J. Math. Anal. Appl. 385 (2012) 797–807

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiplicity results for some fourth-order elliptic equations

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ARTICLE INFO

Article history:
Received 28 February 2011
Available online 12 July 2011
Submitted by P.J. McKenna

Keywords:
Variational method
Mountain pass theorem
Upper–lower solutions

ABSTRACT

This paper deals with some fourth-order elliptic equations with Navier boundary condition. By using the variational method, some existence and multiplicity results are established.

1. Introduction

We study the following fourth-order elliptic equation

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda h(x)|u|^{p-2}u + f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, Δ^2 is the biharmonic operator, c is a constant, $1 < p < 2$, $\lambda \geqslant 0$ is a parameter, $h \in L^\infty(\Omega)$, $h(x) \geqslant 0$, $h(x) \not\equiv 0$, and $f(x,s)$ is a continuous function on $\overline{\Omega} \times \mathbb{R}$.

The fourth-order problem (1.1) is usually used to describe some phenomena appeared in different physical, engineering and other sciences. In [1], Lazer and McKenna studied the problem of nonlinear oscillation in a suspension bridge. They presented a mathematical model for the bridge that takes into account the fact that the coupling provided by the stays connecting the suspension cable to the deck of the road bed is fundamentally nonlinear (see [1–5]). Also, problem (1.1) has been pointed out that this kind of problem furnishes a good model to the static deflection of an elastic plate in a fluid (see [6]). In [7], the authors indicated that problem (1.1) also arises in such as communication satellites, space shuttles, and space stations, which are equipped with large antennas mounted on long flexible masts (beams). Problem (1.1) has been studied extensively in recent years, we refer the reader to [2,6,8–12] and the references therein.

There has been a great deal of interest on second-order elliptic problem with combined nonlinearities since the pioneering work of Ambrosetti, Brezis and Cerami in [13] (see [13–15]). However, to the author's knowledge, it seems that very few results are devoted to the case of the fourth-order elliptic problem (1.1). Our purpose here is to introduce some kind of local analogues to the classical conditions of sublinearity at 0 and of superlinearity at ∞ with respect to u.

The paper is organized as follows. The functional setting and the proof of the (PS) condition are given in Section 2. Section 3 deals with the case $\lambda = 0$. In [6], the authors have obtained at least one nontrivial solution under some strong assumptions (see Remark 3.1). Under weaker assumptions we obtain the existence of a positive solution and a negative solution. Then we consider the case $\lambda > 0$ in Section 4. For $\lambda > 0$ small enough, we give the existence of four solutions.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.07.011

A global result of Ambrosetti–Brezis–Cerami type also be considered, which is motivated by the results of second-order version in [13–15]. Finally, in Section 5, we show that, besides the four solutions given in Theorem 4.1, there exists another different solution, which can change the sign. Our method to obtain the fifth solution follows the ideas developed in [14] for Laplacian operator. However, while in [14] the authors assumed that nonlinearity is superlinear at ∞, we discuss here a local asymptotically linear problem.

2. Preliminary results

Let $H = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$\langle u, v\rangle_H = \int_\Omega (\Delta u \Delta v + \nabla u \cdot \nabla v)\, dx,$$

and the deduced norm

$$\|u\|_H^2 = \int_\Omega |\Delta u|^2\, dx + \int_\Omega |\nabla u|^2\, dx.$$

Denote λ_k $(k \in N)$ the eigenvalues and φ_k $(k \in N)$ the corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where each eigenvalue λ_k is repeated as often as the multiplicity. Recall that $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \to +\infty$ and $\varphi_1 > 0$ for $x \in \Omega$. We can easily see that $\Lambda_k = \lambda_k(\lambda_k - c)$ are eigenvalues of the problem

$$\begin{cases} \Delta^2 u + c\Delta u = \Lambda u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

and the corresponding eigenfunctions are still φ_k.

Assume that $c < \lambda_1$. Let us define a norm of $u \in H$ as follows:

$$\|u\|^2 = \int_\Omega |\Delta u|^2\, dx - c\int_\Omega |\nabla u|^2\, dx.$$

It is easy to show that the norm $\|\cdot\|$ is an equivalent norm on H, and for all $u \in H$, the following Poincaré inequality holds:

$$\|u\|^2 \geqslant \Lambda_1 \|u\|_{L^2}^2. \tag{2.3}$$

We say that $u \in H$ is a weak solution to problem (1.1), if u satisfies

$$\int_\Omega \big(\Delta u \Delta v - c\nabla u \cdot \nabla v - \lambda h(x)|u|^{p-2}uv - f(x,u)v\big)\, dx = 0, \quad \forall v \in H^*,$$

where H^* is the dual space of H.

It is well known that the weak solution of problem (1.1) is equivalent to the critical point of the Euler–Lagrange functional

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_\Omega h(x)|u|^p\, dx - \int_\Omega F(x,u)\, dx, \quad u \in H.$$

Obviously $I_\lambda \in C^1(H, \mathbb{R})$, and

$$\big\langle \nabla I_\lambda(u), v\big\rangle = \int_\Omega \big(\Delta u \Delta v - c\nabla u \cdot \nabla v - \lambda h(x)|u|^{p-2}uv - f(x,u)v\big)\, dx, \quad \forall\, u, v \in H.$$

Let $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$.

Consider the following problem

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda h(x)\big|u^+\big|^{p-2}u^+ + f^+(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where

$$f^+(x,t) = \begin{cases} f(x,t), & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

Define the corresponding functional $I_\lambda^+ : H \to \mathbb{R}$ as follows:

$$I_\lambda^+(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_\Omega h(x)\left|u^+\right|^p dx - \int_\Omega F^+(x,u)\,dx, \quad u \in H,$$

where $F^+(x,u) = \int_0^u f^+(x,s)\,ds$. Obviously, $I_\lambda^+ \in C^1(H,\mathbb{R})$. Let u be a critical point of I_λ^+, which implies that u is a weak solution of (2.4). Furthermore, by the weak maximum principle it follows that $u \geqslant 0$ in Ω. Thus u is also a solution of problem (1.1) and $I_\lambda(u) = I_\lambda^+(u)$.

Similarly, we can define

$$f^-(x,t) = \begin{cases} f(x,t), & t \leqslant 0, \\ 0, & t > 0 \end{cases} \tag{2.5}$$

and

$$I_\lambda^-(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_\Omega h(x)\left|u^-\right|^p dx - \int_\Omega F^-(x,u)\,dx, \quad u \in H,$$

where $F^-(x,u) = \int_0^u f^-(x,s)\,ds$. It is easily seen that $I_\lambda^- \in C^1(H,\mathbb{R})$ and if v is a critical point of I_λ^- then it is a solution of problem (1.1) with $I_\lambda(v) = I_\lambda^-(v)$.

We assume that $f(x,s)$ satisfies the following hypotheses:

(H1) $f(x,0) = 0$.
(H2) $\lim_{s\to 0}\frac{f(x,s)}{s} = \mu$, $\lim_{|s|\to+\infty}\frac{f(x,s)}{s} = l$, uniformly a.e. in $x \in \Omega$,

$$0 \leqslant \mu < \lambda_1(\lambda_1 - c) < l < +\infty, \quad \lambda_1 \text{ is the first eigenvalue of } \left(-\Delta, H_0^1(\Omega)\right).$$

Lemma 2.1. *Suppose that* (H1) *and* (H2) *hold. Then $I_\lambda^\pm$ satisfies the* (PS) *condition.*

Proof. We just prove the case of I_λ^+. The arguments for the case of I_λ^- are similar. Since Ω is bounded and (H2) holds, then if $\{u_n\}$ is bounded in H, by using the Sobolev embedding and the standard procedures, we can get a subsequence converges strongly. So we need only to show that $\{u_n\}$ is bounded in H.

Assume that $\{u_n\} \subset H$ is a (PS) sequence, i.e.,

$$I_\lambda^+(u_n) \to c, \qquad \nabla I_\lambda^+(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{2.6}$$

From (H2) we know that

$$\left|f^+(x,s)s\right| \leqslant C\left(1 + |s|^2\right).$$

(2.6) implies that for all $\varphi \in H$,

$$\int_\Omega \left(\Delta u_n \Delta\varphi - c\nabla u_n \cdot \nabla\varphi - \lambda h(x)\left|u_n^+\right|^{p-2}u_n^+\varphi - f^+(x,u_n)\varphi\right)dx \to 0. \tag{2.7}$$

Setting $\varphi = u_n$ and using the Hölder inequality we have

$$\begin{aligned} \|u_n\|^2 &= \int_\Omega f^+(x,u_n)u_n\,dx + \int_\Omega \lambda h(x)\left|u_n^+\right|^p dx + \left\langle \nabla I_\lambda^+(u_n), u_n\right\rangle \\ &\leqslant \int_\Omega f^+(x,u_n)u_n\,dx + \lambda|h|_\infty \int_\Omega |u_n|^p\,dx + o(1)\|u_n\| \\ &\leqslant C|\Omega| + C\|u_n\|_{L^2}^2 + C_p\|u_n\|_{L^2}^p + o(1)\|u_n\|. \end{aligned} \tag{2.8}$$

We claim that $\|u_n\|_{L^2}$ is bounded. Assume, by contradiction, that passing to a subsequence,

$$\|u_n\|_{L^2}^2 \to +\infty, \quad \text{as } n \to +\infty.$$

We put $\omega_n := \frac{u_n}{\|u_n\|_{L^2}}$. Then $\|\omega_n\|_{L^2} = 1$. Moreover, from (2.8) we know

$$\|\omega_n\|^2 \leqslant o(1) + C + \frac{o(1)}{\|u_n\|_{L^2}} \cdot \frac{\|u_n\|}{\|u_n\|_{L^2}} \leqslant o(1) + C + o(1)\|\omega_n\|.$$

Hence, $\|\omega_n\|$ is bounded. Passing to a subsequence, we may assume that there exists $\omega \in H$, $\|\omega\|_{L^2} = 1$ such that

$$\omega_n \rightharpoonup \omega, \quad \text{weakly in } H,\ n \to +\infty,$$
$$\omega_n \to \omega, \quad \text{strongly in } L^2(\Omega),\ n \to +\infty.$$

From (2.7) we derive

$$\int\limits_\Omega \left(\Delta\omega\Delta\varphi - c\nabla\omega \cdot \nabla\varphi - l\omega^+\varphi\right) dx = 0, \quad \forall \varphi \in H. \tag{2.9}$$

Then $\omega \in H$ is a weak solution of the equation

$$\Delta^2\omega + c\Delta\omega = l\omega^+.$$

The weak maximum principle implies $\omega = \omega^+ \geqslant 0$. Taking $\varphi(x) = \varphi_1(x)$, from (2.9) we have

$$\int\limits_\Omega (\Delta\omega\Delta\varphi_1 - c\nabla\omega \cdot \nabla\varphi_1)\, dx = l \int\limits_\Omega \omega\varphi_1\, dx. \tag{2.10}$$

On the other hand, since $\varphi_1(x) > 0$ is the Λ_1-eigenfunction of (2.2), we have also

$$\int\limits_\Omega (\Delta\omega\Delta\varphi_1 - c\nabla\omega \cdot \nabla\varphi_1)\, dx = \Lambda_1 \int\limits_\Omega \omega\varphi_1\, dx,$$

from which follows that $\omega \equiv 0$ by $\Lambda_1 < l$. But this conclusion contradicts $\|\omega\|_{L^2} = 1$, and hence $\|u_n\|_{L^2}$ is bounded. Then, from (2.8) we know that $\{u_n\}$ is bounded in H. $\square$

3. Asymptotically linear problem

For the case $\lambda = 0$, our main result is as follows:

Theorem 3.1. *Assume that $c < \lambda_1$, $\lambda = 0$. Then under assumptions* (H1) *and* (H2), *problem* (1.1) *has at least two nontrivial solutions, one of which is positive and the other is negative.*

Remark 3.1. Recently, Y. An and R. Liu in [6] have obtained at least one nontrivial solution, under (H1), (H2), and the following two conditions:

(H3*) $f(x,s) \equiv 0$, $\forall x \in \Omega$, $s \leqslant 0$; $f(x,t) \geqslant 0$, $\forall x \in \Omega$, $s > 0$; $\frac{f(x,s)}{s}$ is nondecreasing.
(H4*) $|f(x,s)| \leqslant a(x) + b|s|^p$, where $a(x) \in L^q(\Omega)$, $b \in \mathbb{R}^N$, $1 < p < \frac{N+4}{N-4}$ if $N > 4$ and $1 < p < \infty$ if $N \leqslant 4$, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.1 improves previous results, such as [6].

For convenience we denote $I_\lambda^\pm$ with $\lambda = 0$ by $I^\pm$. Now we prove that the functionals $I^\pm$ have a mountain pass geometry.

Lemma 3.1. *Under the assumption* (H2), *I^+ and I^- are unbounded from below.*

Proof. (H2) implies that, for any $\varepsilon > 0$ there exists $C_1 > 0$, such that

$$F(x,s) \geqslant \frac{1}{2}(l - \varepsilon)s^2 - C_1, \quad \forall x \in \Omega,\ s \neq 0. \tag{3.1}$$

Taking $\varepsilon > 0$ such that $l - \varepsilon > \Lambda_1$, $\phi = -\varphi_1$, from (3.1) we obtain

$$\begin{aligned} I^-(t\phi) &\leqslant \frac{1}{2}\int\limits_\Omega \left(\left|\Delta(t\phi)\right|^2 - c\left|\nabla(t\phi)\right|^2\right) dx - \frac{1}{2}(l-\varepsilon)\int\limits_\Omega t^2\phi^2\, dx + \int\limits_\Omega C_1\, dx \\ &\leqslant \frac{t^2}{2}\|\phi\|^2 - \frac{t^2}{2}(l-\varepsilon)\|\phi\|_{L^2}^2 + C_1|\Omega| \\ &\leqslant \frac{1}{2}\left(1 - \frac{l-\varepsilon}{\Lambda_1}\right)t^2\|\phi\|^2 + C_1|\Omega|, \end{aligned} \tag{3.2}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Then

$$\lim_{t\to+\infty} I^-(t\phi) = -\infty.$$

For I^+ we can choose φ_1, then, by an analogous manner as above we get

$$\lim_{t\to+\infty} I^+(t\varphi_1) = -\infty.$$

The proof is complete. $\square$

Lemma 3.2. *Assume that* (H1) *and* (H2) *hold. Then there exist* $\rho, R > 0$ *such that* $I^{\pm}(u) \geqslant R$, *if* $\|u\| = \rho$.

Proof. From (H2), we can find α, such that $2 < \alpha < 2^*$, where $2^* = \begin{cases} \frac{2N}{N-2}, & N > 2; \\ +\infty, & N \leqslant 2. \end{cases}$ (H1), (H2) imply that for all given $\varepsilon_0 > 0$, there exists $C_0 > 0$, such that

$$F(x,s) \leqslant \frac{1}{2}(\mu + \varepsilon_0)|s|^2 + C_0|s|^\alpha. \tag{3.3}$$

Combining (3.3) and the Poincaré inequality as well as the Sobolev embedding, we have

$$\begin{aligned} I^{\pm}(u) &\geqslant \frac{1}{2}\|u\|^2 - \frac{\mu+\varepsilon_0}{2}\int_\Omega |u|^2\,dx - C_0\int_\Omega |u|^\alpha\,dx \\ &\geqslant \left(\frac{1}{2} - \frac{\mu+\varepsilon_0}{2\Lambda_1}\right)\|u\|^2 - C_s\|u\|^\alpha, \end{aligned} \tag{3.4}$$

where C_s is a constant. In (3.4), by taking $\varepsilon_0 > 0$ such that $\mu + \varepsilon_0 < \Lambda_1$, and choosing $\|u\| = \rho > 0$ small enough, we obtain $I^{\pm}(u) \geqslant R > 0$, if $\|u\| = \rho$. $\square$

Proof of Theorem 3.1. From Lemma 3.1 and Lemma 3.2 we know that there exists $e \in H$, $\|e\| > \rho$, such that $I^{\pm}(e) < 0$.
Define

$$P = \{\gamma : [0,1] \to H:\ \gamma \text{ is continuous and } \gamma(0) = 0,\ \gamma(1) = e\},$$

and

$$c^{\pm} = \inf_{\gamma\in P}\max_{t\in[0,1]} I^{\pm}(\gamma(t)).$$

From Lemma 3.2 it follows that

$$I^{\pm}(0) = 0, \qquad I^{\pm}(e) < 0, \qquad I^{\pm}(u)\big|_{\partial B_\rho} \geqslant R > 0.$$

Moreover, I^+ and I^- satisfy the (PS) condition. By the mountain pass theorem, we know c^+ is a critical value of I^+ and there is at least one nontrivial critical point in H corresponding to this value. This critical point is nonnegative, then the strong maximum principle implies that there is a positive solution of problem (1.1). By an analogous way we know there exists at least one negative solution, which is a nontrivial critical point of I^-. Hence, the problem (1.1) admits at least a positive solution and a negative solution. $\square$

4. Combined nonlinearities

In this section, we discuss the multiplicity of solutions of problem (1.1) for the case $\lambda > 0$, in which the nonlinearity is locally sublinear and asymptotically linear.

Theorem 4.1. *Assume that* $c < \lambda_1$. *Then under assumptions* (H1) *and* (H2), *there exists* $\lambda^* > 0$ *such that for* $\lambda \in (0, \lambda^*)$, *problem* (1.1) *has at least four nontrivial solutions*: u_+, u_-, v_+, *and* v_-, *satisfying* $u_+ > 0$, $u_- < 0$, $v_+ > 0$, $v_- < 0$, *and* $I_\lambda(u_\pm) > 0 > I_\lambda(v_\pm)$.

We first give the following lemmas which will be used to prove Theorem 4.1.

Lemma 4.1. *Under the assumption* (H2), I_λ^+ *and* I_λ^- *are unbounded from below.*

Proof. The proof is similar to the proof of Lemma 3.1. $\square$

Lemma 4.2. *Assume that* (H1) *and* (H2) *hold. Then for* $\lambda > 0$ *small enough, there exist* $\rho, R > 0$ *such that* $I_\lambda^\pm(u) \geqslant R$, *if* $\|u\| = \rho$.

Proof. (H1), (H2) imply that for all given $\varepsilon_0 > 0$, there exists $C_0 > 0$, such that (3.3) hold. Take $\varepsilon_0 > 0$ such that $\mu + \varepsilon_0 < \Lambda_1$. From (3.3), combining the Hölder inequality and the Poincaré inequality as well as the Sobolev embedding, we have

$$\begin{aligned} I_\lambda^\pm(u) &\geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda|h|_\infty}{p}\int_\Omega |u|^p\,dx - \frac{\mu+\varepsilon_0}{2}\int_\Omega |u|^2\,dx - C_0\int_\Omega |u|^\alpha\,dx \\ &\geqslant \left(\frac{1}{2} - \frac{\mu+\varepsilon_0}{2\Lambda_1}\right)\|u\|^2 - \frac{\lambda|h|_\infty}{p}\left(\int_\Omega \left(|u|^p\right)^{\frac{\alpha}{p}}dx\right)^{\frac{p}{\alpha}}\left(\int_\Omega 1\,dx\right)^{\frac{\alpha-p}{\alpha}} - C_s\|u\|^\alpha \\ &\geqslant \left(\frac{1}{2} - \frac{\mu+\varepsilon_0}{2\Lambda_1}\right)\|u\|^2 - \lambda K\|u\|^p - C_s\|u\|^\alpha \\ &= \left(\frac{1}{2}\left(1 - \frac{\mu+\varepsilon_0}{\Lambda_1}\right) - \lambda K\|u\|^{p-2} - C_s\|u\|^{\alpha-2}\right)\|u\|^2, \end{aligned}$$

where C_s, K are constant, $2 < \alpha < 2^*$.

Let

$$Q(t) = \lambda K t^{p-2} + C_s t^{\alpha-2}.$$

We claim that there exists t_0 such that

$$Q(t_0) < \frac{1}{2}\left(1 - \frac{\mu+\varepsilon_0}{\Lambda_1}\right).$$

Indeed,

$$Q'(t) = \lambda K(p-2)t^{p-3} + C_s(\alpha-2)t^{\alpha-3}.$$

Setting

$$Q'(t) = 0$$

we know

$$t_0 = \left(\frac{\lambda K(2-p)}{C_s(\alpha-2)}\right)^{\frac{1}{\alpha-p}}.$$

Obviously, $Q(t)$ has a minimum at $t = t_0$. Let

$$\beta = \frac{K(2-p)}{C_s(\alpha-2)}, \qquad \bar p = \frac{p-2}{\alpha-p}, \qquad \bar q = \frac{\alpha-2}{\alpha-p}, \qquad \kappa = \frac{1}{2}\left(1 - \frac{\mu+\varepsilon_0}{\Lambda_1}\right).$$

Substituting t_0 in $Q(t)$ we have

$$Q(t_0) < \frac{1}{2}\left(1 - \frac{\mu+\varepsilon_0}{\Lambda_1}\right), \quad 0 < \lambda < \lambda^*,$$

where $\lambda^* = (\frac{\kappa}{K\beta^{\bar p} + C_s\beta^{\bar q}})^{1/\bar q}$.

Take $\rho = t_0$. Then there exists $R > 0$ such that the lemma holds. $\square$

Proof of Theorem 4.1. For $I_\lambda^\pm$, we first show the existence of local minimum $v_\pm$, with $I_\lambda^\pm(v_\pm) < 0$. We just prove the case of I_λ^+. The arguments for the case of I_λ^- are similar.

For ρ given in Lemma 4.2, we set

$$\overline{B}(\rho) = \left\{u \in H,\ \|u\| \leqslant \rho\right\}, \qquad \partial B(\rho) = \left\{u \in H,\ \|u\| = \rho\right\}.$$

Then $\overline{B}(\rho)$ is a complete metric space with the distance

$$\operatorname{dist}(u, v) = \|u - v\|, \quad \forall u, v \in \overline{B}(\rho).$$

By Lemma 4.2, we know for $0 < \lambda < \lambda^*$,

$$I_\lambda^+(u)\big|\partial B_\rho \geqslant R > 0.$$

Moreover, it is easy to see that $I_\lambda^+ \in C^1(\overline{B}(\rho), \mathbb{R})$, hence I_λ^+ is lower semi-continuous and bounded from below on $\overline{B}(\rho)$. Let

$$c_1 = \inf\{I_\lambda^+(u),\ u \in \overline{B}(\rho)\}.$$

Taking $\bar{v} \in C_0^\infty(\Omega)$, with $\bar{v} > 0$, and for $t > 0$, we have

$$\begin{aligned} I_\lambda^+(t\bar{v}) &= \frac{t^2}{2}\|\bar{v}\|^2 - \frac{\lambda t^p}{p}\int_\Omega h(x)\bar{v}^p\,dx - \int_\Omega F^+(x, t\bar{v})\,dx \\ &\leqslant \frac{t^2}{2}\|\bar{v}\|^2 - \frac{\lambda t^p}{p}\int_\Omega h(x)\bar{v}^p\,dx \\ &< 0, \end{aligned}$$

for all $t > 0$ small enough. Hence, $c_1 < 0$.

By Ekeland's variational principle, for any $k > 1$, there exists u_k such that

$$c_1 \leqslant I_\lambda^+(u_k) \leqslant c_1 + \frac{1}{k}, \tag{4.1}$$

$$I_\lambda^+(w) \geqslant I_\lambda^+(u_k) - \frac{1}{k}\|u_k - w\|, \quad \forall w \in \overline{B}(\rho). \tag{4.2}$$

Then $\|u_k\| < \rho$ for k large enough. Otherwise, if $\|u_k\| = \rho$ for infinitely many k, without loss of generality, we may assume that for all $k \geqslant 1$, $\|u_k\| = \rho$. Then from Lemma 4.2 it follows $0 < R \leqslant I_\lambda^+(u_k) \leqslant c_1 + \frac{1}{k} < 0$, for k large enough, a contradiction.

Now we prove that $\nabla I_\lambda^+(u_k) \to 0$ in H^*. In fact, for any $u \in H$ with $\|u\| = 1$, let $w_k = u_k + tu$. Then for a fixed $k > 1$, we know $\|w_k\| \leqslant \|u_k\| + t < \rho$, for $t > 0$ small enough. So, (4.2) implies

$$I_\lambda^+(u_k + tu) \geqslant I_\lambda^+(u_k) - \frac{t}{k}\|u\| = I_\lambda^+(u_k) - \frac{t}{k}.$$

Thus,

$$\frac{I_\lambda^+(u_k + tu) - I_\lambda^+(u_k)}{t} \geqslant -\frac{1}{k}.$$

Setting $t \to 0$, we derive that

$$\left|\left\langle \nabla I_\lambda^+(u_k), u\right\rangle\right| \leqslant \frac{1}{k},$$

for any $u \in H$, with $\|u\| = 1$. So, $\nabla I_\lambda^+(u_k) \to 0$ and (4.1) gives $I_\lambda^+(u_k) \to c_1$. Hence, it follows from Lemma 2.1 that there exists $v_+ \in H$ such that $\nabla I_\lambda^+(v_+) = 0$. v_+ is a weak solution of problem (1.1) and $I_\lambda(v_+) < 0$. Moreover, the maximum principle implies that $v_+ > 0$ a.e. in Ω. By an analogous way we know there exists a negative solution v_-, which is a nontrivial critical point of I_λ^-, satisfying $I_\lambda(v_-) < 0$. Using the similar arguments as in the proof of Theorem 3.1, we know there exist two nontrivial solutions $u_+ > 0$, $u_- < 0$, satisfying $I_\lambda(u_\pm) \geqslant R > 0$. $\square$

Theorem 4.1 is local, since λ has to be small enough. A global result of Ambrosetti–Brezis–Cerami type is also given in the following theorem. For convenience we only consider positive solutions.

Theorem 4.2. *Assume that $c < \lambda_1$, $h \geqslant h_0$, where h_0 is a positive constant,* (H1) *and* (H2) *hold. Then there exists $\Lambda > 0$ such that*

1. *for $\lambda \in (0, \Lambda)$, problem* (1.1) *has at least two positive solutions: u_λ and v_λ, satisfying $v_\lambda < u_\lambda$, $I_\lambda(v_\lambda) < 0$. Moreover, v_λ is a minimal solution and is increasing with respect to λ;*
2. *for $\lambda = \Lambda$, problem* (1.1) *has at least one positive solution;*
3. *for all $\lambda > \Lambda$, problem* (1.1) *has no positive solution.*

The above theorem suggests that the structure of the set of positive solutions of (1.1) looks as shown in Fig. 1.

Let us define $\Lambda = \sup\{\lambda > 0\colon\ (1.1) \text{ has a solution}\}$.

Lemma 4.3. $0 < \Lambda < \infty$.

Proof. From Theorem 4.1 it follows that (1.1) has at least two positive solutions whenever $\lambda \in (0, \lambda^*)$ and thus $\Lambda \geqslant \lambda^* > 0$. Let $\bar{\lambda}$ be such that

$$\bar{\lambda} h(x) t^{p-1} + f(x, t) > \Lambda_1 t, \quad \forall t > 0.$$

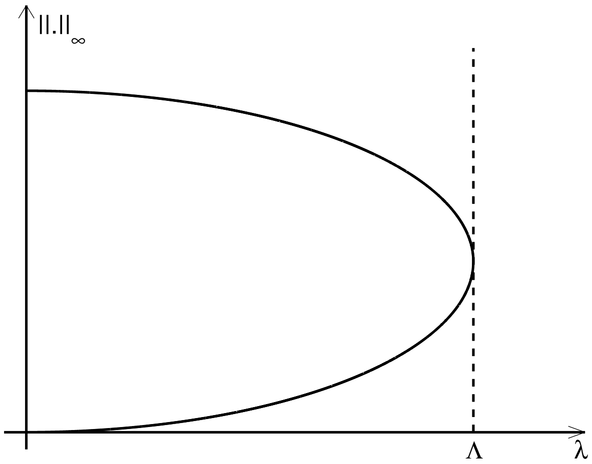

Fig. 1. The structure of the set of positive solutions.

If λ is such that (1.1) has a positive solution u, multiplying (1.1) by φ_1 and integrating over Ω we find

$$\Lambda_1 \int_\Omega u\varphi_1\,dx = \lambda \int_\Omega h(x)u^{p-1}\varphi_1\,dx + \int_\Omega f(x,u)\varphi_1\,dx.$$

This implies that $\lambda < \bar{\lambda}$ and shows that $\Lambda \leqslant \bar{\lambda}$. $\square$

Now, we recall the version of the method of upper–lower solutions which we will use to prove Theorem 4.2. Let $g(x,s)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ with the property that for any $s_0 > 0$, there exists a constant A such that $|g(x,s)| \leqslant A$ for a.e. $x \in \Omega$ and all $s \in [-s_0, s_0]$. A function $u \in H$ is called a (weak) lower solution of the problem

$$\begin{cases} \Delta^2 u + c\Delta u = g(x,u), & \text{in } \Omega,\\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

if $u \leqslant 0 \leqslant \Delta u$ on $\partial\Omega$ and

$$\int_\Omega (\Delta u\Delta\varphi - c\nabla u\cdot\nabla\varphi)\,dx \leqslant \int_\Omega g(x,u)\varphi\,dx$$

for all $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$. An upper solution is defined by reversing the inequality signs.

Lemma 4.4. *Assume that $\underline{u}$ and $\bar{u}$ are respectively lower and upper solutions for* (4.3)*, with $\underline{u} \leqslant \bar{u}$ a.e. in Ω. Consider the associated functional*

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)\,dx,$$

where $G(x,u) = \int_0^s g(x,t)\,dt$, and the interval $M := \{u \in H\colon\ \underline{u} \leqslant u \leqslant \bar{u}$ a.e. in $\Omega\}$. Then the infimum of Φ on M is achieved at some u, and u is a solution of (4.3)*.*

Proof. The proof is adapted from [16] which deals with the p-Laplacian operator. By coercivity and weak lower semicontinuity, the infimum of Φ on M is achieved at some u. Let $\varphi \in C_c^\infty(\Omega)$, $\varepsilon > 0$, and define

$$v_\varepsilon := \min\bigl\{\bar{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\bigr\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon,$$

where $\varphi^\varepsilon := \max\{0, u + \varepsilon\varphi - \bar{u}\}$ and $\varphi_\varepsilon := -\min\{0, u + \varepsilon\varphi - \underline{u}\}$. Since u minimizes Φ on M, it follows $\langle\nabla\Phi(u), v_\varepsilon - u\rangle \geqslant 0$, which gives

$$\bigl\langle\nabla\Phi(u),\varphi\bigr\rangle \geqslant \bigl(\bigl\langle\nabla\Phi(u),\varphi^\varepsilon\bigr\rangle - \bigl\langle\nabla\Phi(u),\varphi_\varepsilon\bigr\rangle\bigr)/\varepsilon. \tag{4.4}$$

Since $\bar{u}$ is an upper solution, one also has

$$\left\langle \nabla\Phi(u), \varphi^{\varepsilon}\right\rangle \geqslant \left\langle \nabla\Phi(u) - \nabla\Phi(\bar{u}), \varphi^{\varepsilon}\right\rangle$$
$$\geqslant \varepsilon \int_{\Omega_\varepsilon} \big(\Delta(u-\bar{u})\Delta\varphi - c\nabla(u-\bar{u})\cdot\nabla\varphi\big)\,dx - \varepsilon \int_{\Omega_\varepsilon} \big|g(x,u) - g(x,\bar{u})\big||\varphi|\,dx$$

where $\Omega_\varepsilon := \{x \in \Omega\colon\ u(x) + \varepsilon\varphi(x) \geqslant \bar{u}(x) > u(x)\}$. Since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$, it implies $\langle \nabla\Phi(u), \varphi^{\varepsilon}\rangle \geqslant o(\varepsilon)$ as $\varepsilon \to 0$. Similarly $\langle \nabla\Phi(u), \varphi_{\varepsilon}\rangle \leqslant o(\varepsilon)$, and by (4.4), $\langle \nabla\Phi(u), \varphi\rangle \geqslant 0$. Replacing φ by $-\varphi$, we conclude that u solves (4.3). $\square$

Proof of Theorem 4.2. 1. For convenience we denote (1.1) by $(1.1)_\lambda$. We first show that for all $\lambda \in (0, \Lambda)$, $(1.1)_\lambda$ has a positive solution v_λ with $I_\lambda(v_\lambda) < 0$. Let $0 < \lambda < \Lambda$ and take $\bar{\lambda}$ with $\lambda < \bar{\lambda} \leqslant \Lambda$ such that $(1.1)_{\bar{\lambda}}$ has a positive solution $\bar{v}$. Then,

$$\int_\Omega (\Delta\bar{v}\Delta\varphi - c\nabla\bar{v}\cdot\nabla\varphi)\,dx = \bar{\lambda}\int_\Omega h(x)|\bar{v}|^{p-2}\bar{v}\varphi\,dx + \int_\Omega f(x,\bar{v})\varphi\,dx$$
$$\geqslant \lambda\int_\Omega h(x)|\bar{v}|^{p-2}\bar{v}\varphi\,dx + \int_\Omega f(x,\bar{v})\varphi\,dx$$

for all $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$. This implies that $\bar{v}$ is an upper solution for $(1.1)_\lambda$. Let $v_\varepsilon = \varepsilon\varphi_1$, $\varepsilon > 0$. From (H2), there exists ε small enough such that

$$\Lambda_1\varepsilon\varphi_1 < \lambda h(x)\varepsilon^{p-1}\varphi_1^{p-1} + f(x, \varepsilon\varphi_1).$$

We can easily obtain

$$\int_\Omega (\Delta v_\varepsilon\Delta\varphi - c\nabla v_\varepsilon\cdot\nabla\varphi)\,dx \leqslant \int_\Omega \Lambda_1 v_\varepsilon\varphi\,dx \leqslant \lambda\int_\Omega h(x)|v_\varepsilon|^{p-2}v_\varepsilon\varphi\,dx + \int_\Omega f(x,v_\varepsilon)\varphi\,dx$$

which shows that v_ε is a lower solution of $(1.1)_\lambda$. Take ε small enough such that $\varepsilon\varphi_1 \leqslant \bar{v}$. From Lemma 4.4 we know $(1.1)_\lambda$ has a positive solution v_λ. Moreover the minimization property provided by Lemma 4.4 leads to $I_\lambda(v_\lambda) \leqslant I_\lambda(\varepsilon\varphi_1)$. By (H2), $I_\lambda(\varepsilon\varphi_1) < 0$ for ε small enough. Assume that v_λ is isolated local minima. From Lemma 3.1 and Lemma 2.1 we can get another solution u_λ by using the mountain pass theorem. It remains to prove that $v_\lambda < v_{\hat{\lambda}}$ whenever $\lambda < \hat{\lambda}$. Indeed, if $\lambda < \hat{\lambda}$ then $v_{\hat{\lambda}}$ is an upper solution of $(1.1)_\lambda$. For $\varepsilon > 0$ small, $\varepsilon\varphi_1$ is a lower solution of $(1.1)_\lambda$ and $\varepsilon\varphi_1 < v_{\hat{\lambda}}$, then $(1.1)_\lambda$ has a positive solution v with $v \leqslant v_{\hat{\lambda}}$. As v_λ is the minimal solution of $(1.1)_\lambda$, we have $v_\lambda \leqslant v \leqslant v_{\hat{\lambda}}$. The strict inequality follows from the strong maximum principle.

2. Let $\{\mu_n\}$ be a sequence such that $\mu_n \uparrow \Lambda$. Since the positive solution $v_n = v_{\mu_n}$ satisfies $\langle \nabla I_{\mu_n}(v_n), v_n\rangle = 0$, we have

$$\|v_n\|^2 = \lambda\int_\Omega h(x)v_n^p\,dx + \int_\Omega f(x,v_n)v_n\,dx$$
$$\leqslant C\|v_n\|_{L^2}^p + C|\Omega| + C\|v_n\|_{L^2}^2. \tag{4.5}$$

As the proof of Lemma 2.1 we obtain that $\|v_n\|_{L^2}$ is bounded, and hence $\{v_n\}$ is bounded in H. Then there exists $v^* \in H$ such that $v_n \to v^* > 0$ a.e. in Ω, strongly in $L^2(\Omega)$ and weakly in H. v^* is a positive solution of $(1.1)_\Lambda$.

3. This follows from the definition of Λ. $\square$

5. Existence of five solutions

This section is devoted to give more information about the multiplicity of the solutions of problem (1.1). Precisely, we give the following multiplicity result which improves Theorem 4.1.

Theorem 5.1. *Assume that $c < \lambda_1$, $f(x,u)$ satisfies* (H1) *and* (H2), *$h \in L^\infty(\Omega)$ with $h \geqslant h_0$, where h_0 is a positive constant. Then there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$ problem* (1.1) *has at least five nontrivial solutions: u_+, u_-, v_+, v_-, and v_3, satisfying $u_+ > 0$, $u_- < 0$, $v_+ > 0$, $v_- < 0$, and $I_\lambda(u_\pm) > 0 > I_\lambda(v_\pm)$, $I_\lambda(v_3) < 0$.*

Proof. First of all, by an analogous argument as in the proof of Theorem 4.1, the existence of u_+, u_-, v_+ and v_- follows. We need only to show the existence of v_3 with $I_\lambda(v_3) < 0$. The energy functional for problem (1.1) is

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_\Omega h(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx, \quad u \in H.$$

Note that, according to the proof of Theorem 4.1, v_+ and v_- are also local minima of I_λ. We can assume that v_+ and v_- are isolated local minima. Let us denote by b_λ the mountain pass critical level of I_λ with base points v_+, v_-:

$$b_\lambda = \inf_{\psi\in\Psi} \max_{t\in[0,1]} I_\lambda\big(\psi(t)\big),$$

where

$$\Psi = \big\{\psi \in C^1\big([0,1], H\big):\ \psi(0) = v_+,\ \psi(1) = v_-\big\}.$$

We claim that $b_\lambda < 0$ if λ is small enough. In fact, by (H2), for any $0 < \varepsilon_* < \mu$, there exists $\rho > 0$, such that for $|s| < \rho$,

$$\mu - \varepsilon_* < \frac{f(x,s)}{s} < \mu + \varepsilon_*,$$

and

$$\frac{1}{2}(\mu - \varepsilon_*)s^2 < F(x,s) < \frac{1}{2}(\mu + \varepsilon_*)s^2. \tag{5.1}$$

We define $\Gamma_\pm, \Upsilon_\pm : [0,1] \to \mathbb{R}$ by

$$\Gamma_\pm(t) = I_\lambda(tv_\pm) = \frac{t^2}{2}\|v_\pm\|^2 - \lambda\frac{t^p}{p}\int_\Omega h(x)|v_\pm|^p\,dx - \int_\Omega F(x,tv_\pm)\,dx,$$

$$\Upsilon_\pm(t) = \frac{t^2}{2}\|v_\pm\|^2 - \lambda\frac{t^p}{p}\int_\Omega h(x)|v_\pm|^p\,dx - \frac{t^2}{2}(\mu - \varepsilon_*)\|v_\pm\|_{L^2}^2.$$

Then from (5.1) we know $\Gamma_\pm(t) < \Upsilon_\pm(t)$, for all $t \in [0,1]$.

If there exists $t_0 \in (0,1)$ such that $\Gamma_\pm(t_0) = \max_{t\in(0,1)} \Gamma_\pm(t)$, then $\Gamma'_\pm(t_0) = 0$. Let

$$M_\pm(t) = \|v_\pm\|^2 - \int_\Omega \frac{f(x,tv_\pm)}{tv_\pm}v_\pm^2.$$

Since

$$\begin{aligned}
\Gamma'_\pm(t) &= t\|v_\pm\|^2 - \lambda t^{p-1}\int_\Omega h(x)|v_\pm|^p\,dx - \int_\Omega f(x,tv_\pm)v_\pm\,dx \\
&= t\|v_\pm\|^2 - \lambda t^{p-1}\int_\Omega h(x)|v_\pm|^p\,dx - \int_\Omega \frac{f(x,tv_\pm)}{tv_\pm}tv_\pm^2\,dx \\
&= \left(\|v_\pm\|^2 - \int_\Omega \frac{f(x,tv_\pm)}{tv_\pm}v_\pm^2\right)t - \lambda t^{p-1}\int_\Omega h(x)|v_\pm|^p\,dx \\
&= M_\pm(t)t - \lambda t^{p-1}\int_\Omega h(x)|v_\pm|^p\,dx,
\end{aligned} \tag{5.2}$$

setting $\Gamma'_\pm(t) = 0$, we know that t_0 satisfies $M_\pm(t_0)t_0^{2-p} = \lambda\int_\Omega h(x)|v_\pm|^p\,dx$. Then noticing $1 < p < 2$ and using the Poincaré inequality (2.3), we have

$$\begin{aligned}
\Gamma_\pm(t_0) &< \Upsilon_\pm(t_0) \\
&= \frac{t_0^2}{2}\|v_\pm\|^2 - \lambda\frac{t_0^p}{p}\int_\Omega h(x)|v_\pm|^p\,dx - \frac{t_0^2}{2}(\mu - \varepsilon_*)\|v_\pm\|_{L^2}^2 \\
&= \left(\frac{1}{2}\|v_\pm\|^2 - \frac{\mu - \varepsilon_*}{2}\|v_\pm\|_{L^2}^2 - \frac{M_\pm(t_0)}{p}\right)t_0^2 \\
&\leqslant \left(\frac{1}{2}\|v_\pm\|^2 - \frac{\mu - \varepsilon_*}{2}\|v_\pm\|_{L^2}^2 - \frac{1}{p}\|v_\pm\|^2 + \frac{\mu + \varepsilon_*}{p}\|v_\pm\|_{L^2}^2\right)t_0^2 \\
&= \left(\left(\frac{1}{2} - \frac{1}{p}\right)\|v_\pm\|^2 + \left(\frac{\mu + \varepsilon_*}{p} - \frac{\mu - \varepsilon_*}{2}\right)\|v_\pm\|_{L^2}^2\right)t_0^2 \\
&\leqslant \left(\left(\frac{1}{2} - \frac{1}{p}\right)\Lambda_1 + \left(\frac{\mu + \varepsilon_*}{p} - \frac{\mu - \varepsilon_*}{2}\right)\right)\|v_\pm\|_{L^2}^2 t_0^2.
\end{aligned} \tag{5.3}$$

Take $\varepsilon_* > 0$, such that $\varepsilon_* < \frac{(2-p)(\Lambda_1-\mu)}{p+2}$. From (5.3) it follows that $\Gamma_\pm(t_0) < 0$. Recall that $\Gamma_\pm(0) = 0$, $\Gamma_\pm(1) < 0$. Hence, for $t \in (0,1)$,

$$\Gamma(t) < 0.$$

This implies that there exists $\delta > 0$ such that

$$I_\lambda(tv_\pm) < 0, \quad \forall t \in (0,1),\ 0 < \lambda < \delta. \tag{5.4}$$

Now let us consider the 2-dimensional plane Π_2 containing the straightlines tv_- and tv_+, and take $v \in \Pi_2$ with $\|v\| = \varepsilon$. Note that for such v one has $\|v\|_{L^p} = C_p\varepsilon$, and $\|v\|_{L^2} = C_2\varepsilon$. Then we get

$$I_\lambda(v) \leqslant \frac{\varepsilon^2}{2} - \frac{\lambda}{p}C_p^p h_0 \varepsilon^p - C_2^2\varepsilon^2.$$

For small ε,

$$I_\lambda(v) < 0. \tag{5.5}$$

Consider the path $\bar{\gamma}$ obtained glueing together the segments $\{tv_-\colon\ \varepsilon\|v_-\|^{-1} \leqslant t \leqslant 1\}$, $\{tv_+\colon\ \varepsilon\|v_+\|^{-1} \leqslant t \leqslant 1\}$ and the arc $\{v \in \Pi_2\colon\ \|v\| = \varepsilon\}$. From (5.4) and (5.5) it follows that

$$b_\lambda \leqslant \max_{v \in \bar{\gamma}} I_\lambda(v) < 0,$$

which verifies the claim. Since the (PS) condition holds because of Lemma 2.1, the level $\{I_\lambda(v) = b_\lambda\}$ carries a critical point v_3 of I_λ, and v_3 is different from $v_\pm$. □

Acknowledgments

The author would like to thank Professor Yong Li and Professor Yanheng Ding for their helpful instructions. The author is grateful to anonymous referees for valuable suggestions.

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