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On the Tau-Cycle Condition

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Abstract—There is a set of equivalence conditions for the orthonormality of the compactly supported scaling functions. Among them, there is the Cohen's τ-cycle condition. In order to answer the question whether it is enough to check this condition by a finite number of points, we study the τ-cycles in more detail.

Cohen's τ-cycle condition is one of the equivalence conditions for the orthonormality of the compactly supported scaling functions. It is known that if the associate function $m_0(\xi) \neq 0$ for all $|\xi| \leq \pi/3$, then the τ-cycle condition is satisfied. We notice that if $m_0(\xi) \neq 0$ for all $|\xi| \leq \pi/5$ and for $\xi = \pi/3$, then the τ-cycle condition is satisfied. This leads us to ask the question that, when $m_0(\xi) \neq 0$ in a certain interval, whether it is enough to check the τ-cycle condition by a finite number of points.

Given a real number ξ, let $[\xi]$ be the (unique) number in $[-\pi, \pi)$ such that $[\xi] = \xi \pmod{2\pi}$, and let τ be the operator such that $\tau\xi = 2\xi \pmod{2\pi}$. The set $\boldsymbol{\xi} = \{\xi_i | \xi_i \in [-\pi, \pi), 0 \leq i \leq k-1\}$ is a τ-cycle of length k if (1) $\xi_{i+1} = \tau\xi_i$, (2) $\xi_i = \tau^k\xi_i$, and (3) $\xi_i \neq \xi_j$ for any $i \neq j$. It is clear that if $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two τ-cycles and their intersection is nonempty, then $\boldsymbol{\xi} = \boldsymbol{\eta}$. The only τ-cycle of length 1 is $([0])$; we call it the trivial cycle.

Let $\phi(x) \in L^2(\mathbb{R})$ be a compactly supported *scaling function* that satisfies the scaling equation $\phi(x) = \sum c_k\phi(2x-k)$ with the *scaling coefficients* c_k. Let

$$m_0(\xi) = \frac{1}{2}\sum_k c_k e^{ik\xi}$$

be the corresponding filter *characteristic function* for $\phi(x)$. Then $m_0(\xi)$ is a 2π-periodic trigonometric polynomial.

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Recall that the *Cohen's τ-cycle condition* is the following statement:

$$\text{There is no nontrivial } \tau\text{-cycle } \boldsymbol{\xi} \text{ such that } m_0([\xi_k - \pi]) = 0 \quad \text{for all } \xi_k \in \boldsymbol{\xi}. \tag{C}$$

This is one of the conditions for orthonormal wavelets of compact supports. (See "Ten Lectures" by Daubechies [1] and Cohen's original works cited therein.)

It is known that every nontrivial τ-cycle has an element in $[-\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \pi]$. Therefore, if $m_0(\xi) \neq 0$ in the interval $[-\pi/3, \pi/3]$, then the cycle condition (C) is satisfied. Let us now consider the intervals of the form

$$[-\alpha\pi, \alpha\pi] \quad \text{for } 0 < \alpha < 1.$$

The above fact amounts to saying that when $m_0(\xi) \neq 0$ inside of $[-\alpha\pi, \alpha\pi]$ and $\frac{1}{3} \leq \alpha \leq 1$, then the τ-cycle condition (C) is satisfied. We now ask the following question: when $m_0(\xi) \neq 0$ inside of $[-\alpha\pi, \alpha\pi]$, are there finitely many points $\xi_1, \xi_2, \ldots, \xi_n$ such that if $m_0(\xi_i) \neq 0$ for $1 \leq i \leq n$ then the τ-cycle condition (C) is satisfied? We will see that the answer is positive if α is not too small, and the answer is negative if α is too small.

Now we claim that any τ-cycle of length $k \geq 3$ has an element $[\xi_i] \in [-\pi, -\frac{4}{5}\pi] \cup [\frac{4}{5}\pi, \pi]$ (see also [2]). Assume the contrary, let $\boldsymbol{\xi} = ([\xi_i])$ be a τ-cycle of length $k \geq 3$ and

$$[\xi_i] \notin \left[-\pi, -\frac{4}{5}\pi\right] \cup \left[\frac{4}{5}\pi, \pi\right] \quad \forall i. \tag{1}$$

Then

$$[\xi_{i+1}] \notin \left[\frac{-2}{5}\pi, \frac{2}{5}\pi\right], \quad [\xi_{i-1}] \notin \left[-\frac{3}{5}\pi, -\frac{2}{5}\pi\right] \cup \left[\frac{2}{5}\pi, \frac{3}{5}\pi\right].$$

Since i is arbitrary, we conclude that $[\xi_i]$ cannot fall into a "wider" region than that claimed in (1):

$$[\xi_i] \notin \left[-\pi, -\frac{4}{5}\pi\right] \cup \left[-\frac{3}{5}\pi, \frac{3}{5}\pi\right] \cup \left[\frac{4}{5}\pi, \pi\right] \quad \forall i. \tag{2}$$

Using (2) and considering the possible locations for $[\xi_{i-1}]$ and $[\xi_{i+1}]$, we see that in fact $[\xi_i]$ cannot fall into an even "wider" region than that claimed in (2). We continue with this process and we find that in addition of (2) we also have

$$[\xi_i] \notin \bigcup_n \left\{\left[-\frac{4}{5}\pi, -b_n\pi\right] \cup \left[-a_n\pi, -\frac{3}{5}\pi\right]\right\},$$

$$[\xi_i] \notin \bigcup_n \left\{\left[\frac{3}{5}\pi, a_n\pi\right] \cup \left[b_n\pi, \frac{4}{5}\pi\right]\right\} \quad \forall i,$$

where

$$a_0 = \frac{3}{5}, \quad b_0 = \frac{4}{5}, \quad b_n = \frac{a_{n-1} + b_{n-1}}{2}, \quad a_n = \frac{a_{n-1} + b_n}{2}.$$

We take the limit and we can conclude that the only points where $[\xi_i]$ can be are $[-\frac{2}{3}\pi]$ and $[\frac{2}{3}\pi]$. But these two points form a τ-cycle of length 2 which contradicts the assumption that $k \geq 3$.

By the property of τ-cycles claimed above and the fact that the only τ-cycle of length 2 is $([-\frac{2}{3}\pi], [\frac{2}{3}\pi])$, we have the following proposition for $\frac{1}{5} \leq \alpha < \frac{1}{3}$.

PROPOSITION A. *If $m_0(\xi) \neq 0$ in the interval $[-\pi/5, \pi/5]$, and if $m_0(\pi/3) \neq 0$, then the τ-cycle condition* (C) *is satisfied.*

Similarly, we have the following proposition for $\frac{3}{17} \leq \alpha < \frac{1}{5}$.

PROPOSITION B. *If $m_0(\xi) \neq 0$ in the interval $[-\frac{3}{17}\pi, \frac{3}{17}\pi]$, and if $m_0(\pi/3) \neq 0$ and $m_0(\pi/5) \neq 0$, then the τ-cycle condition* (C) *is satisfied.*

But, we will see that the parameter α cannot be less than $\frac{1}{7}$. Note that

$$\left(\left[\frac{4}{7}\pi\right], \left[-\frac{6}{7}\pi\right], \left[\frac{2}{7}\pi\right]\right), \quad \left(\left[\frac{6}{7}\pi\right], \left[-\frac{2}{7}\pi\right], \left[-\frac{4}{7}\pi\right]\right)$$

are two τ-cycles of length 3. We let

$$\beta = \frac{2}{7(2^{3m} + 1)}\pi.$$

Consider

$$\boldsymbol{\xi}_m := \left\{\left[2^i\left(\frac{4}{7}\pi + \beta\right)\right] : 0 \leq i \leq 6m - 1\right\}.$$

One can show that $2^i(\frac{4}{7}\pi + \beta)$ are all distinct for $0 \leq i \leq 3m - 1$. Note that

$$\left[2^{3m}\left(\frac{4}{7}\pi + \beta\right)\right] = \left[\frac{4}{7}\pi + \frac{2 \cdot 2^{3m}}{7(2^{3m} + 1)}\right] = \left[\frac{4}{7}\pi + \frac{2}{7}\pi - \beta\right] = \left[\frac{6}{7}\pi - \beta\right].$$

One can also show that $2^i(\frac{6}{7}\pi - \beta)$ are all distinct for $0 \leq i \leq 3m - 1$. But

$$\left[2^{3m}\left(\frac{6}{7}\pi - \beta\right)\right] = \left[\frac{6}{7}\pi - \frac{2 \cdot 2^{3m}}{7(2^{3m} + 1)}\right] = \left[\frac{6}{7}\pi - \frac{2}{7}\pi + \beta\right] = \left[\frac{4}{7}\pi + \beta\right].$$

We conclude that $\boldsymbol{\xi}_m$ is a τ-cycle of length $6m$ for any $m > 0$. The elements of $\boldsymbol{\xi}_m$ are

$$\left[\left(\frac{4}{7}+\frac{2\cdot 2^{3i}}{7(2^{3m}+1)}\right)\pi\right]\left[\left(-\frac{6}{7}+\frac{2\cdot 2^{3i+1}}{7(2^{3m}+1)}\right)\pi\right]\left[\left(\frac{2}{7}+\frac{2\cdot 2^{3i+2}}{7(2^{3m}+1)}\right)\pi\right],$$

$$\left[\left(\frac{6}{7}-\frac{2\cdot 2^{3i}}{7(2^{3m}+1)}\right)\pi\right]\left[\left(-\frac{2}{7}-\frac{2\cdot 2^{3i+1}}{7(2^{3m}+1)}\right)\pi\right]\left[\left(-\frac{4}{7}-\frac{2\cdot 2^{3i+2}}{7(2^{3m}+1)}\right)\pi\right],$$

for $0 \leq i \leq m-1$. Therefore, all elements for all $\boldsymbol{\xi}_m$ are inside of the interval $(-\frac{6}{7}\pi, \frac{6}{7}\pi)$. In other words, all of these τ-cycles of length $6m$ (for any integer $m > 0$) do not fall into the intervals $[-\pi, -\frac{6}{7}\pi]$ and $[\frac{6}{7}\pi, \pi]$.

The foregoing example shows that, after a shift of π, there are infinitely many τ-cycles whose elements do not fall into the interval $[-\frac{1}{7}\pi, \frac{1}{7}\pi]$. We then have the following conclusion.

PROPOSITION C. *If $m_0(\xi) \neq 0$ for $\xi \in [-\alpha\pi, \alpha\pi]$ with $\alpha \leq \frac{1}{7}$, then infinitely many points must be checked to show the validity of the τ-cycle condition* (C).

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