can be written in the form (2.2) can be proved as follows. Define Q by (2.6) and consider a decomposition $Q(\mathrm{d}r, \mathrm{d}u) = Q(\mathrm{d}r|u)\sigma(\mathrm{d}r)$, where $\sigma(S^{d-1}) < \infty$. Then we define $q(r, u)$ by (2.3). The computations given at the beginning of this proof verify (2.2).

Finally, observe that

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(\mathrm{d}x) = Q(\mathbb{R}^d). \tag{2.10}$$

Furthermore, Q has a representation $Q(\mathrm{d}r, \mathrm{d}u) = Q(\mathrm{d}r|u)\sigma(\mathrm{d}u)$, where $Q(\cdot|u)$ are probability measures (equivalently, $q(0+, u) = 1$) and σ is a finite measure if and only if $Q(\mathbb{R}^d) < \infty$. In view of (2.10) the proof is complete. □

Definition 2.4. The unique measure R in (2.7) is called the *spectral measure* of the corresponding $T\alpha S$ distribution.

We would like to mention that the necessity and sufficiency of (2.8) for M to be a Lévy measure was also stated in [2] (without a proof). The next corollary explains the difference between proper and general $T\alpha S$ distributions in terms of Lévy measures.

Corollary 2.5. *Let M be a measure given by (2.7). Then the function $s \mapsto s^\alpha M(\{\|x\| > s\})$, $s > 0$, is decreasing with*

$$\lim_{s\to 0^+} s^\alpha M(\{\|x\| > s\}) = \alpha^{-1} \int_{\mathbb{R}^d} \|x\|^\alpha R(\mathrm{d}x) \quad \textit{and} \quad \lim_{s\to\infty} s^\alpha M(\{\|x\| > s\}) = 0.$$

Hence M is a Lévy measure of a proper $T\alpha S$ distribution if and only if

$$\lim_{s\to 0^+} s^\alpha M(\{\|x\| > s\}) < \infty.$$

Proof. From (2.7) we get for every $s > 0$

$$s^\alpha M(\{\|x\| > s\}) = \int_{\mathbb{R}^d} \int_{\|x\|^{-1}}^\infty v^{-\alpha-1} \mathrm{e}^{-sv}\, \mathrm{d}v R(\mathrm{d}x).$$

Hence the map $s \mapsto s^\alpha M(\{\|x\| > s\})$ is decreasing to zero. By the monotone convergence theorem

$$\lim_{s\to 0^+} s^\alpha M(\{\|x\| > s\}) = \alpha^{-1} \int_{\mathbb{R}^d} \|x\|^\alpha R(\mathrm{d}x).$$

The conclusion comes from the last part of Theorem 2.3. $\square$

Corollary 2.6. *In the subclass of proper tempered stable distributions the parametrization (α, R) of the Lévy measures is identifiable.*

$T\alpha S$ distributions may have moments of any order, even exponential moments of some order. This simply depends on their spectral measures.

Proposition 2.7. *Let μ be a $T\alpha S$ distribution with Lévy measure given by (2.7). Then*

(i) $\int_{\mathbb{R}^d} \|x\|^p \mu(\mathrm{d}x) < \infty$ *for* $p \in (0, \alpha)$;
(ii) $\int_{\mathbb{R}^d} \|x\|^\alpha \mu(\mathrm{d}x) < \infty \iff \int_{\|x\|>1} \|x\|^\alpha \log \|x\| R(\mathrm{d}x) < \infty$;
(iii) $\int_{\mathbb{R}^d} \|x\|^p \mu(\mathrm{d}x) < \infty \iff \int_{\|x\|>1} \|x\|^p R(\mathrm{d}x) < \infty$ *when* $p > \alpha$;
(iv) $\int_{\mathbb{R}^d} \exp(\theta\|x\|)\mu(\mathrm{d}x) < \infty \iff R(\{x : \|x\| > \theta^{-1}\}) = 0$, *where* $\theta > 0$.

Proof. The above moment conditions for μ are equivalent to the corresponding conditions for $M_{|\{\|x\|>1\}}$; cf. [22, p. 159]. We write for $p > 0$

$$\begin{aligned}\int_{\|x\|>1} \|x\|^p M(\mathrm{d}x) &= \int_{\|x\|\le 1} \|x\|^p \int_{\|x\|^{-1}}^\infty t^{p-\alpha-1}\mathrm{e}^{-t}\, \mathrm{d}t R(\mathrm{d}x) \\ &\quad + \int_{\|x\|>1} \|x\|^p \int_{\|x\|^{-1}}^\infty t^{p-\alpha-1}\mathrm{e}^{-t}\, \mathrm{d}t R(\mathrm{d}x) =: I_1 + I_2.\end{aligned}$$

Let $C := \sup_{t\ge 1} t^{p+2-\alpha}\mathrm{e}^{-t}$. Then

$$I_1 \le C \int_{\|x\|\le 1} \|x\|^p \int_{\|x\|^{-1}}^\infty t^{-3}\, \mathrm{d}t R(\mathrm{d}x) \le 2^{-1} C \int_{\|x\|\le 1} \|x\|^2 R(\mathrm{d}x) < \infty$$

by (2.8). Therefore, the finiteness of $\int_{\|x\|>1} \|x\|^p M(\mathrm{d}x)$ is decided by I_2.

If $p < \alpha$ then

$$I_2 \le \int_{\|x\|>1} \|x\|^p \int_{\|x\|^{-1}}^\infty t^{p-\alpha-1}\, \mathrm{d}t R(\mathrm{d}x) = (\alpha - p)^{-1} \int_{\|x\|>1} \|x\|^\alpha R(\mathrm{d}x) < \infty,$$

by (2.8). This proves (i).

If $p > \alpha$, then (ii) follows from the following bounds:

$$\left(\int_1^\infty t^{p-\alpha-1}\mathrm{e}^{-t}\, \mathrm{d}t\right) \int_{\|x\|>1} \|x\|^p R(\mathrm{d}x) \le I_2 \le \Gamma(p-\alpha) \int_{\|x\|>1} \|x\|^p R(\mathrm{d}x).$$

For $p = \alpha$ we have

$$I_2 \le \int_{\|x\|>1} \|x\|^\alpha \int_{\|x\|^{-1}}^1 t^{-1}\,\mathrm{d}t\,R(\mathrm{d}x) + \int_{\|x\|>1} \|x\|^\alpha \int_1^\infty \mathrm{e}^{-t}\,\mathrm{d}t\,R(\mathrm{d}x)$$
$$= \int_{\|x\|>1} \|x\|^\alpha (\log\|x\| + \mathrm{e}^{-1}) R(\mathrm{d}x)$$

and

$$I_2 \ge \mathrm{e}^{-1} \int_{\|x\|>1} \|x\|^\alpha \log\|x\|\,R(\mathrm{d}x).$$

This completes the proof of (i)–(iii).

Suppose now that $R(\{x : \|x\| > \theta^{-1}\}) = 0$. Put $C := \sup_{t\ge 2\theta} \mathrm{e}^{-t/2} t^{2-\alpha}$. Then

$$\begin{aligned}\int_{\|x\|>1} \mathrm{e}^{\theta\|x\|} M(\mathrm{d}x) &= \int_{\|x\|\le\theta^{-1}} \int_{\|x\|^{-1}}^\infty \mathrm{e}^{(\theta\|x\|-1)t} t^{-\alpha-1}\,\mathrm{d}t\,R(\mathrm{d}x)\\ &\le \int_{(2\theta)^{-1}<\|x\|\le\theta^{-1}} \int_{\|x\|^{-1}}^\infty t^{-\alpha-1}\,\mathrm{d}t\,R(\mathrm{d}x)\\ &\quad + \int_{\|x\|\le(2\theta)^{-1}} \int_{\|x\|^{-1}}^\infty \mathrm{e}^{-t/2} t^{-\alpha-1}\,\mathrm{d}t\,R(\mathrm{d}x)\\ &\le \alpha^{-1} \int_{\|x\|>(2\theta)^{-1}} \|x\|^\alpha R(\mathrm{d}x) + 2^{-1} C \int_{\|x\|\le(2\theta)^{-1}} \|x\|^2 R(\mathrm{d}x) < \infty.\end{aligned}$$

Conversely, if $R(\{x : \|x\| > \theta^{-1}\}) > 0$ then there is an $\epsilon > 0$ such that $R(\{x : \|x\| > \theta^{-1} + \epsilon\}) > 0$. We obtain

$$\int_{\|x\|>1} \exp(\theta\|x\|) M(\mathrm{d}x) \ge \int_{\|x\|>\theta^{-1}+\epsilon} \int_{\|x\|^{-1}}^\infty \exp(\epsilon\theta t) t^{-\alpha-1}\,\mathrm{d}t\,R(\mathrm{d}x) = \infty$$

which implies that $\int_{\mathbb{R}^d} \exp(\theta\|x\|)\mu(\mathrm{d}x) = \infty$. □

The "finite variation" case is characterized by the following.

Proposition 2.8. *Let M and R be related by (2.7), where R satisfies (2.8). Then*

$$\int_{\|x\|\le1} \|x\| M(\mathrm{d}x) < \infty \iff \alpha \in (0,1) \quad \textit{and} \quad \int_{\|x\|\le1} \|x\| R(\mathrm{d}x) < \infty.$$

Proof. Suppose $\int_{\|x\|\le1} \|x\| M(\mathrm{d}x) < \infty$. Choose $r \ge 1$ such that $R(\{\|x\| \le r\}) \ne 0$. Then

$$\begin{aligned}\infty > \int_{\|x\|\le1} \|x\| M(\mathrm{d}x) &\ge \int_{\|x\|\le r} \|x\| \int_0^{\|x\|^{-1}} t^{-\alpha}\mathrm{e}^{-t}\,\mathrm{d}t\,R(\mathrm{d}x)\\ &\ge \int_{\|x\|\le r} \|x\| R(\mathrm{d}x) \int_0^{r^{-1}} t^{-\alpha}\mathrm{e}^{-t}\,\mathrm{d}t\end{aligned}$$

which implies $\alpha < 1$ and $\int_{\|x\|\le1} \|x\| R(\mathrm{d}x) < \infty$. The converse follows from the following bounds:

$$\int_{\|x\|\le1} \|x\| M(\mathrm{d}x) = \int_{\|x\|\le1} \|x\| \int_0^{\|x\|^{-1}} t^{-\alpha}\mathrm{e}^{-t}\,\mathrm{d}t\,R(\mathrm{d}x)$$

$$+ \int_{\|x\|>1} \|x\| \int_0^{\|x\|^{-1}} t^{-\alpha} \mathrm{e}^{-t} \, \mathrm{d}t \, R(\mathrm{d}x)$$
$$\leq \Gamma(1-\alpha) \int_{\|x\|\leq 1} \|x\| R(\mathrm{d}x) + (1-\alpha)^{-1} \int_{\|x\|>1} \|x\|^{\alpha} R(\mathrm{d}x). \quad \square$$

The next theorem gives an explicit form of the characteristic function of $T\alpha S$ distributions and justifies the term "spectral measure" for R.

Theorem 2.9. *Let μ be a $T\alpha S$ distribution with Lévy measure given by (2.7), $\alpha \in (0, 2)$. If $\int_{\mathbb{R}^d} \|x\| \mu(\mathrm{d}x) < \infty$, then*

$$\hat{\mu}(y) = \exp\left\{ \int_{\mathbb{R}^d} \psi_\alpha(\langle y, x\rangle) R(\mathrm{d}x) + \mathrm{i}\langle y, b\rangle \right\}, \tag{2.11}$$

where

$$\psi_\alpha(s) = \begin{cases} \Gamma(-\alpha)\left[(1-\mathrm{i}s)^\alpha - 1 + \mathrm{i}\alpha s\right], & \alpha \neq 1 \\ (1-\mathrm{i}s)\log(1-\mathrm{i}s) + \mathrm{i}s, & \alpha = 1 \end{cases} \tag{2.12}$$

and $b = \int_{\mathbb{R}^d} x \mu(\mathrm{d}x)$. This is always the case when $1 < \alpha < 2$, or

$$\alpha = 1 \quad \textit{and} \quad \int_{\|x\|>1} \|x\| \log \|x\| R(\mathrm{d}x) < \infty, \tag{2.13}$$

or

$$0 < \alpha < 1 \quad \textit{and} \quad \int_{\|x\|>1} \|x\| R(\mathrm{d}x) < \infty. \tag{2.14}$$

If $0 < \alpha < 1$ and

$$\int_{\|x\|\leq 1} \|x\| R(\mathrm{d}x) < \infty, \tag{2.15}$$

then

$$\hat{\mu}(y) = \exp\left\{ \int_{\mathbb{R}^d} \psi_\alpha^0(\langle y, x\rangle) R(\mathrm{d}x) + \mathrm{i}\langle y, b_0\rangle \right\}, \tag{2.16}$$

where

$$\psi_\alpha^0(s) = \Gamma(-\alpha)\left[(1-\mathrm{i}s)^\alpha - 1\right] \tag{2.17}$$

and $b_0 \in \mathbb{R}^d$ is the drift vector. In particular, if μ is a proper $T\alpha S$ distribution with $\alpha \in (0, 1)$, then (2.16) applies.

Before the proof, we will state the limiting behavior of ψ_α's at zero and infinity. The computations are elementary, and thus omitted.

Lemma 2.10. *We have*

$$\begin{aligned}
&\lim_{s\to 0} s^{-2}\psi_\alpha(s) = -\frac{1}{2}\Gamma(2-\alpha), \quad \alpha\in(0,2);\\
&\lim_{s\to\infty} s^{-1}\psi_\alpha(s) = -\Gamma(1-\alpha)\mathrm{i}, \quad \alpha\in(0,1);\\
&\lim_{s\to\infty} (s^{-1}\psi_1(s)+\mathrm{i}\log s) = -\frac{\pi}{2}+\mathrm{i}, \quad \alpha=1;\\
&\lim_{s\to\infty} s^{-\alpha}\psi_\alpha(s) = \Gamma(-\alpha)\mathrm{e}^{-\mathrm{i}\alpha\pi/2}, \quad \alpha\in(1,2).
\end{aligned}\tag{2.18}$$

We also have for $\alpha\in(0,1)$

$$\begin{aligned}
&\lim_{s\to 0} s^{-1}\psi^0_\alpha(s) = \Gamma(1-\alpha)\mathrm{i};\\
&\lim_{s\to\infty} s^{-\alpha}\psi^0_\alpha(s) = \Gamma(-\alpha)\mathrm{e}^{-\mathrm{i}\alpha\pi/2}.
\end{aligned}\tag{2.19}$$

Consequently, for each α *there is a finite positive constant* C_α *such that for all* $s\in\mathbb{R}$

$$\begin{aligned}
&C_\alpha^{-1}(s^2\wedge|s|^{\alpha\vee 1}) \le |\psi_\alpha(s)| \le C_\alpha(s^2\wedge|s|^{\alpha\vee 1}), \quad \alpha\neq 1;\\
&C_1^{-1}[s^2\wedge|s|(1+\log^+|s|)] \le |\psi_1(s)| \le C_\alpha[s^2\wedge|s|(1+\log^+|s|)], \quad \alpha=1;\\
&C_\alpha^{-1}(|s|\wedge|s|^\alpha) \le |\psi^0_\alpha(s)| \le C_\alpha(|s|\wedge|s|^\alpha), \quad \alpha\in(0,1).
\end{aligned}\tag{2.20}$$

Proof of Theorem 2.9. Notice that by (2.8) and (2.20) of Lemma 2.10 the integrals in (2.11) and (2.16) are well defined. Now we will verify that

$$\psi^0_\alpha(s) = \int_0^\infty (\mathrm{e}^{\mathrm{i}st}-1)t^{-\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad \alpha\in(0,1) \tag{2.21}$$

and

$$\psi_\alpha(s) = \int_0^\infty (\mathrm{e}^{\mathrm{i}st}-1-\mathrm{i}st)t^{-\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad \alpha\in(0,2). \tag{2.22}$$

Consider $0<\alpha<1$. Let w,z be two complex numbers with $\Re(w),\Re(z)>0$. Integrating by parts we get that

$$\int_0^\infty (\mathrm{e}^{-zt}-\mathrm{e}^{-wt})t^{-\alpha-1}\mathrm{d}t = \Gamma(-\alpha)(z^\alpha-w^\alpha).$$

Putting $z=1-\mathrm{i}s$ and $w=1$ we obtain (2.21). Then

$$\int_0^\infty (\mathrm{e}^{\mathrm{i}st}-1-\mathrm{i}st)t^{-\alpha-1}\mathrm{e}^{-t}\mathrm{d}t = \psi^0_\alpha(s) - \mathrm{i}s\int_0^\infty t^{-\alpha}\mathrm{e}^{-t}\,\mathrm{d}t = \psi_\alpha(s).$$

Now let $1<\alpha<2$. Integrating by parts twice we verify that for $\Re(w),\Re(z)>0$,

$$\int_0^\infty [\mathrm{e}^{-zt}-\mathrm{e}^{-wt}+(z-w)t]t^{-\alpha-1}\,\mathrm{d}t = \Gamma(-\alpha)(z^\alpha-w^\alpha).$$

Taking $z=1-\mathrm{i}s$ and $w=1$ we obtain

$$\begin{aligned}
\int_0^\infty (\mathrm{e}^{\mathrm{i}st}-1-\mathrm{i}st)t^{-\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t &= \int_0^\infty [\mathrm{e}^{-(1-\mathrm{i}s)t}-\mathrm{e}^{-t}-\mathrm{i}st]t^{-\alpha-1}\,\mathrm{d}t\\
&\quad + \mathrm{i}s\int_0^\infty (1-\mathrm{e}^{-t})t^{-\alpha}\,\mathrm{d}t = \Gamma(-\alpha)[(1-\mathrm{i}s)^\alpha-1] + \mathrm{i}s\frac{\Gamma(2-\alpha)}{\alpha-1} = \psi_\alpha(s).
\end{aligned}$$

There remains $\alpha = 1$. Take $\alpha' \in (1, 2)$. We have just shown that

$$\psi_{\alpha'}(s) = \frac{\Gamma(2-\alpha')}{\alpha'} \, \frac{(1-\mathrm{i}s)^{\alpha'} - 1 + \mathrm{i}\alpha' s}{\alpha' - 1}.$$

Using the dominated convergence theorem on the left hand side (see (2.22)) and differentiation on the right hand side we pass to the limit as $\alpha' \searrow 1$. This yields the desired formula for ψ_1.

If $\int_{\mathbb{R}^d} \|x\| \mu(\mathrm{d}x) < \infty$, then $\widehat{\mu}$ can be written as

$$\widehat{\mu}(y) = \exp\left(\int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}\langle y,x\rangle} - 1 - \mathrm{i}\langle y, x\rangle) \, M(\mathrm{d}x) + \mathrm{i}\langle y, b\rangle \right), \tag{2.23}$$

where $b = \int x \mu(\mathrm{d}x)$. Using (2.7), (2.22) and (2.20) we obtain (2.11). The sufficiency of (2.13) and (2.14) follows from Proposition 2.7.

Now consider $\alpha \in (0, 1)$ and $\int_{\|x\|\le 1} \|x\| R(\mathrm{d}x) < \infty$. By Proposition 2.8 $\int_{\|x\|\le 1} \|x\| M(\mathrm{d}x) < \infty$, in which case $\widehat{\mu}$ can be written as

$$\widehat{\mu}(y) = \exp\left(\int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}\langle y,x\rangle} - 1) M(\mathrm{d}x) + \mathrm{i}\langle y, b_0\rangle \right), \tag{2.24}$$

where b_0 is a "natural" drift. Applying (2.7), (2.21) and (2.20) we conclude the proof. □

Definition 2.11. Let X be a random vector having a $T\alpha S$ distribution with the spectral measure R. We will write $X \sim TS_\alpha(R, b)$ to indicate that the characteristic function of X is given by (2.11) and assumptions (2.13) and (2.14) are satisfied in the case $\alpha \in (0, 1]$. If $\alpha \in (0, 1)$ and (2.15) holds, then $X \sim TS^0_\alpha(R, b_0)$ means that the characteristic function of X is of the form (2.16).

Notice that if $\alpha \in (0, 1)$ and $\int_{\mathbb{R}^d} \|x\| R(\mathrm{d}x) < \infty$, then both forms (2.11) and (2.16) of characteristic functions are valid for X, $X \sim TS^0_\alpha(R, b_0)$ and $X \sim TS_\alpha(R, b)$, where $b = b_0 + \Gamma(1-\alpha) \int_{\mathbb{R}^d} x R(\mathrm{d}x)$.

The above parametrization behaves nicely under convolutions and linear transformations.

Corollary 2.12. *Let X, X_1, X_2 be random vectors in $\mathbb{R}^d$ and let $V : \mathbb{R}^d \mapsto \mathbb{R}^k$ be a linear map.*

(i) *If $X_i \sim TS_\alpha(R_i, b_i)$ are independent, then $X_1 + X_2 \sim TS_\alpha(R_1 + R_2, b_1 + b_2)$.*
(ii) *If $X \sim TS_\alpha(R, b)$ then $V(X) \sim TS_\alpha((R \circ V^{-1})_{|\mathbb{R}^d \setminus \{0\}}, V(b))$.*

The analogous properties hold for the TS^0_α parametrization.

Corollary 2.13. *Let $X \sim TS_\alpha(R, 0)$ and $R(\{x : \|x\| > \theta^{-1}\}) = 0$ for some $\theta > 0$. Then for every $y \in \mathbb{R}^d$ with $\|y\| \le \theta$ the moment generating function of X exists and is equal to*

$$E\mathrm{e}^{\langle y,X\rangle} = \begin{cases} \exp\left[\Gamma(-\alpha) \int \left[(1 - \langle y, x\rangle)^\alpha - 1 + \alpha\langle y, x\rangle\right] R(\mathrm{d}x)\right], & \alpha \ne 1 \\ \exp\left[\int \left[(1 - \langle y, x\rangle) \log(1 - \langle y, x\rangle) + \langle y, x\rangle\right] R(\mathrm{d}x)\right], & \alpha = 1. \end{cases} \tag{2.25}$$

If $X \sim TS^0_\alpha(R, 0)$ and $R(\{x : \|x\| > \theta^{-1}\}) = 0$, then

$$E\mathrm{e}^{\langle y,X\rangle} = \exp\left[\Gamma(-\alpha) \int [(1 - \langle y, x\rangle)^\alpha - 1] R(\mathrm{d}x)\right]. \tag{2.26}$$

Proof. By Proposition 2.7(iv), $E\mathrm{e}^{\theta\|X\|} < \infty$. Theorem 25.17 in [22] justifies a formal replacement of $y \in \mathbb{R}^d$ in (2.11) by $-\mathrm{i}y \in \mathbb{C}^d$ such that $\|y\| \le \theta$. This gives (2.25). (2.26) follows by the same argument. □

Example 1. Let $q(r,u) = \mathrm{e}^{-r}$ in (2.2). This is a uniform tilting of a stable Lévy measure M_0 in all directions, $M(\mathrm{d}x) = \mathrm{e}^{-\|x\|}M_0(\mathrm{d}x)$. It is easy to see that the measures Q and R are concentrated on S^{d-1} on which they coincide with σ. Let $X \sim TS_\alpha(R,0)$. By Proposition 2.7(iv), $E\mathrm{e}^{\|X\|} < \infty$. The moment generating function of X is given by (2.25).

Example 2. Let $d = 1$ and $X \sim TS^0_\alpha(c\delta_1, 0)$, where $0 < \alpha < 1$ and $c > 0$. Then X is a non-negative random variable and

$$E\mathrm{e}^{-\lambda X} = \exp\{-\Gamma(-\alpha)c[1-(1+\lambda)^\alpha]\} \quad \lambda \ge -1.$$

When $\alpha = 1/2$, X has the well known inverse Gaussian distribution; see [22, p. 233].

At the conclusion of this section we will relate parameters of proper $T\alpha S$ distributions to the stable ones.

Lemma 2.14. *Let M be a Lévy measure of a proper $T\alpha S$ distribution, as in (2.2), with the spectral measure R. Let M_0 be the Lévy measure of an α-stable distribution given by (2.1). Then*

$$M_0(A) = \int_{\mathbb{R}^d}\int_0^\infty I_A(tx)t^{-\alpha-1}\,\mathrm{d}t\,R(\mathrm{d}x) \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.27}$$

Furthermore,

$$\sigma(B) = \int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right)\|x\|^\alpha R(\mathrm{d}x), \quad B \in \mathcal{B}(S^{d-1}). \tag{2.28}$$

Proof. Using (2.5) and (2.4) we get for every $A \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d}\int_0^\infty I_A(tx)t^{-\alpha-1}\,\mathrm{d}t\,R(\mathrm{d}x) &= \int_{\mathbb{R}^d}\int_0^\infty I_A\left(t\frac{x}{\|x\|^2}\right)t^{-\alpha-1}\|x\|^\alpha\,\mathrm{d}t\,Q(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d}\int_0^\infty I_A\left(s\frac{x}{\|x\|}\right)s^{-\alpha-1}\,\mathrm{d}s\,Q(\mathrm{d}x)\\ &= \int_{S^{d-1}}\int_0^\infty I_A(su)s^{-\alpha-1}\,\mathrm{d}s\sigma(\mathrm{d}u) = M_0(A).\end{aligned}$$

Now we verify the second formula of the lemma:

$$\begin{aligned}\int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right)\|x\|^\alpha R(\mathrm{d}x) &= \int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right)Q(\mathrm{d}x)\\ &= \int_B\int_0^\infty Q(\mathrm{d}s|u)\sigma(\mathrm{d}u) = \sigma(B). \quad \square\end{aligned}$$

Let μ_0 be an α-stable distribution with Lévy measure M_0 given by (2.1). We have

$$\hat{\mu}_0(y) = \begin{cases}\exp\left[-c_\alpha\int_{S^{d-1}}|\langle y,u\rangle|^\alpha\left(1-\mathrm{i}\tan\frac{\pi\alpha}{2}\mathrm{sgn}\langle y,u\rangle\right)\sigma(\mathrm{d}u)+\mathrm{i}\langle y,a\rangle\right]\\ \quad\alpha\ne 1,\\ \exp\left[-c_1\int_{S^{d-1}}\left(|\langle y,u\rangle|+\mathrm{i}\frac{2}{\pi}\langle y,u\rangle\log|\langle y,u\rangle|\right)\sigma(\mathrm{d}u)+\mathrm{i}t\langle y,a\rangle\right]\\ \quad\alpha = 1\end{cases} \tag{2.29}$$

where $c_\alpha = |\Gamma(-\alpha)\cos(\frac{\pi\alpha}{2})|$ when $\alpha \neq 1$ and $c_1 = \pi/2$. See [22, Theorem 14.10].

Definition 2.15. We will write $Y \sim S_\alpha(\sigma, a)$ to indicate that Y is an α-stable random vector with the characteristic function as in (2.29).

Notice that $c_\alpha\sigma$ (not σ) is traditionally called the spectral measure of a stable distribution μ [21, p. 66]. We will use the notation of Definition 2.15, however, for the sake of consistency with the notation of tempered stable distributions.

3. $T\alpha S$ Lévy processes: Short and long time behavior

A Lévy process $\{X(t) : t \geq 0\}$ in $\mathbb{R}^d$ such that $X(1)$ has a $T\alpha S$ distribution will be called a *$T\alpha S$ Lévy process*. Proper $T\alpha S$ Lévy processes are defined analogously. To investigate the short and long time behavior of $X(t)$, we define the time rescaled process

$$X_h(t) = X(ht) \quad h > 0,\ t \geq 0. \tag{3.1}$$

The following theorem justifies and quantifies the statement that a tempered stable process in a short time looks as a stable process while in a large time scale it looks as a Brownian motion. Below, $\xrightarrow{d}$ denotes the weak convergence of processes in the space $D([0,\infty), \mathbb{R}^d)$, of functions from $[0,\infty)$ into $\mathbb{R}^d$ right-continuous with left limits, equipped with the Skorohod topology.

Theorem 3.1. *Let $\{X(t) : t \geq 0\}$ be a $T\alpha S$ Lévy process in $\mathbb{R}^d$ and let R be the spectral measure of $\mathcal{L}(X(1))$.*
(i) Short time behavior. *Suppose that*

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(\mathrm{d}x) < \infty \tag{3.2}$$

and let σ be a finite measure on S^{d-1} given by (2.28). Assume that $X(1) \sim TS_\alpha^0(R, 0)$ when $\alpha \in (0, 1)$ and that $X(1) \sim TS_\alpha(R, 0)$ when $\alpha \in (1, 2)$. Then

$$h^{-1/\alpha} X_h \xrightarrow{d} Y \quad \text{as } h \to 0,$$

where $\{Y(t) : t \geq 0\}$ is a strictly α-stable Lévy process with $Y(1) \sim S_\alpha(\sigma, 0)$. If $\alpha = 1$, assume additionally that $\int_{\mathbb{R}^d} \|x\|| \log \|x\|| R(\mathrm{d}x) < \infty$ and $X(1) \sim TS_\alpha(R, 0)$. Then

$$h^{-1} X_h - a_h \xrightarrow{d} Y \quad \text{as } h \to 0,$$

where

$$a_h(t) = t \log h \int_{\mathbb{R}^d} x R(\mathrm{d}x),$$

and $\{Y(t) : t \geq 0\}$ is a 1-stable Lévy process with $Y(1) \sim S_1(\sigma, b)$ and

$$b = \int_{\mathbb{R}^d} x(1 - \log \|x\|) R(\mathrm{d}x).$$

(ii) Long time behavior. *Suppose that*

$$\int_{\mathbb{R}^d} \|x\|^2 R(\mathrm{d}x) < \infty \tag{3.3}$$

and let $X(1) \sim TS_\alpha(R,0)$, $\alpha \in (0,2)$. *Then*

$$h^{-1/2} X_h \xrightarrow{d} B \quad \text{as } h \to \infty,$$

where $\{B(t) : t \geq 0\}$ *is a Brownian motion with the characteristic function*

$$E e^{i\langle y, B(t)\rangle} = \exp\left\{-\frac{t}{2}\Gamma(2-\alpha)\int_{\mathbb{R}^d} \langle y, x\rangle^2 R(dx)\right\}.$$

Proof. (i). First we consider $\alpha \neq 1$. Since $\{h^{-1/\alpha} X_h(t) : t \geq 0\}$ is a Lévy process, by a theorem due to Skorohod [12, Theorem 15.17], it is enough to show the convergence in distribution of $h^{-1/\alpha} X_h(1)$ to $Y(1)$. We will show the convergence of the respective characteristic functions. For $\alpha \in (0,1)$ we have

$$\begin{aligned} E\exp[i\langle y, h^{-1/\alpha} X_h(1)\rangle] &= E\exp[i\langle h^{-1/\alpha} y, X(h)\rangle] \\ &= \exp\left[\int_{\mathbb{R}^d} h\psi_\alpha^0(h^{-1/\alpha}\langle y, x\rangle) R(dx)\right], \end{aligned} \tag{3.4}$$

and for $\alpha \in (1,2)$ we have

$$E\exp[i\langle y, h^{-1/\alpha} X_h(1)\rangle] = \exp\left[\int_{\mathbb{R}^d} h\psi_\alpha(h^{-1/\alpha}\langle y, x\rangle) R(dx)\right]. \tag{3.5}$$

Using the upper bounds (2.20) of Lemma 2.10 and (3.2) we justify the passage $h \to 0$ under the above integrals. Since $\psi_\alpha^0(-s) = \overline{\psi_\alpha^0(s)}$ and $\psi_\alpha(-s) = \overline{\psi_\alpha(s)}$ (see (2.21) and (2.22)), we get by (2.19)

$$\begin{aligned} \lim_{h\to 0} h\psi_\alpha^0(h^{-1/\alpha}\langle y, x\rangle) &= \Gamma(-\alpha)|\langle y, x\rangle|^\alpha \exp\left\{-i\frac{\alpha\pi}{2}\operatorname{sgn}\langle y, x\rangle\right\} \\ &= \Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)|\langle y, x\rangle|^\alpha\left(1 - \tan\frac{\alpha\pi}{2}\operatorname{sgn}\langle y, x\rangle\right). \end{aligned}$$

Therefore, the limit in (3.4) coincides with (2.29) for $a = 0$ and $\alpha \in (0,1)$. Similarly we get the limit in (3.5) as $h \to 0$. This establishes (i) for $\alpha \neq 1$.

If $\alpha = 1$ then

$$\begin{aligned} &E\exp[i\langle y, h^{-1} X_h(1) - a_h(1)\rangle] \\ &\quad = \exp\left\{\int_{\mathbb{R}^d} [h\psi_1(h^{-1}\langle y, x\rangle) - i\langle y, x\rangle \log h] R(dx)\right\}. \end{aligned} \tag{3.6}$$

Putting $v = \langle y, x\rangle$ we can write

$$\begin{aligned} h\psi_1(h^{-1}v) - iv\log h &= h\log(1 + h^{-2}v^2)^{1/2} - v\tan^{-1}(h^{-1}v) \\ &\quad + i[v - v\log(h^2 + v^2)^{1/2} - h\tan^{-1}(h^{-1}v)]. \end{aligned}$$

Therefore

$$\begin{aligned} |h\psi_1(h^{-1}v) - iv\log h| &\leq h\log(1 + h^{-1}|v|) + |v|\frac{\pi}{2} \\ &\quad + |v|\max\{|\log|v||, |\log(|v|+1)|\} + 2|v| \\ &\leq C|v|(1 + |\log|v||) \end{aligned}$$

uniformly over $h \in (0,1]$, where C is a universal constant. In the first inequality we used the monotonicity of the logarithm and that $|\tan^{-1} u| \leq u$, and in the second one, the bounds

$\log(1+u) \le u$ and $|\log(1+u)| \le |\log u| + \log 2$, for all $u > 0$. We have obtained a bound

$$|h\psi_1(h^{-1}\langle y, x\rangle) - \mathrm{i}\langle y, x\rangle \log h| \le C|\langle y, x\rangle|(1 + |\log|\langle y, x\rangle||).$$

Therefore, by the assumption on the integrability of R we may pass to the limit under the integral sign in (3.6). Applying (2.18) and (2.28) we get

$$\begin{aligned}
&\lim_{h\to 0} E\exp[\mathrm{i}\langle y, h^{-1}X_h(1) - a_h(1)\rangle] \\
&\quad = \exp\left\{\int_{\mathbb{R}^d}\left[-\frac{\pi}{2}|\langle y, x\rangle| + \mathrm{i}(\langle y, x\rangle - \langle y, x\rangle \log|\langle y, x\rangle|)\right] R(\mathrm{d}x)\right\} \\
&\quad = \exp\left\{-c_1\int_{\mathbb{R}^d}\left(|\langle y, x\rangle| + \mathrm{i}\frac{2}{\pi}\langle y, x\rangle \log\left|\left\langle y, \frac{x}{\|x\|}\right\rangle\right|\right) R(\mathrm{d}x)\right. \\
&\qquad \left. + \mathrm{i}\int_{\mathbb{R}^d}\langle y, x\rangle(1 - \log\|x\|)R(\mathrm{d}x)\right\} \\
&\quad = \exp\left\{-c_1\int_{S^{d-1}}\left(|\langle y, u\rangle| + \mathrm{i}\frac{2}{\pi}\langle y, u\rangle \log|\langle y, u\rangle|\right)\sigma(\mathrm{d}u) + \mathrm{i}\langle y, b\rangle\right\}.
\end{aligned}$$

This completes the proof of part (i).

Now we will prove (ii). We have

$$E\exp[\mathrm{i}\langle y, h^{-1/2}X_h(1)\rangle] = \exp\left[\int_{\mathbb{R}^d} h\psi_\alpha(h^{-1/2}\langle y, x\rangle)R(\mathrm{d}x)\right]. \tag{3.7}$$

Using the upper bounds (2.20) of Lemma 2.10 and (3.3) we justify the passage $h \to \infty$ under the integral. Applying (2.18) we get

$$\lim_{h\to\infty} h\psi_\alpha(h^{-1/2}\langle y, x\rangle) = -\frac{1}{2}\Gamma(2-\alpha)\langle y, x\rangle^2$$

which completes the proof. □

Remark 3.2. Under different and more complicated assumptions on the spectral measure R it is possible to obtain β-stable behavior of X at zero and/or infinity, where $\beta \in (\alpha, 2)$. These extensions will be considered elsewhere.

4. Absolute continuity with respect to stable processes

In the previous section we have shown that in a short time a proper tempered stable Lévy process looks like a stable one. In this section we relate the distributions of these two processes.

A process $\{X(t) : t \ge 0\}$ in $\mathbb{R}^d$ is said to be canonical if $X(t, \omega) = \omega(t)$, $t \ge 0$, $\omega \in \Omega$, where $\Omega = D([0,\infty), \mathbb{R}^d)$; Ω is equipped with the σ-field $\mathcal{F} = \sigma\{X(s) : s \ge 0\}$ and the right-continuous natural filtration $\mathcal{F}_t = \bigcap_{s>t}\sigma\{X(u) : u \le s\}$, $t \ge 0$. The canonical process is completely described by a probability measure P on $(\Omega, \mathcal{F})$. As usual, we set $\Delta X(t) = X(t) - X(t-)$. By $P_{|\mathcal{F}_t}$ we will denote the restriction of P to the σ-field $\mathcal{F}_t$.

Theorem 4.1. *In the above setting consider two probability measures P_0 and P on $(\Omega, \mathcal{F})$ such that the canonical process $\{X(t) : t \ge 0\}$ under P_0 is a Lévy α-stable process while under P it is a proper $T\alpha S$ Lévy process. Specifically, assume that under P_0, $X(1) \sim S_\alpha(\sigma, a)$, where σ is related to R by (2.28) and $\alpha \in (0, 2)$, while under P, $X(1) \sim TS^0_\alpha(R, b)$ when $\alpha \in (0, 1)$*

and $X(1) \sim TS_\alpha(R, b)$ *when* $\alpha \in [1, 2)$. *Let* M, *the Lévy measure corresponding to* R, *be as in* (2.2), *where* $q(0^+, u) = 1$ *for all* $u \in S^{d-1}$. *Then*
(i) $P_{0|\mathcal{F}_t}$ *and* $P_{|\mathcal{F}_t}$ *are mutually absolutely continuous for every* $t > 0$ *if and only if*

$$\int_{S^{d-1}} \int_0^1 [1 - q(r, u)]^2 \, r^{-\alpha-1} \, \mathrm{d}r\sigma(\mathrm{d}u) < \infty \tag{4.1}$$

and

$$b - a = \begin{cases} 0, & 0 < \alpha < 1, \\ \displaystyle\int_{\mathbb{R}^d} x(\log \|x\| - 1) R(\mathrm{d}x), & \alpha = 1, \\ \Gamma(1-\alpha) \displaystyle\int_{\mathbb{R}^d} x R(\mathrm{d}x), & 1 < \alpha < 2. \end{cases} \tag{4.2}$$

Condition (4.1) *implies that the integrals in* (4.2) *exist. Furthermore, if either* (4.1) *or* (4.2) *fails, then* $P_{0|\mathcal{F}_t}$ *and* $P_{|\mathcal{F}_t}$ *are singular for all* $t > 0$.
(ii) *If* (4.1) *and* (4.2) *hold, then for each* $t > 0$

$$\frac{\mathrm{d}P}{\mathrm{d}P_0}_{|\mathcal{F}_t} = \mathrm{e}^{Z(t)}, \tag{4.3}$$

where $\{Z(t) : t \geq 0\}$ *is a Lévy process on* $(\Omega, \mathcal{F}, P_0)$ *given by*

$$Z(t) = \lim_{\epsilon \downarrow 0} \left\{ \sum_{\{s \leq t : \|\Delta X(s)\| > \epsilon\}} \log q \left(\|\Delta X(s)\|, \frac{\Delta X(s)}{\|\Delta X(s)\|} \right) \right.$$
$$\left. + t \int_{S^{d-1}} \int_\epsilon^\infty [1 - q(r, u)] \, r^{-\alpha-1} \, \mathrm{d}r\sigma(\mathrm{d}u) \right\}.$$

The above convergence is P_0*-a.s., uniform in* t *on any bounded interval. The Lévy measure* ν *of* $\mathcal{L}\{Z(1)\}$ *is concentrated on* $(-\infty, 0)$ *and determined by*

$$\int_{-\infty}^0 F(s)\nu(\mathrm{d}s) = \int_{S^{d-1}} \int_0^\infty F(\log q(r, u)) r^{-\alpha-1} \, \mathrm{d}r\sigma(\mathrm{d}u) \tag{4.4}$$

for every Borel function F. *The characteristic function of* $Z(1)$ *is of the form*

$$E_{P_0} \mathrm{e}^{\mathrm{i}\theta Z(1)} = \exp\left\{ \mathrm{i}\theta a_0 + \int_{-\infty}^0 [\mathrm{e}^{\mathrm{i}\theta v} - 1 - \mathrm{i}\theta v I_{[-1,0)}(v)] \nu(\mathrm{d}v) \right\}, \tag{4.5}$$

where

$$a_0 = -\int_{-\infty}^0 [\mathrm{e}^v - 1 - v I_{[-1,0)}(v)] \nu(\mathrm{d}v).$$

Proof. Part (i). Let M_0 and M be as in (2.1) and (2.2). We have

$$\frac{\mathrm{d}M}{\mathrm{d}M_0}(x) = q\left(\|x\|, \frac{x}{\|x\|}\right), \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{4.6}$$

Indeed, for every $A \in \mathcal{B}(\mathbb{R}^d)$

$$\int_A q\left(\|x\|, \frac{x}{\|x\|}\right) M_0(\mathrm{d}x) = \int_{S^{d-1}} \int_0^\infty I_A(ru) q(r, u) r^{-\alpha-1} \, \mathrm{d}r\sigma(\mathrm{d}u) = M(A).$$

Put $\phi(x) = \log q(\|x\|, \frac{x}{\|x\|})$. According to Theorem 33.1 in [22], $P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if

$$\int_{\mathbb{R}^d} (\mathrm{e}^{\phi(x)/2} - 1)^2 M_0(\mathrm{d}x) < \infty \tag{4.7}$$

and

$$B_\alpha = 0. \tag{4.8}$$

Here

$$B_\alpha = \begin{cases} b + \int_{\|x\|\le 1} x M(\mathrm{d}x) - \left(a + \int_{\|x\|\le 1} x M_0(\mathrm{d}x)\right) - \int_{\|x\|\le 1} x(M - M_0)(\mathrm{d}x), \\ \quad 0 < \alpha < 1, \\ b - \int_{\|x\|> 1} x M(\mathrm{d}x) - \left(a - c\int_{S^{d-1}} u\sigma(\mathrm{d}u)\right) - \int_{\|x\|\le 1} x(M - M_0)(\mathrm{d}x), \\ \quad \alpha = 1, \\ b - \int_{\|x\|> 1} x M(\mathrm{d}x) - \left(a - \int_{\|x\|> 1} x M_0(\mathrm{d}x)\right) - \int_{\|x\|\le 1} x(M - M_0)(\mathrm{d}x), \\ \quad 1 < \alpha < 2. \end{cases}$$

In the case $\alpha = 1$, $c = 1 - \gamma$, where γ is the Euler constant. (This is the same constant c as in [22, Lemma 14.11]; to see that, use integration by parts and [10, 8.2301–8.2302].)

(4.7) can be written as

$$\int_{\mathbb{R}^d} \left[1 - q^{1/2}\left(\|x\|, \frac{x}{\|x\|}\right)\right]^2 M_0(\mathrm{d}x) < \infty.$$

Since the integrand is bounded by 1, we may consider only integration over $\{\|x\| \le 1\}$. Applying elementary inequalities $\frac{1}{4}(1-x)^2 \le (1-\sqrt{x})^2 \le (1-x)^2$ for $x \in [0, 1]$, we infer that the above condition is equivalent to

$$\int_{\|x\|\le 1} \left[1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right]^2 M_0(\mathrm{d}x) < \infty. \tag{4.9}$$

This yields (4.1), after a change of variable.

Now we will prove that (4.2) is equivalent to (4.8). First we remark that by (4.9)

$$\int_{\|x\|\le 1} \|x\|(M_0 - M)(\mathrm{d}x) < \infty. \tag{4.10}$$

Indeed,

$$\begin{aligned} \int_{\|x\|\le 1} \|x\|(M_0 - M)(\mathrm{d}x) &= \int_{\|x\|\le 1} \|x\| \left[1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right] M_0(\mathrm{d}x) \\ &\le \left(\int_{\|x\|\le 1} \|x\|^2 M_0(\mathrm{d}x)\right)^{1/2} \left(\int_{\|x\|\le 1} \left[1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right]^2 M_0(\mathrm{d}x)\right)^{1/2} < \infty. \end{aligned}$$

If $0 < \alpha < 1$, then $B_\alpha = b - a = 0$ by (4.2). Let $1 < \alpha < 2$. In this case

$$\int_{\|x\|>1} \|x\| M(\mathrm{d}x) \le \int_{\|x\|>1} \|x\| M_0(\mathrm{d}x) < \infty.$$

Combining this with (4.10) we get

$$\int_{\mathbb{R}^d} \|x\|(M_0 - M)(\mathrm{d}x) < \infty.$$

Then, using (2.7) and (2.27) and the integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \|x\|(M_0 - M)(\mathrm{d}x) &= \int_{\mathbb{R}^d}\int_0^\infty \|x\| t^{-\alpha}(1-\mathrm{e}^{-t})\mathrm{d}t\, R(\mathrm{d}x)\\ &= -\Gamma(1-\alpha)\int_{\mathbb{R}^d}\|x\| R(\mathrm{d}x).\end{aligned}$$

Thus $\int_{\mathbb{R}^d}\|x\|R(\mathrm{d}x) < \infty$. Applying the same steps we get

$$B_\alpha = \int_{\mathbb{R}^d} x(M_0 - M)(\mathrm{d}x) + b - a = -\Gamma(1-\alpha)\int_{\mathbb{R}^d} x R(\mathrm{d}x) + b - a = 0,$$

where the last equation follows from (4.2), proving (4.8).

It remains to prove (4.8) in the case $\alpha = 1$. We will evaluate parts of B_1 and then combine them to show that $B_1 = 0$. By (4.10) we have

$$\begin{aligned}\infty &> \int_{\|x\|\le 1}\|x\|(M_0 - M)(\mathrm{d}x) = \int_{\mathbb{R}^d}\|x\|\int_0^{\|x\|^{-1}} t^{-1}(1-\mathrm{e}^{-t})\mathrm{d}t\, R(\mathrm{d}x)\\ &\ge \frac{1}{2}\int_{\|x\|\le 1}\|x\|\int_1^{\|x\|^{-1}} t^{-1}\,\mathrm{d}t\, R(\mathrm{d}x) = \frac{1}{2}\int_{\|x\|\le 1}\|x\||\log\|x\||R(\mathrm{d}x).\end{aligned}$$

Combining this with (2.13), we get $\int_{\mathbb{R}^d}\|x\||\log\|x\||R(\mathrm{d}x) < \infty$. This makes the integral in (4.2) well defined and validates the following computation.

$$\begin{aligned}\int_{\|x\|\le 1} x(M_0 - M)(\mathrm{d}x) &= \int_{\mathbb{R}^d} x\int_0^{\|x\|^{-1}} t^{-1}(1-\mathrm{e}^{-t})\mathrm{d}t\, R(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d} x(E_1(\|x\|^{-1}) - \log\|x\| + \gamma)R(\mathrm{d}x)\end{aligned}$$

where $E_1(v) = \int_v^\infty t^{-1}\exp(-t)\,\mathrm{d}t$ is the exponential integral function [1, 5.1.39]. Next we notice that by (2.13), $\int_{\|x\|>1}\|x\|M(\mathrm{d}x) < \infty$ (see Proposition 2.7(ii)). Moreover,

$$\int_{\|x\|>1} xM(\mathrm{d}x) = \int_{\mathbb{R}^d} x\int_{\|x\|^{-1}}^\infty t^{-1}\mathrm{e}^{-t}\mathrm{d}t\, R(\mathrm{d}x) = \int_{\mathbb{R}^d} xE_1(\|x\|^{-1})R(\mathrm{d}x).$$

Combining these evaluations we get

$$\begin{aligned}B_1 &= b - \int_{\mathbb{R}^d} xE_1(\|x\|^{-1})R(\mathrm{d}x) - a + (1-\gamma)\int_{S^{d-1}} u\sigma(\mathrm{d}u)\\ &\quad + \int_{\mathbb{R}^d} x(-\log\|x\| + E_1(\|x\|^{-1}) + \gamma)R(\mathrm{d}x)\\ &= b - a + (1-\gamma)\int_{\mathbb{R}^d} xR(\mathrm{d}x) - \int_{\mathbb{R}^d} x\log\|x\|R(\mathrm{d}x) + \gamma\int_{\mathbb{R}^d} xR(\mathrm{d}x) = 0.\end{aligned}$$

The conclusion of the proof of part (i) comes from the dichotomy result of [7] which says that, since M and M_0 are mutually absolutely continuous by (4.6), $P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are either mutually absolutely continuous or singular for all $t > 0$.

Part (ii) is a direct application of Theorem 33.2 in [22], where the form of Radon–Nikodym derivative is specified for two mutually absolutely continuous Lévy processes. □

Remark 4.2. Condition (4.1) fails when the function $q(\cdot, u)$ decreases too rapidly near zero. Intuitively, this means that the tempering is too strong to preserve the almost sure local structure of sample paths of the stable process. In the next section we will give a more tangible probabilistic interpretation of this phenomenon.

We will consider two examples as an illustration of Theorem 4.1. The first one is a continuation of Example 1 of Section 2.

Example 3. Let $q(r, u) = \mathrm{e}^{-r}$ and let σ be any finite measure on S^{d-1}. It is easy to see that condition (4.1) holds. Therefore, the density transformation (4.3) changes a Lévy α-stable process $\{X(t) : t \geq 0\}$ with $X(1) \sim S_\alpha(\sigma, a)$ into a $T\alpha S$ Lévy process with $X(1) \sim TS_\alpha(\sigma, b)$ having finite exponential moments. (4.2) yields

$$b - a = \begin{cases} \Gamma(1-\alpha) \displaystyle\int_{S^{d-1}} u\sigma(\mathrm{d}u) & \alpha \neq 1, \\ -\displaystyle\int_{S^{d-1}} u\sigma(\mathrm{d}u) & \alpha = 1. \end{cases}$$

(See Definition 2.11 for the relation between the two parametrizations in the case $\alpha \in (0, 1)$.) The process $\{Z(t) : t \geq 0\}$ of (4.3) is an α-stable process in $\mathbb{R}$ with only negative jumps and its Lévy density specified by (4.4) equals

$$\frac{\mathrm{d}\nu}{\mathrm{d}s} = \sigma(S^{d-1})|s|^{-\alpha-1} I_{(-\infty,0)}(s).$$

Example 4. Let $q(r, u) = \mathrm{e}^{-r^\beta}$, where $0 < \beta \leq \alpha/2$. Then (4.1) fails for any non-zero measure σ. Indeed,

$$\int_0^1 [1 - q(r, u)]^2 \, r^{-\alpha-1} \mathrm{d}r \geq \frac{1}{4} \int_0^1 r^{2\beta-\alpha-1} \mathrm{d}r = \infty.$$

5. Shot noise representation of proper $T\alpha S$ laws and processes

In this section we give probabilistic representations of proper $T\alpha S$ distributions and the corresponding Lévy processes. They reveal the nature of tempering of stable jumps. These are shot-noise-type series based on marked Poisson point processes [19] (cf. [17]). The difficulty of getting such representations for tempered stable laws is that the tail of the radial component of Lévy measure M in (2.2) does not have an explicit inverse. This is the case even in the simplest situation with $d = 1$, $\sigma = \delta_1$ and $q(r, u) = \mathrm{e}^{-r}$, when the tail is of the form $x \mapsto \int_x^\infty \mathrm{e}^{-r} r^{-\alpha-1} \, \mathrm{d}r$. Therefore, the usual method with the inverse of Lévy measure is hard to practically implement (cf. [19]). The representation given below does not require making such an inverse, works for any function $q(r, u)$, and is more revealing about the structure of $T\alpha S$ laws.

We will now fix the notation. Let M be the Lévy measure of a proper tempered α-stable distribution on $\mathbb{R}^d$ as in (2.2). Let Q and R be the measures on $\mathbb{R}^d$ associated with M, given by (2.4) and (2.5). By (2.10) and (2.28) the following holds:

$$\|\sigma\| := \sigma(S^{d-1}) = Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} \|x\|^\alpha R(\mathrm{d}x) < \infty.$$

Let $\{v_j\}$ be an iid sequence of random vectors in $\mathbb{R}^d$ with the common distribution $Q/\|\sigma\|$. Let $\{u_j\}$ be an iid sequence of uniform random variables on (0, 1) and let $\{e_j\}$ and $\{e'_j\}$ be iid sequences of exponential random variables with parameter 1. Assume that $\{v_j\}$, $\{u_j\}$, $\{e_j\}$, and $\{e'_j\}$ are independent. Put $\gamma_j = e'_1 + \cdots + e'_j$; $\{\gamma_j\}$ forms a Poisson point process on $(0, \infty)$ with the Lebesgue intensity measure. As usual, $x \wedge y := \min\{x, y\}$.

Theorem 5.1. *Under the above assumptions we have the following.*
(i) *If* $\alpha \in (0, 1)$*, or if* $\alpha \in [1, 2)$ *and* Q *is symmetric, set*

$$S_0 = \sum_{j=1}^{\infty} \left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|}. \tag{5.1}$$

Then the series converges a.s.; $S_0 \sim TS^0_\alpha(R, 0)$ *for* $\alpha \in (0, 1)$ *and* $S_0 \sim TS_\alpha(R, 0)$ *for* $\alpha \in [1, 2)$.
(ii) *If* $\alpha \in [1, 2)$ *and* Q *is non-symmetric, assume additionally that* $\int_{\mathbb{R}^d} \|x\| |\log \|x\|| R(\mathrm{d}x) < \infty$ *when* $\alpha = 1$ *and that* $\int_{\mathbb{R}^d} \|x\| R(\mathrm{d}x) < \infty$ *when* $\alpha \in (1, 2)$. *Put*

$$S_1 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} - \left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + b \tag{5.2}$$

where

$$x_0 = E \frac{v_j}{\|v_j\|} = \|\sigma\|^{-1} \int_{S^{d-1}} u \sigma(\mathrm{d}u),$$

$$b = \begin{cases} \alpha^{-1/\alpha} \zeta \left(\dfrac{1}{\alpha} \right) \|\sigma\|^{1/\alpha} x_0 - \Gamma(1-\alpha) x_1, & 1 < \alpha < 2, \\ (2\gamma + \log \|\sigma\|) x_1 - \displaystyle\int_{\mathbb{R}^d} x \log \|x\| R(\mathrm{d}x), & \alpha = 1, \end{cases} \tag{5.3}$$

ζ *denotes the Riemann zeta function,* $\gamma = 0.577\ldots$ *is the Euler constant, and*

$$x_1 = \int_{\mathbb{R}^d} x R(\mathrm{d}x).$$

Then the series (5.2) converges a.s. and $S_1 \sim TS_\alpha(R, 0)$.

Before the proof let us comment on a practical issue of a simulation of v_j's.

Remark 5.2. It is easier to use Q for the purpose of simulation than the measure R. Indeed, if $\{(\eta_j, \xi_j)\}$ is an iid sequence such that $\mathcal{L}\{\xi_j\} = \sigma/\|\sigma\|$ and conditionally on $\xi_j = u$, the distribution of η_j is $Q(\cdot|u)$ (as in (2.4)), then $v_j := \eta_j \xi_j$ are iid with the common distribution $Q/\|\sigma\|$.

However, one can also use measure R for simulation in the following way. Define a probability measure R_1 by $R_1(\mathrm{d}x) = \|\sigma\|^{-1} \|x\|^\alpha R(\mathrm{d}x)$, $x \in \mathbb{R}^d$. Let $\{w_j\}$ be an iid sequence with the common distribution R_1. Then $v_j := w_j/\|w_j\|^2$ are iid with the common distribution $Q/\|\sigma\|$. Substituting such v_j's into the corresponding series we obtain representations of $T\alpha S$ random vectors in terms of w_j, as in [20].

Proof of Theorem 5.1. To prove this theorem, we use [19, Theorem 4.1] in the case of

$$H(\gamma_j, (v_j, e_j, u_j)) := \left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|}.$$

To this end we first need to show that for every $0 \notin A \in \mathcal{B}(\mathbb{R}^d)$

$$\int_0^\infty P(H(s, (v_1, e_1, u_1)) \in A)\, \mathrm{d}s = M(A),$$

that is,

$$\int_0^\infty P\left\{\left(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha}\|v_1\|^{-1}\right)\frac{v_1}{\|v_1\|} \in A\right\} \mathrm{d}s = M(A). \tag{5.4}$$

It is enough to verify this equation for the sets of the form $A = \{x \in \mathbb{R}^d : \|x\| > a, \frac{x}{\|x\|} \in B\}$, where $a > 0$ and $B \in \mathcal{B}(S^{d-1})$. For such A the left hand side of (5.4) can be written as

$$\begin{aligned}
&E\int_0^\infty I\left(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} > a, e_1 u_1^{1/\alpha} > a\|v_1\|, \frac{v_1}{\|v_1\|} \in B\right) \mathrm{d}s \\
&= \alpha^{-1}\|\sigma\| a^{-\alpha} EI\left(e_1 u_1^{1/\alpha} > a\|v_1\|,\ \frac{v_1}{\|v_1\|} \in B\right) \\
&= \alpha^{-1} a^{-\alpha}\int_B\int_0^\infty P(e_1 u_1^{1/\alpha} > as) Q(\mathrm{d}s|u)\sigma(\mathrm{d}u) \\
&= \int_B\int_0^\infty\int_a^\infty \mathrm{e}^{-rs} r^{-\alpha-1}\,\mathrm{d}r\, Q(\mathrm{d}s|u)\sigma(\mathrm{d}u) \\
&= \int_B\int_a^\infty q(r, u) r^{-\alpha-1}\,\mathrm{d}r\sigma(\mathrm{d}u) = M(A).
\end{aligned}$$

This proves (5.4).

If $\alpha \in (0, 1)$, then by Proposition 2.8 $\int_{\|x\|\le 1}\|x\|M(\mathrm{d}x) < \infty$. Equivalently, by (5.4),

$$\int_0^\infty E(\|H(s, (v_1, e_1, u_1))\| I(\|H(s, (v_1, e_1, u_1))\| \le 1))\,\mathrm{d}s = \int_{\|x\|\le 1}\|x\|M(\mathrm{d}x) < \infty.$$

Therefore, Theorem 4.1(A) in [19] applies and proves part (i) when $\alpha \in (0, 1)$.

If $\alpha \in [1, 2)$, then

$$\int_0^\infty E(\|H(s, (v_1, e_1, u_1))\| I(\|H(s, (v_1, e_1, u_1))\| > 1))\,\mathrm{d}s = \int_{\|x\|> 1}\|x\|M(\mathrm{d}x) < \infty.$$

Therefore, from Theorem 4.1(B) in [19] we infer that

$$S_2 = \sum_{j=1}^\infty\left[\left(\left(\frac{\alpha\gamma_j}{\|\sigma\|}\right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|} - c_j\right] \tag{5.5}$$

converges a.s. and $S_2 \sim TS_\alpha(R, 0)$, where

$$c_j = \int_{j-1}^j E\left[\left(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha}\|v_1\|^{-1}\right)\frac{v_1}{\|v_1\|}\right]\mathrm{d}s. \tag{5.6}$$

In general, the right hand side in (5.6) does not seem to have a closed form. However, when Q is symmetric then trivially we have $c_j = 0$. In this case (5.5) coincides with (5.1). This completes the proof of (i). To establish (ii), it remains to show that

$$\sum_{j=1}^{\infty}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j\right] = b, \tag{5.7}$$

where b is given by (5.3).

First we consider the case $\alpha \in (1,2)$. Define for $j \geq 1$

$$c'_j = \int_{j-1}^{j} E\left[\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \frac{v_1}{\|v_1\|}\right] \mathrm{d}s = \frac{\alpha^{1-1/\alpha}\|\sigma\|^{1/\alpha}}{\alpha-1}[j^{1-1/\alpha} - (j-1)^{1-1/\alpha}]x_0. \tag{5.8}$$

We have

$$\|c'_j - c_j\| \leq \int_{j-1}^{j} E\left\{\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha}\|v_1\|^{-1}\right]\right\} \mathrm{d}s.$$

Observe that for every $\theta > 0$

$$\int_0^{\infty}\left\{\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge \theta\right]\right\} \mathrm{d}s = \frac{\|\sigma\|}{\alpha(\alpha-1)}\theta^{1-\alpha}. \tag{5.9}$$

Using this identity for $\theta = e_1 u_1^{1/\alpha}\|v_1\|^{-1}$ pointwise, we get

$$\begin{aligned}
\sum_{j=1}^{\infty}\|c'_j - c_j\| &\leq E\int_0^{\infty}\left\{\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha}\|v_1\|^{-1}\right]\right\} \mathrm{d}s \\
&= \frac{\|\sigma\|}{\alpha(\alpha-1)} E[e_1^{1-\alpha} u_1^{-1+1/\alpha}\|v_1\|^{\alpha-1}] \\
&= \frac{\Gamma(2-\alpha)}{\alpha-1}\|\sigma\| E\|v_1\|^{\alpha-1} = \frac{\Gamma(2-\alpha)}{\alpha-1}\int_{\mathbb{R}^d}\|x\| R(\mathrm{d}x) < \infty.
\end{aligned} \tag{5.10}$$

Using (5.9) again we get

$$\begin{aligned}
\sum_{j=1}^{\infty}(c'_j - c_j) &= E\left\{\int_0^{\infty}\left(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha}\|v_1\|^{-1}\right]\right) \mathrm{d}s \frac{v_1}{\|v_1\|}\right\} \\
&= E\left\{\frac{\|\sigma\|}{\alpha(\alpha-1)} e_1^{1-\alpha} u_1^{-1+1/\alpha}\|v_1\|^{\alpha-1}\frac{v_1}{\|v_1\|}\right\} \\
&= \frac{\Gamma(2-\alpha)}{\alpha-1}\int_{\mathbb{R}^d} x\|x\|^{\alpha-2} Q(\mathrm{d}x) \\
&= \frac{\Gamma(2-\alpha)}{\alpha-1}\int_{\mathbb{R}^d} x R(\mathrm{d}x) = -\Gamma(1-\alpha)x_1.
\end{aligned}$$

Then we have

$$\sum_{j=1}^{n}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c'_j\right] = \left(\sum_{j=1}^{n} j^{-1/\alpha} - \frac{\alpha}{\alpha-1}n^{1-1/\alpha}\right)\alpha^{-1/\alpha}\|\sigma\|^{1/\alpha}x_0.$$

From a classical formula [1, 23.2.9],

$$\sum_{j=1}^{n} j^{-z} - \frac{n^{1-z}}{1-z} = \zeta(z) + z\int_n^\infty \frac{s-[s]}{s^{z+1}}\,\mathrm{d}s, \quad \Re(z) > 0,\ \Re(z) \neq 1, \tag{5.11}$$

we obtain

$$\sum_{j=1}^{\infty}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c'_j\right] = \zeta\left(\frac{1}{\alpha}\right)\alpha^{-1/\alpha}\|\sigma\|^{1/\alpha}x_0.$$

Consequently,

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j\right] &= \sum_{j=1}^{\infty}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c'_j\right] + \sum_{j=1}^{\infty}(c'_j - c_j)\\ &= \zeta\left(\frac{1}{\alpha}\right)\alpha^{-1/\alpha}\|\sigma\|^{1/\alpha}x_0 - \Gamma(1-\alpha)x_1 = b\end{aligned}$$

which proves (5.7) in the case $\alpha \in (1,2)$. We will also record the following estimate for later use:

$$\begin{aligned}\sum_{j=1}^{\infty}\left\|\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j\right\| &\le \sum_{j=1}^{\infty}\left\|\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c'_j\right\| + \sum_{j=1}^{\infty}\|c'_j - c_j\|\\ &= \sum_{j=1}^{\infty}\left(\frac{\alpha}{\alpha-1}[j^{1-1/\alpha} - (j-1)^{1-1/\alpha}] - j^{-1/\alpha}\right)\\ &\quad \times \alpha^{-1/\alpha}\|\sigma\|^{1/\alpha}\|x_0\| + \sum_{j=1}^{\infty}\|c'_j - c_j\|\\ &\le \alpha^{-1/\alpha}|\zeta(1/\alpha)|\|\sigma\|^{1/\alpha} + \frac{\Gamma(2-\alpha)}{\alpha-1}\int_{\mathbb{R}^d}\|x\|R(\mathrm{d}x)\\ &= C_1\|\sigma\|^{1/\alpha} + C_2\int_{\mathbb{R}^d}\|x\|R(\mathrm{d}x).\end{aligned} \tag{5.12}$$

The first equality we deduce from the mean value theorem for $s \mapsto \frac{\alpha}{\alpha-1}s^{1-1/\alpha}$; the last inequality holds because $\|x_0\| \le 1$, (5.10) and (5.11).

Now we consider the case $\alpha = 1$. Proceeding like we did above, define for $j \ge 2$

$$c'_j = \int_{j-1}^{j} E\left[\left(\frac{s}{\|\sigma\|}\right)^{-1}\frac{v_1}{\|v_1\|}\right]\mathrm{d}s = (\log j - \log(j-1))\|\sigma\|x_0 \tag{5.13}$$

and put $c'_1 = 0$. Observe that for every $\theta > 0$

$$\begin{aligned}&\int_1^\infty\left\{\left(\frac{s}{\|\sigma\|}\right)^{-1} - \left[\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge\theta\right]\right\}\mathrm{d}s\\ &= \{\theta - \|\sigma\|\log\theta + \|\sigma\|\log\|\sigma\| - \|\sigma\|\}\,I(\theta \le \|\sigma\|)\\ &\le \|\sigma\|\log^+\left(\frac{\|\sigma\|}{\theta}\right).\end{aligned} \tag{5.14}$$

Thus,

$$\begin{aligned}\sum_{j=2}^{\infty}\|c'_j - c_j\| &\le E\int_1^{\infty}\left\{\left(\frac{s}{\|\sigma\|}\right)^{-1} - \left[\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge e_1u_1\|v_1\|^{-1}\right]\right\}\mathrm{d}s\\ &\le \|\sigma\| E\log^+\left(\frac{\|\sigma\|\|v_1\|}{e_1u_1}\right)\\ &\le \|\sigma\|\left(|\log\|\sigma\|| + E|\log\|v_1\|| + E|\log e_1u_1|\right)\\ &= \|\sigma\||\log\|\sigma\|| + \int_{\mathbb{R}^d}|\log\|x\||\|x\|R(\mathrm{d}x) + K\|\sigma\| < \infty \end{aligned} \tag{5.15}$$

where $K = E|\log e_1u_1| < \infty$. We will now compute $\sum_{j=1}^{\infty}(c'_j - c_j)$. Observe that for $\theta > 0$

$$\int_0^1\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge\theta\,\mathrm{d}s = \theta I(\theta\le\|\sigma\|) + \{\|\sigma\| - \|\sigma\|\log\|\sigma\| + \|\sigma\|\log\theta\} I(\theta > \|\sigma\|).$$

Combining this identity with (5.14) we get

$$\begin{aligned}&-\int_0^1\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge\theta\,\mathrm{d}s + \int_1^{\infty}\left\{\left(\frac{s}{\|\sigma\|}\right)^{-1} - \left[\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge\theta\right]\right\}\mathrm{d}s\\ &\quad = \|\sigma\|(\log\|\sigma\| - \log\theta - 1).\end{aligned}$$

Using this formula for $\theta = e_1u_1\|v_1\|^{-1}$ pointwise, we get

$$\begin{aligned}\sum_{j=1}^{\infty}(c'_j - c_j) &= E\left\{\left[-\int_0^1\left(\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge e_1u_1\|v_1\|^{-1}\right)\mathrm{d}s\right.\right.\\ &\quad\left.\left. + \int_1^{\infty}\left(\left(\frac{s}{\|\sigma\|}\right)^{-1} - \left[\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge e_1u_1\|v_1\|^{-1}\right]\right)\mathrm{d}s\right]\frac{v_1}{\|v_1\|}\right\}\\ &= \|\sigma\| E\left\{(\log\|\sigma\| + \log\|v_1\| - \log(e_1u_1) - 1)\frac{v_1}{\|v_1\|}\right\}.\end{aligned}$$

Now we find that the density of e_1u_1 is the exponential integral function $E_1(x) = \int_x^{\infty} t^{-1}\exp(-t)\,\mathrm{d}t$. Hence

$$E\{\log(e_1u_1)\} = \int_0^{\infty}\log x\, E_1(x)\,\mathrm{d}x = -1 - \gamma$$

where γ is the Euler constant [10, 6.234]. Consequently,

$$\begin{aligned}\sum_{j=1}^{\infty}(c'_j - c_j) &= \|\sigma\| E\left\{\frac{v_1}{\|v_1\|}(\log\|\sigma\| + \log\|v_1\| + \gamma)\right\}\\ &= \int_{\mathbb{R}^d}\frac{x}{\|x\|}[\log\|\sigma\| + \log\|x\| + \gamma]Q(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d} x[\gamma + \log\|\sigma\| - \log\|x\|]R(\mathrm{d}x)\\ &= (\gamma + \log\|\sigma\|)x_1 - \int_{\mathbb{R}^d} x\log\|x\| R(\mathrm{d}x).\end{aligned}$$

On the other hand, for every $n \geq 1$

$$\sum_{j=1}^{n}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1} x_0 - c'_j\right] = \left(\sum_{j=1}^{n} j^{-1} - \log n\right) \|\sigma\| x_0.$$

Since

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n} j^{-1} - \log n\right) = \gamma, \tag{5.16}$$

we get

$$\sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1} x_0 - c'_j\right] = \gamma \|\sigma\| x_0 = \gamma x_1.$$

Consequently,

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1} x_0 - c_j\right] &= \sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1} x_0 - c'_j\right] + \sum_{j=1}^{\infty}(c'_j - c_j)\\ &= (2\gamma + \log\|\sigma\|)x_1 - \int_{\mathbb{R}_0^d} x \log\|x\| R(\mathrm{d}x) = b.\end{aligned}$$

This establishes (5.7) and completes the proof. For future use we also record the following estimate:

$$\begin{aligned}\sum_{j=1}^{\infty}\left\|\left(\frac{j}{\|\sigma\|}\right)^{-1} x_0 - c_j\right\| &\leq \sum_{j=1}^{\infty}\left\|\left(\frac{j}{\|\sigma\|}\right)^{-1} x_0 - c'_j\right\| + \sum_{j=1}^{\infty}\|c'_j - c_j\|\\ &= \left(1 + \sum_{j=1}^{\infty}(\log j - \log(j-1) - j^{-1})\right)\|\sigma\|\|x_0\|\\ &\quad + \sum_{j=1}^{\infty}\|c'_j - c_j\|\\ &\leq C\|\sigma\| + \|\sigma\||\log\|\sigma\|| + \int_{\mathbb{R}^d} |\log\|x\||\|x\| R(\mathrm{d}x)\end{aligned} \tag{5.17}$$

where C is a numerical constant. In the last bound we used that $\|x_0\| \leq 1$ and (5.15). □

The main difficulty in part (ii) of Theorem 5.1 was to find an explicit centering of the series. Besides the theoretical interest, explicit centers are needed for practical implementation of the representations, e.g., for simulation. Once we have a shot noise representation of an infinitely divisible law it is easy to give a shot noise representation of the corresponding Lévy process.

Theorem 5.3. *With the notation and assumptions of Theorem 5.1, let $\{\tau_j\}$ be an* iid *sequence of uniform random variables in* $[0, T]$*, where $T > 0$ is fixed. Assume that $\{\tau_j\}$ is independent of the random sequences $\{v_j\}$, $\{u_j\}$, $\{e_j\}$, and $\{\gamma_j\}$. Let x_0, x_1, ζ, and γ be as in Theorem 5.1.*
(i) *If $\alpha \in (0, 1)$, or if $\alpha \in [1, 2)$ and Q is symmetric, set*

$$X_0(t) = \sum_{j=1}^{\infty} I_{(0,t]}(\tau_j)\left(\left(\frac{\alpha\gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|}, \quad t \in [0, T]. \tag{5.18}$$

Then the series converges a.s. uniformly in $t \in [0, T]$ *to a Lévy process such that* $X_0(t) \sim TS^0_\alpha(tR, 0)$ *when* $\alpha \in (0, 1)$ *and* $X_0(t) \sim TS_\alpha(tR, 0)$ *when* $\alpha \in [1, 2)$.

(ii) *If* $\alpha \in [1, 2)$ *and* Q *is non-symmetric, assume additionally that* $\int_{\mathbb{R}^d} \|x\| |\log \|x\|| R(\mathrm{d}x) < \infty$ *when* $\alpha = 1$ *and that* $\int_{\mathbb{R}^d} \|x\| R(\mathrm{d}x) < \infty$ *when* $\alpha \in (1, 2)$. *Put*

$$X_1(t) = \sum_{j=1}^{\infty} \left[I_{(0,t]}(\tau_j) \left(\left(\frac{\alpha \gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} - \frac{t}{T} \left(\frac{\alpha j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + t b_T \tag{5.19}$$

where

$$b_T = \begin{cases} \alpha^{-1/\alpha} \zeta\left(\frac{1}{\alpha}\right) T^{-1} (T\|\sigma\|)^{1/\alpha} x_0 - \Gamma(1-\alpha) x_1, & 1 < \alpha < 2 \\ (2\gamma + \log(T\|\sigma\|)) x_1 - \int_{\mathbb{R}^d} x \log \|x\| R(\mathrm{d}x), & \alpha = 1. \end{cases} \tag{5.20}$$

Then the series (5.19) *converges a.s. uniformly in* $t \in [0, T]$ *to a Lévy process such that* $X_1(t) \sim TS_\alpha(tR, 0)$.

Proof. It is enough to show the convergence in distribution of series (5.18) and (5.19) for a fixed t; see [19, Theorem 5.1]. The proof goes along the same lines as the proof of the previous Theorem 5.1. For every $0 \notin A \in \mathcal{B}(\mathbb{R}^d)$ we have

$$\begin{aligned} &\int_0^\infty P\left\{ I_{(0,t]}(\tau_1) \left[\left(\frac{\alpha s}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha} \|v_1\|^{-1} \right] \frac{v_1}{\|v_1\|} \in A \right\} \mathrm{d}s \\ &= \frac{t}{T} \int_0^\infty P\left\{ \left[\left(\frac{\alpha s}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha} \|v_1\|^{-1} \right] \frac{v_1}{\|v_1\|} \in A \right\} \mathrm{d}s \\ &= t \int_0^\infty P\left\{ \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha} \|v_1\|^{-1} \right] \frac{v_1}{\|v_1\|} \in A \right\} \mathrm{d}s = tM(A). \end{aligned}$$

The last equality comes from (5.4). The case $\alpha \in (0, 1)$ can be proven in exactly the same way as in Theorem 5.1. If $\alpha \in [1, 2)$, then $\int_{\|x\|>1} \|x\| M(\mathrm{d}x) < \infty$. From Theorem 4.1(B) in [19] we infer that

$$X_2(t) = \sum_{j=1}^{\infty} \left[I_{(0,t]}(\tau_j) \left(\left(\frac{\alpha \gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} - a_j^T(t) \right] \tag{5.21}$$

converges a.s. and $X_2(t) \sim TS_\alpha(tR, 0)$, where

$$a_j^T(t) = \int_{j-1}^{j} E\left[I_{(0,t]}(\tau_1) \left(\left(\frac{\alpha s}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_1 u_1^{1/\alpha} \|v_1\|^{-1} \right) \frac{v_1}{\|v_1\|} \right] \mathrm{d}s.$$

If Q is symmetric then $a_j^T(t) = 0$ and in this case (5.21) coincides with (5.18). This concludes the proof of (i).

To establish (ii), it will be helpful to view c_j of (5.6) as a sequence depending on Q, $c_j := c_j(Q)$. With this notation

$$a_j^T(t) = \frac{t}{T} c_j(TQ). \tag{5.22}$$

From (5.7) and (5.3) (with TQ and TR in place of Q and R, respectively) we have

$$\sum_{j=1}^{\infty} \left[\left(\frac{\alpha j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j(TQ) \right] = \begin{cases} \alpha^{-1/\alpha} \zeta \left(\frac{1}{\alpha} \right) (T\|\sigma\|)^{1/\alpha} x_0 - \Gamma(1-\alpha) T x_1, & 1 < \alpha < 2, \\ (2\gamma + \log(T\|\sigma\|)) T x_1 - T \int_{\mathbb{R}^d} x \log \|x\| R(\mathrm{d}x), & \alpha = 1. \end{cases}$$

From (5.22) we now have

$$\sum_{j=1}^{\infty} \left[\frac{t}{T} \left(\frac{\alpha j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 - a_j^T(t) \right] = t b_T$$

which completes the proof. □

If we drop $e_j u_j^{1/\alpha} \|v_j\|^{-1}$ from (5.18) and (5.19), the resulting series represents stable processes. We give their parametrization for the sake of comparison.

Proposition 5.4. *Let σ be given by (2.28). Then, with the notation of Theorem 5.3, we have the following.*
(i) *If $\alpha \in (0, 1)$, or if $\alpha \in [1, 2)$ and Q is symmetric, set*

$$Y_0(t) = \sum_{j=1}^{\infty} I_{(0,t]}(\tau_j) \left(\frac{\alpha \gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \frac{v_j}{\|v_j\|}. \tag{5.23}$$

Then the series converges a.s. uniformly in $t \in [0, T]$ to a strictly α-stable Lévy processes with $Y_0(t) \sim S_\alpha(t\sigma, 0)$.
(ii) *If $\alpha \in [1, 2)$ and Q is non-symmetric, put*

$$Y_1(t) = \sum_{j=1}^{\infty} \left[I_{(0,t]}(\tau_j) \left(\frac{\alpha \gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \frac{v_j}{\|v_j\|} - \frac{t}{T} \left(\frac{\alpha j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + t b_T, \tag{5.24}$$

where b_T is given in (5.20) of Theorem 5.3. Then the series converges a.s. uniformly in $t \in [0, T]$ to a α-stable Lévy processes with $Y_1(t) \sim S_\alpha(t\sigma, ta)$, where

$$a = \begin{cases} -\Gamma(1-\alpha) x_1, & 1 < \alpha < 2, \\ \int_{\mathbb{R}^d} x(1 - \log \|x\|) R(\mathrm{d}x), & \alpha = 1. \end{cases} \tag{5.25}$$

Proof. This result has been known for a long time. It can also be verified along the lines of the proofs of Theorems 5.1 and 5.3. □

Remark 5.5. Jumps of processes X_0 and X_1 in Theorem 5.3 are equal to either $\left(\frac{\alpha \gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \frac{v_j}{\|v_j\|}$ (stable jumps as in (5.23) and (5.24)) or $e_j u_j^{1/\alpha} \frac{v_j}{\|v_j\|^2}$. If T is small then, under the operation minimum in (5.18) and (5.19), stable jumps prevail and we observe a stable process in a short time. Conversely, if T is large then the iid terms $e_j u_j^{1/\alpha} \frac{v_j}{\|v_j\|^2}$ prevail and the central limit theorem explains the Gaussian behavior of $T\alpha S$ processes in a large time frame. Therefore, representations (5.18) and (5.19) capture the interplay between α-stable and Gaussian trends in $T\alpha S$ processes.

Remark 5.6. Theorem 5.3 in conjunction with Proposition 5.4 reveals the nature of tempering of stable jumps that leads to tempered α-stable processes. Stable jumps $\left(\frac{\alpha\gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} \frac{v_j}{\|v_j\|}$ are cut at the level of $e_j u_j^{1/\alpha} \|v_j\|^{-1}$ but their direction is retained. The level of cut (tempering) does not depend on the magnitude of a stable jump, only on its direction, $\frac{v_j}{\|v_j\|}$. Indeed, given that $\frac{v_j}{\|v_j\|} = u$, the conditional distribution of $\|v_j\|$ is $Q(\cdot|u)$.

Remark 5.7. We may also give a probabilistic interpretation of condition (4.1) in Theorem 4.1 (see also Remark 4.2). Roughly speaking, (4.1) fails when $q(\cdot, u)$ decreases rapidly in the neighborhood of zero in which case $Q(\cdot|u)$ is heavy tailed. However, when $Q(\cdot|u)$ is heavy tailed, large values of $\|v_j\|$ will appear often in (5.18) and (5.19) and so these series will have many iid terms. This, in turn, produces strong Gaussian trend (under appropriate conditions on R) which makes the process significantly different from the underlying stable process (5.23) and (5.24). Theorem 4.1 confirms that intuition.

6. Ornstein–Uhlenbeck-type tempered stable processes

Tempered stable laws are self-decomposable, as we have remarked in Section 2. Therefore, if μ is a tempered stable distribution on $\mathbb{R}^d$, then there exists a Lévy process $Z = \{Z(t) : t \in \mathbb{R}\}$ with $E \log^+ \|Z(1)\| < \infty$ such that

$$\mu = \mathcal{L}\left\{\int_0^\infty \mathrm{e}^{-s}\, \mathrm{d}Z(s)\right\}; \tag{6.1}$$

see [11]. Consequently, one can define an Ornstein–Uhlenbeck-type process (OU process)

$$X(t) = \int_{-\infty}^t \mathrm{e}^{-(t-s)}\, \mathrm{d}Z(s) \quad t \in \mathbb{R}, \tag{6.2}$$

such that $\mathcal{L}(X(t)) = \mu$ for every $t \in \mathbb{R}$. The process Z in (6.2) is called the *background driving Lévy process* (BDLP) of the OU process X. Our goal is to obtain a shot noise representation of X which will give an insight into its structure and can be used for simulation of sample paths. Special cases of tempered stable OU processes have been used extensively in mathematical finance; cf. [3,4].

There is a one-to-one correspondence between Lévy characteristics of μ and $\mathcal{L}\{Z(1)\}$ given in [11]. A direct computation gives the following.

Lemma 6.1. *Let μ be a $T\alpha S$ distribution with spectral measure R. Then the Lévy measure M_Z of $\mathcal{L}\{Z(1)\}$ in (6.1) is given by*

$$M_Z(A) = \int_{\mathbb{R}^d} \int_0^\infty I_A(sx)(\alpha s^{-\alpha-1} + s^{-\alpha})\mathrm{e}^{-s}\, \mathrm{d}s\, R(\mathrm{d}x) \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Notice a similarity between M_Z above and M in (2.7). Using estimates similar to those in the proofs of Propositions 2.7 and 2.8 we establish the following.

Lemma 6.2. *Let Z be a Lévy process as in (6.1), where μ is a $T\alpha S$ distribution with the spectral measure R. If $\alpha \in (0,1)$ and $\mu \sim TS^0_\alpha(R,0)$, then $\int_{\|x\|\le 1} \|x\| M_Z(\mathrm{d}x) < \infty$ and*

$$E\mathrm{e}^{\mathrm{i}\langle y, Z(1)\rangle} = \exp\left(\int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}\langle y,x\rangle} - 1) M_Z(\mathrm{d}x)\right). \tag{6.3}$$

If $\alpha \in [1,2)$ and $\mu \sim TS_\alpha(R,0)$, then $\int_{\|x\|>1} \|x\| M_Z(\mathrm{d}x) < \infty$ and

$$E\mathrm{e}^{\mathrm{i}\langle y, Z(1)\rangle} = \exp\left(\int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}\langle y,x\rangle} - 1 - \mathrm{i}\langle y, x\rangle) M_Z(\mathrm{d}x)\right). \tag{6.4}$$

In what follows we will assume the notation of Section 5. It appears that the series representation of the process Z is very similar to (5.18) and (5.19).

Proposition 6.3. *Let Z be a Lévy process as in (6.1), where μ is a proper $T\alpha S$ distribution with the spectral measure R. Assume that $\mu \sim TS^0_\alpha(R,0)$ when $\alpha \in (0,1)$ and $\mu \sim TS_\alpha(R,0)$ when $\alpha \in [1,2)$. Let $T > 0$ be fixed.*
(i) *If $\alpha \in (0,1)$, or if $\alpha \in [1,2)$ and Q is symmetric, set*

$$Z_0(t) = \sum_{j=1}^{\infty} I_{(0,t]}(\tau_j) \left(\left(\frac{\gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|}. \tag{6.5}$$

Then the series converges a.s. uniformly in $t \in [0,T]$ and Z_0 is a version of Z restricted to $[0,T]$.
(ii) *If $\alpha \in [1,2)$ and Q is non-symmetric, assume additionally that $\int_{\mathbb{R}^d} \|x\| |\log \|x\|| R(\mathrm{d}x) < \infty$ when $\alpha = 1$ and that $\int_{\mathbb{R}^d} \|x\| R(\mathrm{d}x) < \infty$ when $\alpha \in (1,2)$. Put*

$$\begin{aligned} Z_1(t) = \sum_{j=1}^{\infty} \Bigg[& I_{(0,t]}(\tau_j) \left(\left(\frac{\gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} \\ & - \frac{t}{T} \left(\frac{j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 \Bigg] + t c_T \end{aligned} \tag{6.6}$$

where

$$c_T = \begin{cases} \zeta\left(\dfrac{1}{\alpha}\right) T^{-1} (T\|\sigma\|)^{1/\alpha} x_0 - \Gamma(1-\alpha) x_1, & 1 < \alpha < 2 \\ (2\gamma - 1 + \log(T\|\sigma\|)) x_1 - \displaystyle\int_{\mathbb{R}^d} x \log\|x\| R(\mathrm{d}x), & \alpha = 1. \end{cases} \tag{6.7}$$

Then series (6.6) converges a.s. uniformly in $t \in [0,T]$ and Z_1 is a version of Z restricted to $[0,T]$.

Proof. The proof follows exactly the same line of arguments as the proofs of Theorems 5.1 and 5.3 and thus is omitted. □

Theorem 6.4. *Let X be an* OU *process in $\mathbb{R}^d$ with $\mathcal{L}(X(t)) = \mu$, where μ is a proper $T\alpha S$ distribution having the spectral measure R. Assume that $\mu \sim TS^0_\alpha(R,0)$ when $\alpha \in (0,1)$ and $\mu \sim TS_\alpha(R,0)$ when $\alpha \in [1,2)$. Let $T > 0$ be fixed,and let $\{\tau_j\}$, $\{v_j\}$, $\{e_j\}$, and $\{\gamma_j\}$ be random*

sequences as in Theorem 5.3. Let ξ_0 be a random vector with $\mathcal{L}(\xi_0) = \mu$ and independent of the random sequences $\{\tau_j\}$, $\{v_j\}$, $\{e_j\}$, and $\{\gamma_j\}$. Let x_0, x_1, ζ, and γ be as in Theorem 5.1.
(i) *If $\alpha \in (0, 1)$, or if $\alpha \in [1, 2)$ and Q is symmetric, set*

$$X_0(t) = \xi_0 + \sum_{j=1}^{\infty} \mathrm{e}^{-(t-\tau_j)} I_{(0,t]}(\tau_j) \left(\left(\frac{\gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|}. \tag{6.8}$$

Then the series converges a.s. uniformly in $t \in [0, T]$ and X_0 is a version of X restricted to $[0, T]$.
(ii) *If $\alpha \in [1, 2)$ and Q is non-symmetric, assume additionally that $\int_{\mathbb{R}^d} \|x\| |\log \|x\|| R(\mathrm{d}x) < \infty$ when $\alpha = 1$ and that $\int_{\mathbb{R}^d} \|x\| R(\mathrm{d}x) < \infty$ when $\alpha \in (1, 2)$. Put*

$$X_1(t) = \xi_0 + \sum_{j=1}^{\infty} \left[\mathrm{e}^{-(t-\tau_j)} I_{(0,t]}(\tau_j) \left(\left(\frac{\gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} - \frac{1-\mathrm{e}^{-t}}{T} \left(\frac{j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + (1-\mathrm{e}^{-t}) c_T \tag{6.9}$$

where c_T is given by (6.7). Then series (6.6) converges a.s. uniformly in $t \in [0, T]$ and X_1 is a version of X restricted to $[0, T]$.

Proof. For any $f \in D([0, T], \mathbb{R}^d)$ we have $\int_0^t \mathrm{e}^{-(t-s)} \,\mathrm{d}f(s) = f(t) - \mathrm{e}^{-t} f(0) - \int_0^t f(s)\mathrm{e}^{-(t-s)} \,\mathrm{d}s$. Hence, if $f_n, f \in D([0, T], \mathbb{R}^d)$ and $f_n \to f$ uniformly on $[0, T]$, then $\int_0^t \mathrm{e}^{-(t-s)} \,\mathrm{d}f_n(s) \to \int_0^t \mathrm{e}^{-(t-s)} \,\mathrm{d}f(s)$ uniformly in $t \in [0, T]$ $(n \to \infty)$. Applying this fact to series (6.5) of Proposition 6.3(i) we get

$$\int_0^t \mathrm{e}^{-(t-s)} \mathrm{d}_s \left[\sum_{j=1}^{n} I_{(0,s]}(\tau_j) \left(\left(\frac{\gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} \right] \to \int_0^t \mathrm{e}^{-(t-s)} \,\mathrm{d}Z_0(s)$$

a.s. uniformly in $t \in [0, T]$ as $n \to \infty$. That is

$$\sum_{j=1}^{\infty} \mathrm{e}^{-(t-\tau_j)} I_{(0,t]}(\tau_j) \left(\left(\frac{\gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge e_j \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} = \int_0^t \mathrm{e}^{-(t-s)} \,\mathrm{d}Z_0(s)$$

a.s. and the series converges a.s. uniformly in $t \in [0, T]$. This proves part (i) of the theorem because Z_0 is a version of Z on $[0, T]$ and by (6.2), $X(0) \sim \mu$ is independent of $\int_0^t \mathrm{e}^{-(t-s)} \,\mathrm{d}Z(s)$.

Using the same argument in conjunction with Proposition 6.3 (ii) we prove (ii) of the theorem and conclude the proof. □

Acknowledgements

This research was supported by a grant from the National Science Foundation.

The author is grateful to Ole E. Barndorff-Nielsen, Christian Houdré, Neil Shephard, and Wojbor A. Woyczyński for stimulating discussions and continuous interest in this work.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables, Dover Publications, 1974.
[2] O.E. Barndorff-Nielsen, M. Maejima, K. Sato, Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations, Bernoulli 12 (2006) 1–33.

[3] O.E. Barndorff-Nielsen, N. Shephard, Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics, J. Roy. Statist. Soc. Ser. B 63 (2001) 1–42.
[4] O.E. Barndorff-Nielsen, N. Shephard, Normal modified stable processes, Theory Probab. Math. Statist. 65 (2002) 1–20.
[5] L. Bondesson, Generalized gamma convolutions and related classes of distributions and densities, in: Lecture Notes in Statist., vol. 76, Springer-Verlag, 1992.
[6] S. Boyarchenko, S. Levendorskii, Option pricing for truncated Lévy processes, Int. J. Theor. Appl. Finance 3 (2000) 549–552.
[7] P.L. Brockett, H.G. Tucker, A conditional dichotomy theorem for stochastic processes with independent increments, J. Multivariate Anal. 7 (1977) 13–27.
[8] P. Carr, H. Geman, D.B. Madan, M. Yor, The fine structure of asset returns: An empirical investigation, J. Business 75 (2002) 303–325.
[9] P. Carr, H. Geman, D.B. Madan, M. Yor, Stochastic volatility for Lévy processes, Math. Finance 13 (2003) 345–382.
[10] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 2000.
[11] Z.J. Jurek, W. Vervaat, An integral representation for selfdecomposable Banach space valued random variables, Z. Wahrscheinlichkeitstheorie verw. Gebiete 62 (1983) 247–262.
[12] O. Kallenberg, Foundations of Modern Probability, 2nd ed., Springer-Verlag, 2002.
[13] I. Koponen, Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, Phys. Rev. E 52 (1995) 1197–1199.
[14] R.N. Mantegna, H.E. Stanley, Stochastic process with ultraslow convergence to a Gaussian: The truncated Lévy flight, Phys. Rev. Lett. 73 (1994) 2946–2949.
[15] E.A. Novikov, Infinitely divisible distributions in turbulence, Phys. Rev. E 50 (1994) R3303–R3305.
[16] J. Pitman, M. Yor, The two parameter Poisson–Dirichlet laws, Ann. Probab. 25 (1997) 855–900.
[17] J. Rosiński, On series representations of infinitely divisible random vectors, Ann. Probab. 18 (1990) 405–430.
[18] J. Rosiński, Contribution to the discussion of "Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics", O.E. Barndorff-Nielsen, N. Shephard, Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics, J. Roy. Statist. Soc. Ser. B 63 (2001) 167–241, 230–231.
[19] J. Rosiński, Series representations of Levy processes from the perspective of point processes, in: O.E. Barndorff-Nielsen, T. Mikosch, S.I. Resnick (Eds.), Lévy Processes — Theory and Applications, Birkhäuser, Boston, 2001, pp. 401–415.
[20] J. Rosiński, Tempered stable processes, in: O.E. Barndorff-Nielsen (Ed.), Second MaPhySto Conference on Lévy Processes: Theory and Applications. http://www.maphysto.dk/publications/MPS-misc/2002/22.pdf, 2002, pp. 215–220.
[21] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes, Chapman & Hall, 1994.
[22] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.