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Numerical existence and uniqueness proof for solutions of nonlinear hyperbolic equations

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Abstract

We consider a numerical method to verify the existence and uniqueness of the solutions of nonlinear hyperbolic problems with guaranteed error bounds. Using a C^1 finite element solution and an inequality constituting a bound on the norm of the inverse operator of the linearized operator, we numerically construct a set of functions which satisfy the hypothesis of Banach's fixed point theorem for a continuous map on L^p-space in a computer. We present detailed verification procedures and give some numerical examples. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the two preceding papers [4,5], we discussed the numerical verification of the existence of solutions to nonlinear parabolic and hyperbolic equations in the one-dimensional case. This verification method is based on Plum's formulation [9] of verification methods and weak formulation for determining a bound on the inverse norm of the linearized operator. In this paper, we describe a numerical verification method that demonstrates existence and uniqueness of solutions to nonlinear hyperbolic equations. In order to ensure existence and uniqueness, we use the idea contained in Nakao's method [6,8,11]. Another method used for hyperbolic equations [7] requires that the nonlinear map in question is retractive in a neighborhood of the solution. Our method is not subjected to such a condition. Thus, our method has the potential of being applicable to a more general class of hyperbolic equations.

In the following section, we introduce the problem considered and its fixed point formulation. In Section 3, a fundamental theorem which contains the verification conditions of our method is presented. In Section 4, using a weak formulation, we estimate the inverse norm of the linearized

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operator and give the algorithm for our method. Section 5 contains some examples that illustrate our method.

2. Problem and the fixed-point formulation

Consider the problem of finding a function u that satisfies the following relations:

$$u \in L^2(J; H_0^1(\Omega)), \quad u_t \in L^2(J; L^2(\Omega))$$

$$\frac{d^2}{dt^2}(u, v) + (\nabla u, \nabla v) = (-f(\cdot, u), v), \quad v \in H_0^1(\Omega), \quad t \in J := (0, T),$$

$$u(\cdot, 0) = 0, \tag{1}$$

where Ω is a bounded open interval on $\boldsymbol{R}$ or a bounded rectangular domain in $\boldsymbol{R}^2$, $(\cdot,\cdot)$ is the usual $L^2(\Omega)$ inner product, and f is a function on $Q \times \boldsymbol{R}$ with $Q = \Omega \times J$. Also, suppose that $\hat{f}$ defined by $(\hat{f}(u))(x,t) := f(x,t,u(x,t))$ maps $L^p(Q)$ into $L^2(Q)$ for some p satisfying $2 \leqslant p \leqslant 6$.

To be precise, the derivative d^2/dt^2 in (1) is treated as the generalized derivative of real functions on (0,T), that is,

$$\int_0^T (u(\cdot,t), v)\varphi''(t)\,dt + \int_0^T (\nabla u(\cdot,t), \nabla v)\varphi(t)\,dt = \int_0^T (-f(\cdot,t,u(\cdot,t)), v)\varphi(t)\,dt,$$

$$\forall \varphi \in C_0^\infty[0,T).$$

Here we note that Eq. (1) is the generalized problem corresponding to the following equations:

$$u_{tt} - \Delta u = -f(x,t,u) \quad (x,t) \in Q,$$

$$u(x,t) = 0 \quad (x,t) \in \partial\Omega \times J,$$

$$u(x,0) = 0 \quad x \in \Omega,$$

$$u_t(x,0) = 0 \quad x \in \Omega, \tag{2}$$

and the derivatives u_{tt}, u_t and Δu are understood in the distributional sense, where $\partial\Omega$ stands for the boundary of Ω.

Next, we define the time-dependent Sobolev space H by

$$H \equiv L^2(J; H_0^1(\Omega)) \cap H^1(J; L^2(\Omega))$$

with norm

$$\|u\|_H^2 = \int_J \|\nabla u(\cdot,t)\|_{L^2(\Omega)}^2\,dt + \int_J \|u_t(\cdot,t)\|_{L^2(\Omega)}^2\,dt,$$

where $\|\cdot\|_{L^p(\Omega)}$ is the usual $L^p(\Omega)$ norm.

Let $\tilde{H} := \{\phi \in H \mid \phi(\cdot,0)=0\}$ and let $u_h \in \tilde{H}$ be an approximate solution of (1). It is most common to think of such a solution as some finite element solution depending on h. Then suppose the following conditions hold for the nonlinear map f in (1) and (2):

(A1) $\hat{f} : L^p(Q) \to L^2(Q)$ is continuous and maps bounded sets into bounded sets.

(A2) $\hat{f}$ is Fréchet differentiable in $L^p(Q)$.

Let $f'(\cdot,u_h)(x,t) := (\partial f/\partial u)(x,t,u_h(x,t))$.

Now, as well known [3], for each $g \in L^2(Q)$, if $a \in C^1([0,T]; L^\infty(\Omega))$, the following problem has a unique solution $\phi \in \tilde{H}$:

$$\frac{d^2}{dt^2}(\phi, v) + (\nabla\phi, \nabla v) + (a\phi, v) = (g, v), \quad v \in H_0^1(\Omega), \quad t \in J. \tag{3}$$

We denote the above correspondence by $Ag = \phi$. Moreover, assuming that $a = f'(\cdot,u_h)$ and $(d/dt)f'(\cdot,u_h) \in L^\infty(Q)$, we define the fixed-point operator T by

$$Tu \equiv A[f'(\cdot,u_h)u - f(\cdot,u)]. \tag{4}$$

Then, from (A1), (A2), and the fact that the operator $A : L^2(Q) \to \tilde{H}$ and the injection $H \hookrightarrow L^p(Q)$ are continuous and bounded (see Lemmas 1 and 2), the operator $T : L^p(Q) \to L^p(Q)$ is Fréchet differentiable in $L^p(Q)$.

The map $f(\cdot,u) = gu^m$ is an example that satisfies assumptions (A1) and (A2), where $g \in L^\infty(Q)$, and m is an arbitrary nonnegative integer satisfying $1 \leqslant m \leqslant 3$. In this case, $f'(\cdot,u_h) = mgu_h^{m-1}$, and if $u_h \in C^1([0,T]; L^\infty(\Omega))$, the operator T in (4) is well defined.

Remark 1. In the one-dimensional case, we can choose $2 \leqslant p < \infty$, which implies $1 \leqslant m < \infty$ in the above example. In any case, we assume that the nonlinearity of f has a polynomial form with respect to u.

3. Verification condition

In order to transform (4) into a “residual-form”, setting $v = u - u_h$, we introduce the operator $\tilde{T} : L^p(Q) \to L^p(Q)$ defined by

$$\tilde{T}v \equiv T(u_h + v) - u_h. \tag{5}$$

Then if we wish to find a solution of the given problem that is close to u_h, we may look for a fixed point of $\tilde{T}$ that is close to 0. To construct a set V which includes the error of a solution to (2), taking some real number α, we set

$$V \equiv \{v \in L^p(Q) \mid \|v\|_{L^p(Q)} \leqslant \alpha\}. \tag{6}$$

Next, we choose the positive real numbers β and γ such that

$$\|\tilde{T}(0)\|_{L^p(Q)} \leqslant \beta, \tag{7}$$

$$\|\tilde{T}'(v_1)v_2\|_{L^p(Q)} \leqslant \gamma \quad \forall v_1, v_2 \in V \tag{8}$$

and define the set $K \subset L^p(Q)$ by

$$K \equiv \{v \in L^p(Q) \mid \|v\|_{L^p(Q)} \leqslant \beta + \gamma\}. \tag{9}$$

Our verification condition is described in the following theorem, which is similar to those in [6,11].

Theorem 1. *If $K \subset V$ holds for V (that is, if $\beta + \gamma \leqslant \alpha$), then there exists a solution to*

$$v = \tilde{T}(v)$$

in K, and this solution is unique within the set V.

Proof. From the definition γ and the mean value theorem,

$$\|\tilde{T}(v)\|_{L^p(Q)} \leqslant \|\tilde{T}(0)\|_{L^p(Q)} + \|\tilde{T}(v) - \tilde{T}(0)\|_{L^p(Q)} \leqslant \beta + \gamma \leqslant \alpha, \quad \forall v \in V.$$

This means that

$$\tilde{T}(V) \subset V. \tag{10}$$

The convexity of V and the mean value theorem give the following relations:

$$\begin{aligned} \|\tilde{T}(v_1) - \tilde{T}(v_2)\|_{L^p(Q)} &\leqslant \sup_{s\in[0,1]} \|\tilde{T}'(sv_1 + (1-s)v_2)(v_1 - v_2)\|_{L^p(Q)} \\ &\leqslant \sup_{v_3 \in V} \|\tilde{T}'(v_3)(v_1 - v_2)\|_{L^p(Q)} \\ &= \sup_{v_3 \in V} \|\tilde{T}'(v_3)w\|_{L^p(Q)} \frac{1}{\alpha} \|v_1 - v_2\|_{L^p(Q)} \\ &\leqslant \frac{\gamma}{\alpha} \|v_1 - v_2\|_{L^p(Q)} \quad \forall v_1, v_2 \in V, \end{aligned} \tag{11}$$

where $w := \alpha(v_1 - v_2)/\|v_1 - v_2\|_{L^p(Q)} \in V$. Thus, the assumption of this theorem provides the relation

$$\frac{\gamma}{\alpha} < \frac{\beta + \gamma}{\alpha} \leqslant 1.$$

Noting that $0 < \gamma < \alpha$ and $0 < \beta$, Banach's fixed-point theorem then gives the desired result. □

4. Constants in the verification condition

In this section we describe how to estimate β and γ introduced in the previous section. We assume that there exist the constants C_1 and C_2 satisfying

$$\|Ar\|_H \leqslant C_1 \|r\|_{L^2(Q)} \quad \forall r \in L^2(Q), \tag{12}$$

$$\|Ar\|_{L^p(Q)} \leqslant C_2 \|Ar\|_H. \tag{13}$$

If we then compute an approximate solution $u_h \in \tilde{H}$ so as to satisfy $d[u_h] \equiv u_{htt} - \Delta u_h + f(\cdot, u_h) \in L^2(Q)$ by using $C^1(Q)$ finite element, for example, the following relations hold:

$$\begin{aligned}\|\tilde{T}(0)\|_{L^p(Q)} &= \|Ad[u_h]\|_{L^p(Q)} \\ &\leqslant C_2\|Ad[u_h]\|_H \\ &\leqslant C_1C_2\|d[u_h]\|_{L^2(Q)}.\end{aligned} \tag{14}$$

Similarly, we can obtain

$$\begin{aligned}\|\tilde{T}'(v_1)v_2\|_{L^p(Q)} &\leqslant C_1C_2\|\hat{f}'(u_h+v_1)v_2 - \hat{f}'(u_h)v_2\|_{L^2(Q)} \\ &\leqslant C_1C_2G_\alpha \quad \forall v_1, v_2 \in V.\end{aligned} \tag{15}$$

Here G_α is a constant depending on α in (6).

In what follows, we consider the open intervals $I_{x_1} = (a_{x_1}, b_{x_1})$ and $I_{x_2} = (a_{x_2}, b_{x_2})$ for real numbers $a_{x_1} < b_{x_1}$ and $a_{x_2} < b_{x_2}$ and define $\Omega = I_{x_1} \times I_{x_2}$ and $d = \max\{b_{x_1} - a_{x_1}, b_{x_2} - a_{x_2}\}$.

Lemma 2. *Let $\underline{a}$ and $\bar{a}$ denote constants satisfying $\underline{a} \leqslant a(x,t) \leqslant \bar{a}$ for almost all $(x,t) \in Q$. Then C_1 in (12) is given by*

$$C_1 = \sqrt{\frac{1}{c}(e^{cT} - 1)},$$

where $c = \max(1 - \underline{a}T, (d^2/n_0\pi^2)\|a_t\|_{L^\infty(Q)})$ for $\underline{a} < 0$ and $c = \max(1, (d^2/n_0\pi^2)\|a_t\|_{L^\infty(Q)})$ for $\underline{a} \geqslant 0$ and n_0 is the dimension.

Proof. We first consider two-dimensional case.

Let $\rho \in C^\infty(\boldsymbol{R})$ denote some function such that

$$\rho \geqslant 0, \quad \operatorname{supp}(\rho) \subset (-1,1), \quad \int_{-1}^{1} \rho(t)\,dt = 1. \tag{16}$$

For fixed $\varepsilon > 0$, let $\rho_\varepsilon(t) := \frac{1}{\varepsilon}\rho(t/\varepsilon)$. Moreover, with given $g \in L^2(Q)$ and $\phi := Ag$, let

$$\phi_\varepsilon(x,t) := \int_0^T \phi(x,s)\rho_\varepsilon(s-t)\,ds \quad \text{for } (x,t) \in \bar{\Omega} \times \boldsymbol{R}. \tag{17}$$

Then, $\phi_\varepsilon \in C^\infty(\boldsymbol{R}; H_0^1(\Omega))$, and $\phi_\varepsilon = 0$ outside $\bar{\Omega} \times (-\varepsilon, T+\varepsilon)$. Moreover, for $(x,t) \in \bar{\Omega} \times [-\varepsilon, T-\varepsilon]$,

$$\begin{aligned}&\phi_{\varepsilon,t}(x,t) = -\int_0^T \phi(x,s)\rho_\varepsilon'(s-t)\,ds = \int_0^T \phi_t(x,s)\rho_\varepsilon(s-t)\,ds, \\ &\phi_{\varepsilon,tt}(x,t) = \int_0^T \phi(x,s)\rho_\varepsilon''(s-t)\,ds, \quad \nabla\phi_\varepsilon(x,t) = \int_0^T \nabla\phi(x,s)\rho_\varepsilon(s-t)\,ds.\end{aligned} \tag{18}$$

Now we use Eq. (3) in distributional form, i.e.,

$$\int_0^T (\phi(\cdot,s),v)\varphi''(s)\,\mathrm{d}s+\int_0^T (\nabla\phi(\cdot,s),\nabla v)\varphi(s)\,\mathrm{d}s+\int_0^T (a(\cdot,s)\phi(\cdot,s),v)\varphi(s)\,\mathrm{d}s$$

$$=\int_0^T (g(\cdot,s),v)\varphi(s)\,\mathrm{d}s,\quad v\in H_0^1(\Omega),\quad \varphi\in C_0^\infty[0,T)$$

with $v:=\phi_{\varepsilon,t}(\cdot,t)$ and $\varphi(s):=\rho_\varepsilon(s-t)$, where $t\in[-\varepsilon,T-\varepsilon]$ is fixed for the moment.

By (18), this yields

$$(\phi_{\varepsilon,tt}(\cdot,t),\phi_{\varepsilon,t}(\cdot,t))+(\nabla\phi_\varepsilon(\cdot,t),\nabla\phi_{\varepsilon,t}(\cdot,t))$$

$$=-\left(\int_0^T a(\cdot,s)\phi(\cdot,s)\rho_\varepsilon(s-t)\,\mathrm{d}s,\phi_{\varepsilon,t}(\cdot,t)\right)+\left(\int_0^T g(\cdot,s)\rho_\varepsilon(s-t)\,\mathrm{d}s,\phi_{\varepsilon,t}(\cdot,t)\right).$$

Since the left-hand side equals $\frac{1}{2}(\mathrm{d}/\mathrm{d}t)[\|\phi_{\varepsilon,t}(\cdot,t)\|^2_{L^2(\Omega)}+\|\nabla\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}]$, and $\phi_\varepsilon(x,t)$ vanishes for t in some neighborhood of $-\varepsilon$ due to (16) and (17), we obtain by integration from $-\varepsilon$ to $t\in(-\varepsilon,T-\varepsilon]$:

$$\frac{1}{2}[\|\phi_{\varepsilon,t}(\cdot,t)\|^2_{L^2(\Omega)}+\|\nabla\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}]$$

$$=-\int_{-\varepsilon}^t\left(\int_0^T a(\cdot,s)\phi(\cdot,s)\rho_\varepsilon(s-\tilde t)\,\mathrm{d}s,\phi_{\varepsilon,t}(\cdot,\tilde t)\right)\,\mathrm{d}\tilde t$$

$$+\int_{-\varepsilon}^t\left(\int_0^T g(\cdot,s)\rho_\varepsilon(s-\tilde t)\,\mathrm{d}s,\phi_{\varepsilon,t}(\cdot,\tilde t)\right)\,\mathrm{d}\tilde t. \tag{19}$$

We only prove the case for $\underline{a}\leqslant 0$, because we may put $\underline{a}=0$ for $\underline{a}\geqslant 0$ in the following.

Defining now $a(\cdot,t):=a(\cdot,0)$ for $t\in[-\varepsilon,0)$ and

$$R_\varepsilon(t):=\int_{-\varepsilon}^t\left(\int_0^T [a(\cdot,\tilde t)-a(\cdot,s)]\phi(\cdot,s)\rho_\varepsilon(s-\tilde t)\,\mathrm{d}s,\phi_{\varepsilon,t}(\cdot,\tilde t)\right)\,\mathrm{d}\tilde t, \tag{20}$$

we find that the first term on the right-hand side of (19) equals

$$-\int_{-\varepsilon}^t (a(\cdot,\tilde t)\phi_\varepsilon(\cdot,\tilde t),\phi_{\varepsilon,t}(\cdot,\tilde t))\,\mathrm{d}\tilde t+R_\varepsilon(t)$$

$$=-\frac{1}{2}(a(\cdot,t)\phi_\varepsilon(\cdot,t),\phi_\varepsilon(\cdot,t))+\frac{1}{2}\int_{-\varepsilon}^t (a_t(\cdot,\tilde t)\phi_\varepsilon(\cdot,\tilde t),\phi_\varepsilon(\cdot,\tilde t))\,\mathrm{d}\tilde t+R_\varepsilon(t)$$

$$\leqslant -\frac{1}{2}\underline{a}\|\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}+\frac{1}{2}\|a_t\|_{L^\infty(Q)}\int_{-\varepsilon}^t \|\phi_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t+R_\varepsilon(t)$$

and

$$\|\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}=\|\phi_\varepsilon(\cdot,t)-\phi_\varepsilon(\cdot,-\varepsilon)\|^2_{L^2(\Omega)}\leqslant\left(\int_{-\varepsilon}^t \|\phi_{\varepsilon,t}(\cdot,\tilde t)\|_{L^2(\Omega)}\,\mathrm{d}\tilde t\right)^2$$

$$\leqslant T\int_{-\varepsilon}^t \|\phi_{\varepsilon,t}(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t,$$

$$\int_{-\varepsilon}^{t}\|\phi_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t\leqslant\frac{d^2}{2\pi^2}\int_{-\varepsilon}^{t}\|\nabla\phi_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t \tag{21}$$

by Poincaré's inequality (cf. [12, Chapter 18]). Furthermore, defining

$$g_\varepsilon(x,t):=\int_0^T g(x,s)\rho_\varepsilon(s-t)\,\mathrm{d}s\quad\text{for }(x,t)\in\bar\Omega\times\boldsymbol{R}, \tag{22}$$

we obtain that the second term on the right-hand side of (19) is bounded by

$$\frac{1}{2}\int_{-\varepsilon}^{t}\|g_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t+\frac{1}{2}\int_{-\varepsilon}^{t}\|\phi_{\varepsilon,t}(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t.$$

Altogether we obtain from (19), using the constant c defined in this lemma, that for $t\in[-\varepsilon,T-\varepsilon]$,

$$\|\phi_{\varepsilon,t}(\cdot,t)\|^2_{L^2(\Omega)}+\|\nabla\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}$$

$$\leqslant c\int_{-\varepsilon}^{t}(\|\phi_{\varepsilon,t}(\cdot,\tilde t)\|^2_{L^2(\Omega)}+\|\nabla\phi_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)})\,\mathrm{d}\tilde t+\int_{-\varepsilon}^{t}\|g_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t+2R_\varepsilon(t).$$

Therefore, Gronwall's Lemma yields

$$\|\phi_{\varepsilon,t}(\cdot,t)\|^2_{L^2(\Omega)}+\|\nabla\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}\leqslant \mathrm{e}^{ct}\left[\int_{-\varepsilon}^{T-\varepsilon}\|g_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t+2\sup_{-\varepsilon\leq\tilde t\leqslant T-\varepsilon}|R_\varepsilon(\tilde t)|\right]$$

and by integration over $[-\varepsilon,T-\varepsilon]$, we obtain

$$\|\phi_{\varepsilon,t}\|^2_{L^2(Q_\varepsilon)}+\|\nabla\phi_\varepsilon\|^2_{L^2(Q_\varepsilon)}\leqslant \mathrm{e}^{-\varepsilon}\frac{\mathrm{e}^{cT}-1}{c}\left[\|g_\varepsilon\|^2_{L^2(Q_\varepsilon)}+2\sup_{-\varepsilon\leqslant t\leqslant T-\varepsilon}|R_\varepsilon(t)|\right], \tag{23}$$

where $Q_\varepsilon:=\Omega\times(-\varepsilon,T-\varepsilon)$.

Now let ε tend to 0. Using $|a(x,\tilde t)-a(x,s)|\leqslant\|a_t\|_{L^\infty(Q)}\cdot\varepsilon$ in (20), we find that $\sup_{-\varepsilon\leqslant t\leqslant T-\varepsilon}|R_\varepsilon(t)|\to 0$ as $\varepsilon\to 0$. Furthermore, by (16)–(18), (22),

$$\phi_\varepsilon\to\phi,\quad \phi_{\varepsilon,t}\to\phi_t,\quad \nabla\phi_\varepsilon\to\nabla\phi,\quad g_\varepsilon\to g\quad\text{in }L^2(Q)$$

as $\varepsilon\to 0$. Thus, (23) provides the assertion by letting ε tend to 0.

In one-dimensional case, noting that

$$\int_{-\varepsilon}^{t}\|\phi_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t\leqslant\frac{d^2}{\pi^2}\int_{-\varepsilon}^{t}\|\nabla\phi_\varepsilon(\cdot,\tilde t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tilde t$$

holds instead of (21), we can get the desired result. □

To calculate C_2 in (13), we slightly modify the proof of the Sobolev Embedding theorem.

Proposition 3 (e.g. Adams [1]). *Let* $\tilde x_1=(x_2,x_3),\tilde x_2=(x_1,x_3),\tilde x_3=(x_1,x_2)$, *and let* I_k $(k=1,2,3)$ *be bounded open intervals. The function* F *is defined by*

$$F(x)=F(x_1,x_2,x_3)=F_1(\tilde x_1)F_2(\tilde x_2)F_3(\tilde x_3),$$

where $F_k\in L^2(\Omega_k)$ $(k=1,2,3)$, *and* $\Omega_1=I_2\times I_3,\ \Omega_2=I_1\times I_3,\ \Omega_3=I_1\times I_2$. *Then* $F\in L^1(\Omega)$, *and the following inequality holds*:

$$\|F\|_{L^1(\Omega)}\leqslant\|F_1\|_{L^2(\Omega_1)}\|F_2\|_{L^2(\Omega_2)}\|F_3\|_{L^2(\Omega_3)},$$

where $\Omega=I_1\times I_2\times I_3$.

Lemma 4. *C_2 in (13) is given by*

$$C_2 = \begin{cases} \frac{p}{4}|Q|^{1/p} & (\Omega \text{ is one dimensional}) \\ (\frac{1}{2})^{2/3}\frac{2\sqrt{3}}{9}p|Q|^{(6-p)/6p} & (\Omega \text{ is two dimensional}). \end{cases}$$

Proof. The proof for the one-dimensional case is found in [2,4].

We consider the two-dimensional case. For any $w \in \tilde{C} := \{v \in C^1([0,T]; C_0^\infty(\Omega)) \mid v(\cdot,0) = 0\}$ and arbitrary $(x_1, x_2, t) \in Q$, we have

$$|w(x_1,x_2,t)| \leqslant \frac{1}{2}\int_{I_{x_1}} |w_{x_1'}(x_1',x_2,t)|\,\mathrm{d}x_1' =: f_1(\tilde{x}_1),$$

$$|w(x_1,x_2,t)| \leqslant \frac{1}{2}\int_{I_{x_2}} |w_{x_2'}(x_1,x_2',t)|\,\mathrm{d}x_2' =: f_2(\tilde{x}_2)$$

and

$$|w(x_1,x_2,t)| \leqslant \int_J |w_{t'}(x_1,x_2,t'))|\,\mathrm{d}t' =: f_3(\tilde{t}).$$

These relations give the following inequality:

$$|w(x_1,x_2,t)|^{3/2} \leqslant f_1^{1/2}(\tilde{x}_1) f_2^{1/2}(\tilde{x}_2) f_3^{1/2}(\tilde{t}). \tag{24}$$

Then applying Proposition 1 to (24), we obtain

$$\int_Q |w(x_1,x_2,t)|^{3/2}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t \leqslant \frac{1}{2}\|w_{x_1}\|_{L^1(Q)}^{1/2}\|w_{x_2}\|_{L^1(Q)}^{1/2}\|w_t\|_{L^1(Q)}^{1/2}.$$

This implies

$$\|w\|_{L^{(3/2)}(Q)} \leqslant (\tfrac{1}{2})^{2/3}\|w_{x_1}\|_{L^1(Q)}^{1/3}\|w_{x_2}\|_{L^1(Q)}^{1/3}\|w_t\|_{L^1(Q)}^{1/3}. \tag{25}$$

By using the fact that $\tilde{C}$ is dense in $\tilde{H}$ (cf. [12, Chapters 23 and 24]), estimate (25) holds for arbitrary $w \in \tilde{H}$.

Substituting $w = |v|^s$ $(1 \leqslant s \leqslant 4)$ into (25) gives

$$\|v\|_{L^{(3/2)s}(Q)}^s \leqslant (\tfrac{1}{2})^{2/3} s\|v^{s-1}\|_{L^2(Q)}\|v_{x_1}\|_{L^2(Q)}^{1/3}\|v_{x_2}\|_{L^2(Q)}^{1/3}\|v_t\|_{L^2(Q)}^{1/3}. \tag{26}$$

Then using Hölder's inequality, we obtain

$$\int_Q |v(x_1,x_2,t)|^{2(s-1)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t \leqslant \|v\|_{L^{(3/2)s}(Q)}^{2(s-1)}|Q|^{(4-s)/3s}. \tag{27}$$

We conclude from (26) and (27) that

$$\begin{aligned} \|v\|_{L^{(3/2)s}(Q)} &\leqslant (\tfrac{1}{2})^{2/3} s|Q|^{(4-s)/6s}\|v_{x_1}\|_{L^2(Q)}^{1/3}\|v_{x_2}\|_{L^2(Q)}^{1/3}\|v_t\|_{L^2(Q)}^{1/3} \\ &\leqslant (\tfrac{1}{2})^{2/3} s|Q|^{(4-s)/6s}\frac{1}{3}(\|v_{x_1}\|_{L^2(Q)} + \|v_{x_2}\|_{L^2(Q)} + \|v_t\|_{L^2(Q)}) \\ &\leqslant (\tfrac{1}{2})^{2/3} s|Q|^{(4-s)/6s}\frac{\sqrt{3}}{3}\|v\|_H. \end{aligned}$$

Setting $s = \frac{2}{3}p$, we obtain the desired conclusion. □

Finally, we describe the algorithm for finding a real number α that satisfies the verification condition in Theorem 1,

$$\beta + \gamma \leqslant \alpha. \tag{28}$$

What we present here is the most basic such as algorithm [6]. Since γ depends on α, we write $\gamma = \gamma(\alpha)$. The algorithm is as follows:

1. Compute a constant β satisfying (7).
2. Set $\alpha = \beta$.
3. Compute $\gamma(\alpha)$ satisfying (8).
4. Check condition (28), $\beta + \gamma(\alpha) \leqslant \alpha$. If this condition is satisfied, then stop. This means that verification is completed.
5. Otherwise, make the replacement

   $$\alpha \leftarrow (1+\delta)\alpha$$

   for a certain positive number δ and return to 3.

If the iteration number exceeds some maximal value that we decide in advance without satisfying (28), the verification fails.

5. Verification procedures and numerical examples

In this section, we describe the computation of the approximate solution u_h and defect $\|d[u_h]\|_{L^2(Q)}$ in (14).

Let S_h be a finite-dimensional subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ depending on h and let N be the dimension of S_h. Then we can represent u_h by

$$u_h(x,t) = \sum_{i=1}^{N} u_i(t)\hat{\phi}_i(x),$$

where $\hat{\phi}_i$ are base functions in S_h. The function $u_i(t)$ constitutes the time-dependent coefficient of the base function $\hat{\phi}_i(x)$.

Now u_h is computed by the following Newton-iteration:

$$(u_{htt}^{(n)}, \hat{\phi}_j) + (\nabla u_h^{(n)}, \nabla \hat{\phi}_j) + (f'(u_h^{(n-1)})u_h^{(n)}, \hat{\phi}_j) = (f'(u_h^{(n-1)})u_h^{(n-1)} - f(u_h^{(n-1)}), \hat{\phi}_j), \tag{29}$$

where n is the iteration number.

For the discretization of time, we take equal time steps of length Δt and define

$$t_k = k\Delta t, \quad k = 0, 1, 2, \ldots \ .$$

We used the Newmark method [10], which generates the following scheme:

$$u_i^{(n)}(t+\Delta t) \approx u_i^{(n)}(t) + \Delta t \dot{u}_i^{(n)}(t) + \Delta t^2[\beta \ddot{u}_i^{(n)}(t+\Delta t) + (\tfrac{1}{2} - \beta)\ddot{u}_i^{(n)}(t)] \tag{30}$$

$$\dot{u}_i^{(n)}(t+\Delta t) \approx \dot{u}_i^{(n)}(t) + \Delta t[\theta \ddot{u}_i^{(n)}(t+\Delta t) + (1-\theta)\ddot{u}_i^{(n)}(t)], \tag{31}$$

where $\dot{u}_i = \mathrm{d}u_i/\mathrm{d}t$ and $\ddot{u}_i = \mathrm{d}^2u_i/\mathrm{d}t^2$, and θ and β are some nonnegative parameters.

We compute an approximate solution by combining the Newton iteration and the Newmark method. From (30),

$$\ddot{u}_i^{(n)}(t+\Delta t) \approx \frac{1}{\beta\,\Delta t^2}[u_i^{(n)}(t+\Delta t) - u_i^{(n)}(t)] - \frac{1}{\beta\,\Delta t}\dot{u}_i^{(n)}(t) - \left(\frac{1}{2\beta} - 1\right)\ddot{u}_i^{(n)}(t) \tag{32}$$

We approximate iteration (29) by carrying out the following procedure (i)–(iii), and by getting an approximate solution in each time step on condition that $u_i^{(n)}(t)$, $\dot{u}_i^{(n)}(t)$, $\ddot{u}_i^{(n)}(t)$ are already known.

(i) Substitute (32) to the next equation

$$(u_{htt}^{(n)}(t+\Delta t), \hat{\phi}_j) + (\nabla u_h^{(n)}(t+\Delta t), \nabla\hat{\phi}_j) + (f'(u_h^{(n-1)}(t+\Delta t))u_h^{(n)}(t+\Delta t), \hat{\phi}_j)$$
$$=(f'(u_h^{(n-1)}(t+\Delta t))u_h^{(n-1)}(t+\Delta t) - f(u_h^{(n-1)}(t+\Delta t)), \hat{\phi}_j) \tag{33}$$

and compute $u_i^{(n)}(t+\Delta t)$.

(ii) Substitute $u_i^{(n)}(t+\Delta t)$ in (i) to (32), and compute $\ddot{u}_i^{(n)}(t+\Delta t)$.

(iii) Substitute $\ddot{u}_i^{(n)}(t+\Delta t)$ in (ii) to (31),and compute $\dot{u}_i^{(n)}(t+\Delta t)$.

For initial value of this scheme, we find $u_i(0) = \dot{u}_i(0) = 0$ from initial condition in (2) for each i, and $\ddot{u}_i(0)$ is computed by solving (33) in $t+\Delta t = 0$ with respect to $\ddot{u}_i(0)$, i.e. by solving the linear system $\sum_{i=1}^{N}(\hat{\phi}_j, \hat{\phi}_i)\ddot{u}_i(0) = -(f(0), \hat{\phi}_j)$ $(j = 1, 2, \ldots, N)$. In particular, $u_i(0), \dot{u}_i(0), \ddot{u}_i(0)$ are independent of n.

Since $u_{htt} - \Delta u_h + f(\cdot, u_h) \in L^2(Q)$ is required, we use the piecewise cubic Hermite function in one-dimensional case and the piecewise bi-cubic Hermite function in two-dimensional case as the base function in space, and also use the piecewise cubic Hermite interpolation in time. Moreover, since f has a polynomial restriction with respect to u and piecewise polynomials are used in space and time, we can compute $\|d[u_h]\|_{L^2(Q)}$ directly, elementwise in each time step when all coefficients appearing in f are simple (typically constants).

One-dimensional case:

In (2), we set

$$f(x,t,u) = -Au^2 - k\sin\pi x(2 + \pi^2 t^2 - Akt^4 \sin\pi x),$$

where A and k are constants, and we let $\Omega = (0,1)$ and $T = 1$. The exact solution is $u(x,t) = kt^2 \sin\pi x$. If we take $p = 4$, then (A1) and (A2) are satisfied. Then, since we have

$$\|\hat{f}'(u_h + v_1)v_2 - \hat{f}'(u_h)v_2\|_{L^2(Q)}^2 = 4A^2 \int\int |v_1 v_2|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant 4A^2 \|v_1\|_{L^4(Q)}^2 \|v_2\|_{L^4(Q)}^2 \leqslant 4A^2\alpha^4,$$

(15) is satisfied for

$$G_\alpha = 2|A|\alpha^2.$$

We illustrate our algorithm with the numerical results of several examples, where NS and NT are the partition numbers of space and time, respectively, and $M = NS \times NT$. Since the cubic Hermite interpolation procedure is fourth-order accurate for all sufficiently smooth functions and the Newmark method is second-order accurate, we choose NS and NT to satisfy the relation $NT = NS \times NS$ when adjusting the accuracy. Generally speaking, it is difficult to describe the stability of the Newmark method for nonlinear problems, but according to [10], if $\theta = \frac{1}{2}$ and $\beta = \frac{1}{4}$, the Newmark method is

Table 1
Results in the one-dimensional-case

	The result in [4]	Our new result
Case 1: $A=1.5, k=1,\ NS=60,\ M=216\,000$		
C_1C_2	7.3890561	3.6945281
$\|d[u_h]\|_{L^2(Q)}$	0.0013745	0.0013745
α	0.0785593	0.0054016
Case 2: $A=0.8,\ k=2,\ NS=100,\ M=1\,000\,000$		
C_1C_2	8.16616992	3.98468291
$\|d[u_h]\|_{L^2(Q)}$	0.00075867	0.00075867
α	0.146602	0.00308368

Table 2
Specification of numerical environment

	Software	CPU time
Case 1	FUJITSU Fortran Compiler FSUNf90cp. V4	937 (s)
Case 2	FUJITSU Fortran Compiler FSUNf90cp. V4	4412 (s)
Case 3	FUJITSU C Compiler FSUNfcc. V4	35376 (s)

unconditionally stable for linear hyperbolic equations. Thus, we choose $\theta=\frac{1}{2}$ and $\beta=\frac{1}{4}$ in these examples (Table 1).

In our numerical examples we used $u(\cdot,(k-1)\Delta t)$ $(k\geqslant 1)$ as the initial value of Newton iteration at $t=k\,\Delta t$. In the following, α represents the verified error bound in Theorem 1.

The computations were carried out on Sun Ultra 60 with UltraSPARC-II 360 MHz. The execution times are shown in Table 2.

We obtain sharper bounds than those in [4] owing to the better constant $C=C_1C_2$ used here. We note that Theorem 1 ensures not only the existence of solutions but also their local uniqueness.

Two-dimensional case:

In (2), we set

$$f(x,t,u)=f(x_1,x_2,t,u)=Au^2-k\sin\pi x_1\sin\pi x_2(2+2t^2\pi^2+Akt^4\sin\pi x_1\sin\pi x_2),$$

and let $\Omega=(0,1)\times(0,1)$ and $T=1$. The exact solution is $u(x_1,x_2,t)=kt^2\sin\pi x_1\sin\pi x_2$. The other conditions are the same as in the one-dimensional case (Table 3).

Remark 2. In these computations, we used the usual floating-point number system with double precision. Therefore, the above results may include some unknown rounding errors. From the author's experiences, however, the order of magnitude of the effect of round-off errors is smaller than 10^{-10}. With this observation, we can assume that the numerical results are sufficiently reliable to at least six digits or so. Of course, we need to use arithmetic system with guaranteed accuracy for a really rigorous verification.

Table 3
Results in two-dimensional case

Case 3: $A = 1.0,\ k = 1.0$	
NS	8
Δt	$\frac{1}{64}$
C_1C_2	1.27137
$\|d[u_h]\|_{L^2(Q)}$	0.01825348
α	0.0247665

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