Topology and its Applications 159 (2012) 1277–1287

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A non-metric perfectly normal hereditarily indecomposable continuum

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A R T I C L E   I N F O

Article history:
Received 2 August 2010
Received in revised form 20 July 2011
Accepted 21 November 2011

MSC:
primary 54F15, 54D35
secondary 54B20

Keywords:
Indecomposable continuum
Hereditarily indecomposable continua
Composant
Pseudo-arc
Non-metric continua
Perfectly normal

A B S T R A C T

We construct an example of a non-metric perfectly normal hereditarily indecomposable continuum. The example is constructed as an inverse limit of non-metric analogues of solenoids. Theorems needed to insure perfect normality are stated and proven. It is shown that the example cannot be embedded in a countable product of Hausdorff arcs.

1. Introduction

In [1] Bellamy answered the long outstanding question of the existence of a (necessarily) non-metric indecomposable continuum with only one composant. A natural question arising from this result is whether or not there exists a hereditarily indecomposable continuum with only one composant. Lewis stated this as Question 36 in his recent paper in *Open Problems in Topology II* (2007) [13]. This is the problem that began this investigation. Bellamy constructed his example by first constructing an indecomposable continuum with exactly two composants as an inverse limit indexed by ω_1 and then identifying two points, one from each composant [1]. One of us has used inverse limits indexed by ω_1 to construct a hereditarily indecomposable continuum with two composants [22]. If the technique of Bellamy, in which he identifies two points from different composants, is used, then a continuum is produced that contains a decomposable subcontinuum. In considering this question the authors have endeavored to develop techniques to construct non-metric hereditarily indecomposable continua and to determine the number of composants of these continua. Results [18,20,21,10] show that, in many situations potentially promising techniques imply that hereditary indecomposable continua so constructed are metric.

Since the continua in these settings were first countable, Gruenhage asked (at the dissertation defense of one of us – Greiwe) if a first countable hereditarily indecomposable continuum is necessarily metric. After some investigation of the problem, it was conjectured that an example constructed by one of us [23] was likely a counterexample. He then asked if this example has the additional property of being perfectly normal. This question led to the main result of this paper.

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.11.051

It addresses a question posed by Daniel, Nikiel, Treybig, Tuncali and Tymchatyn in [8] in which they use inverse limits to construct special non-metric perfectly normal spaces.

We construct an example of a non-metric perfectly normal hereditarily indecomposable continuum; the example will also be shown to be first countable. Non-trivial inverse limits of continua do not typically produce perfectly normal continua simply because the non-perfect normality of ω_1 is carried over into the inverse limit space. Metric spaces are perfectly normal, and in a sense perfectly normal spaces are "close" to metric spaces. Our example then is one answer to the more general question: "How close to metric can a non-metric hereditarily indecomposable continuum be?"

2. Background definitions and theorems

Definition 1. A set is said to be a G_δ set if it is the common part of a countable collection of open sets.

Definition 2. The space X is said to be *perfectly normal* iff it is normal and every closed set is G_δ.

A main component of our example is built upon the so called "double arrow" space which we denote by Z. For ease of notation we will use a subscript notation in lieu of product notation to describe this space; we do this in order to avoid confusion that could occur with a product space one coordinate of which is another product space. Toward that end, for $x \in [0,1]$ and $i \in \{0,1\}$ we let $(x)_i$ denote a point of our space Z; let $Z = \{(x)_i \mid x \in [0,1],\ i = 0,1\} - \{(0)_0, (1)_1\}$. Define the following order on Z: $(x)_i <_Z (y)_j$ iff $x < y$ or $x = y$ and $i < j$. The space Z with the order topology is called the double arrow space. Note that Z is a non-metric compact linearly ordered topological space. For the sake of completeness we justify the needed properties of the space Z relevant to our work.

Theorem 1. *The double arrow space Z is perfectly normal.*

Proof. It is straightforward to verify that with the order topology Z is a compact Hausdorff space; the normality of Z follows from compactness.

Let $h : Z \to [0,1]$ denote the function defined by $h((x)_i) = x$. We claim that h is continuous.

Let $(a,b) = \{y \in [0,1] \mid a < y < b\}$ be a basic open set in $[0,1]$ that contains neither 0 nor 1. Then $h^{-1}((a,b)) = \{z \in Z \mid (a)_1 <_Z z <_Z (b)_0\}$ is open in Z. For the basic open set $[0,b)$ in the interval, we have $h^{-1}([0,b)) = \{z \in Z \mid (0)_1 \leqslant_Z z <_Z (b)_0\}$ is open in Z; similarly for the basic open set $(a,1]$. Therefore h is continuous.

Let M be a closed subset of Z; we will show that M is a G_δ set. Since Z is compact, M is compact and $h(M)$ is compact which is closed in $[0,1]$. Since the unit interval $[0,1]$ is perfectly normal, there is a countable collection $\{U_i\}_{i=1}^\infty$ of open sets in $[0,1]$ so that $h(M) = \bigcap_{i=1}^\infty U_i$. Thus, $M \subset \bigcap_{i=1}^\infty h^{-1}(U_i)$. Let K be the set to which $(t)_j$ belongs if and only if $(t)_j \in M$ and only one member of the set $\{(t)_0, (t)_1\}$ lies in M. Thus $h|_K$ is one-to-one and $h|_{M-K}$ is two-to-one; $h(K)$ is the set to which x belongs if and only exactly one of $(x)_0$ or $(x)_1$ is in M. Let $K_0 = \{(x)_i \in K \mid i = 0\}$ and $K_1 = \{(x)_i \in K \mid i = 1\}$.

We claim that K is countable. Suppose not, then since $K = K_0 \cup K_1$, either K_0 or K_1 is uncountable. Without loss of generality assume K_0 is uncountable, and hence $h(K_0)$ is an uncountable subset of the reals. This set must contain a point y which is a limit point of the set of points of $h(K_0)$ both to the left and to the right of it; that is: y is a limit point of both $[0,y) \cap h(K_0)$ and $(y,1] \cap h(K_0)$. Since $y \in h(K_0)$, $(y)_0 \in M$ but $(y)_1 \notin M$. Since y is a limit point of $(y,1] \cap h(K_0)$ it follows that for every $t > y$, the basic open set $\{z \in Z \mid (y)_1 \leqslant_Z z < (t)_1\}$ contains a point of K_0. So $(y)_1$ is a limit point of K_0 and hence a limit point of M. Since M is closed $(y)_1 \in M$. But then both $(y)_0$ and $(y)_1$ are points of M and that contradicts the definition of K. Hence K is countable.

For each $k \in K$ there is an element $x^k \in [0,1]$ so that only one of $\{(x^k)_0, (x^k)_1\}$ lies in M and let z^k denote whichever of this two element set is *not* in M. Therefore

$$M = \bigcap_{i=1}^{\infty} h^{-1}(U_i) - \{z^k \mid k \in K\} = \bigcap_{i=1}^{\infty} h^{-1}(U_i) \cap \left(\bigcap_{k \in K} \left(Z - \{z^k\}\right) \right).$$

So M is the common part of two countable collections of open sets: namely the collections $\{h^{-1}(U_i) \mid i$ is a positive integer$\}$ and $\{Z - \{z^k\} \mid k \in K\}$. Thus M is G_δ. $\square$

Theorem 2. *The product space $X = Z \times [0,1]$ is perfectly normal.*

Proof. Let π_Z and π_I be the projections of X onto the Z and $[0,1]$ coordinates respectively. Let M be a closed set in X. In order to prove perfect normality we break up the $[0,1]$ coordinate into finer and finer collections of overlapping intervals $\{[\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i+1}{2^n} + \frac{1}{4^n}]\}_{i=1}^{2^n}$ and use these to break up $Z \times [0,1]$ into finer and finer overlapping strips (see Fig. 1).

For each positive integer n and each positive integer $i \leqslant 2^n$, $\pi_Z(M \cap Z \times [\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i}{2^n} + \frac{1}{4^n}])$ is closed in Z. It is the common part of a countable collection of open sets $\{V_{i,j}^n\}_{j=1}^\infty$.

Let S_n be the set of all functions from the finite set $\{1, 2, 3, \ldots, 2^n\}$ into the positive integers.

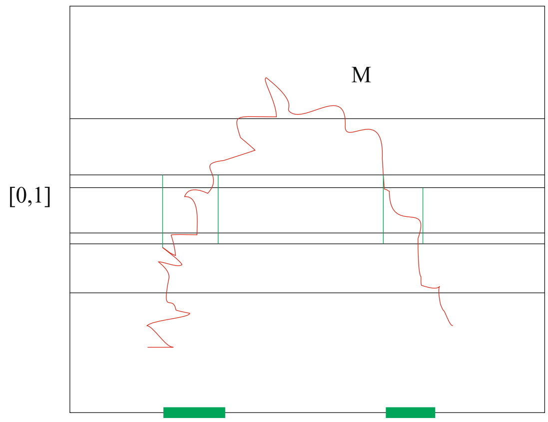

Fig. 1. $\pi_Z(M \cap Z \times [\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i}{2^n} + \frac{1}{4^n}])$.

For each integer n and $\rho \in S_n$ let

$$W_\rho^n = \bigcup_{i=1}^{2^n} V_{i,\rho(i)}^n \times \left(\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i}{2^n} + \frac{1}{4^n}\right).$$

We claim that

$$M = \bigcap_{n=1}^{\infty}\left(\bigcap_{\rho\in S_n} W_\rho^n\right)$$

which represents M as the common part of a countable union of countable open sets in X.

Since $[0,1] = \bigcup_{i=1}^{2^n}[\frac{i-1}{2^n}, \frac{i}{2^n}]$ and for each positive integers n and i and $\rho \in S_n$:

$$M \cap \left(Z \times \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right) \subset V_{i,\rho(i)}^n \times \left(\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i}{2^n} + \frac{1}{4^n}\right)$$

it follows that

$$M \subset \bigcap_{n=1}^{\infty}\left(\bigcap_{\rho\in S_n} W_\rho^n\right).$$

Suppose that the subset relation does not go in the other direction and that there is a point

$$\big((z)_k, t\big) \in \left(\bigcap_{n=1}^{\infty}\left(\bigcap_{\rho\in S_n} W_\rho^n\right)\right) - M.$$

Then, since M is closed, there is a basic open set in the form $S \times (r,s)$ that contains $((z)_k, t)$ and does not intersect M. There is an integer n and an integer $i \leqslant 2^n$ so that $t \in [\frac{i-1}{2^n}, \frac{i}{2^n}] \subset (r,s)$; furthermore, n and i can be chosen so that we also have $[\frac{i-2}{2^n} - \frac{1}{4^n}, \frac{i+1}{2^n} + \frac{1}{4^n}] \subset (r,s)$. There is a compact interval $[a,b]$ in Z so that $(z)_k \in [a,b] \subset S$. Since $S \times (r,s)$ misses M it follows that

$$\pi_Z\left(M \cap \left(Z \times \left[\frac{i-2}{2^n} - \frac{1}{4^n}, \frac{i-1}{2^n} + \frac{1}{4^n}\right]\right)\right) \cap [a,b] = \emptyset,$$
$$\pi_Z\left(M \cap \left(Z \times \left[\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i}{2^n} + \frac{1}{4^n}\right]\right)\right) \cap [a,b] = \emptyset,$$
$$\pi_Z\left(M \cap \left(Z \times \left[\frac{i}{2^n} - \frac{1}{4^n}, \frac{i+1}{2^n} + \frac{1}{4^n}\right]\right)\right) \cap [a,b] = \emptyset.$$

So there exist integers j_1, j_2 and j_3 so that (modulo the case when $i = 1$ or $i = 2^n$):

$$V_{i-1,j_1}^n \cap [a,b] = \emptyset,$$
$$V_{i,j_2}^n \cap [a,b] = \emptyset,$$
$$V_{i+1,j_3}^n \cap [a,b] = \emptyset.$$

Then let $\rho \in S_n$ be chosen such that $\rho(i-1) = j_1$, $\rho(i) = j_2$ and $\rho(i+1) = j_3$. Since the only integers k so that $[\frac{k}{2^n} - \frac{1}{4^n}, \frac{k}{2^n} + \frac{1}{4^n}]$ intersects $[\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i}{2^n} + \frac{1}{4^n}]$ are $k = i-1, i, i+1$ it follows that W^n_ρ does not intersect $[a,b] \times [\frac{i-1}{2^n} - \frac{1}{4^n}, \frac{i}{2^n} + \frac{1}{4^n}]$ and so $x \notin W^n_\rho$. And this verifies

$$M = \bigcap_{n=1}^{\infty} \left(\bigcap_{\rho \in S_n} W^n_\rho \right). \qquad \square$$

Theorem 3. *Let Q denote the Hilbert cube $[0,1]^\infty$ and Z the double arrow space. Then $Z \times Q$ is perfectly normal.*

Proof. Let $f : [0,1] \to Q$ be an onto continuous map. Then $F = \mathrm{id}_Z \times f : Z \times [0,1] \to Z \times Q$ is an onto continuous map. Suppose $M \subset Z \times Q$ is closed, then $W = F^{-1}(Z \times Q - M)$ is open and $Z \times [0,1] - W$ is closed. Since $Z \times [0,1]$ is perfectly normal there is a countable collection $\{U_i\}_{i=1}^\infty$ of open sets so that $Z \times [0,1] - W = \bigcap_{i=1}^\infty U_i$. Then

$$W = \bigcup_{i=1}^{\infty} \big(Z \times [0,1] - U_i\big),$$

$$F(W) = F\left(\bigcup_{i=1}^{\infty} \big(Z \times [0,1] - U_i\big) \right),$$

$$Z \times Q - M = \bigcup_{i=1}^{\infty} F\big(Z \times [0,1] - U_i\big),$$

$$M = Z \times Q - \bigcup_{i=1}^{\infty} F\big(Z \times [0,1] - U_i\big).$$

So M is the complement of an F_σ set and hence is G_δ. $\square$

Corollary 1. *If P is a subcontinuum of the Hilbert cube Q then $Z \times P$ is perfectly normal.*

It follows from this theorem that $Z \times$ pseudo-arc is also perfectly normal.

By a continuum we mean a compact and connected Hausdorff space. We define some special types of continua below. See [16,12,17] for background on the properties of these types of continua.

Definition 3. A continuum is said to be decomposable if it is the union of two proper subcontinua.

Definition 4. A continuum is said to be indecomposable if it is not the union of two proper subcontinua.

The existence of a non-degenerate indecomposable continuum has been known for about 100 years, important properties of these continua as well as additional references can be found in [14,12,16]. It has been shown that there are many non-metric such continua. For example see Bellamy [1] and Smith [19].

Definition 5. A continuum is said to be hereditarily indecomposable if every subcontinuum of it is indecomposable.

The existence of a non-degenerate hereditarily indecomposable continuum was first shown by Knaster in [11]. Bing showed that they are a very common occurrence in the metric case [3]. He showed that all metric continua of dimension 2 or greater contain non-degenerate hereditarily indecomposable continua [2]. He also showed that in the case when X is Euclidean n-space for $n \geqslant 2$ that the set of all such continua form a dense G_δ subset in the space $C(X)$ of subcontinua (with the Vietoris topology). As we have indicated, the existence of non-metric hereditarily indecomposable continua may be a more rare phenomenon.

Definition 6. If X is a continuum and $p \in X$ then the *composant* of X at the point p is the set to which x belongs if and only if there is a proper subcontinuum of X containing p and x. A composant of a continuum is a composant of the continuum at some point of the continuum.

The continuum constructed by Knaster has since been called a pseudo-arc. See Moise [15] and Bing [4] for some essential properties of this continuum. It is now characterized as follows:

Definition 7. A pseudo-arc is a non-degenerate metric chainable hereditarily indecomposable continuum.

Every point of the pseudo-arc is an "end" point [5]. A straightforward application of the techniques developed by Bing [4] and Moise [15] can be used to prove the fact that the pseudo-arc can be mapped onto any metric chainable continuum. Moise's technique can be used to prove the following more specific result: If for $i = 1, 2, 3$, P_i and Q_i are points in two different composants of the pseudo arc M_i and M_1 is glued to M_2 so that Q_1 is identified with P_2, then there is a mapping f from the pseudo-arc M_3 to $M_1 \cup M_2$ so that $f(P_3) = P_1$ and $f(Q_3) = Q_2$.

3. Main results

The example will be constructed in three steps.

Step 1. We construct a product of the double arrow space and a pseudo-arc.

Step 2. Then an upper semi-continuous decomposition of the space constructed in Step 1 is formed. This decomposition space will be obtained by "gluing" components of the space obtained in Step 1.

Step 3. Finally, an inverse limit indexed by ω_0 of copies of the space constructed in Step 2 is formed.

In the example it is only necessary to insure at each step of the construction that perfect normality is preserved. First we state the theorems that insure this is so. The proofs of these theorems are straightforward, and we include them for completeness.

Theorem 4. *If X is a compact perfectly normal space and G is an upper semi-continuous decomposition of X then X/G is perfectly normal.*

Proof. Suppose $M \subset X/G$ is closed in the decomposition space. Then $M^* = \bigcup\{g \in G \mid g \in M\}$ is closed (and compact) in X. Since X is perfectly normal, there is a collection $\{U_i\}_{i=1}^{\infty}$ of open sets in X so that $M^* = \bigcap_{i=1}^{\infty} U_i$. But $\hat{U}_i = \{g \in G \mid g \subset U_i\}$ is open in X/G and $M = \bigcap_{i=1}^{\infty} \hat{U}_i$, so M is a G_δ set. $\square$

Theorem 5. *If for each i the space X_i is perfectly normal, then the inverse limit space with continuous bonding maps $X = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ is also perfectly normal.*

Proof. Let M be a closed subset of X. For $x \in X$, let x_i denote the ith coordinate of x so that $x = \{x_i\}_{i=1}^{\infty}$. If R is an open set in some X_i then $\overleftarrow{R} = \{x \in X \mid x_i \in R\}$ is an open set in X. Then for each integer n let $\{U_i^n\}_{i=1}^{\infty}$ be a collection of open sets so that $\pi_n(M) = \bigcap_{i=1}^{\infty} U_i^n$. Then

$$M = \bigcap_{n=1}^{\infty}\left(\bigcap_{i=1}^{\infty}\overleftarrow{U_i^n}\right).$$

Thus M is a G_δ set. $\square$

3.1. Description of example

The first step of the construction is to form the product of Z and a pseudo-arc. Let P denote the pseudo-arc and a and b be two points of P so that P is irreducible from a to b.

Let $X = Z \times P$. Then since the pseudo-arc is compact metric, X is perfectly normal.

We now define the function $F : Z \to Z$ needed for the identification for the decomposition space of Step 2. This function serves as a non-metric analog to an adding machine as defined by Block and Coppel [6] (see Fig. 2).

Let $F : Z \to Z$ be defined by

$$F\big((t)_k\big) = \left(t + \frac{1}{2}\right)_k \quad \text{for } (0)_1 \leqslant_Z (t)_k \leqslant_Z \left(\frac{1}{2}\right)_0,\ k = 0, 1,$$

$$F\big((t)_k\big) = \left(t - \frac{1}{4}\right)_k \quad \text{for } \left(\frac{1}{2}\right)_1 \leqslant_Z (t)_k \leqslant_Z \left(\frac{3}{4}\right)_0,\ k = 0, 1,$$

$$F\big((t)_k\big) = \left(t - \frac{5}{8}\right)_k \quad \text{for } \left(\frac{3}{4}\right)_1 \leqslant_Z (t)_k \leqslant_Z \left(\frac{7}{8}\right)_0,\ k = 0, 1,$$

$$\vdots$$

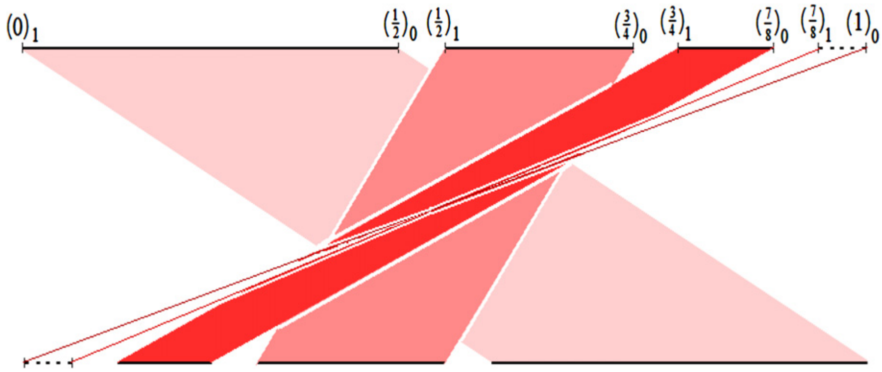

Fig. 2. The "Non-metric Adding Machine" $F : Z \to Z$.

$$F\big((t)_k\big) = \left(t - \frac{2^n-3}{2^n}\right)_k \quad \text{for} \left(\frac{2^{n-1}-1}{2^{n-1}}\right)_1 \leqslant_Z (t)_k \leqslant_Z \left(\frac{2^n-1}{2^n}\right)_0,\ k=0,1,$$

$$\vdots$$

and

$$F\big((1)_0\big) = (0)_1.$$

Claim 1. *The function F is continuous.*

Proof. From the definition of the function we see that F is 1–1, onto and order preserving on each of the defining intervals, thus

$$\left[(0)_1, \left(\frac{1}{2}\right)_0\right] \to \left[\left(\frac{1}{2}\right)_1, (1)_0\right],$$

$$\left[\left(\frac{1}{2}\right)_1, \left(\frac{3}{4}\right)_0\right] \to \left[\left(\frac{1}{4}\right)_1, \left(\frac{1}{2}\right)_0\right],$$

$$\left[\left(\frac{3}{4}\right)_1, \left(\frac{7}{8}\right)_0\right] \to \left[\left(\frac{1}{8}\right)_1, \left(\frac{1}{4}\right)_0\right],$$

$$\vdots$$

$$\left[\left(\frac{2^{n-1}-1}{2^{n-1}}\right)_1, \left(\frac{2^n-1}{2^n}\right)_0\right] \to \left[\left(\frac{1}{2^n}\right)_1, \left(\frac{1}{2^{n-1}}\right)_0\right],$$

$$\vdots$$

$$(1)_0 \to (0)_1.$$

Since for each n, F maps the clopen set $[(\frac{2^{n-1}-1}{2^{n-1}})_1, (\frac{2^n-1}{2^n})_0]$ with an order preserving map onto $[(\frac{1}{2^n})_1, (\frac{1}{2^{n-1}})_0]$ it follows that F restricted to the union of these clopen sets is continuous. It only remains to prove that F is continuous at the point $((1)_0)$. Continuity at this point follows from the fact that for each positive integer n the clopen set

$$\left[\left(\frac{2^n-1}{2^n}\right)_1, (1)_0\right]$$

is mapped onto

$$\left[(0)_1, \left(\frac{1}{2^n}\right)_0\right]. \qquad \square$$

The orbit of each point under F is dense in Z. We will prove this below. In order to assist the reader, we indicate the orbit of $(1)_0$ and $(0)_1$:

$$\cdots\left(\frac{5}{8}\right)_0 \to \left(\frac{3}{8}\right)_0 \to \left(\frac{7}{8}\right)_0 \to \left(\frac{1}{4}\right)_0 \to \left(\frac{3}{4}\right)_0 \to \left(\frac{1}{2}\right)_0 \to (1)_0 \to (0)_1 \to \left(\frac{1}{2}\right)_1 \to \left(\frac{1}{4}\right)_1$$
$$\to \left(\frac{3}{4}\right)_1 \to \left(\frac{1}{8}\right)_1 \to \left(\frac{5}{8}\right)_1 \to \left(\frac{3}{8}\right)_1 \to \left(\frac{7}{8}\right)_1 \cdots.$$

In order to prove this, we examine how F operates on blocks of intervals in Z. Let $H_1^1 = [(0)_1, (\frac{1}{2})_0]$; $H_2^1 = [(\frac{1}{2})_1, (1)_0]$ and in general

$$H_k^n = \left[\left(\frac{k-1}{2^n}\right)_1, \left(\frac{k}{2^n}\right)_0\right] \quad \text{for } k = 1, 2, 3, \ldots, 2^n.$$

Thus: $H_k^n = H_{2k-1}^{n+1} \cup H_{2k}^{n+1}$.

These sets serve the same role in the double arrow space Z as the complementary intervals to the middle thirds segments in the Cantor set. Note that, in an analogous way to how the Cantor set is self-similar, each H_k^n is homeomorphic to Z.

Claim 2. *For each $z \in Z$, $\{F^n(z)\}_{n=1}^{\infty}$ is dense in Z.*

Proof. We note that for each n:

1. F maps each block $H_i^n, i \neq 2^n$ homeomorphically with an order preserving map onto another block H_j^n, $j \neq 1$;
2. F permutes all the blocks $\{H_i^n\}_{i=1}^{2^n}$ at the nth level so by repeated applications of F each block eventually maps onto every block;
3. F maps $H_{2^n}^n$ onto H_1^n;
4. F maps the sub-blocks $H_{2^n-1}^{n+1}, H_{2^n}^{n+1}$ onto the sub-blocks H_1^{n+1}, H_2^{n+1} but reverses their order.

Let $z \in Z$. The consideration of the action of F on these blocks implies that if $z \in H_k^n$ for some k, then $\{F^i(z)\}_{i=1}^n$ will intersect each of the blocks: $\{H_i^n\}_{i=1}^n$. Since every open set in Z contains some block H_i^n for sufficiently large n and F permutes all the blocks, then the set $\{F^i(z)\}_{i=1}^{\infty}$ will be dense in Z. $\square$

Define G to be the upper semi-continuous decomposition of $X = Z \times P$ that identifies the point (z, b) with the point $(F(z), a)$. Recall that a and b are chosen so that P is irreducible from a to b. Thus $G = \{g \mid g = \{(z, b), (F(z), a)\} \text{ for } z \in Z\} \cup \{g \mid g = \{(z, p)\} \text{ for } p \notin \{a, b\}\}$. Then since F is continuous the collection G is an upper semi-continuous decomposition of X.

Definition 8. Let X be a Hausdorff continuum and let $C(X)$ denote the space of subcontinua of X with the Vietoris topology. Then $\mu : C(X) \to [0, \infty)$ is said to be a generalized Whitney map if and only if μ is a continuous function into the non-negative reals so that:

$\mu(H) = 0$ if and only if $H \in C(X)$ is a degenerate continuum,
$\mu(H) < \mu(K)$ if and only if $H \subsetneq K$.

Suppose that in the construction each pseudo-arc is replaced with a metric arc. Then the space is a non-metric indecomposable continuum that admits a Whitney map. A description of the example can be found in Stone [23]. It is a non-metric analog of the solenoid which is built on the double-arrow space rather then the Cantor set. The example described by J.J. Charatonik and W.J. Charatonik in [7] appears to be homeomorphic to it. As in the metric solenoid, each composant is the union of a countable collection of arcs as in a ray. Thus the example X/G is a non-metric solenoid-like continuum each composant of which is a countable collection of pseudo-arcs attached end to end in a long ray going to $\pm\infty$.

We define a map between copies of X/G to produce an inverse limit space. Let $P_1 = \{p_1 \mid p \in P\}$ and $P_2 = \{p_2 \mid p \in P\}$ be two copies of the pseudo arc. Suppose that $Q = Q^1$ is the continuum obtained by identifying the point b_1 with the point a_2.

Since Q is a chainable continuum, there is a mapping $\phi : P \to Q$ which maps a onto a_1 and b onto b_2 and furthermore so that: $\phi^{-1}(b_2) = b$ and $\phi^{-1}(a_1) = a$. [See Fearnley [9] for details on the construction of such a map.] Similarly, define Q^n to be the continuum obtained by gluing together two copies of Q^{n-1} end to end (connecting an "a" end to a "b" end).

Then it is straightforward to verify the four enumerated statements below. In each case the homeomorphisms can be constructed as a product of homeomorphisms on the factor spaces Z and P that preserves the identification of the decomposition:

1. $H_1^1 \times P$ with the identification $(z, b) \simeq (F^2(z), a)$ is homeomorphic to X/G. Let us call the decomposition induced by this identification J. Thus $(H_1^1 \times P)/J$ is homeomorphic to X/G. Call this homeomorphism k_P. Similarly $(H_2^1 \times P)/J$ is homeomorphic to X/G.

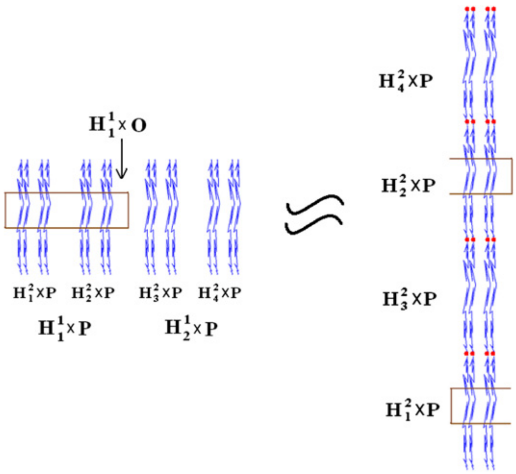

Fig. 3. $H_1^1 \times O$.

2. Also $(H_1^1 \times Q)/J$ is homeomorphic to X/G.
3. The homeomorphism of 2 can be obtained in the form $h \times Id$ where h is a "linear map" of Z onto H_1^1 and Id is the identity on the pseudo-arc elements of the product where the identifications of G are preserved. Call this homeomorphism k_Q.
4. The function $Id_Z \times \phi$ maps $(H_1^1 \times P)/J$ continuously onto $(H_1^1 \times Q)/J$ (where Id_Z is the identity on $H_1^1 \subset Z$).

We encapsulate these facts in the following diagram where $\Phi = k_Q \circ Id_Z \times \phi \circ k_P^{-1}$ is the function that makes the diagram commute

$$\begin{tikzcd} H_1^1 \times Q \arrow[d, "\approx"', "k_Q"] & H_1^1 \times P \arrow[l, "Id_Z \times \phi"'] \arrow[d, "\approx"', "k_P"] \\ X/G & X/G. \arrow[l, "\Phi"] \end{tikzcd}$$

Thus for each $z \in Z$, $\{z\} \times P$ is mapped onto $\{z\} \times Q$.

Although condition 1 is not necessary to the construction, it allows us to see that our inverse limit space is composed of factor spaces all homeomorphic to X/G.

For each integer i let $S_i = X/G$ and $f_i = \Phi$. Let $M = \varprojlim\{S_i, f_i\}_{i=1}^{\infty} = \varprojlim\{X/G, \Phi\}_{i=1}^{\infty}$. Let π_i denote the projection of M onto the ith factor space.

In preparation for the proof that M is hereditarily indecomposable we need to examine how subcontinua of X/G sit in the space. Define the projections π_Z, π_P on the space $(Z \times P)/G = X/G$ as follows

$$\pi_Z\big((z, p)\big) = z,$$
$$\pi_P\big((z, p)\big) = p.$$

Claim 3. *If L is a proper subcontinuum of X/G then $\pi_Z(L)$ is finite and L is a subset of finitely many pseudo-arcs glued end to end.*

Proof. If L is a proper subcontinuum of X/G then there exists an open set O of the pseudo-arc and integers n and i so that $L \cap (H_i^n \times O) = \emptyset$. For ease of understanding, let us examine the case where $H_i^n = H_1^1$. Observe that since H_1^1 is totally disconnected and G only glues points of H_1^1 to points of H_2^1 that $H_1^1 \times O$ separates $H_3^2 \times P$ from $H_4^2 \times P$ and so L can intersect at most one of $H_3^2 \times P$ or $H_4^2 \times P$. Without lose of understanding suppose L does not intersect $H_4^2 \times P$. So $L \subset X - Int(H_4^2 \times P)$. But if P_3 denotes three pseudo-arcs glued b-end to a-end then $X - Int(H_4^2 \times P)$ is homeomorphic to $H_1^2 \times P_3$, and each component of this set is homeomorphic to P_3, so L is a subset of the union of three pseudo arcs glued end to end. In this case, $\pi_Z(L)$ has at most three elements. The same argument works if $L \cap (H_2^1 \times O) = \emptyset$ (see Fig. 3).

If $n > 1$ then without lose of understanding assume there is an open subset O of the pseudo-arc so that $L \cap (H_1^n \times O) = \emptyset$. Since X/G is homeomorphic to $H_1^{n-1} \times Q^{n-1}$ we can repeat the argument of the first paragraph with the continuum P replaced with Q^{n-1}. In this case $\pi_Z(L)$ cannot have more than 2^{n+1} points. The argument works similarly for $i \neq 1$. $\square$

Claim 4. *The continuum M is hereditarily indecomposable.*

Proof. Let π_n denote the projection $X \to S_n$. Suppose that L is a proper decomposable subcontinuum of M. There is an integer N so that $\pi_n(L)$ is a proper subcontinuum of X/G and is decomposable for all $n \geqslant N$. By the above claim it follows that $\pi_N(L)$ lies in a copy of Q^k for some integer k. By the construction of the bonding maps, $\Phi^{-1}(\pi_N(L))$ lies in a copy of Q^{k-1}. So $\pi_{N+1}(L)$ lies in a copy of Q^{k-1}. Repeating this process one eventually obtains $\pi_{N+k}(L)$ lying in a copy of P, contradicting the fact that $\pi_{N+k}(L)$ is supposed to be decomposable. □

Since Z is non-metric and M contains a copy of Z, the above claims together with Theorems 1 and 2 show that:

Theorem 6. *The continuum M is a non-metric perfectly normal hereditarily indecomposable continuum.*

We now consider the separability and first countability of the example.

Theorem 7. *The continuum M is separable and first countable.*

Proof. We need to verify that these properties are preserved at each step of the construction.

Step 1. The product $Z \times P$ of two separable first countable spaces Z and P is separable and first countable.

Separability: Let D_Z be a countable dense set in Z and D_P a countable dense set in P then $D_Z \times D_P$ is dense in $Z \times P$.

First countability: Let B_z be a countable basis at the point $z \in Z$ and let B_p be a countable basis at the point $p \in P$. Then $\{b_1 \times b_2 \mid b_1 \in B_z,\ b_2 \in B_p\}$ is a countable basis for the point $(z, p) \in Z \times P$. So a product of these two spaces is separable and first countable.

Step 2. The decomposition space $Z \times P/G$ is separable and first countable.

Separability: Let D be dense in $Z \times P$. Let $D_G = \{g \in G \mid d \in g \text{ for some } d \in D\}$. Then D_G is dense in the decomposition space.

First countability: First countability at each of the degenerate elements of G follows easily. If $g \in G$ is one of the non-degenerate elements then g is of the form $\{w, F(w)\}$ for some $w \in Z \times P$. Let B_1 be a countable local basis for $w \in Z \times P$ and let B_2 be a countable local basis for $F(w)$. For each $b_1 \in B_1$ and $b_2 \in B_2$ let $C(b_1, b_2) = \{g \in G \mid g \subset b_1 \cup b_2\}$. Then $\{C(b_1, b_2) \mid b_1 \in B_1,\ b_2 \in B_2\}$ is a countable local basis for $(w, F(w))$. (In general the upper semi-continuous decomposition of a first countable space is not first countable. The first uncountable ordinal ω_1 with the order topology is first countable. But the decomposition space formed which has only one non-degenerate element g consisting of all the limit points of ω_1 produces a decomposition space which is not first countable at g.)

Step 3. The inverse limit with onto bonding maps of a separable first countable space is first countable.

Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse limit of separable first countable spaces so that for each positive integer i, f_i is an onto bonding map.

Separability: For each i let D_i be a countable dense set in X_i. For each i and for each $d \in D_i$ construct inductively the sequence $x_{i+1}, x_{i+2}, \ldots$ so that $x_i = d$ and $x_{i+k+1} \in f_{i+k}^{-1}(x_{i+k})$ for each positive integer k; since f_{n+k} is onto this can be done. Then let $w^d = \{w_n\}_{n=1}^{\infty}$ be the point of X defined as follows

$$w_n = f_n^i(d) \quad \text{for } n < i,$$

$$w_n = d \quad \text{for } n = i,$$

$$w_n = x_n \quad \text{for } n > i.$$

Let $W_i = \{w^d \mid d \in D_i\}$. Then the set $W = \bigcup_{i=1}^{\infty} W_i$ is a countable dense set in the inverse limit space. (We note that the onto condition is necessary: the inverse limit constructed with $X_i = \beta([i, \infty))$ with the inclusion maps as bonding maps is an inverse limit of compact separable spaces whose inverse limit is homeomorphic to $\beta([0, \infty)) - [0, \infty)$ which is not separable.)

First countability: Let $x = \{x_i\}_{i=1}^{\infty}$ be a point of X. For each $x_i \in X_i$ let B_i be a countable local basis for x_i in X_i. Then $B = \{\overleftarrow{b} \mid b \in B_i \text{ for some } i\}$ is a countable local basis for x. □

We note that the example described was first constructed by one of us, Stone [23] as an example of a non-metric hereditarily indecomposable continuum that supports a generalized Whitney map. It is also easy to see that if a continuum supports a generalized Whitney map then each order arc in the hyperspace $C(M)$ must be metric and hence separable.

Greiwe and Smith have studied the question, "When is a non-metric hereditarily indecomposable continuum embeddable in the product of Hauddorff arcs?" The following two theorems are relevant to this question. Both theorems are applicable to continua like M.

Theorem 8. *If X is non-metric and separable then X cannot be embedded in a countable product of Hausdorff arcs.*

Proof. Let $\prod_{n=1}^{\infty}\alpha_n$ denote the countable product of Hausdorff arcs and suppose X lies in this product. Let π_n denote the projection onto the nth coordinate space. Let $I_n=\pi_n(X)$. Since X is separable, then so is I_n. So $X\subset\prod_{n=1}^{\infty}I_n$, which is just a copy of the Hilbert cube. Since X embeds in the Hilbert cube, then X is metric which contradicts the fact that X is assumed to be non-metric. □

Definition 9. An order arc in $C(X)$ is a Hausdorff arc α so that one non-cut point is a degenerate continuum and the other non-cut point is X, and so that if $H,K\in\alpha$, then either $H\subset K$ or $K\subset H$.

Theorem 9. *If X is a non-metric continuum so that some order arc in the hyperspace $C(X)$ of subcontinua of X is metric, then X is not embeddable in a countable product of Hausdorff arcs.*

Proof. Let α be a metric order arc in $C(X)$ from the point P to the continuum X in $C(X)$. We can consider α as a function from $[0,1]$ into $C(X)$ so that $\alpha(0)$ is the singleton pointset $\{P\}$ in X and $\alpha(1)=X$. We use the same notation as in the previous theorem. Let $I_n=\pi_n(\alpha)$ and let $<$ denote the order on A_n. By the metrizability of α there is a countable set $\{M_k\}_{k=1}^{\infty}$ of subcontinua of X dense in α. Let $M_0=\{P\}$. For each $k=0,1,2,\ldots,$ let $c_k\leqslant d_k$ denote the endpoints of $\pi_n(M_k)$. (Note that $c_0=d_0$ and that it is possible for $c_j=d_j$ for other j.) Let $C=\{x\mid x=c_k \text{ for some } k\}$ and let $D=\{x\mid x=d_k \text{ for some } k\}$. We claim that the countable set $E=C\cup D$ is dense in I_n.

Claim: $\overline{C}$ is connected. Suppose not, then there exist two points y_1 and y_2 not in $\overline{C}$ and two points c_1 and c_2 in $\overline{C}$ so that

$$c_1<y_1<y_2<c_2.$$

But then

$$\bigl\{M\in\alpha \bigm| \text{the lower end point of } \pi_n(M)\leqslant y_1\bigr\}$$

and

$$\bigl\{M\in\alpha \bigm| \text{the higher end point of } \pi_n(M)\geqslant y_2\bigr\} \text{ form a separation of } \alpha.$$

Similarly $\overline{D}$ is connected and the two sets $\overline{C}$ and $\overline{D}$ both contain $c_0=d_0$.

Claim: $I_n=\overline{C}\cup\overline{D}$. If not then there is a continuum $L\in\alpha$ so that if d is the higher endpoint of $\pi_n(L)$ then $d\notin\overline{C}\cup\overline{D}$ and without lose of generality is above all the points of D. Let (y_1,y_2) be an open segment in I_n containing d and no point of $\overline{C}\cup\overline{D}$. Then $\{M\in\alpha\mid\pi_n(M)\cap(y_1,y_2)\neq\emptyset\}$ is an open set intersecting α and so must contain some M_j. But then $d_j\geqslant d$, which is a contradiction.

Therefore I_n is separable. Since $\alpha(1)=X$ it follows that $\pi_n(X)=I_n$. Thus, as in the previous theorem, X embeds in the Hilbert cube and we have a contradiction. So the theorem holds. □

Theorem 10. *If X is a non-metric continuum and supports a generalized Whitney map then X cannot be embedded in a countable product of lexicographic arcs.*

Proof. For such a continuum, every order arc in $C(X)$ is metric since the generalized Whitney map restricted to the arc is a homeomorphism. If X is non-metric and embeds in a countable product of lexicographic arcs, then the projection on some coordinate is not metric and contains uncountably many disjoint open sets. Thus some order arc must map onto a non-metric order arc of the lexicographic arc and hence cannot be separable. □

We showed that the example is separable and ask the following:

Question. Is every non-metric hereditarily indecomposable perfectly normal continuum separable?

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