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Universal Graphs without Large Cliques*

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INTRODUCTION

The theory of universal graphs originated from the observation of R. Rado [4, 5] that a universal countable graph X exists; i.e., X is countable and isomorphically embeds every countable graph. He also showed that under GCH there is a universal graph in every infinite cardinal. Since then, several results have been proved about the existence of universal elements in different classes of graphs. For example, a construction similar to Rado's shows that for every natural number $n \geqslant 3$, there is a universal $K(n)$-free countable graph, or, if GCH is assumed, there is one in every infinite cardinal ($K(n)$ denotes the complete graph on n vertices). This result, at least for uncountable cardinals, also follows from the existence theorem of universal and special models.

The following folklore observation shows that this cannot be extended to $K(\omega)$. Assume that $X = (V, E)$ is a $K(\omega)$-free graph of some cardinal λ that embeds every $K(\omega)$-free graph of cardinal λ. Let $a \notin V$, and define the new graph X' on $V' = V \cup \{a\}$ as follows. X' on V is identical with X, and a is joined to every vertex of V. Clearly, X' is $K(\omega)$-free. So, by assumption, there is an embedding $g\colon V' \to V$ of X' into X. Put $a_0 = a$, and, by induction, $a_{n+1} = g(a_n)$. As g is edge preserving, we get, by induction on n, that

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a_n is joined to every a_t with $t > n$, so they are distinct and form a $K(\omega)$ in X', a contradiction.

In Section 1 we give some existence/nonexistence statements on universal graphs, which under GCH give a necessary and sufficient condition for the existence of a universal graph of size λ with no $K(\kappa)$, namely, if κ is finite or $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$. The special case when $\lambda^{<\kappa} = \lambda$ was first proved by F. Galvin [6].

In Section 2 we investigate the question, when there is no universal $K(\kappa)$-free graph of size λ, of how many of these graphs embed all the others. It was proved in [1] that if $\lambda^{<\lambda} = \lambda$ (e.g., if λ is regular and the GCH holds below λ) and $\kappa = \omega$, then this number is λ^+. We show that this holds for every $\kappa \leqslant \lambda$ of countable cofinality. On the other hand, even for $\kappa = \omega_1$ and any regular $\lambda \geqslant \omega_1$, it is consistent that the GCH holds below λ, 2^λ is as large as we wish, and this number is either λ^+ or 2^λ, so both extremes can actually occur.

Notation. We use the standard axiomatic set theory notation. If f is a function then $\mathrm{Dom}(f)$, $\mathrm{Ran}(f)$ denote the domain, range of f, respectively. For $A \subseteq \mathrm{Dom}(f)$, $f[A] = \{f(x) : x \in A\}$. If X is a set and κ is a cardinal, $[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\}$, $[X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}$. A *graph* is a pair $X = (V, E)$, where V is some set and $E \subseteq [V]^2$; i.e., we exclude loops and parallel edges. If $|V| = \lambda$, we call X a *λ-graph*, and whenever possible, we assume outright that $V = \lambda$. A graph $X = (V, E)$ is $K(\kappa)$-free, it there is no clique of cardinal κ, i.e., $[T]^2 \nsubseteq E$ holds for every $T \in [V]^\kappa$. A (λ, κ)-graph is a $K(\kappa)$-free λ-graph. If $X_i = (V_i, E_i)$ $(i < 2)$ are graphs, the one-to-one function $f\colon V_0 \to V_1$ is a *weak* (*strong*) *embedding* if $\{x, y\} \in E_0$ implies that $\{f(x), f(y)\} \in E_1$ (if $\{x, y\} \in E_0$ iff $\{f(x), f(y)\} \in E_1$). A *weakly* (*strongly*) *(λ, κ)-universal graph* is a (λ, κ)-graph X that weakly (strongly) embeds every (λ, κ)-graph.

1. When GCH Holds

Lemma 1. *If λ is a strong limit, $\lambda > \kappa \geqslant \omega$, and $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$, then there exists a strongly (λ, κ)-universal graph.*

Proof. Let $\lambda = \sup\{\lambda_\alpha : \alpha < \mathrm{cf}(\lambda)\}$, where the sequence is continuous, $2^{\lambda_\alpha} \leqslant \lambda_{\alpha+1}$, and $\lambda_0 = 0$. Let T be a tree of height $\mathrm{cf}(\lambda)$ in which every α-branch has $\lambda_{\alpha+2}$ extensions on the αth level. Clearly, $|T| = \lambda^{<\mathrm{cf}(\lambda)} = \lambda$. The vertex set of the universal graph X will be the disjoint union of some sets $\{A(t) : t \in T\}$, where $|A(t)| = \lambda_{\alpha+1}$ if $t \in T$ is on the αth level. No edge of X will go between $A(t)$ and $A(t')$ when t, t' are incomparable in T. By induction on $\alpha < \mathrm{cf}(\lambda)$, we determine for each $t \in T$ of height α how to build X on $A(t)$ and how to join the vertices of $A(t)$ into $\bigcup \{A(t') : t' < t\}$. This latter set is of cardinal λ_α, with a graph on it, and we make sure that it will be extended to a set of cardinal $\lambda_{\alpha+1}$, i.e., to some $A(t)$, in all possible ways, such that the graph on $A(t)$ is $K(\kappa)$-free. This is possible, as for every branch we have enough extensions reserved. It is immediately seen that every (λ, κ)-graph embeds into X; one only has to select the right branch.

The vertex set is of cardinal $\leqslant |T| \lambda = \lambda$. Finally, a $K(\kappa)$ could only be produced along a branch $\{A(t) : t \in b\}$, but as $|b| \leqslant \mathrm{cf}(\lambda) < \mathrm{cf}(\kappa)$, some $A(t)$ must contain a $K(\kappa)$, a contradiction; i.e., X is a (λ, κ)-graph. ∎

LEMMA 2. (F. Galvin). *If $\lambda^{<\kappa} = \lambda$, then there is no weakly (λ, κ)-universal graph.*

Proof. Assume that $X = (\lambda, E)$ is (λ, κ)-universal. Let $Y = (V, G)$ be the following graph. The elements of V are those functions f with $\mathrm{Dom}(f)$ an ordinal $< \kappa$ such that $\mathrm{Ran}(f)$ is a clique in E. $\{f, g\} \in G$ iff $f \subset g$; i.e., g end-extends f. Clearly, $|V| = \lambda^{<\kappa} = \lambda$. If $\{f_\alpha : \alpha < \kappa\}$ form a $K(\kappa)$, then they are compatible functions, and their union $f = \bigcup \{f_\alpha : \alpha < \kappa\}$ injects κ into a clique of X, a contradiction, as X is $K(\kappa)$-free.

Assume that $g: V \to \lambda$ is a weak embedding of Y into X. By induction on $\alpha < \kappa$ we define $x_\alpha < \lambda$, $f_\alpha \in V$, such that for $\beta < \alpha$, $\{x_\beta, x_\alpha\} \in E$, $f_\beta \subset f_\alpha$ (so $\{f_\beta, f_\alpha\} \in G$) should hold. If we succeed, we are done, as $\{x_\alpha : \alpha < \kappa\}$ is a clique again. If $\{x_\beta, f_\beta : \beta < \alpha\}$ are defined, let f_α be the following function: $\mathrm{Dom}(f_\alpha) = \alpha$, $f_\alpha(\beta) = x_\beta$ $(\beta < \alpha)$; $f_\alpha \in V$, as its range, $\{x_\beta : \beta < \alpha\}$ is a clique. Put $x_\alpha = g(f_\alpha)$. As by the way f_α is constructed, $f_\beta \subset f_\alpha$ $(\beta < \alpha)$ and g is a weak embedding, x_α will, indeed, be joinded to x_β for $\beta < \alpha$, and so the inductive step is succesfully completed. ∎

LEMMA 3. *If λ is a strong limit, and $\kappa \leqslant \lambda$, $\mathrm{cf}(\kappa) \leqslant \mathrm{cf}(\lambda)$, then there is no weakly (λ, κ)-universal graph.*

Proof. We can assume that $\kappa > \mathrm{cf}(\lambda)$, as otherwise Lemma 2 gives the result. Assume that $X = (\lambda, E)$ is (λ, κ)-universal. Let $\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ be an increasing sequence of regular cardinals, cofinal in κ, with $\kappa_0 > \mathrm{cf}(\lambda)$. Let F be the set of those functions f which satisfy the following requirements: $\mathrm{Dom}(f)$ is an ordinal $< \mathrm{cf}(\kappa)$, for $\alpha \in \mathrm{Dom}(f)$, $f(\alpha)$ is a bounded subset of λ with $|f(\alpha)| = \kappa_\alpha$, and $\bigcup \{f(\alpha) : \alpha < \mathrm{Dom}(f)\}$ is a clique in X. Let V, the vertex set of the graph $Y = (V, G)$, be the disjoint union of the sets $\{A(f) : f \in F\}$, where $|A(f)| = \kappa_\alpha$ if $\mathrm{Dom}(f) = \alpha$. Two distinct vertices are joined iff one of them is in $A(f)$ and the other is in $A(f')$ for some $f \subseteq f'$.

Clearly, $|V| \leqslant \kappa\, |F| = \lambda$. Assume that T spans a clique in Y and that $|T| = \kappa$. Then $T \subseteq \bigcup \{A(f_\gamma) : \gamma \in \Gamma\}$ for a collection of pairwise compatible f_γ's; $\sup(\mathrm{Dom}(f_\gamma)) = \mathrm{cf}(\kappa)$, as otherwise $|T| < \kappa$, but then $\bigcup \{\mathrm{Ran}(f_\gamma) : \gamma \in \Gamma\}$ is a $K(\kappa)$ in X, a contradiction. We have therefore established that Y is a (λ, κ)-graph.

Assume that $g: V \to \lambda$ is a weak embedding of Y into X. By induction on $\alpha < \mathrm{cf}(\kappa)$ we are going to define $f_\alpha \in F$ such that $\mathrm{Dom}(f_\alpha) = \alpha$, $f_{\alpha+1}(\alpha) \subseteq g[A(f_\alpha)]$, and $f_\beta \subset f_\alpha$ whenever $\beta < \alpha$. If this can be carried out, we reach a contradiction, as then $\bigcup \{\mathrm{Ran}(f_\alpha): \alpha < \mathrm{cf}(\kappa)\}$ is a $K(\kappa)$ in X. There is no problem with the definition of f_α if $\alpha = 0$ or is a limit. Assume that f_α is given. $g[A(f_\alpha)]$ is a clique in X of size $\kappa_\alpha = \mathrm{cf}(\kappa_\alpha) > \mathrm{cf}(\lambda)$, so there is a bounded (in λ) subset of it of cardinal κ_α, say, S. We can define $f_{\alpha+1}(\alpha) = S$ and $f_{\alpha+1}(\beta) = f_\alpha(\beta)$ for $\beta < \alpha$. By induction $f_\alpha(\beta) = f_{\beta+1}(\beta) \subseteq g[A(f_\beta)]$. Since $f_\beta \subseteq f_\alpha$, $A(f_\beta)$ is joined to $A(f_\alpha)$. Therefore, since g is a weak embedding, the vertices in $f_\alpha(\beta)$ will be joined to S, and the induction continues. ∎

From the known results and Lemmas 1–3 we can deduce the following.

THEOREM 1. (GCH). *Given $\lambda \geq \kappa$, $\lambda \geq \omega$, there is a weakly/strongly (λ, κ)-universal graph iff $\kappa < \omega$ or* $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$.

2. THE STRUCTURE OF THE CLASS OF (λ, κ)-GRAPHS

In this section we investigate the complexity of the class of (λ, κ)-graphs when there is no universal element in it.

DEFINITION. For $\lambda \geq \kappa$, $\mathrm{CF}(\lambda, \kappa)$ is the minimal cardinal μ such that there is a family $\{X_\alpha : \alpha < \mu\}$ of (λ, κ)-graphs with the property that every (λ, κ)-graph is weakly embedded into some X_α. $\mathrm{CF}^+(\lambda, \kappa)$ is the same but with strong embeddings.

Clearly, $\mathrm{CF}(\lambda, \kappa) \leq \mathrm{CF}^+(\lambda, \kappa) \leq 2^\lambda$. Also, $\mathrm{CF}(\lambda, \kappa) \leq \lambda$ iff $\mathrm{CF}(\lambda, \kappa) = 1$ iff there is a weakly (λ, κ)-universal graph, and likewise for $\mathrm{CF}^+(\lambda, \kappa)$. It was observed in [1] that $\mathrm{CF}^+(\omega, \omega) = \omega_1$. We slightly extend that result.

THEOREM 2. *If $\lambda \geq \kappa$, $\mathrm{cf}(\kappa) = \omega$, and λ is either a strong limit or of the form $\lambda = \mu^+ = 2^\mu$, then* $\mathrm{CF}^+(\lambda, \kappa) = \lambda^+$.

Proof. From Lemmas 2–3, $\mathrm{CF}(\lambda, \kappa) \geq \lambda^+$. Fix an increasing sequence $\kappa_n \to \kappa$, $\kappa_0 = 0$. Call a structure $(A, <, X, R)$ a *ranked graph* if $(A, <)$ is a well-ordered set, X is a graph on A, and R is a function mapping those bounded cliques of X with order-type some κ_n into the ordinals with the property that if clique C' end-extends clique C, then $R(C') < R(C)$. Obviously, then, X will be $K(\kappa)$-free. On the other hand, if a $K(\kappa)$-free graph X is given on a well-ordered set $(A, <)$, then the tree

$$T(X) = \{C \subseteq A : \mathrm{type}(C) = \kappa_n \text{ (some } n), C \text{ clique}\},$$

endowed with end-extension as the partial order, will be ω-branchless, so an ordinal-valued function R as above exists. If $|A| = \lambda$, then $|T| = \lambda$, so only λ ordinals are used; therefore $R(0) < \lambda^+$ holds. We call the minimal possible $R(0)$ the *rank* of X.

Assume first that λ is a strong limit. Fix a continuous, cofinal sequence $\{\lambda_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ of cardinals with $\lambda_0 = 0$ and $2^{\lambda_\alpha} \leqslant \lambda_{\alpha+1}$. For every $\xi < \lambda^+$ we are going to construct a graph that embeds all graphs with rank ξ; this will clearly conclude the proof.

Let T be a tree with height $\mathrm{cf}(\lambda)$, with one root, such that whenever $0 < \alpha < \mathrm{cf}(\lambda)$ every α-branch has $\lambda_{\alpha+2}$ extensions to the αth level. For $t \in T$ on the αth level, let $A(t)$ be an ordered set of order-type $\lambda_{\alpha+1}$ such that the sets $\{A(t) : t \in T\}$ are pairwise disjoint. The vertex set V of our graph will be the union V of these sets. We partially order V by assuming that $A(t) < A(t')$ for $t < t'$; i.e., all elements of $A(t)$ precede all elements of $A(t')$.

For every $t \in T$, put $B(t) = \bigcup \{A(t') : t' < t\}$. By induction on the height of t we define $S(t)$, a ranked graph with ranks $\leqslant \xi$ on $B(t) \cup A(t)$ such that if b is an α-branch, then all possible end-extensions (if there are any) of the already defined structure on $\bigcup \{A(t) : t \in b\}$ actually occur. This is possible, as there are enough extensions of b to be the αth level.

It is now obvious that all (λ, κ)-graphs of rank $\leqslant \xi$ embed into our tree. One only has to select the appropriate branch through T. Also, $|V| = |T|\,\lambda = \lambda^{<\mathrm{cf}(\lambda)} = \lambda$. We need to show that there is no $K(\kappa)$ in the resulting graph. Assume that U is a clique and that $|U| = \kappa$. As we joined vertices only in comparable $A(t)$'s, $U \subseteq \bigcup \{A(t) : t \in b\}$ for some branch b. For some $t_n \in b$ $(n = 0, 1, \ldots)$, it is true that the first κ_n elements of U are bounded in $S(t_n)$, so they get a decreasing sequence of ordinals as ranks, a contradiction.

The case $\lambda = \mu^+ = 2^\mu$ is actually simpler; we need one-element $A(t)$'s, and we need only μ^+ extensions of every branch of length $< \mu^+$. ∎

Finally we show that under $\kappa^{<\kappa} = \kappa$, $\mathrm{CF}(\kappa, \omega_1)$ can be as small as κ^+, or as large as 2^κ, and this latter value can be as large as we wish.

THEOREM 3. *Assume that in V, a model of GCH, $\mu, \kappa > \omega$ are cardinals and* $\mathrm{cf}(\mu) > \kappa = \mathrm{cf}(\kappa)$. *Then in a cardinal and cofinality preserving forcing extension V^P, the GCH holds below κ and* $\mathrm{CF}(\kappa, \omega_1) = 2^\kappa = \mu$.

Proof. If $\kappa = \lambda^+$, with $\mathrm{cf}(\lambda) = \omega$, then we first add a $\square_\lambda$-sequence, i.e., a sequence $\{C_\alpha : \alpha < \kappa, \text{ limit}\}$ with the following properties:

(1) $C_\alpha \subseteq \alpha$ is closed, unbounded;

(2) if γ is a limit point of C_α, then $C_\gamma = \gamma \cap C_\alpha$;

(3) $|C_\alpha| < \lambda$.

It is well known that such a sequence can be added by a cardinal and cofinality preserving forcing of size κ, so we may assume that it exists in V. Fix such a sequence, a sequence of cardinals $\lambda_n \to \lambda$, and a one-to-one mapping $\varphi_{\alpha,\beta}: [\alpha, \beta) \to \lambda$ for each $\alpha < \beta < \kappa$. Here $[\alpha, \beta)$ denotes the ordinal interval $\{\xi: \alpha \leqslant \xi < \beta\}$.

We call a countable set $A \subseteq \kappa$ *low*, if $\mathrm{tp}(A)$ is a limit, and if we put $\delta = \sup(A)$, $C_\delta = \{c_\xi: \xi < \mathrm{tp}(C_\delta)\}$, the increasing enumeration of C_δ, then for some $n < \omega$, $\varphi_{c_\xi, c_{\xi+1}}(a) < \lambda_n$ holds for $a \in A$ satisfying $c_\xi \leqslant a < c_{\xi+1}$.

If $\kappa > \omega_1$ is not of the form $\kappa = \lambda^+$ with $\mathrm{cf}(\lambda) = \omega$, then we call every countable subset of limit type low.

Claim 1. *The number of low subsets of any $\alpha < \kappa$ is $< \kappa$.*

Proof. If κ is not of the form λ^+ with $\mathrm{cf}(\lambda) = \omega$, then $|\alpha|^\omega < \kappa$. In the other case the statement follows from property (3). ∎

Claim 2. *If $B \subseteq \kappa$ is of order-type ω_1, then for some cofinal subset $B' \subseteq B$ it is true that if $\gamma < \sup(B')$ is a limit point of B', then $B' \cap \gamma$ is low.*

Proof. If κ is of the form $\kappa = \lambda^+$ with $\mathrm{cf}(\lambda) = \omega$, put $\delta = \sup(B)$. Shrink B to a cofinal $B' \subseteq B$, such that the elements of B' are separated by C_δ, and there is an $n < \omega$, such that if $c_\xi \leqslant b < c_{\xi+1}$ for some ξ, then $\varphi_{c_\xi, c_{\xi-1}}(b) < \lambda_n$ $(b \in B')$. Then the claim follows from property (2) of the $\square$-sequence.

If κ is not of the form $\kappa = \lambda^+$ with $\mathrm{cf}(\lambda) = \omega$ then the choice $B' = B$ works. ∎

The poset $(P, \leqslant)$ of the proof of the theorem will be the $< \kappa$ support product of μ copies of some poset $(Q, \leqslant)$ to be described below.

$q \in Q$ if $q = (\delta, X, \mathscr{A})$, where $\delta < \kappa$, $X \subseteq [\delta]^2$, X is $K(\omega_1)$-free; if $\kappa > \omega_1$ then $\mathscr{A}$ is a family of low subsets of δ; if $\kappa = \omega_1$, then $\mathscr{A}$ is a countable family of countable subsets of δ of limit type. Moreover, we require that if $A \in \mathscr{A}$, $\sup(A) \leqslant x < \delta$, then $A \times \{x\} \nsubseteq X$.

We define extensions as follow: $q' = (\delta', X', \mathscr{A}') \leqslant q = (\delta, X, \mathscr{A})$ iff $\delta' \geqslant \delta$, $X = X' \cap [\delta]^2$, $\mathscr{A} = \mathscr{A}' \cap [\delta]^{\aleph_0}$.

Claim 3. $|Q| = \kappa$.

Proof. For every $\delta < \kappa$ there are at most κ many possibilities of selecting X, $\mathscr{A}$, such that $(\delta, X, \mathscr{A}) \in Q$. ∎

Claim 4. *Forcing with $(Q, \leqslant)$ does not introduce new sequences of ordinals of length $< \kappa$.*

Proof. If $\kappa = \omega_1$, then $(Q, \leqslant)$ is $< \omega_1$-closed. If $\kappa > \omega_1$, assume that $q \Vdash f: \tau \to \mathrm{OR}$, $\tau < \kappa$. We construct the decreasing sequence of conditions

$\{q_\alpha = (\delta_\alpha, X_\alpha, \mathscr{A}_\alpha): \alpha \leqslant \tau\}$, such that $q_0 = q$, $q_{\alpha+1} \Vdash f(\alpha) = g(\alpha)$, and if α is limit, then $\delta_\alpha = \sup\{\delta_\beta : \beta < \alpha\}$, $X_\alpha = \bigcup \{X_\beta : \beta < \alpha\}$. If $\mathrm{cf}(\alpha) \neq \omega$ then $\mathscr{A}_\alpha = \bigcup \{\mathscr{A}_\beta : \beta < \alpha\}$; otherwise we add the low subsets that are cofinal in δ_α, as well. If we can carry out the construction, we are done; q_τ determines all the values of f. The only problem is if one of the X_α's is not $K(\omega_1)$-free. Let $\alpha \leqslant \tau$ be minimal such that there exists an uncountable clique $T \subseteq \delta_\alpha$. Clearly, $\mathrm{cf}(\alpha) = \omega_1$. For some cofinal $T' \subseteq T$, if $\gamma < \delta_\alpha$ is a limit point of T', then $T' \cap \gamma$ is low. There is a limit $\beta < \alpha$ such that δ_β is a limit point of T', so by our construction $T' \cap \delta_\beta \in \mathscr{A}_\beta$, so $T' \cap \delta_\beta$ may not have been later extended to an ω_1-clique. ∎

We now start investigating $(P, \leqslant)$.

Claim 5. *Forcing with $(P, \leqslant)$ does not introduce new sequences of ordinals of length $< \kappa$.*

Proof. Similar to the previous proof. ∎

Claim 6. *$(P, \leqslant)$ is $\kappa^+ - c.c.$*

Proof. By Claim 3 and Δ-system arguments. ∎

As every factor of $(P, \leqslant)$ adds a subset of κ, $2^\kappa \geqslant \mu$ will hold in the extension. On the other hand, by Claim 6 and the fact that $|P| = \mu$, $2^\kappa \leqslant \mu$ will also hold.

If, in V^P, $\mathrm{CF}(\kappa, \omega_1) < \mu$, then a family of graphs witnessing this is in a $< \mu$-sized subproduct of P. By the product lemma we only need to show that forcing with $(Q, \leqslant)$ introduces a (κ, ω_1)-graph that cannot be embedded into any ground model (κ, ω_1)-graph. If $G \subseteq Q$ is generic, put $Y = \bigcup \{X: (\delta, X, \mathscr{A}) \in G\}$.

Claim 7. *Y is $K(\omega_1)$-free.*

Proof. If $\kappa = \omega_1$, $q \Vdash T$ is an ω_1-clique, select a decreasing sequence $q = q_0 \geqslant q_1 \geqslant \cdots$ such that $q_{n+1} = (\delta_{n+1}, X_{n+1}, \mathscr{A}_{n+1}) \Vdash t_n \in T$, $\delta_n < t_n < \delta_{n+1}$, and then put $q' = (\delta, X, \mathscr{A})$, where $\delta = \lim \delta_n$, $X = \bigcup \{X_n : n < \omega\}$, and $\mathscr{A} = \bigcup \{\mathscr{A}_n : n < \omega\} \cup \{\{t_n : n < \omega\}\}$. Then $q' \Vdash T \subseteq \delta$, a contradiction.

If $\kappa > \omega_1$, then by Claim 4 some $q = (\delta, X, \mathscr{A})$ determines all elements of T, the alleged ω_1-clique. We can assume that $T \subseteq \delta$, but then X is not $K(\omega_1)$-free, a contradiction. ∎

Claim 8. *Y does not embed into any ground model (κ, ω_1)-graph.*

Proof. Assume that $q \Vdash f: \kappa \to \kappa$ is an embedding of Y into some ground model (κ, ω_1)-graph, Z. By induction on $\alpha < \omega_1$ construct the decreasing sequence $q_\alpha = (\delta_\alpha, X_\alpha, \mathscr{A}_\alpha)$ such that $q_0 = q$, $q_{\alpha+1} \Vdash f(\delta_\alpha) = g(\alpha)$, for α limit $\delta_\alpha = \lim\{\delta_\beta : \beta < \alpha\}$, $X_\alpha = \bigcup \{X_\beta : \beta < \alpha\}$, $\{\delta_\beta, \delta_\alpha\} \in X_{\alpha+1}$ for $\beta < \alpha$, and $\mathscr{A}_\alpha = \bigcup \{\mathscr{A}_\beta : \beta < \alpha\}$. The only problem with the definition could be that $A \subseteq \{\delta_\beta : \beta < \alpha\}$ for some $A \in \mathscr{A}_\alpha$. But then, $\sup(A)$ is of the form δ_γ for some limit $\gamma \leqslant \alpha$, and no set set of that form was added to $\mathscr{A}_\gamma$.

We can therefore define the sequence, but then the range of g will be a $K(\omega_1)$ in Z, a contradiction. ▌

THEOREM 4. *If, in a model of GCH, $\mu, \kappa > \omega$ are cardinals with $\mathrm{cf}(\mu) > \kappa = \mathrm{cf}(\kappa)$, then, in some cardinal and cofinality preserving extension the GCH holds below κ, $2^\kappa = \mu$ and $\mathrm{CF}^+(\kappa, \omega_1) = \kappa^+$.*

Proof. Again, as in the proof of Theorem 3, we can assume that if $\kappa = \lambda^+$, with $\lambda > \mathrm{cf}(\lambda) = \omega$, then $\square_\lambda$ holds in the ground model. We also assume that the GCH holds below κ and $2^\kappa = \mu$.

In a $<\kappa$-support iteration of length κ^+, we add a family witnessing $\mathrm{CF}^+(\kappa, \omega_1) = \kappa^+$. Factor Q_α will add a (κ, ω_1)-graph that strongly embeds every (κ, ω_1)-graph of V^{P_α}. Note that if the forcing does not collapse cardinals, then $\square_\lambda$ will still hold at every stage.

We first define and investigate one step of the iteration. Let $(Q, \leqslant)$ be the following poset. $q = (\delta, X, \mathscr{A}, \mathscr{Z}, F) \in Q$, if $\delta < \kappa$, $X \subseteq [\delta]^2$ is a $K(\omega_1)$-free graph, $\mathscr{A} \subseteq [\delta]^{\aleph_0}$ is a family of low sets ($\kappa > \omega_1$) is a countable family of limit-type subsets of δ ($\kappa = \omega_1$). $\mathscr{Z}$ is a family of $<\kappa$ many (κ, ω_1)-graphs; $F\colon \mathscr{Z} \times \delta \to \delta$ is a function such that if $Z \in \mathscr{Z}$; then the mapping $x \mapsto F(Z, x)$ is a strong embedding of $Z \mid \delta$ into X; and the following further conditions hold:

(1) If $A \in \mathscr{A}$, $\sup(A) \leqslant x < \delta$, then $A \times \{x\} \nsubseteq X$;

(2) if $A \in \mathscr{A}$, $Z \in \mathscr{Z}$, then $A \nsubseteq F[\{Z\} \times \delta]$.

$q' = (\delta', X', \mathscr{A}', \mathscr{Z}', F') \leqslant q = (\delta, X, \mathscr{A}, \mathscr{Z}, F)$ if $\delta' \geqslant \delta$, $X = X' \cap [\delta]^2$, $\mathscr{Z}' \supseteq \mathscr{Z}$, $\mathscr{A} = \mathscr{A}' \cap [\delta]^{\aleph_0}$, $F' \supseteq F$; and, moreover,

(3) if $Z_0 \neq Z_1 \in \mathscr{Z}$, $\delta \leqslant x, y < \delta'$, then $F'(Z_0, x) \neq F'(Z_1, y)$.

CLAIM 1. *$(Q, \leqslant)$ is transitive.*

Proof. Assume that $q_0 \geqslant q_1 \geqslant q_2$, $q_i = (\delta_i, X_i, \mathscr{A}_i, \mathscr{Z}_i, F_i)$ $(i < 3)$. In establishing $q_0 \geqslant q_2$ only condition (3) could cause problems, but it will not: if $Z_0 \neq Z_1 \in \mathscr{Z}_0$, $\delta_0 \leqslant x < \delta_1 \leqslant y < \delta_2$, then $F_2(Z_0, x) \neq F_2(Z_1, y)$ as the first element is in $[\delta_0, \delta_1)$ and the second is in $[\delta_1, \delta_2)$. ▌

CLAIM 2. *If $\varepsilon < \kappa$, $D = \{(\delta, X, \mathscr{A}, \mathscr{Z}, F) : \delta \geqslant \varepsilon\}$ is dense.*

Proof. We can extend a given $(\delta, X, \mathscr{A}, \mathscr{Z}, F)$ to a large enough δ' by mapping $Z \mid [\delta, \delta')$ $(Z \in \mathscr{Z})$ onto disjoint sets, not extending $\mathscr{A}$, $\mathscr{Z}$, and adjusting X. Condition (1) will not cause a problem, as by (2) no $A \in \mathscr{A}$ will be forced to be joined to a vertex. ∎

CLAIM 3. *If Z is a (κ, ω_1)-graph, then $D = \{(\delta, X, \mathscr{A}, \mathscr{Z}, F) : Z \in \mathscr{Z}\}$ is dense.*

Proof. A similar argument works. ∎

CLAIM 4. *Forcing with $(Q, \leqslant)$ does not introduce new sequences of ordinals of length $< \kappa$.*

Proof. $(Q, \leqslant)$ is $< \omega_1$-closed, and this is enough if $\kappa = \omega_1$. Assume that $\kappa > \omega_1$. Let $q \Vdash f : \tau \to \mathrm{OR}$, $\tau < \kappa$. By induction on $\alpha \leqslant \tau$ we define the decreasing sequence $\{q_\alpha = (\delta_\alpha, X_\alpha, \mathscr{A}_\alpha, \mathscr{Z}_\alpha, F_\alpha) : \alpha \leqslant \tau\}$ such that $q_{\alpha+1} \Vdash f(\alpha) = g(\alpha)$, and for limit α, $\delta_\alpha = \sup\{\delta_\beta : \beta < \alpha\}$, $X_\alpha = \bigcup \{X_\beta : \beta < \alpha\}$, $\mathscr{Z}_\alpha = \bigcup \{\mathscr{Z}_\beta : \beta < \alpha\}$, $F_\alpha = \bigcup \{F_\beta : \beta < \alpha\}$. If $\mathrm{cf}(\alpha) > \omega$, we take $\mathscr{A}_\alpha = \bigcup \{\mathscr{A}_\beta : \beta < \alpha\}$; otherwise we add all the cofinal in δ_α low subsets A for which there is no $Z \in \mathscr{Z}_\alpha$ with $A \subseteq F_\alpha[\{Z\} \times \delta_\alpha]$. The only thing we have to show is that no $K(\omega_1)$ wil be created. We may assume that $\alpha \leqslant \tau$ is a limit, that $T \subseteq \delta_\alpha$ is cofinal, and that T is a clique in X_α. We can assume that segments of T of limit type are low sets. As T could grow for a club subset $C \subseteq \alpha$ of order type ω_1, it is true that if $\beta \in C$, then $T \cap \delta_\beta \subseteq F_\beta[\{Z\} \times \delta_\beta]$ for some $Z \in \mathscr{Z}_\beta$. By condition (3), there can be only one such Z. If, moreover, β is a limit point of the limit points of C, then there is an $h(\beta) < \beta$, such that for $h(\beta) < \gamma \leqslant \beta$ this Z for γ is the same. By the pressing down lemma, h is bounded on an unbounded subset, so $T \cap \delta_\beta \subseteq F_\alpha[\{Z\} \times \delta_\beta]$ for uncountably many $\beta < \alpha$, but then the inverse image of T will be a $K(\omega_1)$ in Z, a contradiction. ∎

Let Y be the graph added by Q; i.e., if $G \subseteq Q$ is generic, then $Y = \bigcup \{X : (\delta, X, \mathscr{A}, \mathscr{Z}, F) \in G\}$.

CLAIM 5. *Y is $K(\omega_1)$-free.*

Proof. If $\kappa = \omega_1$, $q \Vdash T$ is an ω_1-clique in Y, then an argument as above shows that there is a decreasing sequence $\{q_\alpha : \alpha < \omega_1\}$ determining more and more elements of T, and we can freeze T unless it is covered by $\bigcup \{F_\alpha[\{Z\} \times \delta_\alpha] : \alpha < \omega_1\}$ for some Z, which again gives a $K(\omega_1)$ in Z.

If $\kappa > \omega_1$, by the above claim, the supposed clique T is in the ground model and some $q \in G$ contains it in its X-part, a contradiction. ∎

The iteration $(P_\alpha, Q_\alpha : \alpha \leqslant \kappa^+)$ is defined as a $< \kappa$-support iteration, with Q_α as the above Q, defined in V^{P_α}. In Q_α, let D_α be the set of those conditions of the form $q = (\delta, X, \mathscr{A}, \mathscr{Z}, F)$ for which it is true that $Z_0 \neq Z_1 \in \mathscr{Z}$ implies that $Z_0 \mid \delta \neq Z_1 \mid \delta$.

CLAIM 6. *D_α is dense in Q_α.*

Proof. Using Claim 2, with ε large enough. ∎

If $q=(\delta, X, \mathscr{A}, \mathscr{X}, F)\in Q_\alpha$ we put $l(q)=(\delta, X, \mathscr{A}, \mathscr{X} \mid \delta, F)$. Let E_α be the following subset of P_α; $p\in P_\alpha$ if for all $\beta<\alpha$, $p \mid \beta$ determines $l(p(\beta))$ and forces that $p(\beta)\in D_\beta$.

CLAIM 7. *For every $\alpha\leqslant\kappa^+$*

(a) *E_α is dense in P_α;*

(b) *forcing with P_α does not add sequences of ordinals of length $<\kappa$.*

Proof. Assume first that $\kappa>\omega_1$. The proof is by induction on $\alpha\leqslant\kappa^+$. If (b) holds for α, then it holds for $\alpha+1$, by Claim 4. Assume that (a) and (b) hold for α and that $p\in P_{\alpha+1}$. We may assume that $p \mid \alpha \Vdash p(\alpha)\in D_\alpha$. As (b) holds for α, there is a $q\leqslant p \mid \alpha$ which determines $p(\alpha)$. Extend q to an $r\in E_\alpha$; then take $r\cup p(\alpha)\in E_{\alpha+1}$.

Assume that α is a limit, $p\in P_\alpha$. In order to prove (a) for α, we may assume that supp(p) is cofinal in α and let $\{\alpha_\xi : \xi<\tau\}$ converge to α. We define $\{p_\xi : \xi<\tau\}$, a decreasing sequence of conditions. Put $p_0=p$. Define $p_\xi\in P_\alpha$ in such a way that $p_\xi \mid \alpha_\xi\in E_{\alpha_\xi}$, and $p_\xi\leqslant p_\zeta$, $p_\xi \mid [\alpha_\xi,\alpha)= p_\zeta \mid [\alpha_\xi,\alpha)$ hold for $\zeta<\xi$. If ξ is a limit and $\beta\geqslant\alpha_\xi$, the names $p_\zeta(\beta)$ are identical ($\zeta<\xi$), so we can take it as $p_\xi(\beta)$. If $\beta<\alpha_\xi$, we take $p_\xi(\beta)$ as $\bigcup\{p_\zeta(\beta)\}$ by adding all the low subsets which can be added, as in Claim 4. We show that p_ξ is a condition. To this end, we show by induction on $\beta<\alpha$ that $p_\xi \mid \beta$ is a condition. The limit case is trivial. The problem with $p_\xi(\beta)$ can only be that its X part contains a $K(\omega_1)$, but then, as in the proof of Claim 4, we get that $p_\xi \mid \beta \Vdash Z$ is not $K(\omega_1)$-free for some $Z\in\mathscr{X}$.

If α is a limit and we are to show (b) for α and $p\Vdash f:\tau\to$ OR for some $\tau<\kappa$, we can define a decreasing, continuous sequence $\{p_\xi : \xi\leqslant\tau\}$ with $p_\xi \Vdash f(\xi)=g(\xi)$, $p_\xi\in E_\alpha$. This can be carried out, as above, and then p_τ decides f.

For $\kappa=\omega_1$, (b) follows from the fact that we iterate a countably closed poset with countable supports, and for (a) an easy inductive proof can be given, as for the other case above. ∎

CLAIM 8. *P_{κ^+} is κ^+ – c.c.*

Proof. Given κ^+ conditions, we can assume that they are from E_{κ^+}. By the usual Δ-system arguments we can find two of them, p and p', such that $l(p(\alpha))=l(p'(\alpha))$ holds for every $\alpha\in$ supp$(p)\cap$ supp(p'). We show that $p\cup p'$ is a condition (though not necessarily in E_{κ^+}).

To this end, we show that $(p\cup p') \mid \alpha\in P_\alpha$ by induction on α. All cases are trivial, except when $\alpha=\beta+1$, $\beta\in$ supp$(p)\cap$ supp(p'). What we have to show is that the F part of $(p \cup p')(\beta)$ is well defined; i.e., if $Z = Z'$ are from the $\mathscr{X}$ part, then $F(Z, x) = F(Z', x)$ $(x < \delta)$. But this will hold (or, more precisely, will be forced to hold by $(p \cup p') \mid \beta)$ as $F(Z, x)$ is determined by $Z \mid \delta$ and by x, and it is determined the same way in p and p'. ∎

From the last claim, every (κ, ω_1)-graph appears in some intermediate extension, and so it is embedded into the next graph, Y_α, by Q_α. We still have to show that Y_α remains $K(\omega_1)$-free under the further extensions. This follows from Claim 7(b) if $\kappa > \omega_1$ and from the following statement for $\kappa = \omega_1$.

CLAIM 9. *If, in V, Y is a $K(\omega_1)$-free graph and P is an $< \omega_1$-closed forcing, then, in V^P, Y is still $K(\omega_1)$-free.*

Proof. If $p \Vdash T$ is an uncountable clique, select $\{p_\alpha : \alpha < \omega_1\}$ fixing more and more elements of T, $p_0 = p$. ∎

Remark. With the technique of Theorem 4 it is possible to show that if $\mu \geqslant \nu > \kappa$, $\mathrm{cf}(\mu) > \kappa$, and ν, κ are regular, then it is consistent that $2^\kappa = \mu$, $\mathrm{CF}(\kappa, \omega_1) = \nu$, and GCH holds below κ. Add a sequence $\{Y_\alpha : \alpha < \nu\}$, rather than of length κ^+, as in Theorem 4. One has only to observe that Y_α does not embed into any $K(\omega_1)$-free graph in V^{P_α}; this can be proved similarly to Claim 8 in Theorem 3.

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