Positive Solutions of Quasilinear Boundary Value Problems

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This paper deals with the existence of positive solutions for the boundary value problem

\[
(p(t)\phi(u'))' + \lambda p(t)f(t, u) = 0; \quad a < t < b
\]

\[
u(a) = 0 = u(b),
\]

where \( f \) is either \( \phi \)-superlinear or \( \phi \)-sublinear at \( \infty \) and \( f(t, 0) \) may be negative and \( p \) is a positive continuous function. The results extend several known results for semilinear equations. Our approach is based on fixed point theory for completely continuous operators which leave invariant a suitable cone in a Banach space of continuous functions.
1. Introduction

The results in this paper are motivated by the search for positive radial solutions for the quasilinear elliptic boundary value problem

$$\text{div}(a(|\nabla u|^2) \nabla u) + \lambda f(|x|, u) = 0, \quad a < |x| < b$$
$$u = 0, \quad |x| \in \{a, b\},$$

(1.1)

where $\phi(s) = \alpha(s^2)s$ is an increasing homeomorphism of the real line and $\lambda$ is a positive parameter. Such radial solutions are solutions to boundary value problems of the form

$$\left( p(t) \phi(u') \right)' + \lambda p(t) f(t, u) = 0, \quad a < t < b$$
$$u(a) = 0 = u(b),$$

(1.2)

with $p(t) = t^{N-1}$, $t = |x|$, and $N$ is the dimension of $x$. The case where $a(|\nabla u|^2) \nabla u = |\nabla u|^{p-2} \nabla u, \quad p > 1$, i.e., perturbations of the $p$-Laplacian, has received much attention in the recent literature. Also problem (1.2) with $f(t, 0) \geq 0$ has been studied by several authors in recent years (see [15] and the references therein). Here, we are interested in the case when $f(t, 0)$ may be negative (the so-called semipositone case) (see [6] and its references for a review). Since our results only depend upon the positivity and continuity of the coefficient function $p$, we shall consider (1.2) in this generality.

We first consider the case when $f$ is $\phi$-superlinear at $\infty$. In particular, we make the assumptions:

(A.1) $\phi$ is an odd, increasing homeomorphism on $\mathbb{R}$ with $\phi^{-1}$ concave on $\mathbb{R}^+$.

(A.2) For each $c > 0$, there exists $A_c > 0$ such that $\phi^{-1}(cu) \geq A_c \phi^{-1}(u), \quad u \in \mathbb{R}^+$ and $\lim_{c \to \infty} A_c = \infty$ (note that (A.2) implies the existence of $B_c > 0$ such that $\phi^{-1}(cu) \leq B_c \phi^{-1}(u), \quad u \in \mathbb{R}^+$ with $\lim_{c \to 0} B_c = 0$).

(A.3) $f: [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{u \to \infty} (f(t, u)/\phi(u)) = \infty$ uniformly for $t \in [a, b]$.

(A.4) $p: [a, b] \to (0, \infty)$ is continuous.

Our first result is:

**Theorem 1.1.** Let (A.1)–(A.4) hold. Then there exists $\lambda^* > 0$ such that problem (1.2) has a positive solution $u_\lambda$ for $0 < \lambda < \lambda^*$ with $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0$. 
We also consider the case when $f$ is $\phi$-sublinear at $\infty$, i.e., we consider nonlinearities $f$ that satisfy

\[(A.5) \quad f: [a, b] \times \mathbb{R} \to \mathbb{R} \text{ is continuous, } \lim_{t \to \infty} f(t, u) = \infty \text{ uniformly for } t \in [a, b], \text{ and } \lim_{u \to \infty} (f(t, u)/\phi(u)) = 0 \text{ uniformly for } t \in [a, b].\]

In this case we establish the following result:

**Theorem 1.2.** Let (A.1), (A.2), (A.4), and (A.5) hold. Then there exists $\lambda > 0$ such that (1.2) has a positive solution $u_\lambda$ for $\lambda > \bar{\lambda}$ with $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$.

In the above we adhere to the notation

\[\|u\|_\infty = \sup\{|u(t)| : a \leq t \leq b\}.\]

Theorems 1.1 and 1.2 are extensions of results in [2, 5, 10], to quasilinear equations (in particular, p-Laplacian like equations), and of results in [1, 3, 4, 7, 8, 13, 16]. The paper also serves to provide a unified treatment to a variety of results having been obtained by a host of methods.

## 2. Preliminary Results

We establish some preliminary results.

**Lemma 2.1.** Let $p_0, p_1 > 0$ be such that $p_0 \leq p(t) \leq p_1$, $t \in [a, b]$, and let $M$ be a positive number. Assume $w$ is the solution of

\[
\begin{align*}
(p(t)\phi(u'))' &= -\lambda M p(t), \quad a < t < b \\
u(a) &= 0 = u(b).
\end{align*}
\]

Then $w > 0$ on $(a, b)$ and

\[\|w'\|_\infty \leq \phi^{-1}(\lambda M \delta),\]

where $\delta = (p_1/p_0)(b - a)$.

**Proof.** By integrating, it follows that (2.1) has the unique solution given by

\[w(t) = \int_a^t \phi^{-1}\left(\frac{1}{p(s)} \left(C - \lambda M \int_s^t p(r) \, dr\right)\right) \, ds,\]
where $C$ is such that $w(b) = 0$. Hence we must have $0 < C < \lambda M \int_a^b p(r) \, dr$. Further, using
\[
\phi(w'(t)) = \frac{1}{p(t)} \left( C - \int_a^t \lambda M p(r) \, dr \right),
\]
we obtain
\[
-\lambda M \delta \leq \phi(w'(t)) \leq \lambda M \delta,
\]
$t \in [a, b]$, and (2.2) follows. Finally, since $w(a) = 0 = w(b)$, there exits $z \in (a, b)$ such that $w'(z) = 0$ and, since $p(t) \phi(w')$ is decreasing, $w'(t) < 0$ for $t > z$ while $w'(t) > 0$ for $t < z$. Thus $w > 0$ on $(a, b)$.

**Lemma 2.2.** For $x \geq -\mu$, $y \geq 0$, $\mu \geq 0$,
\[
\phi^{-1}(x + y) \leq \phi^{-1}(x) + \phi^{-1}(y) + \phi^{-1}(\mu).
\]

**Proof.** Since $\phi^{-1}$ is concave and $\phi^{-1}(0) = 0$, it follows that $\phi^{-1}(x + y) \leq \phi^{-1}(x) + \phi^{-1}(y)$ if $x \geq 0$, $y \geq 0$. If $-\mu \leq x < 0$, $y \geq 0$ and $\mu \geq 0$, then, since $\phi^{-1}$ is increasing $\phi^{-1}(x + y) \leq \phi^{-1}(y)$ and $\phi^{-1}(-\mu) \leq \phi^{-1}(x)$. Thus $\phi^{-1}(x) + \phi^{-1}(\mu) \geq 0$, since $\phi^{-1}$ is odd. Hence the conclusion follows.

We next state the fixed point theorems which will be used to prove our results.

**Theorem 2.1** [11]. Let $E$ be a Banach space and let $\mathcal{K}$ be a cone in $E$ such that $\| \cdot \|$ is monotone with respect to the cone $\mathcal{K}$. Let $T: \mathcal{K} \to \mathcal{K}$ be a completely continuous operator $T$. Assume there exist positive constants $r$, $R$, and $k \in \mathcal{K}$, $h \in \mathcal{K}$ with $0 < r < R$, $\|k\| < r$, $\|h\| > R$ such that
(a) For each $0 < \theta < 1$, all solutions $u \in \mathcal{K}$ of
\[
u = \theta T\nu + (1 - \theta)k
\]
satisfy $\|u\| \neq r$.
(b) For each $0 < \theta < 1$, all solutions $u \in \mathcal{K}$ of
\[
u = T\nu + \theta h
\]
satisfy $\|u\| \neq R$.

Then $T$ has at least one fixed point $u \in \mathcal{K}$ with $r \leq \|u\| \leq R$. 
**Theorem 2.2** [11]. Let $E$, $K$, and $T$ be as in Theorem 2.1. Assume there exist positive constants $r$, $R$, and $k \in K$ with $0 < r < R$, $\|k\| = 1$ such that

(a) For each $\gamma > 0$, all solutions $y \in K$ of

$$y = Ty + \gamma k$$

satisfy $\|y\| \neq r$.

(b) For each $0 < \gamma < 1$, all solutions $z \in K$ of

$$z = \gamma Tz$$

satisfy $\|z\| \neq R$.

Then $T$ has a fixed point $x \in K$ with $r \leq \|x\| \leq R$.

**3. Proof of Theorem 1.1**

Let $M > 0$ be such that $g(t, u) := f(t, u) + M > 0$ for $t \in [a, b], u \geq 0$. Define $g(t, u) = g(t, u)$ if $u \geq 0$, $g(t, u) = g(t, 0)$ if $u \leq 0$. Let $w$ be defined as in Lemma 2.1. Then $u$ is a positive solution of (1.2) if $\tilde{u} = u + w$ is a solution of

$$
(p(t)(\phi(\tilde{u}' - w') + \phi(w')))' = -\lambda p(t) \tilde{g}(t, \tilde{u} - w), \quad a < t < b
$$

$$
\tilde{u}(a) = 0 = \tilde{u}(b)
$$

with $\tilde{u} > w$ on $(a, b)$.

For each $v \in C[a, b]$, let $u = Tw$ be the solution of

$$
(p(t)(\phi(u' - w') + \phi(w'))) = -\lambda p(t) \tilde{g}(t, v - w), \quad a < t < b
$$

$$
u(a) = 0 = u(b).
$$

Note that $u$ satisfies

$$u(t) = \int_a^t \phi^{-1} \left( C_1 - \lambda \int_a^t p(\tau) \tilde{g}(\tau, v - w) \frac{d\tau - \phi(w'(s))}{p(s)} \right) ds + w(t),$$

where $C_1$ is such that $u(b) = 0$. Hence

$$0 \leq C_1 \leq \lambda \int_a^b p(\tau) \tilde{g}(\tau, v - w) d\tau,$$
since otherwise,
\[ u(b) > \int_a^b \phi^{-1}( - \phi'(w(s)) \right) ds = 0, \]
which is a contradiction.

Now let \( \mathcal{K} = \{ v \in C[a, b] \mid v(x) \geq 0 \} \). It can be verified that \( T : \mathcal{K} \to \mathcal{K} \) is completely continuous. Let \( \lambda* > 0 \) be such that
\[ \frac{b - a}{M_1} \phi^{-1}(\lambda*M) < \frac{1}{8} \]
and
\[ \lambda* < \left[ \frac{h(1)}{\phi(1/2(b - a))} \right]^{-1} \frac{1}{2\delta}, \]
where \( M_1 = A_{(p_b/p_1)} \leq 1, h(t) = \sup_{a \leq t \leq b, 0 \leq s \leq \tilde{\tau}(t, s)} \tilde{\tau}(t, s) \) and \( \delta \) is given by Lemma 2.1.

We shall now apply Theorem 2.1 to show that \( T \) has a fixed point \( \tilde{u} \) in \( \mathcal{K} \) with \( \|\tilde{u}\| \geq C_\lambda \), where \( C_\lambda \to \infty \) as \( \lambda \to 0 \). Let \( 0 < \lambda < \lambda* \). Then there exists \( C_\lambda > 1 \) such that
\[ \frac{h(C_\lambda)}{\phi(C_\lambda/2(b - a))} = \frac{1}{2\lambda\delta} \]
for
\[ \frac{h(1)}{\phi(1/2(b - a))} < \frac{1}{2\lambda\delta} \quad \text{and} \quad \lim_{t \to \infty} \frac{h(t)}{\phi(t/2(b - a))} = \infty. \]

Let \( u \in \mathcal{K} \) be such that \( u = \theta Tu, 0 < \theta < 1 \). We claim that \( \|u\| \neq C_\lambda \).

Indeed, if \( \|u\| = C_\lambda \) then we have by Lemma 2.2,
\begin{align*}
 u(t) &\leq \int_a^t \phi^{-1} \left( \frac{\lambda \int_s^b p(\tau) \tilde{g}(\tau, u - w) \right) d\tau}{p(s)} \right) ds + w(t) \\
 &\leq \int_a^t \phi^{-1} \left( \frac{\lambda \int_s^b p(\tau) \tilde{g}(\tau, u - w) \right) d\tau}{p(s)} \right) ds \\
 &\quad - \int_a^t \phi^{-1}(\phi(w')) ds + \int_a^t \phi^{-1}(\lambda M\delta) ds + w(t). 
\end{align*}
Here we have used the fact that $\phi(w') < \lambda M \delta$ (see Lemma 2.1). Then

$$C_\lambda = \|u\|_\infty \leq (b - a) \phi^{-1}(\lambda \delta h(C_\lambda)) + (b - a) \phi^{-1}(\lambda^* M \delta),$$

which implies

$$\phi\left(\frac{C_\lambda}{2(b - a)}\right) \leq \lambda \delta h(C_\lambda)$$

or

$$\frac{h(C_\lambda)}{\phi(C_\lambda/2(b - a))} \geq \frac{1}{\lambda \delta}$$

a contradiction to (3.2). Hence $\|u\|_\infty \neq C_\lambda$.

Note that since

$$h(C_\lambda) \geq \frac{1}{2\lambda \delta} \phi\left(\frac{1}{2(b - a)}\right),$$

it follows that $C_\lambda \to \infty$ as $\lambda \to 0$.

Next, we verify that there exist constants $R > C_\lambda$, $h > R$ such that for given $0 < \theta < 1$ all solutions $u \in \mathcal{K}$ of $u = Tu + \theta h$ satisfy $\|u\|_\infty \neq R$. Such $u$ satisfy

$$[p(t)(\phi(u' - w') + \phi(w'))]' = -\lambda p(t) \tilde{g}(t, u - w), \quad a < t < b$$

$$u(a) = \theta h = u(b).$$

Let $\|u\|_\infty = u(t_0)$ and suppose that $\|u\|_\infty > C_\lambda$. Let $v$ be the solution of

$$[p(t)(\phi(v' - w') + \phi(w'))]' = 0, \quad a < t < t_0$$

$$v(a) = \theta h, \quad v(t_0) = \|u\|_\infty.$$ 

Then we have

$$[p(t)(\phi(u' - w') - \phi(v' - w'))]' = -\lambda M \tilde{g}(t, u - w), \quad a < t < t_0$$

$$(u - v)(a) = 0 = (u - v)(t_0),$$

and by a comparison argument it follows that $u > v$ on $(a, t_0)$. Note that

$$v(t) = \|u\|_\infty - \int_t^{t_0} \phi^{-1}\left(\frac{C}{p(s)} - \phi(w'(s))\right) + w'(s) \right) ds,$$
where \( C \) is such that \( v(a) = \theta h \), and hence
\[
\|u\|_\infty = \theta h + \int_a^t \left( \phi^{-1} \left( \frac{C}{p(s)} - \phi(w') \right) \right) ds. \tag{3.3}
\]

If \( C > p_1 \phi(2\|u\|_\infty/(t_0 - a)) \), then
\[
\frac{C}{p(s)} > \frac{p_1}{p_0} \phi \left( \frac{2\|u\|_\infty}{t_0 - a} \right) \geq \phi \left( \frac{2}{b - a} \right) > \lambda M \delta \geq \phi(w'(s))
\]
by Lemma 2.1. This implies
\[
\phi^{-1} \left( \frac{C}{p(s)} \right) = \phi^{-1} \left( \frac{C}{p(s)} - \phi(w'(s)) + \phi(w'(s)) \right)
\leq \phi^{-1} \left( \frac{C}{p(s)} - \phi(w'(s)) + w'(s) + \phi^{-1}(\lambda M \delta) \right)
\]
by Lemma 2.2. Hence
\[
\phi^{-1} \left( \frac{C}{p(s)} - \phi(w'(s)) + w'(s) \right) \geq \frac{2\|u\|_\infty}{t_0 - a} - \phi^{-1}(\lambda M \delta) \geq \frac{\|u\|_\infty}{t_0 - a},
\]
(since \( 1/(b - a) \geq \phi^{-1}(\lambda M \delta) \)) which is a contradiction to (3.3). Thus
\[
0 \leq C \leq p_1 \phi(2\|u\|_\infty/(t_0 - a)) \)
and using Lemma 2.2 it follows that
\[
v(t) \geq \|u\|_\infty - \int_t^{t_0} \left( \phi^{-1} \left( \frac{p_1}{p_0} \phi \left( \frac{2\|u\|_\infty}{t_0 - a} \right) - \phi(w'(s)) \right) \right) ds \]
\geq \|u\|_\infty - (t_0 - t) \left[ \phi^{-1} \left( \frac{p_1}{p_0} \phi \left( \frac{2\|u\|_\infty}{t_0 - a} \right) \right) + \phi^{-1}(\lambda M \delta) \right]
\geq \|u\|_\infty - \frac{2B(p_1/p_0)(t_0 - t)}{t_0 - a} \|u\|_\infty - (b - a) \phi^{-1}(\lambda M \delta)
\geq \frac{1}{2} \|u\|_\infty - (b - a) \phi^{-1}(\lambda M \delta) \geq \frac{3}{8} \|u\|_\infty, \quad t \in [t_1, t_0],
\]
where \( t_1 = t_0 - (t_0 - a)/4B \) and \( B > 1 \) is such that \( \phi^{-1}(p_1/p_0)x \leq B\phi^{-1}(x) \) for \( x \geq 0 \).
Consequently \( u(t) - w(t) \geq v(t) - w(t) \geq \frac{3}{2} \|u\|_\infty - w(t) \geq \frac{1}{4} \|u\|_\infty \), for \( t \in [t_1, t_0] \), since \( w(t) = \int_a^t w'(s) \, ds \leq (b - a) \phi^{-1}(\lambda M \delta) \leq \frac{1}{4} \).

Now

\[
u(t) = w(t) + \theta h + \int_a^t \phi^{-1} \left( \tilde{C} - \lambda \int_a^t \frac{p(\tau) \tilde{g}(\tau, u - w)}{p(s)} \, d\tau \right)
- \phi(w') \right) \, ds,
\]

where \( \tilde{C} \) is such that \( u'(t_0) = 0 \). Thus

\[
w'(t_0) + \phi^{-1} \left( \tilde{C} - \lambda \int_a^{t_0} \frac{p(\tau) \tilde{g}(\tau, u - w)}{p(t_0)} \, d\tau \right) = 0,
\]

which implies that \( \tilde{C} = \lambda \int_a^{t_0} p(\tau) \tilde{g}(\tau, u - w) \, d\tau \). Hence

\[
u(t) = w(t) + \theta h + \int_a^t \phi^{-1} \left( \frac{\lambda \int_s^{t_0} p(\tau) \tilde{g}(\tau, u - w)}{p(s)} \, d\tau \right) - \phi(w') \right) \, ds.
\]

If \( t_0 \geq (b + a)/2 \), then

\[
\|u\|_\infty \geq \int_a^{t_1} \phi^{-1} \left( \frac{\lambda \int_s^{t_0} p(\tau) \tilde{g}(\tau, u - w)}{p(s)} \, d\tau \right) - \phi(w') \right) \, ds
\]

\[
\geq (t_1 - a) \phi^{-1} \left( \frac{\lambda (t_0 - t_1) p_0}{p_1} \inf_{a \leq \tau \leq b, s \geq \|u\|_\infty / 4} \frac{\tilde{g}(\tau, s)}{\phi(s)} \right)
\]

\[
\geq \left( \frac{b - a}{2} \right) \left( 1 - \frac{1}{4B} \right) \phi^{-1} \left( \frac{\lambda p_0 (b - a)}{8Bp_1} \inf_{a \leq \tau \leq b, s \geq \|u\|_\infty / 4} \frac{\tilde{g}(\tau, s)}{\phi(s)} \right)
\times \phi \left( \frac{\|u\|_\infty}{4} \right) - \lambda M \delta
\]

\[
\geq \left( \frac{b - a}{2} \right) \left( 1 - \frac{1}{4B} \right) \phi^{-1} \left( \frac{\|u\|_\infty}{4} \right)
\]

for \( \|u\|_\infty \) large

\[
\geq \left( \frac{b - a}{2} \right) \left( 1 - \frac{1}{4B} \right) \frac{\|u\|_\infty}{4}
\]

by (A.2),
where
\[
\mathcal{C} = \frac{\lambda p_0(b - a)}{16Bp_1} \inf_{\substack{a \leq \tau \leq b, \\ s \geq \|u\|_\infty/4}} \tilde{g}(\tau, s) / \phi(s).
\]

This implies \( A_\tau \leq 8/(b - a)(1 - 1/4B) \). Since \( A_\tau \to \infty \) as \( \|u\|_\infty \to \infty \) (by A.4), there exists \( R_1 > C_\lambda \) independent of \( u \), \( \theta \), and \( h \) such that \( \|u\|_\infty < R_1 \).

If, on the other hand, \( t_0 \leq (b + a)/2 \), we let \( \tilde{v} \) be such that
\[
[p(t)(\phi(\tilde{v}' - w') + \phi(w'))]' = 0, \quad t_0 < t < b \\
\tilde{v}(t_0) = \|u\|_\infty, \quad \tilde{v}(b) = \theta h.
\]
Then \( \tilde{v}(t) = \|u\|_\infty - \int_{t_0}^{b} \phi^{-1}[C/p(s) + \phi(w')] ds + \int_{t_0}^{b} w'(s) ds \), where \( C > 0 \) is such that \( \tilde{v}(b) = \theta h \), i.e.,
\[
\|u\|_\infty = \theta h + \int_{t_0}^{b} \phi^{-1}\left( \frac{C}{p(s)} + \phi(w') \right) ds - \int_{t_0}^{b} w'(s) ds. \quad (3.4)
\]

If \( C > \frac{p_1}{p_0} \phi(2\|u\|_\infty/(b - t_0)) \), then
\[
\frac{C}{p(s)} > \frac{p_1}{p_0} \phi\left( \frac{2\|u\|_\infty}{b - t_0} \right) \geq \phi\left( \frac{2}{b - a} \right) > \lambda M \delta \geq \phi(-w'(s)),
\]
by Lemma 2.1.

This implies
\[
\phi^{-1}\left( \frac{C}{p(s)} \right) = \phi^{-1}\left( \frac{C}{p(s)} + \phi(w'(s)) + \phi(-w'(s)) \right) \\
\leq \phi^{-1}\left( \frac{C}{p(s)} + \phi(w'(s)) \right) - w'(s) + \phi^{-1}(\lambda M \delta),
\]
by Lemma 2.2. Hence
\[
\phi^{-1}\left( \frac{C}{p(s)} + \phi(w'(s)) \right) - w'(s) \geq \phi^{-1}\left( \frac{C}{p(s)} \right) - \phi^{-1}(\lambda M \delta) \\
\geq \frac{2\|u\|_\infty}{b - t_0} - \phi^{-1}(\lambda M \delta) \\
\geq \frac{\|u\|_\infty}{b - t_0},
\]
(since $1/(b - a) \geq \phi^{-1}(\lambda M \delta)$) which is in contradiction to (3.4). Thus

$$0 \leq C \leq p_2 \phi \left( \frac{2 \|u\|_\infty}{b - t_0} \right)$$

and, using similar arguments as before, we have $u \geq \tilde{v}$ on $(t_0, b)$.

Hence

$$\tilde{v}(t) \geq \|u\|_\infty - \int_{t_0}^{t} \phi^{-1} \left( \frac{p_1}{p_0} \phi \left( \frac{2 \|u\|_\infty}{b - t_0} \right) + \phi(w') \right) ds + \int_{t_0}^{t} w'(s) ds$$

$$\geq \|u\|_\infty - \left[ \phi^{-1} \left( \frac{p_1}{p_0} \phi \left( \frac{2 \|u\|_\infty}{b - t_0} \right) \right) + \phi^{-1}(\lambda M \delta) \right] (t - t_0)$$

$$\geq \|u\|_\infty - \frac{2B(p_1/p_0)(t - T)}{b - t_0} \|u\|_\infty - (b - a) \phi^{-1}(\lambda M \delta)$$

$$\geq \frac{1}{2} \|u\|_\infty - (b - a) \phi^{-1}(\lambda M \delta)$$

$$\geq \frac{3}{8} \|u\|_\infty, \quad t \in [t_0, t_2], \quad (3.5)$$

where $t_2 = t_1 + (b - t_0)/4B$. By rewriting $u$ as

$$u(t) = w(t) + \theta h + \int_{t}^{b} \phi^{-1} \left( \frac{\lambda \int_{t_0}^{\tau} p(\tau) \tilde{g}(\tau, u - w) d\tau}{p(s)} - \phi(w') \right) ds$$

and using (3.5), it follows that there is $R_2 > C_{\lambda}$ independent of $u$, $\theta$, and $h$ such that $\|u\|_\infty < R_2$. Thus by Theorem 2.1, $T$ has a fixed point $\tilde{u}$ in $K$ with $\|u\|_\infty \geq C_{\lambda}$.

We now establish that $\tilde{u} > w$ on $(a, b)$. Let $\|\tilde{u}\|_\infty = \tilde{u}(t_0)$, $a < t_0 < b$. From (3.1) we obtain for $a < t < t_0$,

$$p(t)(\phi(\tilde{u}' - w') + \phi(w')) = \int_{t}^{t_0} \lambda p(t) \tilde{g}(t, \tilde{u} - w) dt$$
which implies

\[
\tilde{u}(t_0) = \int_a^{t_0} \phi^{-1} \left( \frac{\int_s^{t_0} \lambda p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau}{p(s)} - \phi(w'(s)) \right) \, ds
\]

\[+ \int_a^{t_0} w'(s) \, ds\]

\[= \int_a^{t_0} \phi^{-1} \left( \frac{\int_s^{t_0} \lambda p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau}{p_0} + \phi(-w'(s)) + w'(s) \right) \, ds\]

\[\leq \int_a^{t_0} \phi^{-1} \left( \frac{\lambda}{p_0} \int_s^{t_0} p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau \right) + \phi^{-1}(\lambda M\delta) \right) \, ds,
\]

where we have used Lemma 2.2.

This implies

\[1 \leq \int_a^{t_0} \phi^{-1} \left( \frac{\lambda}{p_0} \int_s^{t_0} p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau \right) \, ds + (b - a) \phi^{-1}(\lambda M\delta)\]

and hence

\[
\int_a^{t_0} \phi^{-1} \left( \frac{\lambda}{p_0} \int_s^{t_0} p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau \right) \, ds \geq \frac{1}{2}\]

(3.6)

by our choice of \( \lambda^* \). Now, using Lemma 2.2 and (3.6), we obtain

\[
\tilde{u}(t) - w(t) = \int_a^{t} \phi^{-1} \left( \frac{\lambda \int_s^{t_0} p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau}{p(s)} - \phi(w'(s)) \right) \, ds
\]

\[
\geq \int_a^{t} \phi^{-1} \left( \frac{\lambda}{p_1} \int_s^{t_0} p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau - \lambda M\delta \right) \, ds
\]

\[
\geq \int_a^{t} \phi^{-1} \left( \frac{\lambda}{p_1} \int_s^{t_0} p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau \right) \, ds
\]

\[\geq 2(t - a) \phi^{-1}(\lambda M\delta).
\]
Here we have used the fact that
\[ \phi^{-1}(x - y) \geq \phi^{-1}(x) - 2\phi^{-1}(y), \]
for \( x \geq 0, y \geq 0 \) which follows from Lemma 2.2, since
\[ \phi^{-1}(x) = \phi^{-1}(x - y + y) \leq \phi^{-1}(x - y) + \phi^{-1}(y) + \phi^{-1}(y). \]

Hence
\[
\tilde{u}(t) - w(t) \geq M_1 \int_a^t \phi^{-1}\left( \frac{\lambda}{p_0} \int_s^t p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, d\tau \right) \, ds \\
- 2(t - a) \phi^{-1}(\lambda M\delta) \\
\geq \frac{M_1}{2} \frac{(t - a)}{(t_0 - a)} - 2(t - a) \phi^{-1}(\lambda M\delta) > 0, \quad a < t \leq t_0
\]
since \( \int a^t \phi^{-1}(\lambda/p_0)^p(\tau) \tilde{g}(\tau, \tilde{u} - w) \, ds \) is concave on \([a, t_0]\) and (3.6) holds.
Similarly, \( \tilde{u} > w \) on \((t_0, b)\) and thus \( u = \tilde{u} - w \) is a positive solution of (1.2).

4. PROOF OF THEOREM 1.2

Define \( M, \tilde{g}, w, T, \) and \( \kappa \) as in the proof of Theorem 1.1. Let \( k = 1 \) and let \( u \) satisfy
\[ u = Tu + \gamma k, \quad 0 < \gamma. \]

We claim that \( \|u\|_\infty \neq r = (8/M_1)(b - a)\phi^{-1}(\lambda M\delta) \), where \( M_1 = A_{(p_0/p_1)} \) and \( \delta \) is given by Lemma 2.1. Indeed, proceeding as in the proof of Theorem 1.1., we obtain
\[ u(t) - w(t) \geq \frac{1}{4} \|u\|_\infty, \quad t \in [t_1, t_0], \]
where \( t_1 = t_0 - (t_0 - a)/4B. \)
Hence, if \( t_0 \geq (a + b)/2 \), we have by (3.3)

\[
\|u\|_\infty \geq \int_a^{t_1} \phi^{-1} \left( \frac{\lambda (t_0 - t_1) p_0}{p_1} G \left( \frac{\|u\|_\infty}{4} \right) - \lambda M \delta \right) ds
\]

\[
\geq \frac{b - a}{2} \phi^{-1} \left( \frac{\lambda p_0 b - a}{8B} G \left( \frac{\|u\|_\infty}{4} \right) - \lambda M \delta \right),
\]

where

\[
G(t) = \inf_{s \leq t \leq b, s \geq t} \tilde{g}(\tau, s).
\]

Since \( G(t) \to \infty \) as \( t \to \infty \), it follows from (A.1) that

\[
\|u\|_\infty \neq \frac{8}{M_1} (b - a) \phi^{-1}(\lambda M \delta),
\]

if \( \lambda \) is sufficiently large.

The case where \( t_0 < (a + b)/2 \) can be treated in a similar way, by using (3.4).

Next we verify that there exists \( R > r \) such that if \( u \) is a solution of

\[
u = \gamma Tu, \quad 0 < \gamma < 1,
\]

then \( \|u\|_\infty \neq R \).

Using Lemma 2.2, we see that

\[
u(t) \leq \int_a^t \phi^{-1} \left( \frac{\lambda \int_s^b p(\tau) \tilde{g}(\tau, u - w) \ d\tau}{p(s)} \right) ds + (b - a) \phi^{-1}(\lambda M \delta)
\]

\[
\leq (b - a) \phi^{-1} \left( \frac{\lambda (b - a) p_1}{p_0} \tilde{G}(\|u\|_\infty) \frac{\phi(\|u\|_\infty)}{\phi(\|u\|_\infty)} \right)
\]

\[
+ (b - a) \phi^{-1}(\lambda M \delta)
\]

\[
= (b - a) \phi^{-1} \left( \lambda C(\|u\|_\infty) \phi(\|u\|_\infty) \right) + (b - a) \phi^{-1}(\lambda M \delta)
\]

\[
\leq (b - a) B_C(\|u\|_\infty) \|u\|_\infty + (b - a) \phi^{-1}(\lambda M \delta),
\]

where \( \tilde{G}(t) = \sup_{a \leq \tau \leq b, 0 \leq \tau \leq t} \tilde{g}(\tau, s) \).

Since \( C(\|u\|_\infty) \to 0 \) as \( \|u\|_\infty \to \infty \), it follows that there exists \( R > r \) such that

\( \|u\|_\infty \neq R \). Hence \( T \) has a fixed point \( \tilde{u} \) in \( K \) with \( r \leq \|\tilde{u}\|_\infty \leq R \).

Proceeding as in the proof of Theorem 1.1, we deduce that \( \tilde{u} > w \) which completes the proof.
REFERENCES