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Classification of Moran fractals

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ABSTRACT

In the paper, we try to classify Moran fractals by using the quasi-Lipschitz equivalence, and prove that two regular homogeneous Moran sets are quasi-Lipschitz equivalent if and only if they have the same Hausdorff dimension.

1. Introduction

How to classify fractals in Euclidean spaces? A natural approach is the classification by using the Lipschitz equivalence:

Definition 1. Suppose that A and B are compact sets in Euclidean spaces. We say that E and F are *Lipschitz equivalent* if there are a bijection $f : A \to B$ and a constant $c > 0$ such that

$$c^{-1}|x-y| \leqslant \left|f(x)-f(y)\right| \leqslant c|x-y| \quad \text{for all } x, y \in A. \tag{1.1}$$

The Lipschitz equivalence of fractals is an interesting topic. For example, it is proved in [2] that two quasi-self-similar circles are Lipschitz equivalent if and only if they have the same Hausdorff dimension.

It is well known that the Lipschitz equivalence of sets A and B implies that $\dim_H A = \dim_H B$. However, self-similar sets with the same Hausdorff dimensions need not be Lipschitz equivalent, as in the following two examples.

(1) Let $\beta \in (0,1)$ with $3\beta^{\log 2/\log 3} = 1$, and $E = \beta E \cup (\beta E + \frac{1-\beta}{2}) \cup (\beta E + 1 - \beta)$ the self-similar set. Then $\dim_H E = \dim_H C$, where C is the Cantor ternary set. However E and C are not Lipschitz equivalent (see [1]).

(2) In [6], two self-similar arcs are constructed such that they have the same Hausdorff dimension but they are not Lipschitz equivalent.

Remark 1. Please refer to [3,9] to find the conditions for self-similar sets to be Lipschitz equivalent. For nearly Lipschitz equivalence, see [3,7].

To classify self-conformal sets satisfying the strong separation condition, Xi introduced in [8] the notion of the quasi-Lipschitz equivalence.

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Definition 2. Two compact sets E and F of Euclidean spaces are said to be *quasi-Lipschitz equivalent*, if there is a bijection $f : E \to F$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\left| \frac{\log|f(x) - f(y)|}{\log|x - y|} - 1 \right| < \varepsilon, \tag{1.2}$$

whenever $x, y \in E$ with $0 < |x - y| < \delta$.

If E and F are quasi-Lipschitz equivalent, then $\dim_H E = \dim_H F$ (see [8]). It is proved in [8] that two self-conformal sets satisfying the strong separation condition are quasi-Lipschitz equivalent if and only if they have the same Hausdorff dimension.

In this paper, we will try to classify homogeneous Moran sets by using the quasi-Lipschitz equivalence. Here, homogeneous Moran set (see [5]) is defined as bellow.

Definition 3. For a given number sequence $\{c_k\}_{k\geqslant 1}$ and a positive integer sequence $\{n_k\}_{k\geqslant 1}$, we suppose that $n_k \geqslant 2$ and $n_k c_k \in (0, 1)$ for all k. Suppose I is a closed interval of $\mathbb{R}^1$. Let $\mathcal{D}^k = \prod_{i=1}^{k}\{1, \dots, n_i\}$, $\mathcal{D}^0 = \{\emptyset\}$ and $I_\emptyset = I$ where $\emptyset$ is the empty word. Suppose that for any $i_1 \cdots i_{k-1} \in \mathcal{D}^{k-1}$, there are closed interval $I_{i_1\cdots i_{k-1}}$ and its n_k closed sub-intervals

$$I_{i_1\cdots i_{k-1}1}, I_{i_1\cdots i_{k-1}2}, \dots, I_{i_1\cdots i_{k-1}n_k} \subset I_{i_1\cdots i_{k-1}}, \tag{1.3}$$

such that $\{\mathrm{int}(I_{i_1\cdots i_{k-1}j})\}_{j=1}^{n_k}$ are pairwise disjoint and

$$|I_{i_1\cdots i_{k-1}j}|/|I_{i_1\cdots i_{k-1}}| = c_k, \quad \text{for } j = 1, \dots, n_k, \tag{1.4}$$

where $\mathrm{int}(I')$ and $|I'|$ are the interior and length of interval I' respectively. Then

$$E = \bigcap_{k=0}^{\infty} \bigcup_{i_1\cdots i_k \in \mathcal{D}^k} I_{i_1\cdots i_k} \tag{1.5}$$

is called a *homogeneous Moran set* with structure $(I, \{n_k\}, \{c_k\})$. We say that $I_{i_1\cdots i_k}$ is a *basic interval* of rank k.

Definition 4. A structure $(I, \{n_k\}, \{c_k\})$ is said to be *regular*, if there exists $s \in (0, 1)$ such that

$$\lim_{k\to\infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k} = s \quad \text{and} \quad \sup_k n_k < +\infty. \tag{1.6}$$

We say that a homogeneous Moran set is regular, if it has a regular structure.

Remark 2. For any regular homogeneous Moran set E, $\dim_H E = s$ as in (1.6) (see [5]).

In this paper, we discuss the quasi-Lipschitz equivalence of regular homogeneous Moran sets and the main result is stated as follows.

Theorem 1. *Two regular homogeneous Moran sets E and F are quasi-Lipschitz equivalent if and only if* $\dim_H E = \dim_H F$.

Remark 3. Examples of [4] show that Theorem 1 does not work for homogeneous Moran sets which are not regular, i.e., one of the following three conditions fails:

(1) $\lim_{k\to\infty} \frac{\log n_1\cdots n_k}{-\log c_1\cdots c_k}$ exists;
(2) $\sup_k n_k < +\infty$;
(3) $\lim_{k\to\infty} \frac{\log n_1\cdots n_k}{-\log c_1\cdots c_k} \in (0, 1)$, i.e., $s \notin \{0, 1\}$.

The paper is organized as follows. Section 2 includes a technical lemma. Section 3 is the proof of Theorem 1, which is based on constructing a bijection from a regular homogeneous Moran set to symbolic system Σ_2 (Proposition 1).

2. Preliminary

Without loss of generality, we assume $I = [0, 1]$. We denote that

$$\alpha_k = \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}, \qquad \beta_k = \sup_{m\geqslant k} |\alpha_m - s|, \qquad \rho_k = (c_1 \cdots c_{k-1}) \frac{1 - n_k c_k}{n_k + 1}.$$

Lemma 1. *Suppose the structure* $([0,1],\{n_k\},\{c_k\})$ *is regular. Then there is an integer sequence* $\{k_t\}_t$ *satisfying*:

(1) $k_{t+1} > k_t$, $k_{t+1} \geqslant \frac{t(t+1)}{2}$ *for all t, and* $\lim_{t\to\infty}|k_{t+1}-k_t| = \infty$;
(2) $\lim_{t\to\infty}\frac{k_{t+1}}{k_t} = 1$;
(3) $\lim_{t\to\infty}\frac{\log\rho_{k_t}}{\log c_1\cdots c_{k_t-1}} = 1$;
(4) $\lim_{t\to\infty}\frac{\log\rho_{k_t}}{\log c_1\cdots c_{k_{(t-1)}-1}} = \lim_{t\to\infty}\frac{\log n_1\cdots n_{k_t}}{-s\log c_1\cdots c_{k_{(t-1)}-1}} = 1$.

Proof. We can verify the following estimation to replace condition (3) in the lemma,

$$\frac{\log n_{k_t}}{-\log c_{k_t}} \leqslant \frac{1}{2}+\frac{s}{2}(>s) \quad (\text{for all } t\in\mathbb{N}). \tag{2.1}$$

We conclude that (2.1) implies condition (3). In fact, since $n_{k_t}c_{k_t} < 1$ and $n_{k_t} \geqslant 2$, we have $c_{k_t} < \frac{1}{2}$, and estimation (2.1) implies that $n_{k_i}c_{k_i}^{(1+s)/2} \leqslant 1$. Consequently,

$$n_{k_t}c_{k_t} = \left(n_{k_t}c_{k_t}^{(1+s)/2}\right)c_{k_t}^{1/2-s/2} \leqslant \left(\frac{1}{2}\right)^{1/2-s/2},$$

which implies

$$1-n_{k_t}c_{k_t} \geqslant 1-\left(\frac{1}{2}\right)^{1/2-s/2} > 0.$$

Letting $t\to\infty$, we get condition (3),

$$\frac{\log\rho_{k_t}}{\log c_1\cdots c_{k_t-1}} = 1+\frac{\log(1-n_{k_t}c_{k_t})-\log(n_{k_t}+1)}{\log c_1\cdots c_{k_t-1}} \to 1,$$

where $\sup_k n_k < \infty$ and $\lim_{t\to\infty} c_1\cdots c_{k_t-1} = 0$ since

$$0 \leqslant \lim_{t\to\infty} c_1\cdots c_{k_t-1} \leqslant \lim_{t\to\infty}\left(\frac{1}{2}\right)^{k_t-1} \leqslant \lim_{t\to\infty}\left(\frac{1}{2}\right)^{t-1} = 0$$

due to $k_t \geqslant t$, here $k_t > k_{t-1} > k_{t-2} > \cdots > k_1$ by condition (1).

By induction, we will construct the sequence $\{k_t\}_t$.

We take k_1 large so enough that $\beta_{k_1} \leqslant \frac{3}{32}s(1-s)$. By the induction, assume that k_t has already been defined, satisfying $\beta_{k_t} \leqslant \beta_{k_1} \leqslant \frac{3}{32}s(1-s) < \frac{1}{4}s$. Let

$$\begin{aligned} p_t &= k_t + t, \\ q_t &= \left[p_t\left(1+\frac{32}{3s(1-s)}\frac{\log(\sup_k n_k)}{\log 2}\beta_{k_t}\right)\right]+2, \end{aligned} \tag{2.2}$$

where $[x]$ is the integer part of x.

Then $\alpha_{q_t} = \frac{\log n_1\cdots n_{p_t}+\sum_{p_t<k\leqslant q_t}\log n_k}{-\log c_1\cdots c_{p_t}+\sum_{p_t<k\leqslant q_t}(-\log c_k)}$. Write

$$\Delta = \frac{\sum_{p_t<k\leqslant q_t}(-\log c_k)}{-\log c_1\cdots c_{p_t}+\sum_{p_t<k\leqslant q_t}(-\log c_k)}.$$

We conclude that there exists an integer $k_{t+1}\in(p_t,q_t]$ such that

$$\frac{\log n_{k_{t+1}}}{-\log c_{k_{t+1}}} \leqslant \frac{1}{2}+\frac{s}{2}. \tag{2.3}$$

Otherwise, we assume that $\frac{\log n_k}{-\log c_k} > \frac{1}{2}+\frac{s}{2}$ for $k\in(p_t,q_t]$. It follows from the definition of β_{k_t} that

$$\begin{aligned} s+\beta_{k_t} \geqslant \alpha_{q_t} &= \frac{\log n_1\cdots n_{p_t}+\sum_{p_t<k\leqslant q_t}\log n_k}{-\log c_1\cdots c_{p_t}+\sum_{p_t<k\leqslant q_t}(-\log c_k)} = \frac{\log n_1\cdots n_{p_t}}{-\log c_1\cdots c_{p_t}}\cdot\frac{-\log c_1\cdots c_{p_t}}{-\log c_1\cdots c_{p_t}+\sum_{p_t<k\leqslant q_t}(-\log c_k)} \\ &\quad + \sum_{p_t<k\leqslant q_t}\frac{\log n_k}{-\log c_k}\cdot\frac{-\log c_k}{-\log c_1\cdots c_{p_t}+\sum_{p_t<k\leqslant q_t}(-\log c_k)} \geqslant \alpha_{p_t}(1-\Delta)+\left(\frac{1}{2}+\frac{s}{2}\right)\Delta \\ &\geqslant (s-\beta_{k_t})(1-\Delta)+\left(\frac{1}{2}+\frac{s}{2}\right)\Delta. \end{aligned} \tag{2.4}$$

Then inequality (2.4) implies

$$\Delta \leqslant \frac{2\beta_{k_t}}{\frac{1}{2} - \frac{s}{2} + \beta_{k_t}} \leqslant \frac{4}{1-s}\beta_{k_t}. \tag{2.5}$$

Notice that

$$\frac{-\log c_1 \cdots c_{q_t}}{\log n_1 \cdots n_{q_t}} \leqslant \frac{1}{s - \beta_{k_t}} \leqslant \frac{1}{s - s/4} = \frac{4}{3s} \tag{2.6}$$

due to $\beta_{k_t} < s/4$. Hence, by (2.6) and inequality $\frac{\log n_k}{-\log c_k} \leqslant 1$, we have

$$\left(\frac{\sum_{p_t < k \leqslant q_t} \log n_k}{\log n_1 \cdots n_{q_t}}\right) / \Delta = \frac{\sum_{p_t < k \leqslant q_t} \log n_k}{\sum_{p_t < k \leqslant q_t} (-\log c_k)} \cdot \frac{-\log c_1 \cdots c_{q_t}}{\log n_1 \cdots n_{q_t}} \leqslant \frac{4}{3s}. \tag{2.7}$$

It follows from (2.5) and (2.7) that

$$\frac{\sum_{p_t < k \leqslant q_t} \log n_k}{\log n_1 \cdots n_{q_t}} = \left(\frac{\sum_{p_t < k \leqslant q_t} \log n_k}{\log n_1 \cdots n_{q_t}} / \Delta\right) \cdot \Delta \leqslant \frac{16}{3s(1-s)}\beta_{k_t}\left(< \frac{1}{2}\right). \tag{2.8}$$

On the other hand,

$$\frac{\sum_{p_t < k \leqslant q_t} \log n_k}{\log n_1 \cdots n_{q_t}} \geqslant \frac{\sum_{p_t < k \leqslant q_t} \log 2}{\log n_1 \cdots n_{p_t} + \sum_{p_t < k \leqslant q_t} \log 2} \geqslant \frac{\frac{q_t - p_t}{p_t \log(\sup_k n_k)} \log 2}{1 + \frac{q_t - p_t}{p_t \log(\sup_k n_k)} \log 2}. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\frac{\frac{q_t - p_t}{p_t \log(\sup_k n_k)} \log 2}{1 + \frac{q_t - p_t}{p_t \log(\sup_k n_k)} \log 2} \leqslant \frac{16}{3s(1-s)}\beta_{k_t} < 1/2,$$

which implies $\lambda \hat{=} \frac{q_t - p_t}{p_t \log(\sup_k n_k)} \log 2 < 1$, and thus

$$\frac{\lambda}{2} < \frac{\lambda}{1+\lambda} \leqslant \frac{16}{3s(1-s)}\beta_{k_t},$$

i.e.,

$$\frac{q_t - p_t}{p_t} \leqslant \frac{32}{3s(1-s)} \cdot \frac{\log(\sup_k n_k)}{\log 2}\beta_{k_t}$$

which contradicts to (2.2). Therefore, we can take $k_{t+1} \in (p_t, q_t]$ such that

$$\frac{\log n_{k_{t+1}}}{-\log c_{k_{t+1}}} \leqslant \frac{1}{2} + \frac{s}{2},$$

then (2.1) follows, and thus condition (3) yields. Here

$$k_{t+1} \geqslant k_t + t \geqslant k_{t-1} + (t-1) + t \geqslant \frac{t(t+1)}{2},$$

which implies condition (1) holds. We also obtain that $\frac{t}{k_t} \to 0$. Consequently,

$$\frac{p_t}{k_t} = 1 + \frac{t}{k_t} \to 1 \quad \text{as } t \to \infty.$$

Letting $t \to \infty$, we notice that

$$\frac{p_t}{k_t} \leqslant \frac{k_{t+1}}{k_t} \leqslant \frac{p_t}{k_t} \cdot \frac{q_t}{p_t} \quad \text{and} \quad \frac{q_t}{p_t} \to 1$$

which implies $\frac{k_{t+1}}{k_t} \to 1$. Then condition (2) is proved.

For condition (4), we notice that

$$\lim_{t\to\infty} \frac{\log n_1 \cdots n_{k_t - 1}}{\log n_1 \cdots n_{k_{(t-1)}-1}} = 1 + \lim_{t\to\infty} \frac{\log n_{k_{(t-1)}} \cdots n_{k_t - 1}}{\log n_1 \cdots n_{k_{(t-1)}-1}},$$

where $|\frac{\log n_{k_{(t-1)}}\cdots n_{k_t-1}}{\log n_1\cdots n_{k_{(t-1)}-1}}| \leqslant \frac{k_t-k_{(t-1)}}{k_{(t-1)}}\frac{\log(\sup_k n_k)}{\log 2} \to 0$ due to condition (2). That means

$$\lim_{t\to\infty}\frac{\log n_1\cdots n_{k_t-1}}{\log n_1\cdots n_{k_{(t-1)}-1}}=1. \tag{2.10}$$

It follows from (2.10) and condition (3) that

$$\begin{aligned}\lim_{t\to\infty}\frac{\log\rho_{k_t}}{\log c_1\cdots c_{k_{(t-1)}-1}} &= \lim_{t\to\infty}\frac{\log c_1\cdots c_{k_t-1}}{\log c_1\cdots c_{k_{(t-1)}-1}} = \lim_{t\to\infty}\frac{\log c_1\cdots c_{k_t-1}}{\log c_1\cdots c_{k_{(t-1)}-1}}\cdot\lim_{t\to\infty}\frac{\log n_1\cdots n_{k_{(t-1)}-1}}{\log n_1\cdots n_{k_t-1}}\\ &= \lim_{t\to\infty}\frac{\log n_1\cdots n_{k_{(t-1)}-1}}{-\log c_1\cdots c_{k_{(t-1)}-1}}\Big/\lim_{t\to\infty}\frac{\log n_1\cdots n_{k_t-1}}{-\log c_1\cdots c_{k_t-1}} = s/s = 1.\end{aligned}$$

Since $\sup_k n_k<\infty$, we have $\lim_{t\to\infty}\frac{\log(n_1\cdots n_{k_t})}{\log(n_1\cdots n_{k_t-1})}=1$. By (2.10), we have

$$\lim_{t\to\infty}\frac{\log(n_1\cdots n_{k_t})}{-s\log(c_1\cdots c_{k_{(t-1)}-1})}=\lim_{t\to\infty}\frac{\log(n_1\cdots n_{k_{(t-1)}-1})}{-s\log(c_1\cdots c_{k_{(t-1)}-1})}\frac{\log(n_1\cdots n_{k_t-1})}{\log(n_1\cdots n_{k_{(t-1)}-1})}\frac{\log(n_1\cdots n_{k_t})}{\log(n_1\cdots n_{k_t-1})}=1. \qquad \square$$

3. Proof of Theorem 1

In the following proposition, we will construct a suitable bijection from a regular homogeneous Moran set to a symbolic system $\Sigma_2=\{0,1\}^\infty$ equipped with a metric D satisfying

$$D(x_1x_2\cdots,y_1y_2\cdots)=2^{-\min\{k:\ x_k\neq y_k\}}.$$

Given a finite word $x_1\cdots x_n$ with $x_i=0$ or 1 for all i, the set $[x_1\cdots x_n]=\{y_1y_2\cdots\in\Sigma_2:\ y_i=x_i \text{ for } i\leqslant n\}$ is called a cylinder (of length n) with respect to word $x_1\cdots x_n$. For subsets A,B of metric space (X,d_X), let $\mathrm{d}(A,B)$ denote the least distance between A and B defined by

$$\mathrm{d}(A,B)=\inf_{a\in A,b\in B}\mathrm{d}_X(a,b).$$

Proposition 1. *Suppose E is a regular homogeneous Moran set with structure (I,n_k,c_k). There exists a bijection φ from E to Σ_2 such that when $|x-y|\to 0$,*

$$\frac{\log D(\varphi(x),\varphi(y))}{s\log|x-y|}\to 1 \quad \textit{uniformly}. \tag{3.1}$$

Proof. Without loss of generality, let $I=[0,1]$. We will construct inductively φ through the sequence $\{k_i\}_i$ mentioned in Lemma 1.

Firstly we introduce the notion of *basic element* by induction. Let $[0,1]$ be the basic element of order 0 and $k_0=0$. We say that $\mathcal{J}$ is a basic element of order i, if

$$\mathcal{J}=I_1\cup\cdots\cup I_{\chi(\mathcal{J})},$$

where $\{I_u\}_{u=1}^{\chi(\mathcal{J})}$ are basic intervals (of rank k_i) with length $c_1\cdots c_{k_i}$ such that:

(1) $\mathcal{J}$ is contained in a basic element of order $(i-1)$;
(2) $I_1,\ldots,I_{\chi(\mathcal{J})}$ are arranged from left to right;
(3) $\mathrm{d}(I_u,I_{u+1})<\rho_{k_i}$ for all $u<\chi(\mathcal{J})$;
(4) for any other basic interval $I'\notin\{I_1,\ldots,I_{\chi(\mathcal{J})}\}$ of rank k_i,

$$\mathrm{d}(\mathcal{J},I')\geqslant\rho_{k_i}.$$

Denote by $\mathrm{ord}(\mathcal{J})=i$ the order of basic element. When $\chi(\mathcal{J})=1$, $\mathcal{J}$ is a basic interval of rank k_i. The basic elements are well defined by above conditions (1)–(4).

Notice that $\chi(\mathcal{J})$ is uniformly bounded. In fact,

$$\chi(\mathcal{J})\leqslant 2n_{k_{\mathrm{ord}(\mathcal{J})}}\leqslant 2(\sup_k n_k). \tag{3.2}$$

Suppose $\mathrm{ord}(\mathcal{J})=i$. Since

$$(n_{k_i}+1)\rho_{k_i}+n_{k_i}c_1\cdots c_{k_i}=c_1\cdots c_{k_i-1}, \tag{3.3}$$

the above condition $\mathrm{d}(I_u, I_{u+1}) < \rho_{k_i}$ implies that there are at most two basic intervals (of rank $k_i - 1$) with length $c_1 \cdots c_{k_i-1}$ which intersects $I_1 \cup \cdots \cup I_\chi$. Otherwise, we assume that there are basic intervals (with length $c_1 \cdots c_{k_i-1}$) intersecting $\mathcal{J}$,

$$L_1, L_2, \ldots, L_m \text{ arranged from left to right with } m \geqslant 3.$$

Set $L_2 = [a, b]$ and denote

$$\{I_1, \ldots, I_{\chi(\mathcal{J})}\} \cap \{I_u\colon\ I_u \subset L_2\} = \big\{[a_1, b_1], \ldots, [a_{n_{k_i}}, b_{n_{k_i}}]\big\},$$

which are arranged from left to right. Therefore, we have

$$|a - a_1| < \rho_{k_i}, \qquad |b_j - a_{j+1}| < \rho_{k_i} \quad \text{for all } j \quad \text{and} \quad |b_{n_{k_i}} - b| < \rho_{k_i}.$$

Then

$$\begin{aligned} c_1 \cdots c_{k_i-1} = |a - b| &= \left(|a - a_1| + \sum_j |b_j - a_{j+1}| + |b_{n_{k_i}} - b|\right) + \sum_j |a_j - b_j| \\ &< (n_{k_i} + 1)\rho_{k_i} + n_{k_i}(c_1 \cdots c_{k_i}) = c_1 \cdots c_{k_i-1}. \end{aligned}$$

This is a contradiction. Then (3.2) is proved. By (3.2) and (3.3), the diameter

$$\operatorname{diam}(\mathcal{J}) \leqslant \chi(\mathcal{J}) c_1 \cdots c_{k_i} + \big(\chi(\mathcal{J}) - 1\big)\rho_{k_i} \leqslant 2c_1 \cdots c_{k_{\operatorname{ord}(\mathcal{J})}-1}. \tag{3.4}$$

Let J be a basic element with $\operatorname{ord}(J) = i - 1$. We ask that how many basic elements of order i which are contained in J. Suppose that there are $Q_i(J)$ basic elements of order i in J. Then by (3.2), we have

$$\frac{n_{(k_{(i-1)}+1)} \cdots n_{k_i}}{2(\sup_k n_k)} \leqslant Q_i(J) \leqslant 2(\sup_k n_k) \cdot n_{(k_{(i-1)}+1)} \cdots n_{k_i}. \tag{3.5}$$

We let Γ denote the collection of all the cylinders in Σ_2, and Ω_i the collection of all the basic elements of order i. Let $\psi_0 : \Omega_i \to \Gamma$ defined by

$$\psi_0\big([0, 1]\big) = \Sigma_2.$$

By induction, we assume that $\psi_{i-1} : \Omega_{i-1} \to \Gamma$ has already been defined and $\psi_{i-1}(J) = [x_1 \cdots x_{\omega(J)}]$, where J is a basic element of order $i - 1$. Let $\sigma(J)$ is the unique integer such that

$$2^{\sigma(J)} < Q_i(J) \leqslant 2^{\sigma(J)+1}. \tag{3.6}$$

Set $T = Q_i(J) - 2^{\sigma(J)}$.

We can take $Q_i(J)$ pairwise disjoint sub-cylinders of $[x_1 \cdots x_{\omega(J)}]$ so that their lengths are

$$\text{either} \quad \omega(J) + \sigma(J) \quad \text{or} \quad \omega(J) + \sigma(J) + 1. \tag{3.7}$$

For this, we let $w = [x_1 \cdots x_{\omega(J)}]$. By adding $\sigma(J)$ digits (0 or 1) after the word w, we get $2^{\sigma(J)}$ words of length $\omega(J) + \sigma(J)$, which have the same prefix w. We denote them by

$$w_1, w_2, \ldots, w_{2^{\sigma(J)}}.$$

By adding 0 or 1 after the first T words, we get $Q_i(J)$ words of lengths $\omega(J) + \sigma(J)$ or $\omega(J) + \sigma(J) + 1$,

$$w_1 * 0, w_1 * 1, w_2 * 0, w_2 * 1, \ldots, w_T * 0, w_T * 1, w_{T+1}, \ldots, w_{2^{\sigma(J)}},$$

where $w' * a$ stand for a new word generated by add digit a after word w'. Then

$$[x_1 \cdots x_{\omega(J)}] = \left(\bigcup_{j \leqslant T} \big([w_j * 0] \cup [w_j * 1]\big)\right) \cup \left(\bigcup_{j > T} [w_j]\right) \tag{3.8}$$

is a disjoint union and there are $Q_i(J)$ sub-cylinders on the right hand of (3.8). Then by induction, there is a bijection from $\{J' \subset J\colon\ \operatorname{ord}(J') = i\}$ to $Q_i(J)$ pairwise disjoint sub-cylinders of $[x_1 \cdots x_{\omega(J)}]$ as in (3.8). Then $\psi_i : \Omega_i \to \Gamma$ is well defined by induction.

For any $x \in E$, there exists a unique sequence of basic elements $\{I_i\}_i$, where I_i is of order i, such that $\{x\} = \bigcap_i I_i$. Let $\varphi(x) (\in \Sigma_2)$ be defined by

$$\big\{\varphi(x)\big\} = \bigcap_{i=0}^{\infty} \psi_i(I_i). \tag{3.9}$$

We shall verify formula (3.1) for φ.

Given different points $x, y \in E$, we suppose that J is the smallest basic element containing both x and y. Assume J is of order $(i-1)$, and J_1 and J_2 are distinct basic elements of order i contained in J such that

$$x \in J_1, \qquad y \in J_2. \tag{3.10}$$

By the condition (4) in the definition of basic element, we have

$$|x-y| \geqslant \rho_{k_i}. \tag{3.11}$$

On the other hand, by (3.4), we have

$$|x-y| \leqslant \operatorname{diam}(J) \leqslant 2c_1 \cdots c_{k_{(i-1)}-1}. \tag{3.12}$$

It follows from (3.11), (3.12) and condition (4) of Lemma 1 that

$$\frac{\log(n_1 \cdots n_{k_i})}{-s \log|x-y|} \to 1 \quad \text{uniformly as } i \to \infty. \tag{3.13}$$

On the other hand, it follows from (3.7) that

$$2^{-\omega(J)-\sigma(J)-1} \leqslant D\big(\varphi(x), \varphi(y)\big) \leqslant 2^{-\omega(J)}.$$

By (3.5) and (3.6), we have

$$\frac{1}{4(\sup_k n_k)} n_{(k_{(i-1)}+1)} \cdots n_{k_i} \leqslant 2^{\sigma(J)} \leqslant 2(\sup_k n_k) n_{(k_{(i-1)}+1)} \cdots n_{k_i}. \tag{3.14}$$

Denote by $\omega(J)$ the length of cylinder $\psi_{\operatorname{ord}(J)}(J)$. Set $k_0 = 0$, and

$$\psi_0\big([0,1]\big) = \Sigma_2, \qquad \omega\big([0,1]\big) = 0 \quad \text{and} \quad \operatorname{ord}\big([0,1]\big) = 0.$$

We shall check the following inductive assumption:

$$\frac{n_1 \cdots n_{k_{\operatorname{ord}(J)}}}{(4 \sup_k n_k)^{\operatorname{ord}(J)}} \leqslant 2^{\omega(J)} \leqslant (4 \sup_k n_k)^{\operatorname{ord}(J)} \cdot n_1 \cdots n_{k_{\operatorname{ord}(J)}}. \tag{3.15}$$

For ord([0, 1])= 0, we have $(4 \sup_k n_k)^{-0} \leqslant 2^0 \leqslant (4 \sup_k n_k)^0$. This is true.

Assume (3.15) is true for ord$(J)= i-1$, we shall check it for i. In fact, for basic element $J' \subset J$ with $\operatorname{ord}(J') = i$, we have $\omega(J') = \omega(J) + \sigma(J)$ or $\omega(J) + \sigma(J) + 1$. Then by (3.14),

$$\frac{1}{(4 \sup_k n_k)^i} n_1 \cdots n_{k_i} \leqslant 2^{\omega(J)+\sigma(J)} \leqslant 2^{\omega(J')} \leqslant 2^{\omega(J)+\sigma(J)+1} \leqslant (4 \sup_k n_k)^i \cdot n_1 \cdots n_{k_i}.$$

Then (3.15) is proved.

For $x \in J_1$, $y \in J_2$ mentioned in (3.10), we notice that $\omega(J_1), \omega(J_2) \in [\omega(J) + \sigma(J), \omega(J) + \sigma(J) + 1]$ and $\psi_i(J_1)$, $\psi_i(J_2)$ are disjoint. Using (3.15), we have

$$(4 \sup_k n_k)^{-i} (n_1 \cdots n_{k_i})^{-1} \leqslant 2^{-\max(\omega(J_1), \omega(J_2))} \leqslant D\big(\varphi(x), \varphi(y)\big) \leqslant 2^{-\omega(J)} \leqslant (4 \sup_k n_k)^{i-1} (n_1 \cdots n_{k_{i-1}})^{-1}.$$

Notice that $\frac{\log n_1 \cdots n_{k_i}}{n_1 \cdots n_{k_{i-1}}} \to 1$ due to (2.10) and $\sup_k n_k < \infty$. By condition (1) of Lemma 1, we have $i/k_i \to 0$. Therefore,

$$\frac{\log D(\varphi(x), \varphi(y))}{-\log n_1 \cdots n_{k_i}} \to 1 \quad \text{uniformly as } i \to \infty. \tag{3.16}$$

It follows from (3.13) and (3.16) that

$$\lim_{|x-y| \to 0} \frac{\log D(\varphi(x), \varphi(y))}{s \log|x-y|} = \lim_{i \to \infty} \frac{\log n_1 \cdots n_{k_i}}{\log n_1 \cdots n_{k_i}} = 1,$$

where $i \to \infty$ uniformly when $|x-y| \to 0$ due to (3.13). □

Proof of Theorem 1. We only need prove that if E and F are regular homogeneous Moran sets with $\dim_H E = \dim_H F = s$, then they are quasi-Lipschitz equivalent.

It follows from Proposition 1 that there are two bijections $\varphi_1 : E \to \Sigma_2$ and $\varphi_2 : F \to \Sigma_2$ such that for $x_1, x_2 \in E$ and $\theta_1, \theta_2 \in \Sigma_2$,

$$\frac{\log D(\varphi_1(x_1), \varphi_1(x_2))}{s \log|x_1 - x_2|} \to 1 \quad \text{uniformly},$$

and

$$\frac{s \log|\varphi_2^{-1}(\theta_1) - \varphi_2^{-1}(\theta_2)|}{\log D(\theta_1, \theta_2)} \to 1 \quad \text{uniformly},$$

as $|x_1 - x_2| \to 0$ and $D(\theta_1, \theta_2) \to 0$. Therefore, $\varphi_2^{-1} \circ \varphi_1 : E \to F$ is a bijection satisfying

$$\frac{\log|\varphi_2^{-1}(\varphi_1(x_1)) - \varphi_2^{-1}(\varphi_1(x_2))|}{\log|x_1 - x_2|} \to 1 \quad \text{uniformly},$$

i.e., $\varphi_2^{-1} \circ \varphi_1$ is the bijection desired. □

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