JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **217,** 672–686 (1998)
ARTICLE NO. AY975762

Positive Solutions of Quasilinear Boundary Value Problems

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Submitted by John Lavery

Received January 27, 1997

This paper deals with the existence of positive solutions for the boundary value problem

$$(p(t)\phi(u'))' + \lambda p(t)f(t,u) = 0; \qquad a < t < b$$
$$u(a) = 0 = u(b),$$

where f is either ϕ-superlinear or ϕ-sublinear at ∞ and $f(t,0)$ may be negative and p is a positive continuous function. The results extend several known results for semilinear equations. Our approach is based on fixed point theory for completely continuous operators which leave invariant a suitable cone in a Banach space of continuous functions. © 1998 Academic Press

0022-247X/98 $25.00

1. INTRODUCTION

The results in this paper are motivated by the search for positive radial solutions for the quasilinear elliptic boundary value problem

$$\operatorname{div}\big(a(|\nabla u|^2)\,\nabla u\big) + \lambda f(|x|, u) = 0, \qquad a < |x| < b$$
$$u = 0, \qquad |x| \in \{a, b\}, \tag{1.1}$$

where $\phi(s) = \alpha(s^2)s$ is an increasing homeomorphism of the real line and λ is a positive parameter. Such radial solutions are solutions to boundary value problems of the form

$$(p(t)\phi(u'))' + \lambda p(t) f(t, u) = 0, \qquad a < t < b$$
$$u(a) = 0 = u(b), \tag{1.2}$$

with $p(t) = t^{N-1}$, $t = |x|$, and N is the dimension of x. The case where

$$\alpha(|\nabla u|^2)\,\nabla u = |\nabla u|^{p-2}\,\nabla u, \qquad p > 1,$$

i.e., perturbations of the p-Laplacian, has received much attention in the recent literature. Also problem (1.2) with $f(t,0) \geq 0$ has been studied by several authors in recent years (see [15] and the references therein). Here, we are interested in the case when $f(t,0)$ may be negative (the so-called semipositone case) (see [6] and its references for a review). Since our results only depend upon the positivity and continuity of the coefficient function p, we shall consider (1.2) in this generality.

We first consider the case when f is ϕ-superlinear at ∞. In particular, we make the assumptions:

(A.1) ϕ is an odd, increasing homeomorphism on $\mathbb{R}$ with ϕ^{-1} concave on $\mathbb{R}^+$.

(A.2) For each $c > 0$, there exists $A_c > 0$ such that $\phi^{-1}(cu) \geq A_c\phi^{-1}(u)$, $u \in \mathbb{R}^+$ and $\lim_{c\to\infty} A_c = \infty$ (note that (A.2) implies the existence of $B_c > 0$ such that $\phi^{-1}(cu) \leq B_c\phi^{-1}(u)$, $u \in \mathbb{R}^+$ with $\lim_{c\to 0} B_c = 0$).

(A.3) $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{u\to\infty}(f(t,u)/\phi(u)) = \infty$ uniformly for $t \in [a,b]$.

(A.4) $p\colon [a,b] \to (0,\infty)$ is continuous.

Our first result is:

THEOREM 1.1. *Let* (A.1)–(A.4) *hold. Then there exists* $\lambda^* > 0$ *such that problem* (1.2) *has a positive solution* u_λ *for* $0 < \lambda < \lambda^*$ *with* $\|u_\lambda\|_\infty \to \infty$ *as* $\lambda \to 0$.

We also consider the case when f is ϕ-sublinear at ∞, i.e, we consider nonlinearities f that satisfy

(A.5) f: $[a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\lim_{u \to \infty} f(t, u) = \infty$ uniformly for $t \in [a, b]$, and $\lim_{u \to \infty}(f(t, u)/\phi(u)) = 0$ uniformly for $t \in [a, b]$.

In this case we establish the following result:

THEOREM 1.2. *Let* (A.1), (A.2), (A.4), *and* (A.5) *hold. Then there exists* $\bar{\lambda} > 0$ *such that* (1.2) *has a positive solution* u_λ *for* $\lambda > \bar{\lambda}$ *with* $\|u_\lambda\|_\infty \to \infty$ *as* $\lambda \to \infty$.

In the above we adhere to the notation

$$\|u\|_\infty = \sup\{|u(t)|: a \le t \le b\}.$$

Theorems 1.1 and 1.2 are extensions of results in [2, 5, 10], to quasilinear equations (in particular, p-Laplacian like equations), and of results in [1, 3, 4, 7, 8, 13, 16]. The paper also serves to provide a unified treatment to a variety of results having been obtained by a host of methods.

2. PRELIMINARY RESULTS

We establish some preliminary results.

LEMMA 2.1. *Let* $p_0, p_1 > 0$ *be such that* $p_0 \le p(t) \le p_1$, $t \in [a, b]$, *and let M be a positive number. Assume w is the solution of*

$$\begin{aligned} (p(t)\phi(u'))' &= -\lambda M p(t), \qquad a < t < b \\ u(a) &= 0 = u(b). \end{aligned} \tag{2.1}$$

Then $w > 0$ *on* (a, b) *and*

$$\|w'\|_\infty \le \phi^{-1}(\lambda M \delta), \tag{2.2}$$

where $\delta = (p_1/p_0)(b - a)$.

Proof. By integrating, it follows that (2.1) has the unique solution given by

$$w(t) = \int_a^t \phi^{-1}\left\{\frac{1}{p(s)}\left(C - \lambda M \int_a^s p(r)\,dr\right)\right\} ds,$$

where C is such that $w(b) = 0$. Hence we must have $0 < C < \lambda M \int_a^b p(r)\,dr$. Further, using

$$\phi(w'(t)) = \frac{1}{p(t)}\left(C - \int_a^t \lambda M p(r)\,dr\right),$$

we obtain

$$-\lambda M \delta \leq \phi(w'(t)) \leq \lambda M \delta,$$

$t \in [a, b]$, and (2.2) follows. Finally, since $w(a) = 0 = w(b)$, there exits $z \in (a, b)$ such that $w'(z) = 0$ and, since $p(t)\phi(w')$ is decreasing, $w'(t) < 0$ for $t > z$ while $w'(t) > 0$ for $t < z$. Thus $w > 0$ on (a, b).

LEMMA 2.2. *For* $x \geq -\mu$, $y \geq 0$, $\mu \geq 0$,

$$\phi^{-1}(x + y) \leq \phi^{-1}(x) + \phi^{-1}(y) + \phi^{-1}(\mu).$$

Proof. Since ϕ^{-1} is concave and $\phi^{-1}(0) = 0$, it follows that $\phi^{-1}(x + y) \leq \phi^{-1}(x) + \phi^{-1}(y)$ if $x \geq 0$, $y \geq 0$. If $-\mu \leq x < 0$, $y \geq 0$ and $\mu \geq 0$, then, since ϕ^{-1} is increasing $\phi^{-1}(x + y) \leq \phi^{-1}(y)$ and $\phi^{-1}(-\mu) \leq \phi^{-1}(x)$. Thus $\phi^{-1}(x) + \phi^{-1}(\mu) \geq 0$, since ϕ^{-1} is odd. Hence the conclusion follows.

We next state the fixed point theorems which will be used to prove our results.

THEOREM 2.1 [11]. *Let E be a Banach space and let $\mathbb{K}$ be a cone in E such that $\|\cdot\|$ is monotone with respect to the cone $\mathbb{K}$. Let $T\colon \mathbb{K} \to \mathbb{K}$ be a completely continuous operator. Assume there exist positive constants r, R, and $k \in \mathbb{K}$, $h \in \mathbb{K}$ with $0 < r < R$, $\|k\| < r$, $\|h\| > R$ such that*

(a) *For each $0 < \theta < 1$, all solutions $u \in \mathbb{K}$ of*

$$u = \theta T u + (1 - \theta) k$$

satisfy $\|u\| \neq r$.

(b) *For each $0 < \theta < 1$, all solutions $u \in \mathbb{K}$ of*

$$u = T u + \theta h$$

satisfy $\|u\| \neq R$.

Then T has at least one fixed point $u \in \mathbb{K}$ with $r \leq \|u\| \leq R$.

THEOREM 2.2 [11]. *Let* E, $\mathbb{K}$, *and* T *be as in Theorem* 2.1. *Assume there exist positive constants* r, R, *and* $k \in \mathbb{K}$ *with* $0 < r < R$, $\|k\| = 1$ *such that*

(a) *For each* $\gamma > 0$, *all solutions* $y \in \mathbb{K}$ *of*

$$y = Ty + \gamma k$$

satisfy $\|y\| \neq r$.

(b) *For each* $0 < \gamma < 1$, *all solutions* $z \in \mathbb{K}$ *of*

$$z = \gamma Tz$$

satisfy $\|z\| \neq R$.

Then T has a fixed point $x \in \mathbb{K}$ with $r \leq \|x\| \leq R$.

3. PROOF OF THEOREM 1.1

Let $M > 0$ be such that $g(t,u) := f(t,u) + M > 0$ for $t \in [a,b]$, $u \geq 0$. Define $\tilde{g}(t,u) = g(t,u)$ if $u \geq 0$, $\tilde{g}(t,u) = g(t,0)$ if $u \leq 0$. Let w be defined as in Lemma 2.1. Then u is a positive solution of (1.2) if $\tilde{u} = u + w$ is a solution of

$$\begin{gathered}\big(p(t)(\phi(\tilde{u}' - w') + \phi(w'))\big)' = -\lambda p(t)\tilde{g}(t, \tilde{u} - w), \qquad a < t < b\\ \tilde{u}(a) = 0 = \tilde{u}(b)\end{gathered} \tag{3.1}$$

with $\tilde{u} > w$ on (a,b).

For each $v \in C[a,b]$, let $u = Tv$ be the solution of

$$\begin{gathered}\big(p(t)(\phi(u' - w') + \phi(w'))\big)' = -\lambda p(t)\tilde{g}(t, v - w), \qquad a < t < b\\ u(a) = 0 = u(b).\end{gathered}$$

Note that u satisfies

$$u(t) = \int_a^t \phi^{-1}\left(\frac{C_1 - \lambda\int_a^s p(\tau)\tilde{g}(\tau, v - w)}{p(s)}\, d\tau - \phi(w'(s))\right) ds + w(t),$$

where C_1 is such that $u(b) = 0$.

Hence

$$0 \leq C_1 \leq \lambda\int_a^b p(\tau)\tilde{g}(\tau, v - w)\, d\tau,$$

since otherwise,

$$u(b) > \int_a^b \phi^{-1}(-\phi(w'(s)))\,ds = 0,$$

which is a contradiction.

Now let $\mathbb{K} = \{v \in C[a,b] \mid v \geq 0\}$. It can be verified that $T\colon \mathbb{K} \to \mathbb{K}$ is completely continuous. Let $\lambda^* > 0$ be such that

$$\frac{b-a}{M_1}\phi^{-1}(\lambda^* M\delta) < \frac{1}{8}$$

and

$$\lambda^* < \left[\frac{h(1)}{\phi(1/2(b-a))}\right]^{-1}\frac{1}{2\delta},$$

where $M_1 = A_{(p_0/p_1)} \leq 1$, $h(t) = \sup_{a \leq \tau \leq b,\, 0 \leq s \leq t} \tilde{g}(\tau, s)$ and δ is given by Lemma 2.1.

We shall now apply Theorem 2.1 to show that T has a fixed point $\tilde{u}$ in $\mathbb{K}$ with $\|\tilde{u}\|_\infty \geq C_\lambda$, where $C_\lambda \to \infty$ as $\lambda \to 0$. Let $0 < \lambda < \lambda^*$. Then there exists $C_\lambda > 1$ such that

$$\frac{h(C_\lambda)}{\phi(C_\lambda/2(b-a))} = \frac{1}{2\lambda\delta} \tag{3.2}$$

for

$$\frac{h(1)}{\phi(1/2(b-a))} < \frac{1}{2\lambda\delta} \qquad \text{and} \qquad \lim_{t\to\infty}\frac{h(t)}{\phi(t/2(b-a))} = \infty.$$

Let $u \in \mathbb{K}$ be such that $u = \theta T u$, $0 < \theta < 1$. We claim that $\|u\|_\infty \neq C_\lambda$. Indeed, if $\|u\|_\infty = C_\lambda$ then we have by Lemma 2.2,

$$\begin{aligned} u(t) &\leq \int_a^t \phi^{-1}\left(\frac{\lambda\int_s^b p(\tau)\tilde{g}(\tau, u-w)\,d\tau}{p(s)} - \phi(w')\right) ds + w(t) \\ &\leq \int_a^t \phi^{-1}\left(\frac{\lambda\int_s^b p(\tau)\tilde{g}(\tau, u-w)\,d\tau}{p(s)}\right) ds \\ &\quad - \int_a^t \phi^{-1}(\phi(w'))\,ds + \int_a^t \phi^{-1}(\lambda M\delta)\,ds + w(t). \end{aligned}$$

Here we have used the fact that $\phi(w') < \lambda M\delta$ (see Lemma 2.1). Then

$$C_\lambda = \|u\|_\infty \le (b-a)\phi^{-1}\big(\lambda\delta h(C_\lambda)\big) + (b-a)\phi^{-1}(\lambda^* M\delta),$$

which implies

$$\phi\left(\frac{C_\lambda}{2(b-a)}\right) \le \lambda\delta h(C_\lambda)$$

or

$$\frac{h(C_\lambda)}{\phi(C_\lambda/2(b-a))} \ge \frac{1}{\lambda\delta}$$

a contradiction to (3.2). Hence $\|u\|_\infty \ne C_\lambda$.

Note that since

$$h(C_\lambda) \ge \frac{1}{2\lambda\delta}\phi\left(\frac{1}{2(b-a)}\right),$$

it follows that $C_\lambda \to \infty$ as $\lambda \to 0$.

Next, we verify that there exist constants $R > C_\lambda$, $h > R$ such that for given $0 < \theta < 1$ all solutions $u \in \mathbb{K}$ of $u = Tu + \theta h$ satisfy $\|u\|_\infty \ne R$. Such u satisfy

$$\big[p(t)(\phi(u' - w') + \phi(w'))\big]' = -\lambda p(t)\tilde{g}(t, u - w), \qquad a < t < b$$
$$u(a) = \theta h = u(b).$$

Let $\|u\|_\infty = u(t_0)$ and suppose that $\|u\|_\infty > C_\lambda$. Let v be the solution of

$$\big[p(t)(\phi(v' - w') + \phi(w'))\big]' = 0, \qquad a < t < t_0$$
$$v(a) = \theta h, \qquad v(t_0) = \|u\|_\infty.$$

Then we have

$$\big[p(t)(\phi(u' - w') - \phi(v' - w'))\big]' = -\lambda M\tilde{g}(t, u - w), \qquad a < t < t_0$$
$$(u - v)(a) = 0 = (u - v)(t_0),$$

and by a comparison argument it follows that $u > v$ on (a, t_0). Note that

$$v(t) = \|u\|_\infty - \int_t^{t_0}\left\{\phi^{-1}\left(\frac{C}{p(s)} - \phi(w'(s))\right) + w'(s)\right\} ds,$$

where C is such that $v(a) = \theta h$, and hence

$$\|u\|_\infty = \theta h + \int_a^{t_0} \left\{ \phi^{-1}\left(\frac{C}{p(s)} - \phi(w') \right) + w' \right\} ds. \tag{3.3}$$

If $C > p_1 \phi(2\|u\|_\infty/(t_0 - a))$, then

$$\frac{C}{p(s)} > \frac{p_1}{p_0} \phi\left(\frac{2\|u\|_\infty}{t_0 - a} \right) \geq \phi\left(\frac{2}{b - a} \right) > \lambda M \delta \geq \phi(w'(s))$$

by Lemma 2.1. This implies

$$\begin{aligned} \phi^{-1}\left(\frac{C}{p(s)} \right) &= \phi^{-1}\left(\frac{C}{p(s)} - \phi(w'(s)) + \phi(w'(s)) \right) \\ &\leq \phi^{-1}\left(\frac{C}{p(s)} - \phi(w'(s)) \right) + w'(s) + \phi^{-1}(\lambda M \delta) \end{aligned}$$

by Lemma 2.2. Hence

$$\begin{aligned} \phi^{-1}\left(\frac{C}{p(s)} - \phi(w'(s)) \right) + w'(s) &\geq \phi^{-1}\left(\frac{C}{p(s)} \right) - \phi^{-1}(\lambda M \delta) \\ &\geq \frac{2\|u\|_\infty}{t_0 - a} - \phi^{-1}(\lambda M \delta) \geq \frac{\|u\|_\infty}{t_0 - a}, \end{aligned}$$

(since $1/(b - a) \geq \phi^{-1}(\lambda M \delta)$) which is a contradiction to (3.3). Thus $0 \leq C \leq p_1 \phi(2\|u\|_\infty/(t_0 - a))$ and using Lemma 2.2 it follows that

$$\begin{aligned} v(t) &\geq \|u\|_\infty - \int_t^{t_0} \left\{ \phi^{-1}\left(\frac{p_1}{p_0} \phi\left(\frac{2\|u\|_\infty}{t_0 - a} \right) - \phi(w'(s)) \right) + w'(s) \right\} ds \\ &\geq \|u\|_\infty - (t_0 - t)\left[\phi^{-1}\left(\frac{p_1}{p_0} \phi\left(\frac{2\|u\|_\infty}{t_0 - a} \right) \right) + \phi^{-1}(\lambda M \delta) \right] \\ &\geq \|u\|_\infty - \frac{2B_{(p_1/p_2)}(t_0 - t)}{t_0 - a} \|u\|_\infty - (b - a)\phi^{-1}(\lambda M \delta) \\ &\geq \frac{1}{2}\|u\|_\infty - (b - a)\phi^{-1}(\lambda M \delta) \geq \frac{3}{8}\|u\|_\infty, \qquad t \in [t_1, t_0], \end{aligned}$$

where $t_1 = t_0 - (t_0 - a)/4B$ and $B > 1$ is such that $\phi^{-1}((p_1/p_0)x) \leq B\phi^{-1}(x)$, $x \geq 0$.

Consequently $u(t) - w(t) \geq v(t) - w(t) \geq \frac{3}{8}\|u\|_\infty - w(t) \geq \frac{1}{4}\|u\|_\infty$, for $t \in [t_1, t_0]$, since $w(t) = \int_a^t w'(s)\,ds \leq (b-a)\phi^{-1}(\lambda M\delta) \leq \frac{1}{8}$.

Now

$$u(t) = w(t) + \theta h + \int_a^t \phi^{-1}\left(\frac{\tilde{C} - \lambda \int_a^s p(\tau)\tilde{g}(\tau, u-w)\,d\tau}{p(s)} - \phi(w') \right) ds,$$

where $\tilde{C}$ is such that $u'(t_0) = 0$. Thus

$$w'(t_0) + \phi^{-1}\left(\frac{\tilde{C} - \lambda \int_a^{t_0} p(\tau)\tilde{g}(\tau, u-w)\,d\tau}{p(t_0)} - \phi(w'(t_0)) \right) = 0,$$

which implies that $\tilde{C} = \lambda \int_a^{t_0} p(\tau)\tilde{g}(\tau, u-w)\,d\tau$. Hence

$$u(t) = w(t) + \theta h + \int_a^t \phi^{-1}\left(\frac{\lambda \int_s^{t_0} p(\tau)\tilde{g}(\tau, u-w)\,d\tau}{p(s)} - \phi(w') \right) ds.$$

If $t_0 \geq (b+a)/2$, then

$$\begin{aligned}
\|u\|_\infty &\geq \int_a^{t_1} \phi^{-1}\left(\frac{\lambda \int_s^{t_0} p(\tau)\tilde{g}(\tau, u-w)\,d\tau}{p(s)} - \phi(w') \right) ds \\
&\geq (t_1 - a)\phi^{-1}\left(\frac{\lambda (t_0 - t_1) p_0 \inf\limits_{a \leq \tau \leq b,\, s \geq \|u\|_\infty/4} \tilde{g}(\tau, s)}{p_1 \phi(\|u\|_\infty/4)} \phi\left(\frac{\|u\|_\infty}{4}\right) - \lambda M\delta \right) \\
&\geq \left(\frac{b-a}{2}\right)\left(1 - \frac{1}{4B}\right)\phi^{-1}\left(\frac{\lambda p_0 (b-a)}{8Bp_1} \left(\inf_{a \leq \tau \leq b,\, s \geq \|u\|_\infty/4} \frac{\tilde{g}(\tau, s)}{\phi(s)} \right) \right. \\
&\qquad\qquad \left. \times \phi\left(\frac{\|u\|_\infty}{4}\right) - \lambda M\delta \right) \\
&\geq \frac{(b-a)}{2}\left(1 - \frac{1}{4B}\right)\phi^{-1}\left(\overline{C}\phi\left(\frac{\|u\|_\infty}{4}\right)\right) \qquad \text{for } \|u\|_\infty \text{ large} \\
&\geq \frac{(b-a)}{2}\left(1 - \frac{1}{4B}\right) A_{\overline{C}} \frac{\|u\|_\infty}{4} \qquad \text{by (A.2)},
\end{aligned}$$

where

$$\overline{C} = \frac{\lambda p_0(b-a)}{16Bp_1} \inf_{\substack{a \le \tau \le b,\\ s \ge \|u\|_\infty/4}} \frac{\tilde{g}(\tau, s)}{\phi(s)}.$$

This implies $A_{\overline{C}} \le 8/(b-a)(1-1/4B)$. Since $A_{\overline{C}} \to \infty$ as $\|u\|_\infty \to \infty$ (by A.4), there exists $R_1 > C_\lambda$ independent of u, θ, and h such that $\|u\|_\infty < R_1$.

If, on the other hand, $t_0 \le (b+a)/2$, we let $\tilde{v}$ be such that

$$[p(t)(\phi(\tilde{v}' - w') + \phi(w'))]' = 0, \qquad t_0 < t < b$$

$$\tilde{v}(t_0) = \|u\|_\infty, \qquad \tilde{v}(b) = \theta h.$$

Then $\tilde{v}(t) = \|u\|_\infty - \int_{t_0}^b \phi^{-1}[C/p(s) + \phi(w')]\,ds + \int_{t_0}^t w'(s)\,ds$, where $C > 0$ is such that $\tilde{v}(b) = \theta h$, i.e.,

$$\|u\|_\infty = \theta h + \int_{t_0}^b \phi^{-1}\left(\frac{C}{p(s)} + \phi(w')\right) ds - \int_{t_0}^b w'(s)\,ds. \quad (3.4)$$

If $C > p_1\phi(2\|u\|_\infty/(b-t_0))$, then

$$\frac{C}{p(s)} > \frac{p_1}{p_0}\phi\left(\frac{2\|u\|_\infty}{b-t_0}\right) \ge \phi\left(\frac{2}{b-a}\right) > \lambda M\delta \ge \phi(-w'(s)),$$

by Lemma 2.1.

This implies

$$\begin{aligned}\phi^{-1}\left(\frac{C}{p(s)}\right) &= \phi^{-1}\left(\frac{C}{p(s)} + \phi(w'(s)) + \phi(-w'(s))\right)\\ &\le \phi^{-1}\left(\frac{C}{p(s)} + \phi(w'(s))\right) - w'(s) + \phi^{-1}(\lambda M\delta),\end{aligned}$$

by Lemma 2.2. Hence

$$\begin{aligned}\phi^{-1}\left(\frac{C}{p(s)} + \phi(w'(s))\right) - w'(s) &\ge \phi^{-1}\left(\frac{C}{p(s)}\right) - \phi^{-1}(\lambda M\delta)\\ &\ge \frac{2\|u\|_\infty}{b-t_0} - \phi^{-1}(\lambda M\delta)\\ &\ge \frac{\|u\|}{b-t_0}\end{aligned}$$

(since $1/(b-a) \geq \phi^{-1}(\lambda M\delta)$) which is in contradiction to (3.4). Thus

$$0 \leq C \leq p_1 \phi\left(\frac{2\|u\|_\infty}{b-t_0}\right)$$

and, using similar arguments as before, we have $u \geq \tilde{v}$ on (t_0, b).

Hence

$$\begin{aligned}
\tilde{v}(t) &\geq \|u\|_\infty - \int_{t_0}^{t} \phi^{-1}\left(\frac{p_1}{p_0}\phi\left(\frac{2\|u\|_\infty}{b-t_0}\right) + \phi(w')\right) ds + \int_{t_0}^{t} w'(s)\, ds \\
&\geq \|u\|_\infty - \left[\phi^{-1}\left(\frac{p_1}{p_0}\phi\left(\frac{2\|u\|_\infty}{b-t_0}\right)\right) + \phi^{-1}(\lambda M\delta)\right](t-t_0) \\
&\geq \|u\|_\infty - \frac{2B_{(p_1/p_0)}(t-T)\|u\|_\infty}{b-t_0} - (b-a)\phi^{-1}(\lambda M\delta) \\
&\geq \frac{1}{2}\|u\|_\infty - (b-a)\phi^{-1}(\lambda M\delta) \\
&\geq \frac{3}{8}\|u\|_\infty, \qquad t \in [t_0, t_2],
\end{aligned} \tag{3.5}$$

where $t_2 = t_1 + (b-t_0)/4B$. By rewriting u as

$$u(t) = w(t) + \theta h + \int_t^b \phi^{-1}\left\{\frac{\lambda \int_{t_0}^{s} p(\tau)\tilde{g}(\tau, u-w)\, d\tau}{p(s)} - \phi(w')\right\} ds$$

and using (3.5), it follows that there is $R_2 > C_\lambda$ independent of u, θ, and h such that $\|u\|_\infty < R_2$. Thus by Theorem 2.1, T has a fixed point $\tilde{u}$ in $\mathbb{K}$ with $\|u\|_\infty \geq C_\lambda$.

We now establish that $\tilde{u} > w$ on (a, b). Let $\|\tilde{u}\|_\infty = \tilde{u}(t_0)$, $a < t_0 < b$. From (3.1) we obtain for $a < t < t_0$,

$$p(t)(\phi(\tilde{u}' - w') + \phi(w')) = \int_t^{t_0} \lambda p(t)\tilde{g}(t, \tilde{u} - w)\, dt$$

which implies

$$\tilde{u}(t_0) = \int_a^{t_0} \phi^{-1}\left(\frac{\int_s^{t_0} \lambda p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau}{p(s)} - \phi(w'(s))\right) ds$$

$$+ \int_a^{t_0} w'(s)\, ds$$

$$= \int_a^{t_0} \left[\phi^{-1}\left(\frac{\int_s^{t_0} \lambda p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau}{p_0} + \phi(-w'(s))\right) + w'(s)\right] ds$$

$$\leq \int_a^{t_0} \left[\phi^{-1}\left(\frac{\lambda}{p_0}\int_s^{t_0} p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau\right) + \phi^{-1}(\lambda M\delta)\right] ds,$$

where we have used Lemma 2.2.

This implies

$$1 \leq \int_a^{t_0} \phi^{-1}\left(\frac{\lambda}{p_0}\int_s^{t_0} p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau\right) ds + (b-a)\phi^{-1}(\lambda M\delta)$$

and hence

$$\int_a^{t_0} \phi^{-1}\left(\frac{\lambda}{p_0}\int_s^{t_0} p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau\right) ds \geq \frac{1}{2} \tag{3.6}$$

by our choice of λ^*. Now, using Lemma 2.2 and (3.6), we obtain

$$\tilde{u}(t) - w(t) = \int_a^{t} \phi^{-1}\left(\frac{\lambda \int_s^{t_0} p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau}{p(s)} - \phi(w'(s))\right) ds$$

$$\geq \int_a^{t} \phi^{-1}\left(\frac{\lambda}{p_1}\int_s^{t_0} p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau - \lambda M\delta\right) ds$$

$$\geq \int_a^{t} \phi^{-1}\left(\frac{\lambda}{p_1}\int_s^{t_0} p(\tau)\tilde{g}(\tau, \tilde{u} - w)\, d\tau\right) ds$$

$$- 2(t-a)\phi^{-1}(\lambda M\delta).$$

Here we have used the fact that

$$\phi^{-1}(x-y) \geq \phi^{-1}(x) - 2\phi^{-1}(y),$$

for $x \geq 0$, $y \geq 0$ which follows from Lemma 2.2, since

$$\phi^{-1}(x) = \phi^{-1}(x-y+y) \leq \phi^{-1}(x-y) + \phi^{-1}(y) + \phi^{-1}(y).$$

Hence

$$\begin{aligned}\tilde{u}(t) - w(t) &\geq M_1 \int_a^t \phi^{-1}\left(\frac{\lambda}{p_0}\int_s^{t_0} p(\tau)\tilde{g}(\tau,\tilde{u}-w)\,d\tau\right) ds \\ &\quad - 2(t-a)\phi^{-1}(\lambda M\delta) \\ &\geq \frac{M_1}{2}\frac{(t-a)}{(t_0-a)} - 2(t-a)\phi^{-1}(\lambda M\delta) > 0, \qquad a < t \leq t_0\end{aligned}$$

since $\int_a^t \phi^{-1}((\lambda/p_0)\int_s^{t_0} p(\tau)\tilde{g}(\tau,\tilde{u}-w))\,ds$ is concave on $[a, t_0]$ and (3.6) holds.

Similarly, $\tilde{u} > w$ on (t_0, b) and thus $u = \tilde{u} - w$ is a positive solution of (1.2).

4. PROOF OF THEOREM 1.2

Define M, $\tilde{g}$, w, T, and $\mathbb{K}$ as in the proof of Theorem 1.1. Let $k \equiv 1$ and let u satisfy

$$u = Tu + \gamma k, \qquad 0 < \gamma.$$

We claim that $\|u\|_\infty \neq r = (8/M_1)(b-a)\phi^{-1}(\lambda M\delta)$, where $M_1 = A_{(p_0/p_1)}$ and δ is given by Lemma 2.1. Indeed, proceeding as in the proof of Theorem 1.1., we obtain

$$u(t) - w(t) \geq \frac{1}{4}\|u\|_\infty, \qquad t \in [t_1, t_0],$$

where $t_1 = t_0 - (t_0 - a)/4B$.

Hence, if $t_0 \geq (a+b)/2$, we have by (3.3)

$$\begin{aligned}\|u\|_\infty &\geq \int_a^{t_1} \phi^{-1}\left\{ \frac{\lambda(t_0 - t_1) p_0}{p_1} G\left(\frac{\|u\|_\infty}{4} \right) - \lambda M \delta \right\} ds \\ &\geq \frac{b-a}{2} \phi^{-1}\left\{ \frac{\lambda p_0}{p_1} \frac{b-a}{8B} G\left(\frac{\|u\|_\infty}{4} \right) - \lambda M \delta \right\},\end{aligned}$$

where

$$G(t) = \inf_{\substack{a \leq \tau \leq b, \\ s \geq t}} \tilde{g}(\tau, s).$$

Since $G(t) \to \infty$ as $t \to \infty$, it follows from (A.1) that

$$\|u\|_\infty \neq \frac{8}{M_1}(b-a)\phi^{-1}(\lambda M \delta),$$

if λ is sufficiently large.

The case where $t_0 < (a+b)/2$ can be treated in a similar way, by using (3.4).

Next we verify that there exists $R > r$ such that if u is a solution of

$$u = \gamma T u, \qquad 0 < \gamma < 1,$$

then $\|u\|_\infty \neq R$.

Using Lemma 2.2, we see that

$$\begin{aligned}u(t) &\leq \int_a^t \phi^{-1}\left\{ \frac{\lambda \int_s^b p(\tau) \tilde{g}(\tau, u - w)\, d\tau}{p(s)} \right\} ds + (b-a)\phi^{-1}(\lambda M \delta) \\ &\leq (b-a)\phi^{-1}\left\{ \frac{\lambda(b-a)p_1}{p_0} \frac{\tilde{G}(\|u\|_\infty)}{\phi(\|u\|_\infty)} \phi(\|u\|_\infty) \right\} \\ &\qquad + (b-a)\phi^{-1}(\lambda M \delta) \\ &= (b-a)\phi^{-1}\{\lambda C(\|u\|_\infty)\phi(\|u\|_\infty)\} + (b-a)\phi^{-1}(\lambda M \delta) \\ &\leq (b-a) B_{\lambda C(\|u\|_\infty)} \|u\|_\infty + (b-a)\phi^{-1}(\lambda M \delta),\end{aligned}$$

where $\tilde{G}(t) = \sup_{a \leq \tau \leq b,\, 0 \leq s \leq t} \tilde{g}(\tau, s)$.

Since $C(\|u\|_\infty) \to 0$ as $\|u\|_\infty \to \infty$, it follows that there exists $R > r$ such that $\|u\|_\infty \neq R$. Hence T has a fixed point $\tilde{u}$ in $\mathbb{K}$ with $r \leq \|\tilde{u}\|_\infty \leq R$. Proceeding as in the proof of Theorem 1.1, we deduce that $\tilde{u} > w$ which completes the proof.

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