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Cayley's hyperdeterminant: A combinatorial approach via representation theory

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ABSTRACT

Cayley's hyperdeterminant is a homogeneous polynomial of degree 4 in the 8 entries of a $2 \times 2 \times 2$ array. It is the simplest (nonconstant) polynomial which is invariant under changes of basis in three directions. We use elementary facts about representations of the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ to reduce the problem of finding the invariant polynomials for a $2 \times 2 \times 2$ array to a combinatorial problem on the enumeration of $2 \times 2 \times 2$ arrays with non-negative integer entries. We then apply results from linear algebra to obtain a new proof that Cayley's hyperdeterminant generates all the invariants. In the last section we discuss the application of our methods to general multidimensional arrays.

1. Introduction

In his famous 1845 paper on the theory of linear transformations, which became the foundation of classical invariant theory, Cayley [5] introduced the concept of the hyperdeterminant of a multidimensional array. He explicitly calculated the hyperdeterminant for the simplest case, an array of size $2 \times 2 \times 2$, which can be represented in two dimensions by its two frontal slices:

$$X = \left[\begin{array}{cc|cc} x_{000} & x_{010} & x_{001} & x_{011} \\ x_{100} & x_{110} & x_{101} & x_{111} \end{array} \right]. \tag{1}$$

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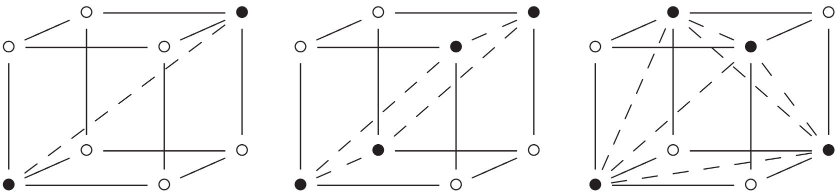

Fig. 1. Geometric configurations in Cayley's hyperdeterminant.

Definition 1. Cayley's hyperdeterminant is the following homogeneous polynomial of degree 4 in the 8 entries x_{ijk} of the $2 \times 2 \times 2$ array of Eq. (1):

$$
\begin{aligned}
C = {} & x_{000}^2 x_{111}^2 + x_{001}^2 x_{110}^2 + x_{010}^2 x_{101}^2 + x_{011}^2 x_{100}^2 \\
& - 2\big(x_{000}x_{001}x_{110}x_{111} + x_{000}x_{010}x_{101}x_{111} + x_{000}x_{011}x_{100}x_{111} \\
& \quad + x_{001}x_{010}x_{101}x_{110} + x_{001}x_{011}x_{100}x_{110} + x_{010}x_{011}x_{100}x_{101}\big) \\
& + 4\left(x_{000}x_{011}x_{101}x_{110} + x_{001}x_{010}x_{100}x_{111}\right).
\end{aligned}
$$

This polynomial has an interesting combinatorial-geometric interpretation. The first four terms have coefficient 1, and the subscripts correspond to the vertices of diagonals of the cube (configurations of dimension 1). The next six terms have coefficient −2, and the subscripts correspond to rectangles in the cube (configurations of dimension 2). The last two terms have coefficient 4, and the subscripts correspond to tetrahedra in the cube (configurations of dimension 3). These three configurations are illustrated by the dashed lines in Fig. 1.

Cayley's hyperdeterminant C is the simplest (nonconstant) polynomial in the entries of the $2 \times 2 \times 2$ array X of Eq. (1) which is invariant under unimodular changes of basis along the three directions. To make this idea more precise, we regard X as an element of the tensor cube $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of the 2-dimensional complex vector space $\mathbb{C}^2$. The group $SL_2(\mathbb{C})$ of 2×2 matrices of determinant 1 acts on $\mathbb{C}^2$ by matrix–vector multiplication, and this gives a component-wise action of the direct product $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. This action extends to the algebra of polynomials in the entries of X, and C is the simplest polynomial which is fixed by every element of the direct product.

Ordinary determinants of square matrices can be characterized by a similar invariance property. Matrices $U \in SL_m(\mathbb{C})$ act on rectangular $m \times n$ matrices A by left multiplication: $A \mapsto UA$. The First Fundamental Theorem of Classical Invariant Theory states that there exist nonconstant invariant polynomials in the entries of A if and only if $m \leq n$, and every invariant is a polynomial in the determinants of the $m \times m$ submatrices obtained by choosing m columns of A; see Procesi [23, §11.1.2]. If we combine the left action of $U \in SL_m(\mathbb{C})$ with the right action of $V \in SL_n(\mathbb{C})$, so that $A \mapsto UAV$, then invariants exist for $SL_m(\mathbb{C}) \times SL_n(\mathbb{C})$ if and only if $m = n$, and every invariant is a polynomial in $\det(A)$.

We now summarize the results of this paper. In Section 2 we recall some elementary results in the representation theory of the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We explain how the 9-dimensional semisimple Lie algebra

$$\mathfrak{sl}_{2,2,2}(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}),$$

acts on the 8-dimensional vector space

$$M_{2,2,2}(\mathbb{C}) = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2,$$

the tensor cube of the natural representation of $\mathfrak{sl}_2(\mathbb{C})$. We describe, using what are essentially the power and product rules from elementary calculus, the action of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$ on the algebra of polynomials on $M_{2,2,2}(\mathbb{C})$. The invariant polynomials are those which are annihilated by all Lie algebra elements (equivalently, fixed by all Lie group elements). For each degree d, the homogeneous polynomials form a finite-dimensional representation of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$, and a well-known theorem implies that this representation is the direct sum of irreducible representations. We express the invariant polynomials as the elements in the kernel of a linear differential operator which represents the action of the Lie algebra on homogeneous polynomials, and from this we obtain the invariant polynomials as the nullspace of a matrix. The domain of this linear map has a monomial basis in bijection with the set of all $2 \times 2 \times 2$ arrays with non-negative integer entries summing to d and having equal sums over the parallel 2×2 slices in the three directions. This reduces the computation of invariants to elementary combinatorics and linear algebra.

In Section 3 we present explicit calculations for degrees 2 and 4. In degree 2, the matrix has size 6×4 and rank 4, so there are no invariants. In degree 4, the matrix has size 24×12 and rank 11; Cayley's hyperdeterminant C is a basis for the nullspace. Considering the powers of C, it follows that the dimension of the space of invariants is at least 1 in each degree d which is a multiple of 4.

In Section 4 we compute the dimensions of certain weight spaces in the representation of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$ on the homogeneous polynomials of degree d. This is equivalent to the enumeration of $2 \times 2 \times 2$ arrays with non-negative integer entries and constraints on the entry sums over the parallel 2×2 slices in the three directions.

In Section 5 we apply a result on subspaces, reminiscent of the inclusion–exclusion principle, to a commutative diagram of injective linear maps between weight spaces in representations of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$. This provides a different proof that the space of invariant polynomials has dimension at least 1 in each degree d which is a multiple of 4. We then use the representation theory of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$ to prove that the algebra of invariants is a polynomial algebra and is generated by Cayley's hyperdeterminant in degree 4. Hence there are no new invariants in higher degrees.

In Section 6 we consider invariant polynomials in the entries of an array of size $n_1 \times n_2 \times \cdots \times n_k$ under the action of $SL_{n_1}(\mathbb{C}) \times SL_{n_2}(\mathbb{C}) \times \cdots \times SL_{n_k}(\mathbb{C})$. The corresponding combinatorial objects are k-dimensional arrays with non-negative integer entries and equal sums over the parallel slices in the k directions.

In Section 7 we briefly summarize recent applications of Cayley's hyperdeterminant and provide some suggestions for further research.

2. Representations of $\mathfrak{sl}_2(\mathbb{C})$

In this section we recall some elementary results in the representation theory of Lie algebras. Standard references are Jacobson [16], Humphreys [15], de Graaf [6], Erdmann and Wildon [10]. For an introduction to Lie theory, by which is meant the relation between Lie groups and Lie algebras, see Stillwell [25]. For the connection with classical invariant theory, see Procesi [23].

The 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ consists of the 2×2 matrices of trace 0 over $\mathbb{C}$ with the Lie bracket $[A, B] = AB - BA$. This operation satisfies anticommutativity and the Jacobi identity:

$$[A, A] \equiv 0, \quad [[A, B], C] + [[B, C], A] + [[C, A], B] \equiv 0.$$

The standard basis of $\mathfrak{sl}_2(\mathbb{C})$ consists of these three matrices:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

In its natural representation, $\mathfrak{sl}_2(\mathbb{C})$ acts by matrix–vector multiplication on the two-dimensional vector space $\mathbb{C}^2$ with this standard basis:

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$ is given by the following equations:

$$H \cdot x_0 = x_0, \quad H \cdot x_1 = -x_1. \quad E \cdot x_0 = 0, \quad E \cdot x_1 = x_0, \quad F \cdot x_0 = x_1, \quad F \cdot x_1 = 0.$$

In particular, x_0 and x_1 are eigenvectors for H and $H \cdot x_i = (-1)^i x_i$ $(i = 0, 1)$. If we regard x_0 and x_1 as indeterminates, then we can express the action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$ by partial differential operators as follows:

$$H = x_0 \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial x_1}, \quad E = x_0 \frac{\partial}{\partial x_1}, \quad F = x_1 \frac{\partial}{\partial x_0}.$$

We identify a $2 \times 2 \times 2$ array $X = (x_{ijk})$ with an element of the tensor cube $M_{2,2,2}(\mathbb{C}) = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We identify the entries with simple tensors, $x_{ijk} = x_i \otimes x_j \otimes x_k$ $(i, j, k = 0, 1)$. (Strictly speaking, since we regard x_{ijk} as a coordinate function on $M_{2,2,2}(\mathbb{C})$, we should use dual basis vectors and write $x_{ijk} = x_i^* \otimes x_j^* \otimes x_k^*$, but this distinction will not be important for us.) The Lie group $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ acts on the vector space $M_{2,2,2}(\mathbb{C})$; the action is defined on simple tensors and extended linearly:

$$(X, Y, Z) \cdot (u \otimes v \otimes w) = (X \cdot u) \otimes (Y \cdot v) \otimes (Z \cdot w).$$

As usual, we linearize the group action by considering the action of the Lie algebra $\mathfrak{sl}_{2,2,2}(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ on $M_{2,2,2}(\mathbb{C})$ defined by this equation:

$$(A, B, C) \cdot (u \otimes v \otimes w) = (A \cdot u) \otimes v \otimes w + u \otimes (B \cdot v) \otimes w + u \otimes v \otimes (C \cdot w).$$

Lemma 2. *The 8-dimensional vector space $M_{2,2,2}(\mathbb{C})$ is an irreducible representation of the 9-dimensional semisimple Lie algebra $\mathfrak{sl}_{2,2,2}(\mathbb{C})$.*

Proof. A representation of a semisimple Lie algebra is irreducible if and only if it is isomorphic to the tensor product of irreducible representations of its simple summands. See Proposition 1.1 of Neher et al. [22]. $\square$

We write H_ℓ, E_ℓ, F_ℓ $(\ell = 1, 2, 3)$ for the standard basis of the ℓth copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sl}_{2,2,2}(\mathbb{C})$; see Eq. (2). The basis of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$ acts on the basis of $M_{2,2,2}(\mathbb{C})$ as follows:

$$\begin{array}{lll}
H_1 \cdot x_{0jk} = x_{0jk}, & H_2 \cdot x_{i0k} = x_{i0k}, & H_3 \cdot x_{ij0} = x_{ij0}, \\
H_1 \cdot x_{1jk} = -x_{1jk}, & H_2 \cdot x_{i1k} = -x_{i1k}, & H_3 \cdot x_{ij1} = -x_{ij1}, \\
E_1 \cdot x_{0jk} = 0, & E_2 \cdot x_{i0k} = 0, & E_3 \cdot x_{ij0} = 0, \\
E_1 \cdot x_{1jk} = x_{0jk}, & E_2 \cdot x_{i1k} = x_{i0k}, & E_3 \cdot x_{ij1} = x_{ij0}, \\
F_1 \cdot x_{0jk} = x_{1jk}, & F_2 \cdot x_{i0k} = x_{i1k}, & F_3 \cdot x_{ij0} = x_{ij1}, \\
F_1 \cdot x_{1jk} = 0, & F_2 \cdot x_{i1k} = 0, & F_3 \cdot x_{ij1} = 0.
\end{array}$$

These equations can be proved by straightforward calculation. We give the details for $\ell = 1$; the other cases are similar:

$$\begin{array}{l}
H_1 \cdot x_{0jk} = H_1 \cdot (x_0 \otimes x_j \otimes x_k) = (H_1 \cdot x_0) \otimes x_j \otimes x_k = x_{0jk}, \\
H_1 \cdot x_{1jk} = H_1 \cdot (x_1 \otimes x_j \otimes x_k) = (H_1 \cdot x_1) \otimes x_j \otimes x_k = -x_{1jk}, \\
E_1 \cdot x_{0jk} = E_1 \cdot (x_0 \otimes x_j \otimes x_k) = (E_1 \cdot x_0) \otimes x_j \otimes x_k = 0, \\
E_1 \cdot x_{1jk} = E_1 \cdot (x_1 \otimes x_j \otimes x_k) = (E_1 \cdot x_1) \otimes x_j \otimes x_k = x_{0jk}, \\
F_1 \cdot x_{0jk} = F_1 \cdot (x_0 \otimes x_j \otimes x_k) = (F_1 \cdot x_0) \otimes x_j \otimes x_k = x_{1jk}, \\
F_1 \cdot x_{1jk} = F_1 \cdot (x_1 \otimes x_j \otimes x_k) = (F_1 \cdot x_1) \otimes x_j \otimes x_k = 0.
\end{array}$$

We consider the polynomial algebra on $M_{2,2,2}(\mathbb{C})$:

$$P = \mathbb{C}[x_{000}, x_{010}, x_{100}, x_{110}, x_{001}, x_{011}, x_{101}, x_{111}].$$

A basis of P over $\mathbb{C}$ consists of the monomials,

$$\prod_{i,j,k=0,1} x_{ijk}^{e_{ijk}} = x_{000}^{e_{000}} x_{001}^{e_{001}} x_{010}^{e_{010}} x_{011}^{e_{011}} x_{100}^{e_{100}} x_{101}^{e_{101}} x_{110}^{e_{110}} x_{111}^{e_{111}},$$

where the exponents e_{ijk} are arbitrary non-negative integers. The degree of a monomial is the sum of its exponents:

$$d = \sum_{i,j,k=0,1} e_{ijk}.$$

We write P_d for the homogeneous subspace of P spanned by the monomials of degree d. We identify P_1 with $M_{2,2,2}(\mathbb{C})$, so that a basis of P_1 consists of the monomials of degree 1, namely $x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111}$. There is a bijection between the monomials of degree d and the $2 \times 2 \times 2$ arrays $E = (e_{ijk})$ of non-negative integers summing to d. The polynomial algebra P is graded by the degree:

$$P = \bigoplus_{d \geq 0} P_d, \quad P_d P_e \subseteq P_{d+e}.$$

We have $P_d = S^d P_1$, the dth symmetric power of P_1. The action of an element $D \in \mathfrak{sl}_{2,2,2}(\mathbb{C})$ extends to all basis monomials of P by the derivation rule $D \cdot (fg) = (D \cdot f)g + f(D \cdot g)$. It follows by induction that

$$D \cdot x_{ijk}^{e_{ijk}} = e_{ijk} x_{ijk}^{e_{ijk}-1} (D \cdot x_{ijk}),$$

and hence that

$$\begin{aligned} D \cdot \prod_{i,j,k} x_{ijk}^{e_{ijk}} &= \sum_{i',j',k'} x_{000}^{e_{000}} \cdots \left(D \cdot x_{i'j'k'}^{e_{i'j'k'}} \right) \cdots x_{111}^{e_{111}} \\ &= \sum_{i',j',k'} x_{000}^{e_{000}} \cdots \left(e_{i'j'k'} x_{i'j'k'}^{e_{i'j'k'}-1} (D \cdot x_{i'j'k'}) \right) \cdots x_{111}^{e_{111}}. \end{aligned}$$

In particular, $D \cdot P_d \subseteq P_d$ for all $D \in \mathfrak{sl}_{2,2,2}(\mathbb{C})$; hence for every $d \geq 1$, the subspace P_d is a finite-dimensional representation of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$, and is isomorphic to a direct sum of irreducible representations. For every non-negative integer n, there is (up to isomorphism) a unique irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with dimension $n+1$, denoted $V(n)$. This representation is generated by a vector v_n with $H \cdot v_n = n v_n$. With respect to the basis v_{n-2i} ($i = 0, 1, \ldots, n$), the action of $\mathfrak{sl}_2(\mathbb{C})$ on $V(n)$ is given by

$$\begin{aligned} H \cdot v_{n-2i} &= (n-2i) v_{n-2i}, \\ E \cdot v_{n-2i} &= (n-i+1) v_{n-2i+2} \quad (i = 1, 2, \ldots, n), \qquad E \cdot v_n = 0, \\ F \cdot v_{n-2i} &= (i+1) v_{n-2i-2} \quad (i = 0, 1, \ldots, n-1), \qquad F \cdot v_{-n} = 0. \end{aligned}$$

Definition 3. In the irreducible representation $V(n)$ of $\mathfrak{sl}_2(\mathbb{C})$, the basis vector v_{n-2i} is a **weight vector** (that is, H-eigenvector) of weight $n-2i$.

An irreducible representation of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$ is isomorphic to the tensor product $V(a) \otimes V(b) \otimes V(c)$ for some non-negative integers a, b, c. The polynomials invariant under the group $SL_2(\mathbb{C})^3$ coincide with the polynomials annihilated by the Lie algebra $\mathfrak{sl}_{2,2,2}(\mathbb{C})$; hence a polynomial $f \in P_d$ is invariant if and only if $D \cdot f = 0$ for all $D \in \mathfrak{sl}_{2,2,2}(\mathbb{C})$. Equivalently, the invariant polynomials correspond to the summands of P_d isomorphic to $V(0) \otimes V(0) \otimes V(0)$. Therefore, a polynomial $f \in P_d$ is invariant if and only if $H_\ell \cdot f = 0$ and $E_\ell \cdot f = 0$ for $\ell = 1, 2, 3$.

Lemma 4. *The basis monomial*

$$\prod_{i,j,k} x_{ijk}^{e_{ijk}},$$

is a simultaneous eigenvector for H_1, H_2, H_3 with eigenvalues

$$\sum_{j,k} e_{0jk} - \sum_{j,k} e_{1jk}, \quad \sum_{i,k} e_{i0k} - \sum_{i,k} e_{i1k}, \quad \sum_{i,j} e_{ij0} - \sum_{i,j} e_{ij1}.$$

Proof. For $\ell = 1$ we have

$$H_1 \cdot x_{ijk}^{e_{ijk}} = e_{ijk} x_{ijk}^{e_{ijk}-1} (H_1 \cdot x_{ijk}) = e_{ijk} x_{ijk}^{e_{ijk}-1} (-1)^i x_{ijk} = (-1)^i e_{ijk} x_{ijk}^{e_{ijk}},$$

and therefore

$$H_1 \cdot \prod_{i,j,k} x_{ijk}^{e_{ijk}} = \left(\sum_{j,k} e_{0jk} - \sum_{j,k} e_{1jk} \right) \prod_{i,j,k} x_{ijk}^{e_{ijk}}.$$

The other cases are similar. $\square$

Definition 5. The **weight space** $W(d; a, b, c)$ is the subspace of P_d spanned by the monomials which have eigenvalues a, b, c for H_1, H_2, H_3 respectively. The **zero weight space** is $W(d; 0, 0, 0)$.

The basis monomial

$$\prod_{i,j,k} x_{ijk}^{e_{ijk}},$$

belongs to $W(d; 0, 0, 0)$ if and only if

$$\sum_{i,j,k} e_{ijk} = d, \quad \sum_{j,k} e_{0jk} = \sum_{j,k} e_{1jk}, \quad \sum_{i,k} e_{i0k} = \sum_{i,k} e_{i1k}, \quad \sum_{i,j} e_{ij0} = \sum_{i,j} e_{ij1}.$$

That is, the $2 \times 2 \times 2$ array (e_{ijk}) of exponents satisfies the condition that in each of the three directions, the parallel 2×2 slices have equal sums. If d is odd then the zero weight space $W(d; 0, 0, 0)$ is the zero subspace. In particular, there are no invariant polynomials in odd degrees. The actions of E_1, E_2, E_3 induce these linear maps on weight spaces:

$$E_1 \colon W(d; 0, 0, 0) \to W(d; 2, 0, 0),$$
$$E_2 \colon W(d; 0, 0, 0) \to W(d; 0, 2, 0),$$
$$E_3 \colon W(d; 0, 0, 0) \to W(d; 0, 0, 2).$$

We define a linear map

$$\mathcal{E}_d \colon W(d; 0, 0, 0) \longrightarrow W(d; 2, 0, 0) \oplus W(d; 0, 2, 0) \oplus W(d; 0, 0, 2),$$

by the equation $\mathcal{E}_d(f) = (\, E_1 \cdot f, \; E_2 \cdot f, \; E_3 \cdot f \,)$ for all $f \in W(d; 0, 0, 0)$. The invariant polynomials in P_d coincide with the kernel of $\mathcal{E}_d$. We can represent the linear map $\mathcal{E}_d$ by the matrix $[\mathcal{E}_d]$ with respect to the ordered monomial bases of the weight spaces. The size $[\mathcal{E}_d]$ is

$$\big(\dim W(d; 2, 0, 0) + \dim W(d; 0, 2, 0) + \dim W(d; 0, 0, 2) \big) \times \dim W(d; 0, 0, 0).$$

In fact the three dimensions in parentheses are equal; this follows by considering the automorphisms of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$ which permute the three summands.

3. Cayley's hyperdeterminant via linear algebra

In this section we show by direct calculation that every (nonconstant) invariant polynomial of degree ≤ 4 is a scalar multiple of Cayley's hyperdeterminant.

We identify monomials with sequences of exponents lexicographically ordered by their triples of subscripts:

$$\prod_{i,j,k} x_{ijk}^{e_{ijk}} \longleftrightarrow [e_{000}, e_{001}, e_{010}, e_{011}, e_{100}, e_{101}, e_{110}, e_{111}].$$

Within each weight space, we order the basis monomials lexicographically.

Lemma 6. *There are no invariant polynomials in degree 2.*

Proof. A basis of the zero weight space $W(2; 0, 0, 0)$ consists of four monomials,

$$00011000, \quad 00100100, \quad 01000010, \quad 10000001$$

which label the columns of the matrix $[\mathcal{E}_2]$. Each nonzero weight space $W(2; 0, 0, 2)$, $W(2; 0, 2, 0)$, $W(2; 2, 0, 0)$ has a basis of two monomials which label the rows of $[\mathcal{E}_2]$:

$$\begin{array}{c} 01100000 \\ 10010000 \\ \hline 01001000 \\ 10000100 \\ \hline 00101000 \\ 10000010 \end{array} \qquad [\mathcal{E}_2] = \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The matrix has full rank, and so its nullspace is $\{0\}$. $\square$

Theorem 7. *In degree 4, the space of invariant polynomials has dimension 1; every invariant is a scalar multiple of Cayley's hyperdeterminant C.*

Proof. A basis of the zero weight space $W(4; 0, 0, 0)$ consists of these 12 monomials:

$$\begin{array}{cccccc} 00022000, & 00111100, & 00200200, & 01011010, & 01100110, & 01101001, \\ 02000020, & 10010110, & 10011001, & 10100101, & 11000011, & 20000002 \end{array}$$

Each nonzero weight space $W(4; 0, 0, 2)$, $W(4; 0, 2, 0)$, $W(4; 2, 0, 0)$ has a basis of 8 monomials; see Fig. 2, which also displays the 24×12 matrix $[\mathcal{E}_4]$ (we use dot for zero). Fig. 3 gives the row canonical form of $[\mathcal{E}_4]$ (we omit zero rows). The rank is 11, and Cayley's hyperdeterminant is a basis of the nullspace. $\square$

Corollary 8. *The dimension of the space of invariant polynomials is at least 1 in each degree d congruent to 0 modulo 4.*

Proof. The existence of Cayley's hyperdeterminant C in degree 4 implies that there is at least one invariant polynomial C^e in each degree $d = 4e$. $\square$

4. Dimension formulas for weight spaces

Our next goal is to prove that there are no new invariants in higher degrees; in other words, that every invariant is a polynomial in C. To do this, we need to prove that the lower bound of Corollary 8 is also an upper bound. The first step is to obtain dimension formulas for certain weight spaces in the representation of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$ on the space P_d of homogeneous polynomials of degree d.

$$
\begin{array}{c}
01111000 \\ 01200100 \\ 02100010 \\ 10021000 \\ 10110100 \\ 11010010 \\ 11100001 \\ 20010001 \\ \hline
01012000 \\ 01101100 \\ 02001010 \\ 10011100 \\ 10100200 \\ 11000110 \\ 11001001 \\ 20000101 \\ \hline
00112000 \\ 00201100 \\ 01101010 \\ 10011010 \\ 10100110 \\ 10101001 \\ 11000020 \\ 20000011
\end{array}
\qquad
\left[\begin{array}{cccccccccccc}
. & 1 & . & 1 & . & 1 & . & . & . & . & . & . \\
. & . & 2 & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & 2 & . & . & . & . & . \\
2 & . & . & . & . & . & . & . & 1 & . & . & . \\
. & 1 & . & . & . & . & . & 1 & . & 1 & . & . \\
. & . & . & 1 & . & . & . & 1 & . & . & 1 & . \\
. & . & . & . & . & 1 & . & . & . & 1 & 1 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & 2 \\ \hline
2 & . & . & 1 & . & . & . & . & . & . & . & . \\
. & 1 & . & . & 1 & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & 2 & . & . & . & . & . \\
. & 1 & . & . & . & . & . & 1 & 1 & . & . & . \\
. & . & 2 & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & 1 & . & . & 1 & . & . & 1 & . \\
. & . & . & . & . & 1 & . & . & 1 & . & 1 & . \\
. & . & . & . & . & . & . & . & . & 1 & . & 2 \\ \hline
2 & 1 & . & . & . & . & . & . & . & . & . & . \\
. & 1 & 2 & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & 1 & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & 1 & 1 & . & . & . \\
. & . & . & . & 1 & . & . & 1 & . & 1 & . & . \\
. & . & . & . & . & 1 & . & . & 1 & 1 & . & . \\
. & . & . & . & . & . & 2 & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 2
\end{array}\right]
$$

Fig. 2. The matrix $[\mathcal{E}_4]$.

$$
\left[\begin{array}{cccccccccccc}
1 & . & . & . & . & . & . & . & . & . & . & -1 \\
. & 1 & . & . & . & . & . & . & . & . & . & 2 \\
. & . & 1 & . & . & . & . & . & . & . & . & -1 \\
. & . & . & 1 & . & . & . & . & . & . & . & 2 \\
. & . & . & . & 1 & . & . & . & . & . & . & 2 \\
. & . & . & . & . & 1 & . & . & . & . & . & -4 \\
. & . & . & . & . & . & 1 & . & . & . & . & -1 \\
. & . & . & . & . & . & . & 1 & . & . & . & -4 \\
. & . & . & . & . & . & . & . & 1 & . & . & 2 \\
. & . & . & . & . & . & . & . & . & 1 & . & 2 \\
. & . & . & . & . & . & . & . & . & . & 1 & 2
\end{array}\right]
$$

Fig. 3. The row canonical form of $[\mathcal{E}_4]$.

Theorem 9. *The dimension of the zero weight subspace* $W(d; 0, 0, 0)$ *equals*

$$\frac{1}{384}(d+4)^2(d^2+8d+24) \qquad \text{if } d \equiv 0 \ (\text{mod } 4), \tag{000-0}$$

$$\frac{1}{384}(d+2)(d+6)(d^2+8d+28) \qquad \text{if } d \equiv 2 \ (\text{mod } 4). \tag{000-2}$$

The dimensions of $W(d; 2, 0, 0)$, $W(d; 0, 2, 0)$ *and* $W(d; 0, 0, 2)$ *equal*

$$\frac{1}{384}d(d+4)^2(d+8) \qquad \text{if } d \equiv 0 \ (\text{mod } 4), \tag{200-0}$$
$$\frac{1}{384}(d+2)(d+6)(d^2+8d+4) \qquad \text{if } d \equiv 2 \ (\text{mod } 4). \tag{200-2}$$

The dimensions of $W(d; 2, 2, 0)$, $W(d; 2, 0, 2)$ *and* $W(d; 0, 2, 2)$ *equal*

$$\frac{1}{384}d(d+4)(d^2+12d+8) \qquad \text{if } d \equiv 0 \ (\text{mod } 4), \tag{220-0}$$
$$\frac{1}{384}(d+2)(d^3+14d^2+28d-24) \qquad \text{if } d \equiv 2 \ (\text{mod } 4). \tag{220-2}$$

The dimension of $W(d; 2, 2, 2)$ *equals*

$$\frac{1}{384}d(d^3+16d^2+32d+32) \qquad \text{if } d \equiv 0 \ (\text{mod } 4), \tag{222-0}$$
$$\frac{1}{384}(d+2)(d^3+14d^2+4d+24) \qquad \text{if } d \equiv 2 \ (\text{mod } 4). \tag{222-2}$$

In all cases, the dimension is 0 if d is odd.

Given non-negative integers d (the degree) and a, b, c (the weights), we consider $2 \times 2 \times 2$ arrays $E = (e_{ijk})$ with $i, j, k \in \{0, 1\}$ of non-negative integer exponents satisfying the following equations:

$$e_{000} + e_{001} + e_{010} + e_{011} + e_{100} + e_{101} + e_{110} + e_{111} = d, \tag{D}$$
$$(e_{000} + e_{001} + e_{010} + e_{011}) - (e_{100} + e_{101} + e_{110} + e_{111}) = a, \tag{M1}$$
$$(e_{000} + e_{001} + e_{100} + e_{101}) - (e_{010} + e_{011} + e_{110} + e_{111}) = b, \tag{M2}$$
$$(e_{000} + e_{010} + e_{100} + e_{110}) - (e_{001} + e_{011} + e_{101} + e_{111}) = c. \tag{M3}$$

These equations hold if and only if the corresponding monomial belongs to the weight space $W(d; a, b, c)$; that is, the number of arrays E satisfying Eqs. (D)–(M3) equals the dimension of $W(d; a, b, c)$. Theorem 9 gives formulas for these dimensions for certain values of a, b, c. These formulas are polynomials of degree 4, as expected since we have eight exponents and four constraints.

Lemma 10. *Consider* 2×2 *matrices* (e_{ij}) *with non-negative integer entries and specified row sums* r_0, r_1 *and column sums* c_0, c_1*; clearly* $r_0 + r_1 = c_0 + c_1$*:*

$$\begin{bmatrix} e_{00} & e_{01} \\ e_{10} & e_{11} \end{bmatrix}, \quad \begin{matrix} e_{00} + e_{01} = r_0, & e_{00} + e_{10} = c_0, \\ e_{10} + e_{11} = r_1, & e_{01} + e_{11} = c_1. \end{matrix} \tag{3}$$

The number of such matrices equals $\min(r_0, r_1, c_0, c_1) + 1$.

Proof. We have four variables and four constraints, but one dependence relation among the constraints, so we expect a 1-dimensional solution set. Without loss of generality, we can interchange the rows (resp. columns) and assume that $r_0 \leq r_1$ (resp. $c_0 \leq c_1$); we can also transpose the matrix and assume that $r_0 \leq c_0$. It is clear that since $c_1 - r_0 \geq c_1 - c_0 \geq 0$ we have the particular solution

$$\begin{bmatrix} e_{00} & e_{01} \\ e_{10} & e_{11} \end{bmatrix} = \begin{bmatrix} 0 & r_0 \\ c_0 & c_1 - r_0 \end{bmatrix}.$$

If u is any integer then we can preserve the constraints by adding u to the diagonal entries and subtracting u from the off-diagonal entries:

$$\begin{bmatrix} e_{00} & e_{01} \\ e_{10} & e_{11} \end{bmatrix} = \begin{bmatrix} u & r_0 - u \\ c_0 - u & c_1 - r_0 + u \end{bmatrix}.$$

This is another solution if and only if $0 \leq u \leq r_0$. Hence the number of solutions is $r_0 + 1 = \min(r_0, r_1, c_0, c_1) + 1$. $\square$

Lemma 11. *For any integer $k \geq 1$ we have*

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\big(\min(i,j)\big)^2 = \frac{1}{6}k(k+1)(k^2+k+1).$$

Proof. By induction on k; the result is clear for $k = 1$. We have

$$\begin{aligned}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\big(\min(i,j)\big)^2 &= \sum_{i=1}^{k}\sum_{j=1}^{k}\big(\min(i,j)\big)^2 + \sum_{i=1}^{k+1}\big(\min(i,k{+}1)\big)^2 + \sum_{j=1}^{k}\big(\min(k{+}1,j)\big)^2 \\ &= \frac{1}{6}k(k{+}1)(k^2{+}k{+}1) + \frac{1}{6}(k{+}1)(k{+}2)(2k{+}3) + \frac{1}{6}k(k{+}1)(2k{+}1) \\ &= \frac{1}{6}(k+1)(k+2)(k^2+3k+3),\end{aligned}$$

using the well-known formula for the sum of squares. $\square$

We now come to the proof of Theorem 9. We prove the first two Eqs. (000-0) and (000-2); the proofs of the others are similar but slightly more complicated, and the details are not particularly enlightening.

Proof. Eqs. (M1)–(M3) imply that d is even, since if $a = b = c = 0$ then each of the sums in parentheses equals $d/2$. Hence we assume that $d = 2m$.

In Eqs. (D)–(M3) we write w, x, y, z for the row and column sums of the 2×2 slice (e_{0jk}) with $i = 0$. We then have $w + x = m$ and $y + z = m$, and

$$\begin{aligned} e_{000} + e_{001} &= w, & e_{100} + e_{101} &= m - w, \\ e_{010} + e_{011} &= x, & e_{110} + e_{111} &= m - x, \\ e_{000} + e_{010} &= y, & e_{100} + e_{110} &= m - y, \\ e_{001} + e_{011} &= z, & e_{101} + e_{111} &= m - z. \end{aligned}$$

Suppose that $w \leq x$ and $y \leq z$. Lemma 10 shows that

- the number of 2×2 slices (e_{0jk}) is $\min(w, y) + 1$, and
- the number of 2×2 slices (e_{1jk}) is $\min(m{-}x, m{-}z) + 1 = \min(w, y) + 1$.

Hence the number of $2 \times 2 \times 2$ arrays is $(\min(w, y){+}1)^2$. Any solution with $w < x$ has a corresponding solution with $w > x$ obtained by interchanging the slices (e_{i0k}) and (e_{i1k}). Any solution with $y < z$ has a corresponding solution with $y > z$ obtained by interchanging (e_{ij0}) and (e_{ij1}).

We first prove Eq. (000-2): the case $d \equiv 2 \pmod 4$. We have $m = 2k-1$ where $k = (d+2)/4$. Since m is odd, we cannot have either $w = x$ or $y = z$; hence all solutions are doubly paired. Thus the number of solutions is four times the number with $w < x$ and $y < z$, and for this we apply Lemma 11:

$$\begin{aligned} 4\sum_{w=0}^{k-1}\sum_{y=0}^{k-1}\big(\min(w,y)+1\big)^2 &= 4\sum_{w=0}^{k-1}\sum_{y=0}^{k-1}\min(w{+}1,y{+}1)^2 \\ &= \frac{2}{3}k(k+1)(k^2+k+1) = \frac{1}{384}(d+2)(d+6)(d^2+8d+28). \end{aligned}$$

We next prove Eq. (000-0): the case $d \equiv 0 \pmod 4$. We have $m = 2k$ where $k = d/4$. In this case we must also consider $w = x$ and $y = z$, so we add

$$\begin{aligned} &2\sum_{w=0}^{k-1}\min(w{+}1,k{+}1)^2 + 2\sum_{y=0}^{k-1}\min(k{+}1,y{+}1)^2 + \min(k{+}1,k{+}1)^2 \\ &= \frac{2}{3}k(k+1)(2k+1) + (k+1)^2 = \frac{1}{3}(k+1)(4k^2+5k+3), \end{aligned}$$

to the previous result, obtaining

$$\frac{2}{3}k(k+1)(k^2+k+1) + \frac{1}{3}(k+1)(4k^2+5k+3) = \frac{1}{384}(d+4)^2(d^2+8d+24).$$

This completes the proof. $\square$

5. Inclusion–exclusion for subspaces

We recall a familiar formula from elementary linear algebra. If U_1 and U_2 are finite-dimensional subspaces of a vector space then

$$\dim(U_1+U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2). \tag{4}$$

The next result generalizes Eq. (4) to an arbitrary finite number of subspaces, and is similar to the combinatorial formula for inclusion–exclusion on finite sets.

Lemma 12. *If $U_1, \ldots, U_n$ are finite-dimensional subspaces of a vector space then*

$$\dim\left(\sum_{i=1}^{n} U_i\right) \le \sum_{r=1}^{n}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le n}\dim(U_{i_1}\cap\cdots\cap U_{i_r}),$$

where the inner sum on the right is over all $\binom{n}{r}$ subsets $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$.

Proof. The statement is false if the inequality is replaced by an equality: consider three distinct lines through the origin in the plane. The statement is clear for $n \le 2$, and can easily be proved by induction on n. $\square$

We now consider a reformulation of this problem, in which we have a positive integer n and a collection of 2^n finite-dimensional vector spaces,

$$\{\, V_{i_1,i_2,\ldots,i_n} \mid 0 \le i_1, i_2, \ldots, i_n \le 1 \,\},$$

corresponding to the vertices of an n-dimensional cube. We also have $n2^{n-1}$ injective linear maps corresponding to the edges of the cube,

$$f^{(k)}_{i_1,\ldots,\widehat{i_k},\ldots,i_n} : V_{i_1,\ldots,1,\ldots,i_n} \longrightarrow V_{i_1,\ldots,0,\ldots,i_n},$$

where the hat indicates omission and the values of the indices are

$$1 \leq k \leq n, \quad (i_1, \ldots, \widehat{i_k}, \ldots, i_n) \in \{0, 1\}^{n-1}.$$

Given any two of these vector spaces, we assume that all compositions of linear maps between the spaces give the same result; that is, the diagram is commutative. We can therefore identify each space $V_{i_1,i_2,\ldots,i_n}$ with its image in $V_{0,0,\ldots,0}$, and so all of the spaces $V_{i_1,i_2,\ldots,i_n}$ can be identified with subspaces of $V_{0,0,\ldots,0}$.

We define n vector spaces $U_1, \ldots, U_n$ by starting at the vertex $(0, \ldots, 0)$ of the n-dimensional cube and following the n edges to the vertices

$$U_i = V_{0,\ldots,1,\ldots,0} \quad (1 \leq i \leq n),$$

in which the subscripts on the right are 0 except for 1 in position i. Given any r-element subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$, we write $\chi(i_1, \ldots, i_r)$ for the element of $\{0, 1\}^n$ which has 1 in positions $i_1, \ldots, i_r$ and 0 elsewhere. Our assumptions allow us to make the following identifications:

$$U_{i_1} \cap \cdots \cap U_{i_r} = V_{\chi(i_1,\ldots,i_r)}.$$

Lemma 12 then implies that

$$\dim\left(\operatorname{im}(f^{(1)}_{0,\ldots,0}) + \cdots + \operatorname{im}(f^{(n)}_{0,\ldots,0})\right) \leq \sum_{r=1}^{n}(-1)^{r+1} \sum_{1\leq i_1<\cdots<i_r\leq n} \dim\left(V_{\chi(i_1,\ldots,i_r)}\right). \tag{5}$$

Theorem 13. *Every polynomial in the entries x_{ijk} of the $2 \times 2 \times 2$ array $X = (x_{ijk})$ $(i, j, k = 0, 1)$, which is invariant under changes of basis with determinant 1 along all the three directions, is a polynomial in Cayley's hyperdeterminant.*

Proof. We consider $n = 3$ and identify the 8 vertices of the cube with the following weight spaces in degree d defined in Section 4:

$$\begin{array}{llll} W(d; 0, 0, 0), & W(d; 2, 0, 0), & W(d; 0, 2, 0), & W(d; 0, 0, 2), \\ W(d; 2, 2, 0), & W(d; 2, 0, 2), & W(d; 0, 2, 2), & W(d; 2, 2, 2). \end{array}$$

The representation theory of $\mathfrak{sl}_2(\mathbb{C})$ shows that the action of the basis elements F_1, F_2, F_3 on the homogeneous polynomials of degree d gives injective linear maps between these weight spaces as illustrated in Fig. 4. The invariant polynomials are the nonzero elements in the irreducible summands $V(0) \otimes V(0) \otimes V(0)$, and the number of these summands equals the codimension, in the zero weight space $W(d; 0, 0, 0)$, of the sum of the images of the weight spaces $W(d; 2, 0, 0)$, $W(d; 0, 2, 0)$, $W(d; 0, 0, 2)$ under the actions of F_1, F_2, F_3 respectively. That is,

- We start with the entire zero weight zero space $W(d; 0, 0, 0)$.
- We factor out the images of vectors of weight (2,0,0) or (0,2,0) or (0,0,2) by the action of F_1 or F_2 or F_3.
- The vectors that come from weight (2,2,0) or (2,0,2) or (0,2,2) by the action of F_1, F_2 or F_1, F_3 or F_2, F_3 have then been factored out twice, so we must add those dimensions back in.
- But then the vectors that come from weight (2,2,2) by the action of F_1, F_2, F_3 must be factored out again.

$$\begin{tikzcd}
& W(d;0,2,2) \arrow[dl, "F_3"] \arrow[dd, "F_2"] & & W(d,2,2,2) \arrow[ll, "F_1"'] \arrow[dl, "F_3"] \arrow[dd, "F_2"] \\
W(d;0,2,0) \arrow[dd, "F_2"] & & W(d;2,2,0) \arrow[ll, "F_1"'] \arrow[dd, "F_2"] & \\
& W(d;0,0,2) \arrow[dl, "F_3"] & & W(d;2,0,2) \arrow[ll, "F_1"'] \arrow[dl, "F_3"] \\
W(d;0,0,0) & & W(d;2,0,0) \arrow[ll, "F_1"'] &
\end{tikzcd}$$

Fig. 4. Linear maps among weight spaces in degree d.

The dimension formulas from Section 4 with Eq. (5) give

$$\begin{aligned}
&\dim W(d;\,0,0,0) - \dim W(d;\,2,0,0) - \dim W(d;\,0,2,0) - \dim W(d;\,0,0,2) \\
&\quad + \dim W(d;\,2,2,0) + \dim W(d;\,2,0,2) + \dim W(d;\,0,2,2) - \dim W(d;\,2,2,2) \\
&= \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Combining this with Lemma 12, this gives another proof of Corollary 8: the dimension of the space of invariants is at least 1 in degrees $d \equiv 0 \pmod 4$.

It remains to use the representation theory of Lie algebras to show that inequality (5) becomes in fact an equality in the present situation. We know that the space P_d of homogeneous polynomials of degree d is completely reducible as a representation of the semisimple Lie algebra $\mathfrak{sl}_{2,2,2}(\mathbb{C})$, and that the irreducible summands are tensor products $V(a) \otimes V(b) \otimes V(c)$ of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. Since the weight spaces in the tensor factors have dimension 1 as representations of $\mathfrak{sl}_2(\mathbb{C})$, it follows that the weight spaces in the tensor product have dimension 1 as representations of $\mathfrak{sl}_{2,2,2}(\mathbb{C})$. Inequality (5) is obviously an equality when all the dimensions are 1, and this completes the proof. □

6. General multidimensional arrays

We consider a k-dimensional array of size $n_1 \times n_2 \times \cdots \times n_k$:

$$X = (x_{i_1 i_2 \cdots i_k}) \quad (1 \le i_1 \le n_1,\ 1 \le i_2 \le n_2,\ \ldots,\ 1 \le i_k \le n_k).$$

(The smallest index is now 1, not 0.) We consider an extension of determinants to these arrays, using a combinatorial approach based on the representation theory of the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. As usual we write $\mathbb{C}^{n_1}, \mathbb{C}^{n_2}, \ldots, \mathbb{C}^{n_k}$ for the complex vector spaces with dimensions $n_1, n_2, \ldots, n_k$ and standard bases

$$e^{(1)}_{i_1}\ (i_1 = 1, \ldots, n_1), \quad e^{(2)}_{i_2}\ (i_2 = 1, \ldots, n_2), \quad \ldots, \quad e^{(k)}_{i_k}\ (i_k = 1, \ldots, n_k).$$

A tensor of order k is an element of the tensor product

$$\mathbb{C}^{n_1, n_2, \ldots, n_k} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_k}.$$

Lemma 14. *Every element of $\mathbb{C}^{n_1, n_2, \ldots, n_k}$ is a finite sum of elements of the form*

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k \quad (v_1 \in \mathbb{C}^{n_1}, v_2 \in \mathbb{C}^{n_2}, \ldots, v_k \in \mathbb{C}^{n_k}).$$

A basis for $\mathbb{C}^{n_1,n_2,\ldots,n_k}$ *over* $\mathbb{C}$ *consists of the* $n_1 n_2 \cdots n_k$ *simple tensors*

$$e_{i_1,i_2,\ldots,i_k} = e^{(1)}_{i_1} \otimes e^{(2)}_{i_2} \otimes \cdots \otimes e^{(k)}_{i_k}.$$

Every tensor of order k can be expressed uniquely in the form

$$\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_k=1}^{n_k} x_{i_1,i_2,\ldots,i_k} e_{i_1,i_2,\ldots,i_k} \quad (x_{i_1,i_2,\ldots,i_k} \in \mathbb{C}).$$

A k-dimensional array consists of the coefficients of a tensor of order k with respect to the basis of simple tensors:

$$X = (x_{i_1,i_2,\ldots,i_k}) \quad (i_1 = 1, \ldots, n_1;\ i_2 = 1, \ldots, n_2;\ \ldots;\ i_k = 1, \ldots, n_k).$$

If $M_1, M_2, \ldots, M_k$ are linear operators on $\mathbb{C}^{n_1}, \mathbb{C}^{n_2}, \ldots, \mathbb{C}^{n_k}$ then, with respect to the standard bases, we identify M_ℓ with an $n_\ell \times n_\ell$ matrix for $\ell = 1, 2, \ldots, k$:

$$M_\ell = \left(m^{(\ell)}_{ij}\right) \quad \left(m^{(\ell)}_{ij} \in \mathbb{C};\ i,j = 1, \ldots, n_\ell\right).$$

The action of a k-tuple of operators $M = (M_1, M_2, \ldots, M_k)$ on a simple tensor in $\mathbb{C}^{n_1,n_2,\ldots,n_k}$ is given by the equation

$$(M_1, M_2, \ldots, M_k) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = M_1 v_1 \otimes M_2 v_2 \otimes \cdots \otimes M_k v_k. \tag{6}$$

We introduce $n_1 n_2 \cdots n_k$ indeterminates corresponding to the entries of X:

$$x_{i_1,i_2,\ldots,i_k} \quad (i_1 = 1, \ldots, n_1;\ i_2 = 1, \ldots, n_2;\ \ldots;\ i_k = 1, \ldots, n_k).$$

We consider the polynomial algebra in these indeterminates over $\mathbb{C}$:

$$\mathbb{C}[\, x_{i_1,i_2,\ldots,i_k} \mid i_1 = 1, \ldots, n_1;\ i_2 = 1, \ldots, n_2;\ \ldots;\ i_k = 1, \ldots, n_k \,].$$

For $\ell = 1, 2, \ldots, k$ the action of M_ℓ on an indeterminate corresponds to its action on the standard basis vectors in $\mathbb{C}^{n_\ell}$:

$$M_\ell\, e^{(\ell)}_j = \sum_{i=1}^{n_\ell} m^{(\ell)}_{ij} e^{(\ell)}_i \implies M_\ell \cdot x_{j_1,\ldots,j_\ell,\ldots,j_k} = \sum_{i=1}^{n_\ell} m^{(\ell)}_{ij_\ell} x_{i_1,\ldots,i,\ldots,i_k}. \tag{7}$$

From this we obtain the action of $M = (M_1, M_2, \ldots, M_k)$ on an indeterminate:

$$(M_1, M_2, \ldots, M_k) \cdot x_{j_1,j_2,\ldots,j_k} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_k=1}^{n_k} m^{(1)}_{i_1 j_1} m^{(2)}_{i_2 j_2} \cdots m^{(k)}_{i_k j_k} x_{i_1,i_2,\ldots,i_k}.$$

This action of $M = (M_1, M_2, \ldots, M_k)$ extends to an action on polynomials:

$$M \cdot f\left(x_{11\ldots1}, \ldots, x_{j_1 j_2 \ldots j_k}, \ldots, x_{n_1 n_2 \ldots n_k}\right) = f\left(M \cdot x_{11\ldots1}, \ldots, M \cdot x_{j_1 j_2 \ldots j_k}, \ldots, M \cdot x_{n_1 n_2 \ldots n_k}\right).$$

Definition 15. The polynomial $f \in \mathbb{C}[\, x_{i_1,i_2,\ldots,i_k}]$ is **invariant** if

$$\det(M_\ell) = 1 \ (\ell = 1, \ldots, k) \implies M \cdot f = f, \ M = (M_1, M_2, \ldots, M_k).$$

The $n \times n$ complex matrices of determinant 1, with the usual operation of matrix multiplication, form the special linear group $SL_n(\mathbb{C})$. Finite-dimensional representations of $SL_n(\mathbb{C})$ can be studied in terms of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, which consists of all $n \times n$ complex matrices of trace 0; the bilinear product is the Lie bracket $[A, B] = AB - BA$. The standard basis of $\mathfrak{sl}_n(\mathbb{C})$ consists of

- the matrix units $U_{i,j}$ for $i \neq j$ with (i, j) entry 1 and other entries 0,
- the diagonal matrices $H_i = U_{i,i} - U_{i+1,i+1}$ for $i = 1, 2, \ldots, n-1$.

The simple root vectors are the matrix units $E_i = U_{i,i+1}$ for $i = 1, 2, \ldots, n-1$. The natural representation of $\mathfrak{sl}_n(\mathbb{C})$ is its action on $\mathbb{C}^n$ by matrix–vector multiplication.

Lemma 16. *In the natural representation of $\mathfrak{sl}_n(\mathbb{C})$ we have*

$$H_i \cdot e_j = \begin{cases} e_j & \text{if } j = i \\ -e_j & \text{if } j = i+1 \\ 0 & \text{otherwise}, \end{cases} \quad E_i \cdot e_j = \begin{cases} e_{j-1} & \text{if } j = i+1 \\ 0 & \text{otherwise}. \end{cases}$$

We consider the action of the semisimple Lie algebra

$$\bigoplus_{\ell=1}^{k} \mathfrak{sl}_{n_\ell}(\mathbb{C}) = \mathfrak{sl}_{n_1}(\mathbb{C}) \oplus \mathfrak{sl}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_{n_k}(\mathbb{C}), \tag{8}$$

on its irreducible representation $\mathbb{C}^{n_1,n_2,\ldots,n_k}$, the tensor product of the natural representations of its simple summands. For $\ell = 1, 2, \ldots, k$ we write $H_i^{(\ell)}, E_i^{(\ell)}$ for the elements $H_i, E_i \in \mathfrak{sl}_{n_\ell}(\mathbb{C})$. Combining Eqs. (6) and (7) with Lemma 16 we obtain the action of $H_i^{(\ell)}$ and $E_i^{(\ell)}$ on the indeterminates $x_{j_1 j_2 \ldots j_k}$.

Lemma 17. *For $\ell = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, n_\ell - 1$ we have*

$$H_i^{(\ell)} \cdot x_{j_1,j_2,\ldots,j_k} = \begin{cases} x_{j_1,j_2,\ldots,j_k} & \text{if } j_\ell = i \\ -x_{j_1,j_2,\ldots j_k} & \text{if } j_\ell = i+1 \\ 0 & \text{otherwise}, \end{cases}$$

$$E_i^{(\ell)} \cdot x_{j_1,j_2,\ldots,j_k} = \begin{cases} x_{j_1,j_2,\ldots,j_\ell - 1,\ldots,j_k} & \text{if } j_\ell = i+1 \\ 0 & \text{otherwise}. \end{cases}$$

The action of a Lie algebra L on a tensor product $V \otimes W$ of representations is given by the derivation rule:

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w) \quad (x \in L,\ v \in V,\ w \in W).$$

We identify the dth symmetric power $S^d V$ of the representation V with the space of homogeneous polynomials of degree d on a basis of V. It follows by induction on d that the action of L on $S^d V$ is given by the following equation:

$$x \cdot (v_1^{e_1} v_2^{e_2} \cdots v_p^{e_p}) = \sum_{i=1}^{p} v_1^{e_1} \cdots (x \cdot v_i^{e_i}) \cdots v_p^{e_p}$$

$$= \sum_{i=1}^{p} v_1^{e_1} \cdots \left(e_i v_i^{e_i - 1} (x \cdot v_i)\right) \cdots v_p^{e_p} = \sum_{i=1}^{p} e_i\, v_1^{e_1} \cdots v_i^{e_i - 1} \cdots v_p^{e_p} (x \cdot v_i).$$

We apply this to

$$L = \bigoplus_{\ell=1}^{k} \mathfrak{sl}_{n_\ell}(\mathbb{C}), \quad V = \bigoplus_{j_1=1}^{n_1} \bigoplus_{j_2=1}^{n_2} \cdots \bigoplus_{j_k=1}^{n_k} \mathbb{C} x_{j_1 j_2 \ldots j_k}.$$

Some equations will be clearer if we write a monomial as follows:

$$\prod_{j_1}^{n_1}\prod_{j_2}^{n_2}\cdots\prod_{j_k}^{n_k} x_{j_1 j_2 \dots j_k}^{e_{j_1 j_2 \dots j_k}} = x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{j_1\dots j_k}^{e_{j_1\dots j_k}} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}}$$

Lemma 18. *For $\ell = 1, 2, \dots, k$ and $i = 1, 2, \dots, n_\ell - 1$ we have*

$$H_i^{(\ell)} \cdot \left(x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{j_1\dots j_k}^{e_{j_1\dots j_k}} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}} \right)$$
$$= \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} (\delta_{j_\ell,i} - \delta_{j_\ell,i+1}) e_{j_1\dots j_\ell \dots j_k} \, x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{j_1\dots j_\ell \dots j_k}^{e_{j_1\dots j_\ell\dots j_k}} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}},$$

$$E_i^{(\ell)} \cdot \left(x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{j_1\dots j_k}^{e_{j_1\dots j_k}} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}} \right)$$
$$= \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} \delta_{j_\ell,i+1} \, e_{j_1\dots j_\ell \dots j_k} \, x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{j_1\dots j_\ell - 1 \dots j_k}^{e_{j_1\dots j_\ell-1\dots j_k}+1} \cdots x_{j_1\dots j_\ell \dots j_k}^{e_{j_1\dots j_\ell\dots j_k}-1} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}},$$

where δ_{ij} is the Kronecker delta ($\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$).

Lemma 19. *For every $\ell = 1, 2, \dots, k$ and $i = 1, 2, \dots, n_\ell - 1$, the monomial*

$$x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}},$$

is an eigenvector for $H_i^{(\ell)}$ with eigenvalue

$$\sum_{j_1=1}^{n_1} \cdots \widehat{\sum_{j_\ell}} \cdots \sum_{j_k=1}^{n_k} e_{j_1\dots i \dots j_k} - \sum_{j_1=1}^{n_1} \cdots \widehat{\sum_{j_\ell}} \cdots \sum_{j_k=1}^{n_k} e_{j_1\dots i+1 \dots j_k},$$

where the hat denotes omission.

The space of homogeneous polynomials of degree d has the basis

$$x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}}, \quad \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} e_{j_1\dots j_k} = d.$$

Definition 20. A monomial $x_{1\dots 1}^{e_{1\dots 1}} \cdots x_{n_1\dots n_k}^{e_{n_1\dots n_k}}$ has **weight zero** if it has eigenvalue 0 for every $H_i^{(\ell)}$ with $\ell = 1, 2, \dots, k$ and $i = 1, 2, \dots, n_\ell - 1$; that is,

$$\sum_{j_1=1}^{n_1} \cdots \widehat{\sum_{j_\ell}} \cdots \sum_{j_k=1}^{n_k} e_{j_1\dots i \dots j_k} = \sum_{j_1=1}^{n_1} \cdots \widehat{\sum_{j_\ell}} \cdots \sum_{j_k=1}^{n_k} e_{j_1\dots i+1 \dots j_k}.$$

The **zero weight space** of degree d consists of the monomials of weight zero.

Definition 21. Let $E = (e_{i_1 i_2 \dots i_k})$ be an array of size $n_1 \times n_2 \times \cdots \times n_k$ with non-negative integer entries. A **slice** of E is a $(k-1)$-dimensional subarray obtained by fixing one subscript; for every $\ell = 1, 2, \dots, k$ we can set $i_\ell = 1, 2, \dots, n_\ell$ and obtain n_ℓ slices of size $n_1 \times \cdots \widehat{n_\ell} \cdots \times n_k$. We call E an **equal parallel slice (EPS) array** if for every $\ell = 1, 2, \dots, k$ the n_ℓ slices in direction ℓ have the same entry sum. That is, for each ℓ the following sum does not depend on j:

$$\sum_{i_1=1}^{n_1} \cdots \widehat{\sum_{i_\ell}} \cdots \sum_{i_k=1}^{n_k} e_{i_1\dots j \dots i_k}.$$

Lemma 22. *A basis for the zero weight space in degree d consists of the monomials whose arrays of exponents are EPS arrays.*

We write $W(d; a_1, \ldots, a_{n-1})$ for the vector space with basis consisting of the monomials with degree d and eigenvalues $(a_1, \ldots, a_{n-1})$ for $H_1, \ldots, H_{n-1}$ as in Lemma 19. In $\mathfrak{sl}_n(\mathbb{C})$ the brackets of H_i and E_j are given by the formulas

$$[H_i, E_j] = \begin{cases} 2E_j & \text{if } i = j \\ -E_j & \text{if } j = i-1 \text{ or } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the actions of $E_1, \ldots, E_{n-1}$ induce the following linear maps:

$$\begin{aligned} E_1 &: W(d; 0, \ldots, 0) \longrightarrow W(d; 2, -1, 0, \ldots, 0, 0), \\ E_2 &: W(d; 0, \ldots, 0) \longrightarrow W(d; -1, 2, -1, \ldots, 0, 0), \\ E_3 &: W(d; 0, \ldots, 0) \longrightarrow W(d; 0, -1, 2, \ldots, 0, 0), \\ &\qquad\qquad\qquad \vdots \\ E_{n-1} &: W(d; 0, \ldots, 0) \longrightarrow W(d; 0, 0, 0, \ldots, -1, 2). \end{aligned}$$

The weights appearing on the right are the rows of the Killing–Cartan matrix,

$$K^{(n-1)} = (\kappa_{ij}), \quad \kappa_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } j = i-1 \text{ or } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

We write $w_1^{(n-1)}, \ldots, w_{n-1}^{(n-1)}$ for the rows of $K^{(n-1)}$ and form the linear map

$$E = (E_1, \ldots, E_{n-1}) \colon W(d; 0, \ldots, 0) \longrightarrow \bigoplus_{i=1}^{n-1} W(d; w_i^{(n-1)}).$$

We apply this to the semisimple Lie algebra (8). We first combine the spaces $W(d; 0, \ldots, 0)$ for each summand into the zero weight space of Definition 20:

$$Z = W(d; \overbrace{0, \ldots, 0}^{n_1-1}) \cap \cdots \cap W(d; \overbrace{0, \ldots, 0}^{n_k-1}).$$

We then combine the linear maps E for each summand into the single linear map

$$\mathcal{E} = \left(E^{(n_1)}, \ldots, E^{(n_k)}\right) \colon Z \longrightarrow \bigoplus_{\ell=1}^{k} \bigoplus_{i=1}^{n_\ell-1} W(d; w_i^{(n_\ell-1)}). \tag{9}$$

Theorem 23. *The invariant polynomials in degree d for the $n_1 \times \cdots \times n_k$ array $X = (x_{i_1 \cdots i_k})$ are the (nonzero) elements of the kernel of the linear map (9).*

In degree d there are no monomials of weight zero unless d is a multiple of $N = \mathrm{LCM}(n_1, \ldots, n_k)$; hence invariants can only exist in degrees $d \equiv 0 \pmod N$. Since we have $n_1 \cdots n_k$ exponents, with one constraint on the degree and $(n_1-1) + \cdots + (n_k-1)$ constraints on the parallel slices, we make the following conjecture.

Conjecture 24. *Let k and $n_1, n_2, \ldots, n_k$ be positive integers. The dimension of the zero weight space in degree d is given by a family of polynomials of degree*

$$\prod_{\ell=1}^{k} n_\ell - \sum_{\ell=1}^{k} n_\ell + k - 1.$$

For the application of these methods to arrays of size $2 \times 2 \times 3$, see Bremner [4].

7. Conclusion

Modern interest in Cayley's hyperdeterminant and its generalizations was revived by the famous paper of Gelfand et al. [11]; see especially Proposition 1.9 on page 234. The same authors developed this subject in great depth, using the techniques of algebraic geometry, in their monograph [12].

A closely related topic, of great importance in applied numerical linear algebra, is the problem of computing the rank of a k-dimensional array. When $k = 2$, this problem has an efficient solution using Gaussian elimination, but for $k \geq 3$ it has been shown by Hastad [13] to be NP-complete. A comprehensive survey on tensor rank and algorithms for tensor decomposition has been given recently by Kolda and Bader [17]. Cayley's hyperdeterminant was rediscovered in the 1970's by Kruskal [18], and is sometimes called Kruskal's polynomial by applied mathematicians; see ten Berge [27] and Martin [21] for an explanation of how it can be used to compute the rank of a $2 \times 2 \times 2$ array. Two recent related papers are de Silva and Lim [7] and Stegeman and Comon [24].

Invariant polynomials on arrays of size $2 \times 2 \times \cdots \times 2$ (k factors) have been studied by theoretical physicists working on quantum computing; see Luque and Thibon [19,20], Djokovic and Osterloh [9]. For the combinatorial-geometric aspects of this problem, see Huggins et al. [14]. These invariants can be regarded as noncommutative analogues of classical invariant theory (for a survey see Dixmier [8]): the 19th century invariant theorists studied the irreducible representations $V(k) \cong S^k V(1)$ of $\mathfrak{sl}_2(\mathbb{C})$, and replacing the symmetric power by the full tensor power gives the vector space of arrays of size 2^k. It is an open problem to extend the methods of the present paper to these arrays. It would be very useful to have a complete description of the structure of the space of homogeneous polynomials as a sum of irreducible representations of the semisimple Lie algebra; one possible approach to this problem has been developed by Adsul and Subrahmanyam [1].

The objects that we call equal parallel slice (EPS) arrays are examples of contingency tables, which are important in combinatorics and statistics. For asymptotic formulas for the enumeration of these objects, see Barvinok [3]. The recent preprint by Sturmfels and Zwiernik [26] uses methods of algebraic statistics to obtain a more compact expression for Cayley's hyperdeterminant in terms of cumulants. In closing, we mention the intriguing applications of Gröbner bases and hyperdeterminants to mathematical genetics; see Allman and Rhodes [2], especially page 146.

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References

[1] B. Adsul, K.V. Subrahmanyam, A geometric approach to the Kronecker problem. I. The two row case, Proc. Indian Acad. Sci. Math. Sci. 118 (2) (2008) 213–226.
[2] E.S. Allman, J.A. Rhodes, Phylogenetic invariants for stationary base composition, J. Symbolic Comput. 41 (2) (2006) 138–150.
[3] A. Barvinok, Brunn–Minkowski inequalities for contingency tables and integer flows, Adv. Math. 211 (1) (2007) 105–122.
[4] M.R. Bremner, On the hyperdeterminant for $2 \times 2 \times 3$ arrays, Linear and Multilinear Algebra, in press, doi:10.1080/03081087.2011.634412.
[5] A. Cayley, On the theory of linear transformations, Cambridge Math. J. 4 (1845) 193–209. Available from: <www.archive.org/details/collectedmathema01cayluoft>
[6] W.A. de Graaf, Lie Algebras: Theory and Algorithms, North-Holland, Amsterdam, 2000.

[7] V. de Silva, L.-H. Lim, Tensor rank and the ill-posedness of the best low-rank approximation problem, SIAM J. Matrix Anal. Appl. 30 (3) (2008) 1084–1127.
[8] J. Dixmier, Quelques aspects de la théorie des invariants, Gaz. Math. 43 (1990) 39–64.
[9] D.Z. Djokovic, A. Osterloh, On polynomial invariants of several qubits, J. Math. Phys. 50 (3, 033509) (2009) 23pp.
[10] K. Erdmann, M.J. Wildon, Introduction to Lie Algebras, Springer-Verlag, London, 2006.
[11] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Hyperdeterminants, Adv. Math. 96 (2) (1992) 226–263.
[12] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston, 1994.
[13] J. Hastad, Tensor rank is NP-complete, J. Algorithms 11 (4) (1990) 644–654.
[14] P. Huggins, B. Sturmfels, J. Yu, D.S. Yuster, The hyperdeterminant and triangulations of the 4-cube, Math. Comput. 77 (263) (2008) 1653–1679.
[15] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer, New York, 1972.
[16] N. Jacobson, Lie Algebras, Interscience Publishers, New York, 1962.
[17] T.G. Kolda, B.W. Bader, Tensor decompositions and applications, SIAM Rev. 51 (3) (2009) 455–500.
[18] J.B. Kruskal, Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics, Linear Algebra Appl. 18 (2) (1977) 95–138.
[19] J.-G. Luque, J.-Y. Thibon, Polynomial invariants of four qubits, Phys. Rev. A (3) 67 (4, 042303) (2003) 5pp.
[20] J.-G. Luque, J.-Y. Thibon, Algebraic invariants of five qubits, J. Phys. A Math. Theoret. 39 (2) (2006) 371–377.
[21] C.D. Martin, The rank of a $2 \times 2 \times 2$ tensor, Linear and Multilinear Algebra 59 (8) (2011) 943–950.
[22] E. Neher, A. Savage, P. Senesi, Irreducible finite-dimensional representations of equivariant map algebras, Trans. Amer. Math. Soc. 364 (2012) 2619–2646.
[23] C. Procesi, Lie Groups: An Approach through Invariants and Representations, Springer, New York, 2007.
[24] A. Stegeman, P. Comon, Subtracting a best rank-1 approximation may increase tensor rank, Linear Algebra Appl. 433 (7) (2010) 1276–1300.
[25] J. Stillwell, Naive Lie Theory, Undergraduate Texts in Mathematics, Springer, New York 2008.
[26] B. Sturmfels, P. Zwiernik, Binary cumulant varieties. <arXiv:1103.0153v2 [math.CO]>.
[27] J.M F. ten Berge, Kruskal's polynomial for $2 \times 2 \times 2$ arrays and a generalization to $2 \times n \times n$ arrays, Psychometrika 56 (4) (1991) 631–636.