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A generalized notion of weak interpretability and the corresponding modal logic

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Abstract

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A tree $Tr(T_1, \ldots, T_n)$ of theories $T_1, \ldots, T_n$ is called *tolerant*, if there are consistent extensions $T_1^+, \ldots, T_n^+$ of $T_1, \ldots, T_n$, where each T_i^+ interprets its successors in the tree $Tr(T_1^+, \ldots, T_n^+)$. We consider a propositional language with the following modal formation rule: if Tr is a (finite) tree of formulas, then $\Diamond Tr$ is a formula, and axiomatically define in this language the decidable logics TLR and TLRω. It is proved that TLR (resp. TLRω) yields exactly the schemata of PA-provable (resp. true) sentences, if $\Diamond Tr(A_1, \ldots, A_n)$ is understood as (a formalization of) "$Tr(\mathrm{PA}+A_1, \ldots, \mathrm{PA}+A_n)$ is tolerant". In fact, TLR axiomatizes a considerable fragment of provability logic with quantifiers over Σ_1-sentences, and many relations that have been studied in the literature can be expressed in terms of tolerance. We introduce and study two more relations between theories: cointerpretability and cotolerance which are, in a sense, dual to interpretability and tolerance. Cointerpretability is a characterization of Σ_1-conservativity for essentially reflexive theories in terms of translations.

Introduction

In [15] the notion of relative[1] interpretability between theories was introduced. Intuitively, "T interprets S" means that the language of S can be translated into the language of T in such a way that T proves the translation of every theorem of S. In [4, 17] a model theory of the propositional logic ILM with the binary modality $\rhd$ was studied. As was proved later, ILM is the complete logic of interpretability over PA (Peano arithmetic). That is, ILM yields exactly the

[1] Henceforth we usually omit the word 'relative(ly)'.

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schemata of PA-provable arithmetical sentences, if we understand $A \rhd B$ as a formalization of the assertion "PA + A interprets PA + B". This result, the arithmetical completeness of ILM, was independently obtained by Berarducci [2] and Shavrukov [13].

Another interesting binary relation between theories, weak interpretability, was introduced in [15]: S is weakly interpretable in T, if S is interpretable in some consistent extension of T in the language of T. Intuitively, "T weakly interprets S" means that the language of S can be translated into the language of T in such a way that the translations of theorems of S are consistent with T.

Unlike interpretability, weak interpretability does not naturally have to be restricted to the binary case, and we introduce the notion of tolerance that is a natural generalization of that of weak interpretability:

A tree $Tr(T_1, \ldots, T_n)$ of theories $T_1, \ldots, T_n$ is said to be *tolerant*, if there are consistent extensions $T_1^+, \ldots, T_n^+$ of $T_1, \ldots, T_n$ such that each T_i^+ interprets its successors in the tree $Tr(T_1^+, \ldots, T_n^+)$.

Consider a particular example. Let $Tr = Tr(T, R, S, T)$ be the tree of theories displayed in Fig. 1. Then the intuitive gist of the statement "Tr is tolerant" is the following: (the theorems of) the theories Q and T can be translated into the language of S and added to S; the augmented S and the theory R can be translated into the language of T and added to T; moving downward in this way, we obtain a kind of 'avalanche' of information contained in these theories. The tolerance of Tr means that this 'avalanche' is consistent (i.e., there is a way of translating that leads to the consistent 'avalanche').

In Section 4 we axiomatically define the decidable modal propositional logic TLR with the following modal formation rule: if Tr is a (finite) tree of formulas, then $\Diamond Tr$ is a formula. It is proved that TLR is sound (Section 7) and complete (Section 8) as the logic of tolerance over PA. That is, TLR yields exactly the schemata of PA-provable arithmetical sentences if $\Diamond Tr(A_1, \ldots, A_n)$ is understood as a formalization of the assertion "$Tr(\mathrm{PA} + A_1, \ldots, \mathrm{PA} + A_n)$ is tolerant". In Section 9 we define the decidable extension TLRω of TLR which yields exactly the schemata of true arithmetical sentences.

De Jongh and Veltman [4] introduced a Kripke-like semantics for ILM and proved the soundness and the completeness of ILM with respect to it. Visser [17]

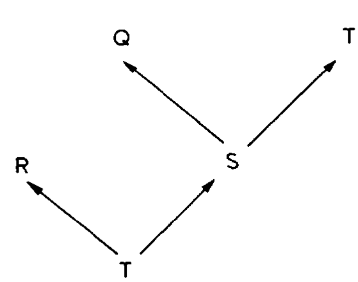

Fig. 1.

simplified this semantics and proved the corresponding soundness and completeness theorems. We call the simplified de Jongh's and Veltman's models 'Visser models'. As we show in Section 4, TLR, too, is sound and complete with respect to Visser models (of course, with a different kind of forcing relation).

Each of Berarducci's and Shavrukov's proofs of the arithmetical completeness of ILM, as well as our proof of the arithmetical completeness of TLR, consists in an 'embedding' of Visser models into arithmetic by means of a Solovay-like [14] function. In fact any one of the three versions of this function, given in [2, 13] and in the present paper, can be used to prove the arithmetical completeness of both ILM and TLR.

Some more background information on the present work.

In [5] the decidable propositional logic HGL ('the logic of arithmetical hierarchy') with infinitely many unary modalities: $\Box$, Σ_1, Σ_1^+, Σ_2, $\Sigma_2^+, \ldots$ was introduced. There the arithmetical completeness of HGL was proved, when $\Box A$ is understood as "A is provable (in PA)", $\Sigma_n A$ as "A is (PA-equivalent to) a Σ_n-sentence", and $\Sigma_n^+ A$ as "A is (PA-equivalent to) a Boolean combination of Σ_n-sentences".

Ignatiev [10] strengthened the results of [5] concerning the fragment of the logic of arithmetical hierarchy obtained by restricting its language to the two modalities $\Box$ and Σ_1. Namely, he changed the unary modality Σ_1 for the more general binary modality $\twoheadrightarrow$, interpreting $A \twoheadrightarrow B$ as "there is a Σ_1-sentence F such that $\mathrm{PA} \vdash (A \rightarrow F) \wedge (F \rightarrow B)$" (for comparison: $\Sigma_1 A$ is understood as "there is a Σ_1-sentence F such that $\mathrm{PA} \vdash (A \rightarrow F) \wedge (F \rightarrow A)$"); there the arithmetically complete decidable logic ELH was constructed, called 'the logic of Σ_1-interpolability'.

The author of the logic of Σ_1-interpolability did not suspect that in the language of ELH, a metarelation that seems much more interesting than Σ_1-interpolability,was expressible. Only later, in [6], it was shown that the relation "there is a Σ_1-sentence F such that $\mathrm{PA} \vdash (A \rightarrow F) \wedge (F \rightarrow B)$" is equivalent to "$\mathrm{PA} + \neg B$ is not weakly interpretable in $\mathrm{PA} + A$". It means that Ignatiev's logic can be regarded as the logic of the binary relation of weak interpretability (in its original, non-generalized version) over PA.

In [6] I introduced the decidable logic TOL with the modality $\Diamond$; the arity of $\Diamond$ is not fixed: if $A_1, \ldots, A_n$ are formulas, then $\Diamond(A_1, \ldots, A_n)$ is a formula, too. There the arithmetical completeness of TOL was proved, when $\Diamond(A_1, \ldots, A_n)$ is understood as "$LTr(\mathrm{PA} + A_1, \ldots, \mathrm{PA} + A_n)$ is tolerant", where $LTr(\mathrm{PA} + A_1, \ldots, \mathrm{PA} + A_n)$ is the linear tree $\mathrm{PA} + A_1 < \cdots < \mathrm{PA} + A_n$. We see that the language of TOL is a fragment of TLR; on the other hand, Ignatiev's modality $A \twoheadrightarrow B$ can be expressed in the language of TLR by $\neg\Diamond(A, \neg B)$; it means that the language of ELH is a fragment of the language of TOL. In fact in each case the inclusion of the languages is essentially proper.

In [6], TOL was called 'the logic of tolerance'. But now we prefer to call TOL 'the logic of linear tolerance', maintaining the name 'the logic of tolerance' for TLR.

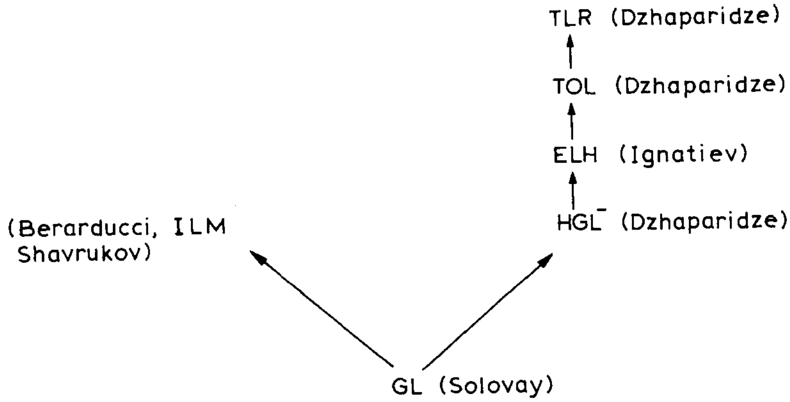

Fig. 2.

The arrows in Fig. 2 that summarizes the above-said, demonstrate the 'more general'-relation between logics GL (the logic of provability, with the modality $\Box$ for the provability predicate), ILM, the $\Box$, Σ_1-fragment of HGL (denoted by HGL^-), ELH, TOL and TLR; in parentheses the names of the authors of the corresponding arithmetical completeness theorems are indicated.

The logics HGL^-, ELH and TOL are described in Appendix A.

In Section 10 we show that the language of TLR is strong enough to express any n-ary metarelation of the type "for all Σ_1-sentences $F_1, \ldots, F_k$ there are Σ_1-sentences $F_{k+1}, \ldots, F_m$ such that $\text{PA} \vdash Bl$", where Bl is any Boolean combination of $F_1, \ldots, F_m, A_1, \ldots, A_n$. It means that TLR axiomatizes a considerable fragment of provability logic with quantifiers over Σ_1-propositions. The strength of the language of TOL is not enough for this. Note that Σ_1-interpolability is a typical example of a metarelation of the above-mentioned type (with $k = 0$, $m = 1$ and $n = 2$).

In Section 3 we introduce two more relations between theories called 'cointerpretability' and 'cotolerance'. They are in a sense dual to the interpretability and tolerance relations.

Intuitively, "S is *cointerpretable* in T" means that the language of T can be translated into the language of S in such a way that T proves every formula the translation of which is provable in S.

And "*Tr* is *cotolerant*", where *Tr* is determined by Fig. 1, means the following. We translate the language of S into the languages of Q and T and then add to S every sentence the Q-translation of which is provable in Q or the T-translation of which is provable in T; denote the augmented S by S^+. Then we translate the language of T into the languages of R and S and add to T every sentence the R-translation of which is provable in R or the S-translation of which is provable in S^+; denote the augmented T by T^+. If there is a way of doing translations that leads to the consistent T^+, then (and only then) we say that *Tr* is cotolerant.

We show in Appendix B that for essentially reflexive theories cointerpretability and Σ_1-conservativity are the same. This is a solution of the problem of finding a characterization of Σ_1-conservativity in terms of translations, which was left open in [8] as a presumably difficult question.

The logic of cointerpretability is not studied at all, and this task seems to be much more difficult than studying the logic of interpretability. As for cotolerance, we show in Sections 2 and 3 that this relation is expressible in terms of linear tolerance, i.e., tolerance of linear trees; it means that TOL can be regarded as the logic of cotolerance (but not as the logic of the nonrestricted relation of tolerance) over PA.

1. Trees

Definitions, notation and terminology

1.1. A *finite irreflexive tree* is a pair $[m, \prec]$, where M is a finite nonempty set, $\prec$ is a transitive irreflexive relation on $\mathcal{M}$, and the following conditions are satisfied:

(a) there is $d \in M$ such that for any $d \neq a \in M$, $d \prec a$;

(b) for all $a, b, c \in M$, if $a \prec c$ and $b \prec c$, then either $a \prec b$ or $a = b$ or $b \prec c$.

Since no other kinds of trees will be considered in this paper, we shall usually omit the words 'finite irreflexive' and say simply 'a tree'.

1.2. Let $[M, \prec]$ be a tree. Then:

1. a is said to be the *immediate predecessor of* b in $[M, \prec]$, iff $a \prec B$ and there is no c with $a \prec c \prec b$. And b is said to be an *immediate successor* of a in $[M, \prec]$, iff a is the immediate predecessor of b in $[M, \prec]$.

2. We say that n is the *depth* of $a \in M$ in $[M, \prec]$, iff there are $b_1, \ldots, b_n$ with $a \prec b_1 \prec \cdots \prec b_n$ and there are no $c_1, \ldots, c_{n+1}$ with $a \prec c_1 \prec \cdots \prec c_{n+1}$; if there is no b with $a \prec b$, then the depth of a in $[M, \prec]$ is 0.

3. We say that n is the *height* of $a \in M$ in $[M, \prec]$, iff there are $b_1, \ldots, b_n$ with $b_1 \prec \cdots \prec b_n \prec a$ and there are no $c_1, \ldots, c_{n+1}$ with $c_1 \prec \cdots \prec c_{n+1} \prec a$; if there is no b with $b \prec a$, then the height of a in $[M, \prec]$ is 0.

4. $a \preccurlyeq b$ means that $a \prec b$ or ($a, b \in M$ and) $a = b$.

5. $\{a\preccurlyeq\}$ (where $a \in M$) denotes the set $\{b: a \preccurlyeq b\}$.

Note that the above signs $\prec$ and $\preccurlyeq$ are, in fact, metavariables for relations. The signs $<$ and $\leq$ are reserved for the usual relations 'is less than' and 'is less than or equal to' on natural numbers.

1.3. We say that α is an *evaluator* of a tree $[m, \prec]$ (or, simply, of M), iff α is a function : $M \to S$ for some set S; when we want to indicate that S is the range of α, we say "α is an evaluator of $[M, \prec]$ (or, of M) in S". Usually, S will be a set of theories, formulas or 'possible worlds'.

1.4. An *evaluated tree* is a triple $[M, <, \alpha]$, where $[M, <]$ is a tree and α is its evaluator; when we want to indicate that S is the range of α, we say "$[M, <, \alpha]$ is a tree of elements of S"; if, e.g., S is the set of theories, we can simply say "$[M, <, \alpha]$ is a tree of theories".

1.5. Let α be an evaluator of some tree $[M, <]$, and let $d \in M$. Then $\alpha =_d \beta$ means that β is an evaluator of M such that for each $d \neq a \in M$, $\alpha(a) = \beta(a)$.

1.6. The signs $\subseteq$ and $\subset$ are used in their usual meaning for "is a subset" and "is a proper subset" relations between sets. Besides, we use these signs to denote relations between trees and evaluated trees:

$[M', <'] \subseteq [M, <]$ (resp. $[M', <'] \subset [M, <]$) means that $M' \subseteq M$ (resp. $M' \subset M$) and $<'$ is the restriction of $<$ to M';

$[M', <', \alpha'] \subseteq [M, <, \alpha]$ (resp. $[M', <', \alpha'] \subset [M, <, \alpha]$) means that $[M', <'] \subseteq [M, <]$ (resp. $[M', <'] \subset [M, <)$ and α' is the evaluator of M' such that $\alpha'(a) = \alpha(a)$ for each $a \in M'$.

1.7. Let $[M, <, \alpha]$ be an evaluated tree. Then:

1. We say that a tree $[M', <']$ (resp. an evaluated tree $[M', <', \alpha']$) is an *initial part* of $[M, <]$ (resp. of $[M, <, \alpha]$), iff $[M', <'] \subseteq [M, <]$ (resp. $[M', <', \alpha'] \subseteq [M, <, \alpha]$) and for all $a, b \in M$, if $a < b$ and $b \in M'$, then $a \in M'$.

2. For any $d \in M$, $[M, <]_d$ and $[M, <, \alpha]_d$ denote $[M', <']$ and $[M', <', \alpha']$, respectively, where $M' = \{d \leqslant\}$, and $<'$ and α' are the restrictions of $<$ and α to M'.

1.8. Suppose $[M, <, \alpha]$ and $[M', <', \alpha']$ are evaluated trees. Then:

We write $[M, <] \approx_i [M', <']$, iff i is an isomorphism between $[m, <]$ and $[M', <']$, i.e., i is a 1–1 function: $M \to M'$ such that for all $a, b \in M$, $a < b \Leftrightarrow ia <' ib$.

$[M, <, a] \approx_i [M', <', a']$ means that $[M, <] \approx_i [M', <']$ and for each $a \in M$, $\alpha(a) = \alpha'(ia)$.

$[M, <] \approx [M', <']$ (resp. $[M, <, \alpha] \approx [M', <', \alpha']$) means that $[M, <] \approx_i [M', <']$ (resp. $[M, <, \alpha] \approx_i [M', <', \alpha']$) for some i.

Clearly $\approx$ is an equivalence relation.

1.9. Sometimes we denote trees and evaluated trees shortly by $[Tr]$, $[Tr']$, $[Tr_1]$, ...

1. If $[Tr]$ is a tree or an evaluated tree, then (Tr) denotes the $\approx$-equivalence class of $[Tr]$, i.e., the set $\{[Tr'] : [Tr'] \approx [Tr]\}$; instead of the short notation '(Tr)', we can use the complete form $(M, <)$ (when $[Tr] = [M, <]$) or $(M, <, \alpha)$ (when $[Tr] = [M, <, \alpha]$).

2. $(Tr_1) \subseteq (Tr_2)$ means that $[Tr_1'] \subseteq [Tr_2']$ for some $(Tr_1') = (Tr_1)$ and $(Tr_2') = (Tr_2)$; similarly for $\subset$.

3. Suppose $[Tr]$ is a tree $[M, <]$ or an evaluated tree $[M, <, \alpha]$, and $d \in M$. Then $(Tr)_d$ denotes the $\approx$-equivalence class $\{[Tr']: [Tr'] \approx [Tr]_d\}$ (see 1.7.2).

1.10. Suppose $[M, <, \alpha]$ is an evaluated tree, and $[Tr]$ is $[M, <]$ or $[M, <, \alpha]$. Then ROOT$[Tr]$ (as well as ROOT$[M, <]$ and ROOT$[M, <, \alpha]$) denotes the very $d \in M$, for which 1.1(a) holds.

1.11. Suppose $[Tr] = [M, <, \alpha]$ is an evaluated tree. Then root(Tr) (as well as root$(M, <, \alpha)$) denotes α(ROOT$[Tr]$).

Note that if $[Tr]$ and $[Tr']$ are $\approx$-equivalent evaluated trees, then root(Tr) = root(Tr'), but possibly ROOT$[Tr] \neq$ ROOT$[Tr']$.

1.12. Suppose $[Tr_1] = [M_1, <_1, \alpha_1]$, $[Tr_2] = [M_2, <_2, \alpha_2]$ are evaluated trees with $M_1 \cap M_2 = \emptyset$, and $d \in M_1$. Then $[Tr_1] +_d [Tr_2]$ is the evaluated tree $[M, <, a]$, where:

(a) $M = M_1 \cup M_2$;

(b) $< \; = \; <_1 \cup <_2 \cup \{(a, b): a \leqslant_1 d \text{ and } b \in M_2\}$;

(c) for any $a \in M_1$, $\alpha(a) = \alpha_1(a)$ and for any $a \in M_2$, $\alpha(a) = \alpha_2(a)$.

1.13. Suppose $[Tr_1] = [M_1, <_1, \alpha_1]$, $[Tr_2] = [M_2, <_2, \alpha_2]$ are evaluated trees (possibly $M_1 \cap M_2 \neq \emptyset$) and $d \in M_1$. Then $(Tr_1) +_d (Tr_2)$ is the $\approx$-equivalence class of $[Tr_1] +_d [M, <, \alpha]$, where $[M, <, \alpha]$ is an arbitrary evaluated tree $\approx$-equivalent to $[Tr_2]$ with $M_1 \cap M = \emptyset$.

1.14. Let $[Tr_2]$ and $[Tr_1] = [M_1, <_1, \alpha_1]$ be evaluated trees. We say that (Tr_2) is a *duplicate* of (Tr_1), iff $(Tr_2) = (Tr_1) +_d (Tr_1)_d$ for some $d \in M_1$.

1.15. A tree $[M, <]$ or an evaluated tree $[M, <, \alpha]$ is said to be *linear*, iff $<$ is linear (i.e., for all $a, b \in M$, $a < b$ or $b < a$, unless $a = b$).

1.16. We use a special notation for $\approx$-equivalence classes of evaluated linear trees. Namely:

Suppose $[M, <, \alpha]$ is an evaluated linear tree. Obviously $[M, <, \alpha]$ is $\approx$-equivalent to $[M', <', \alpha']$, where $M' = \{1, \ldots, n\}$ for some $n \geqslant 1$ and $<'$ is the usual relation 'is less than' on natural numbers (α' is determined uniquely). Then, instead of $(M, <, \alpha)$, we can write $\langle v_1, \ldots, v_n \rangle$, where $v_1, \ldots, v_n$ are the values of $\alpha'(1), \ldots, \alpha'(n)$.

2. Π_1- and Σ_1-consistency

2.1. Terminology. 1. By a 'sentence' we mean a closed first-order formula.

2. An 'arithmetical formula (sentence)' means a formula (sentence) of the language of PA (Peano Arithmetic, cf. [3]).

3. By a 'theory' we mean an arithmetically definable theory formulated in first-order logic with identity. Each theory is determined by a language and a set of sentences in this language, regarded as its extra-logical axioms. For simplicity we assume that the languages of the theories we consider (including PA) contain only a finite number of predicate constants and do not contain functional or individual constants.

4. "A theory T contains a theory S" means that the language of T contains the language of S and T proves every theorem of S.

5. A 'finite extension' of a theory T means an extension of T by one additional axiom in the language of T.

6. By a 'superarithmetical theory' we mean a r.e. (recursively enumerable) theory in the language of PA and containing PA.

7. A sequence s is said to be an *ending segment* of a finite sequence $\langle a_1, \dots, a_n \rangle$, iff s is empty or $s = \langle a_i, \dots, a_n \rangle$ for some $1 \leqslant i \leqslant n$.

8. A finite sequence $\langle a_1, \dots, a_n \rangle$ is said to be an *end-extension* of a sequence s, iff s is empty or $s = \langle a_1, \dots, a_i \rangle$ for some $1 \leqslant i \leqslant n$. "$s'$ is a proper end-extension of s" means that s' is an end-extension of s and $s' \neq s$.

2.2. Notation. 1. $\Pi_1!$ (resp. $\Sigma_1!$) is the set of all arithmetical sentences of the form $\forall x\, F$ (resp. $\exists x\, F$), where F is a p.r. (primitive recursive) formula; and Π_1 (resp. Σ_1) is the set of all arithmetical sentences that are PA-equivalent to some element of $\Pi_1!$ (resp. $\Sigma_1!$).

2. If M is a set of sentences in the language of a theory T, then $T + M$ denotes the extension of T by M as the set of additional axioms; if M consists of only one sentence F, we write $T + F$ for $T + \{F\}$.

3. If M is a finite set of formulas, then $\bigwedge M$ (resp. $\bigvee M$) denotes the conjunction (resp. the disjunction) of all the elements of M. The empty conjunction (resp. disjunction) is identified with $\top$ (resp. $\bot$).

4. If M is a set of sets, then $\bigcup M$ denotes the union of the elements of M.

5. If n is a natural number, then $\bar{n}$ denotes the numeral for n.

6. If F is an arithmetical formula, then $\ulcorner F \urcorner$ denotes the numeral for the Gödel number of F (the Gödel numbering is supposed to be fixed).

2.3. Definition. Suppose $[Tr] = [M, <, \alpha]$ is a tree of theories and Γ is a set of sentences, where each sentence from Γ is common for the language of each theory $\alpha(a)\colon a \in M$. Let β be the evaluator of M such that for each $a \in M$, $\beta(a) = \alpha(a) + \{F\colon F \in \Gamma \text{ and } \beta(b) \vdash F \text{ for some } a < b\}$. (Note that since $[M, <]$ is a finite irreflexive tree, β is defined correctly.) Then:

(a) root$(M, <, \beta)$ is said to be the *Γ-avalanche* on $[Tr]$. Observe that if $[Tr'] \approx [Tr]$, then the Γ-avalanche on $[Tr']$ coincides with the Γ-avalanche on $[Tr]$. Therefore we can use 'the Γ-avalanche on (Tr)' as a synonym of 'the Γ-avalanche on $[Tr]$'.

(b) $[Tr]$ and (Tr) are said to be *Γ-consistent* iff the Γ-avalanche on $[Tr]$ is consistent; otherwise $[Tr]$ and (Tr) are *Γ-inconsistent*.

(c) $[Tr]$ and (Tr) are said to be *Γ-conservative* iff root(Tr) contains the Γ-avalanche on $[Tr]$; if $(Tr) = \langle T, S \rangle$ (recall 1.16), then "S is Γ-conservative over T" is a synonym of "(Tr) is Γ-conservative".

The following lemma is an immediate consequence of Definition 2.3.

2.4. Lemma. (PA$\vdash$:) *Suppose $[M, <, \alpha]$ is a tree of r.e. theories and Γ is a recursive set of sentences, where each sentence from Γ is common for the language of each theory $\alpha(a)$: $a \in M$. Then $(M, <, \alpha)$ is Γ-consistent iff there is an evaluator β of M such that*:

(a) *for each $a \in M$, $\beta(a)$ is a consistent r.e. extension of $\alpha(a)$;*

(b) *for all $a < b$, $\beta(b)$ is Γ-conservative over $\beta(a)$.*

Taking into account that the sets Π_1 and Σ_1 are closed under conjunctions, the proof of the following lemma is quite simple:

2.5. Lemma. (PA$\vdash$:) *A tree $[M, <, \alpha]$ of superarithmetical theories is Π_1- (resp. Σ_1-) inconsistent iff there are Π_1- (resp. Σ_1-) sentences F_a: $a \in M$ such that*

(a) $\mathrm{PA} \vdash \neg F_{\mathrm{ROOT}[M,<]}$;

(b) *for each $a \in M$, $\alpha(a) \vdash \bigwedge \{F_b : a < b\} \to F_a$.*

2.6. Lemma. (PA$\vdash$:) *A linear tree $\langle T_1, \ldots, T_n \rangle$ of superarithmetical theories is Π_1-consistent iff $\langle T_n, \ldots, T_1 \rangle$ is Σ_1-consistent.*

Proof. Argue in PA. According to 2.5, it is enough to show that the following two assertions are equivalent.

(i) there are Π_1-sentences $F_1, \ldots, F_n$ such that $\mathrm{PA} \vdash \neg F_1$ and for each $1 \leq i \leq n$, $T_i \vdash \bigwedge \{F_j : i < j\} \to F_i$;

(ii) there are Σ_1-sentences $E_1, \ldots, E_n$ such that $\mathrm{PA} \vdash \neg E_n$ and for each $1 \leq i \leq n$, $T_i \vdash \bigwedge \{E_j : i > j\} \to E_i$.

It is easy to check that (i) implies (ii), if we set $E_i = \neg \bigwedge \{F_j : i < j\}$, and (ii) implies (i), if we set $\{F_i = \neg \bigwedge \{E_j : i > j\}$. $\square$

2.7. Definition. Let us fix $\exists x\, \Theta(x, y)$ as a formalization of the predicate "y is the Gödel number of a true Σ_1!-sentence", with the primitive recursive Θ. We suppose that:

(a) $\mathrm{PA} \vdash y \neq y' \rightarrow \neg(\Theta(x, y) \wedge \Theta(x, y'))$;

(b) $\mathrm{PA} \vdash \Theta(x, y) \rightarrow \exists x' > x\, \Theta(x', y)$;

(c) for any Σ_1!-sentence F, $\mathrm{PA} \vdash F \leftrightarrow \exists x\, \Theta(x, \ulcorner F \urcorner)$;

(d) PA proves the fact (c).

Then a number n is said to be a *regular witness* for a Σ_1!-sentence F, iff $\Theta(\bar{n}, \ulcorner F \urcorner)$ is true; and n is a *regular counterwitness* for a Π_1!-sentence $\forall z\, E$ iff n is a regular witness for the Σ_1!-sentence $\exists z \neg E$.

2.8. Definition. A *linear analog* of an evaluated tree $[M, <, \alpha]$ is a linear evaluated tree $[M, <', \alpha]$, where for all $a, b \in M$, if $a < b$, then $a <' b$.

The results of Section 10 together with Lemma 2.5 will imply that Σ_1-consistency of trees of finite extensions of PA is expressible in terms of Π_1-consistency. But the reverse doesn't hold: in general (unlike the situation we have in the linear case), Π_1-consistency cannot be defined in terms of Σ_1-consistency. A proof of this negative fact is given in Appendix A.

The following Theorem 2.9 together with Lemma 2.6 imply something more than that Σ_1-consistency is expressible in terms of Π_1-consistency. Namely, we have that Σ_1-consistency is expressible in terms of Π_1-consistency of linear trees of theories:

2.9. Theorem. ($\mathrm{PA} \vdash$:) *A tree of superarithmetical theories is Σ_1-consistent iff (at least) one of its linear analogs is Σ_1-consistent.*

(It follows from 2.6 and the above comments that the theorem doesn't hold if 'Σ_1' is replaced by 'Π_1'.)

Proof. Argue in PA.

Fix a tree $[M, <, \alpha]$ of superarithmetical theories.

($\Rightarrow$): Let S be the set of all the sequences $s = \langle a_1, \ldots, a_n \rangle$ (including the empty sequence $\langle\ \rangle$) of elements of M such that: (1) for all $i \leq i < j \leq n$, $a_i \neq a_j$ and (2) for all $1 \leq i \leq n$ and $b \in M$, if $a_i < b$, then $b = a_j$ for some $i < j$.

For a sequence $s = \langle a_1, \ldots a_n \rangle \in S$, "$b \in s$" means that $b \in \{a_1, \ldots, a_n\}$ and "$U \subseteq s$" means that $U \subseteq \{a_1, \ldots, a_n\}$.

We say that an element s of S is *complete*, iff $M \subseteq s$.

Let SC be the set of all complete elements of S.

For each $r \in SC$ of the form $\langle a_1, \ldots, a_n \rangle$, let $<^r$ be the binary relation on M defined by: $b <^r c$ iff $b = \alpha_i$ and $c = \alpha_j$ for some $i \leq i < j \leq n$. So $\{[M, <^r, \alpha]: r \in SC\}$ is just the set of all linear analogs of $[M, <, \alpha]$.

Suppose every linear analog of $[M, <, \alpha]$ is Σ_1-inconsistent. It means by 2.5 that there are Σ_1-sentences F^r_a: $r \in SC$, $a \in M$ such that:

(1) $\quad \mathrm{PA} \vdash \neg F^r_{\mathrm{ROOT}[M, <^r]}$ (all $r \in SC$);

(2) $\quad \alpha(a) \vdash \bigwedge \{F^r_b : a <^r b\} \rightarrow F^r_a$ (all $r \in SC$, $a \in M$).

We can suppose here that each F_a^r is a Σ_1!-sentence and if $r \neq r'$ or $a \neq a'$, then $F_a^r \neq F_{a'}^{r'}$ and thus, PA proves that F_a^r and $F_{a'}^{r'}$ cannot have common regular witnesses (see 2.7).

Let us define a p.r. function $e: \omega \to S$ as follows:

$e(0) = \langle\ \rangle$.

Suppose $e(x) = \langle b_1, \ldots, b_n \rangle \in S$. Then:

(1) $e(x+1) = s$, if $s = \langle a, b_1, \ldots, b_n \rangle \in S$ and for each $r \in SC$ such that s is an ending segment of r, F_a^r has a $\leqslant x$ regular witness;

(2) otherwise, $e(x+1) = e(x)$.

It is easy to see that:

2.9.1. (PA $\vdash$:) *If $e(x) = s$, then, for all $y > x$, s is an ending segment of $e(y)$.*

It follows from 2.9.1 that:

2.9.2. (PA $\vdash$:) *e has a limit.*

Let *Lim* denote this limit.

2.9.3. *For each $a \in M$, $\alpha(a)$ proves the following: Suppose $e(x) = \langle b_1, \ldots, b_n \rangle$, $r \in SC$ and $\langle a, b_1, \ldots, b_n \rangle$ is an ending segment of r. Then F_a^r is true.*

Proof. Argue in $\alpha(a)$. Assume the conditions of 2.9.3.

Let us consider any $c \in M$ with $a \prec^r c$. We have $c = b_i$ for some $1 \leqslant i \leqslant n$. Therefore, since $e(x) = \langle b_1, \ldots, b_n \rangle$, there is $y < x$ such that $e(y) \neq e(y+1) = \langle b_i, \ldots, b_n \rangle$. By the definition of e (taking into account that $\langle b_i, \ldots, b_n \rangle$, too, is an ending segment of r), this is possible only if F_c^r has a $\leqslant y$ regular witness.

Thus, for all $a \prec^r c$, F_c^r is true. Then, by (2), F_a^r is true. □

2.9.4. *For all $a \in M$ and $s \in S$ such that $a \notin s$ and $\{b: a \prec b\} \subseteq s$, $\alpha(a)$ proves that $s \neq Lim$.*

Proof. Assume $s = \langle b_1, \ldots, b_n \rangle \in S$, $a \in M$, $a \notin s$ and $\{b: a \prec b\} \subseteq s$. Observe that then $\langle a, b_1, \ldots, b_n \rangle \in S$.

Argue in $\alpha(a)$. Suppose $s = Lim$. Fix a number x such that $e(y) = s$ for all $y \geqslant x$. By 2.9.3, for each $r \in SC$ such that $\langle a, b_1, \ldots, b_n \rangle$ is an ending segment of r, F_a^r is true; it means that there is $z \geqslant x$ for which every such F_a^r has a $\leqslant z$ regular witness (for, by 2.7(b), every true Σ_1!-sentence has arbitrary large regular witnesses). Then, by the definition of e, $e(z+1) = \langle a, b_1, \ldots, b_n \rangle \neq s$, a contradiction. □

For each $a \in M$, let E_a be a Σ_1-formalization of the assertion "there is x such that $a \in e(x)$".

2.9.5. *For each $a \in M$, $\alpha(a) \vdash \bigwedge\{E_b: a \prec b\} \to E_a$.*

Proof. Argue in $\alpha(a)$. Suppose $\bigwedge\{E_b: a \prec b\}$. Then if follows easily from 2.9.1–2 that $\{b: a \prec b\} \subseteq Lim$, whence, by 2.9.4, $a \in Lim$. And $a \in Lim$ clearly implies E_a. □

2.9.6. $\mathrm{PA} \vdash \neg E_{\mathrm{ROOT}[M,\,<]}$.

Proof. Let $d = \mathrm{ROOT}[M, <]$. Argue in PA. Suppose $e(x) = s$ and $d \in s$. Clearly $d \in s$ implies that $s \in SC$ and s has the form $\langle d, b_1, \ldots, b_n \rangle$. We may suppose that $e(x-1) = \langle b_1, \ldots, b_n \rangle$. Then, according to the definition of e, there is a $\leqslant(x-1)$ regular witness for F_d^s. But, according to (1), this is impossible because F_d^s is false. □

Now, 2.9.5–6 imply by 2.5 that $[M, <, \alpha]$ is Σ_1-inconsistent. 2.9($\Rightarrow$) is thus proved.

($\Leftarrow$): Suppose $[M, <, \alpha]$ is Σ_1-inconsistent, i.e., by 2.5, there are Σ_1-sentences $F_a\colon a \in M$ such that: (1) $\mathrm{PA} \vdash \neg F_{\mathrm{ROOT}[M,\,<]}$ and (2) for each $a \in M$, $\alpha(a) \vdash \bigwedge\{F_b\colon a < b\} \to F_a$. Consider an arbitrary linear analog $[M, <', \alpha]$ of $[M, <, \alpha]$. Observe that $\mathrm{ROOT}[M, <'] = \mathrm{ROOT}[M, <]$; and, since $a < b$ implies $a <' b$, the above two conditions with $<$ replaced by $<'$ continue to be satisfied. It means by 2.5 that $[M, <', \alpha]$ is Σ_1-inconsistent. □

3. What is so interesting about Π_1- and Σ_1-consistency?

3.1. Definition. Let L and L' be first-order languages (without functional symbols). A *translation* from L into L' is a function t which assigns to each formula F of L a formula tF of L' with exactly the same free variables, such that for some fixed formula $\delta(x)$ (with only x free), we have:

1. $t(\forall x\, F)$ is $\forall x\, (\delta(x) \to tF)$ and $t(\exists x\, F)$ is $\exists x\, (\delta(x) \wedge tF)$;
2. t commutes with the operation of substitution of free variables: if $tF(x_1, \ldots, x_n)$ is $F'(x_1, \ldots, x_n)$, then $tF(y_1, \ldots, y_n)$ is $F'(y_1, \ldots, y_n)$;
3. t commutes with Boolean connectives;
4. $t(x = y)$ is $x = y$.[2]

3.2. Notation. Suppose t is a translation from the language of a theory S into the language of a theory T. Then:

(a) $t(S)$ denotes the set of all sentences tF with $S \vdash F$;

(b) $t^{-1}(T)$ denotes the set of all sentences F with $T \vdash tF$.

3.3. Definition. Let T and S be theories.

(a) S is *interpretable* in T iff there is a translation t from the language of S into the language of T such that T proves every $F \in t(S)$.

(b) S is *cointerpretable* in T iff there is a translation t from the language of T into the language of S such that T proves every $F \in t^{-1}(S)$.

[2] In fact all the lemmas and theorems of the present section will continue to hold if we take Definition 3.1 without clause 4, or if we demand the formula $\delta(x)$ to be 'vacuous', e.g. $x = x$.

3.4. Definition. Let $[Tr] = [m, <, \alpha]$ be a tree of theories.

(a) $[Tr]$ is *tolerant*, iff for each $a < b$ there is a translation t_{ba} from the language of $\alpha(b)$ into the language of $\alpha(a)$ such that root$(M, <, \alpha')$ is consistent, where for each $a \in M$, $\alpha'(a) = \alpha(a) + \bigcup\{t_{ba}\alpha'(b): a < b\}$.

(b) [Tr] is *cotolerant*, iff for each $a < b$ there is a translation t_{ab} from the language of $\alpha(a)$ into the language of $\alpha(b)$ such that root$(M, <, \alpha')$ is consistent, where for each $a \in M$, $\alpha'(a) = \alpha(a) + \bigcup\{t_{ab}^{-1}\alpha'(b): a < b\}$.

The following lemma is in fact an immediate consequence of Definitions 3.3 and 3.4.

3.5. Lemma. (PA ⊢:) *A tree* $[M, <, \alpha]$ *of r.e. theories is tolerant* (*resp. cotolerant*) *iff there is an evaluator* α' *of* M *such that for each* $a \in M$, $\alpha'(a)$ *is a consistent r.e. extension of* $\alpha(a)$ *and for each* $a < b$, $\alpha'(b)$ *is interpretable* (*resp. cointerpretable*) *in* $\alpha'(a)$.

The clause (a) of the following Theorem 3.6 is due to Hajek [9] (see also in [8]). The proof uses Orey's [12] (see also [7]) theorem, according to which S is interpretable in T iff T proves the consistency of every finite subtheory of S.

As for the clause (b), up to now it has not been known. We give a proof of it in Appendix B.

3.6. Theorem. (PA ⊢:) *For all superarithmetical theories* T *and* S,

(a) S *is interpretable in* T *iff* S *is* Π_1*-conservative over* T;

(b) S *is cointerpretable in* T *iff* S *is* Σ_1*-conservative over* T.

It follows immediately from 2.4, 3.5 and 3.6 that:

3.7. Theorem. (PA ⊢:) *A tree of superarithmetical theories is tolerant* (*resp. cotolerant*) *iff it is* Π_1- (*resp.* Σ_1-) *consistent.*

3.8. Remark. Non-tolerance can be regarded as a generalization of the notion of inconsistency and hence of provability. The following argument shows that in certain cases this approach enables us to prove new 'truths' which weren't derivable in initial theories.

Suppose $[Tr] = [M, <, \alpha]$ is a tree of theories and, for each $a \in M$, $\lambda(a)$ is the language of $\alpha(a)$ and $\mu(a)$ is a model of $\alpha(a)$ (all axioms of $\alpha(a)$ are true in $\mu(a)$). Suppose also that for all $a < b$, there is a translation t_{ba} from $\lambda(b)$ into $\lambda(a)$ preserving the truth, i.e., for any sentence $F \in \lambda(b)$, $\mu(b) \models F$ iff $\mu(a) \models t_{ba}F$. It is possible that we only believe in the existence of such translations, but cannot build them constructively.

For $d \in M$, say that a sentence F of $\lambda(d)$ is *([Tr], d)-provable*, iff $[M, <, \alpha']$ is not tolerant, where α' is the evaluator of M such that $\alpha' =_d \alpha$ (recall 1.5) and $\alpha'(d) = \alpha(d) + \neg F$. In general, $([Tr], d)$-provability is weaker than the predicate of (usual) provability in $\alpha(d)$. Nevertheless, as is easy to see, all $([Tr], d)$-provable sentences are true in $\mu(d)$. This is a nice fact because, in many cases (at least, when each $\alpha(a)$ is a superarithmetical theory), the set of $([Tr], d)$-provable sentences is recursively enumerable.

The above approach can prove to be useful in case of non-cotolerance, too.

4. Logic TLR

4.1. *Description* of the language of Logic TLR

4.1.1. Let us fix an r.e. list

$$[M^1, <^1_1], \ldots, [M^1, <^1_{k_1}], [M^2, <^2_1], \ldots, [M^2, <^2_{k_2}], [M^3, <^3_1], \ldots, [M^3, <^3_{k_3}], \ldots$$

of trees such that:

(1) for each $n \geqslant 1$, $M^n = \{1, \ldots, n\}$;

(2) for each $1 \leqslant n$ and $1 \leqslant i < j \leqslant k_n$, *not* $[M^n, <^n_i] \approx [M^n, <^n_j]$;

(3) for any tree $[M, <]$ there are $n \geqslant 1$ and i with $1 \leqslant i \leqslant k_n$ such that $[M, <] \approx [M^n, <^n_i]$ (namely, n is the cardinality of M).

Notice that the above-mentioned list is in fact a complete and non-redundant enumeration of all $\approx$-equivalence classes of trees, if we change the square brackets $[\cdots]$ to $(\cdots)$.

One can easily calculate that, e.g., $k_1 = k_2 = 1$, $k_3 = 2$, $k_4 = 3$, $k_5 = 5$.

4.1.2. The alphabet of the propositional polymodal logic TLR consists of:

- propositional letters: $p_1, p_2, p_3, \ldots$;
- Boolean connectives including $\top$ and $\bot$;
- modal operators: $\Diamond^1_1, \ldots, \Diamond^1_{k_1}, \Diamond^2_1, \ldots, \Diamond^2_{k_2}, \Diamond^3_1, \ldots, \Diamond^3_{k_3}, \ldots$; the arity of each operator $\Diamond^n_i$ is just n;
- technical signs: the usual brackets and the comma.

4.1.3. Formulas of the language of TLR will be called 'TLR-*formulas*'. The class of TLR-formulas is defined as the smallest one such that:

(1) propositional letters are TLR-formulas;

(2) if A and B are TLR-formulas, then $\neg(A)$, $(A) \to (B)$ and the other Boolean combinations of A and B (including the 'empty' ones $\bot$ and $\top$) are TLR-formulas;

(3) for each $n \geqslant 1$ and $1 \leqslant i \leqslant k_n$, if $A_1, \ldots, A_n$ are TLR-formulas, then $\Diamond^n_i(A_1, \ldots, A_n)$ is a TLR-formula.

4.1.4. For each TLR-formula of the form $\Diamond^n_i(A_1, \ldots, A_n)$, let $g(\Diamond^n_i(A_1, \ldots, A_n)) = (M^n, <^n_i, \alpha)$, where α is the evaluator of M^n such that for each $j \in M^n$, $\alpha(j) = A_j$. It is easy to see that g establishes a 1–1 correspondence between the set of all

TLR-formulas of the form $\Diamond_i^n(A_1, \ldots, A_n)$ and the set of all $\approx$-equivalence classes of the trees of TLR-formulas.

4.1.5. Suppose D denotes an $\approx$-equivalence class of some tree of TLR-formulas (usually, D looks like (Tr), $(M, <, \alpha)$ or $\langle A_1, \ldots, A_m \rangle$, recall 1.9.1 and 1.16). Then, taking 4.1.4 into account, we'll use the expression $\Diamond D$ to denote the very TLR-formula $\Diamond_i^n(A_1, \ldots, A_n)$, for which $D = g(\Diamond_i^n(A_1, \ldots, A_n))$.

4.2. Definition (of Logic TLR). The *axioms* of TLR are all TLR-formulas that are tautologies or have one of the following forms:

1. $\Diamond(M, <, \alpha) \to \Diamond(M, <, \alpha_1) \vee \Diamond(M, <, \alpha_2)$, where for some $d \in M$, we have $\alpha_1(d) = \alpha(d) \wedge \neg A$, $\alpha_2(d) = A$ and $\alpha_1 =_d \alpha =_d \alpha_2$ (recall 1.5);
2. $\Diamond\langle A \rangle \to \Diamond\langle A \wedge \neg\Diamond\langle A \rangle\rangle$;
3. $\Diamond\langle A, \Diamond(Tr)\rangle \to \Diamond\langle A \wedge \Diamond(Tr)\rangle$;
4. $\Diamond(Tr_1) \to \Diamond(Tr_2)$, where one of the following holds:
   (a) $(Tr_2) \subseteq (Tr_1)$ (recall 1.9.2);
   (b) (Tr_2) is a duplicate of (Tr_1) (recall 1.14);
   (c) $[TR_1] = [M_1, <_1, \alpha_1]$, there is $d \in M_1$ with $\alpha_1(d) = A \wedge \Diamond(Tr)$ and $(Tr_2) = (Tr_1) +_d (Tr)$ (recall 1.13).

The *rules* of inference of TLR are:

5. Modus ponens;
6. $\vdash \neg A \Rightarrow \vdash \neg\Diamond\langle A \rangle$.

4.3. Lemma. *TLR* $\vdash \Diamond(M, <, \alpha_1) \to \Diamond(M, <, \alpha_2)$, *if there is* $\alpha \in M$ *such that* $\alpha_1 =_a \alpha_2$ *and* TLR $\vdash \alpha_1(a) \to \alpha_2(a)$.

Proof. Suppose TLR $\vdash \alpha_1(a) \to \alpha_2(a)$, i.e., TLR $\vdash \neg(\alpha_1(a) \wedge \neg\alpha_2(a))$. Then, by 4.2.6,

(1) TLR $\vdash \neg\Diamond(\alpha_1(a) \wedge \neg\alpha_2(a))$.

Let $\alpha_3 =_a \alpha_1$ $(=_a \alpha_2)$ and $\alpha_3(a) = \alpha_1(a) \wedge \neg\alpha_2(a)$.

(2) TLR $\vdash \Diamond(M, <, \alpha_1) \to \Diamond(M, <, \alpha_3) \vee \Diamond(M, <, \alpha_2)$ (4.2.1);
(3) TLR $\vdash \Diamond(M, <, \alpha_3) \to \Diamond\langle\alpha_1(a) \wedge \neg\alpha_2(a)\rangle$ (4.2.4a);
TLR $\vdash \Diamond(M, <, \alpha_1) \to \Diamond(M, <, \alpha_2)$
from (1), (2), (3) by propositional logic. □

4.4. Lemma. TLR $\vdash \Diamond(M, <, \alpha) \to \Diamond(M, <, \alpha_1) \vee \Diamond(M, <, \alpha_2)$, *if there is* $a \in M$ *such that* $\alpha_1 =_a \alpha =_a \alpha_2$ *and* $\alpha(a) = \alpha_1(a) \vee \alpha_2(a)$.

Proof. Assume the above conditions.

Let $\alpha_3 =_a \alpha$ and $\alpha_3(a) = \alpha(a) \wedge \neg\alpha_2(a)$. We have:

(1) TLR $\vdash \Diamond(M, <, \alpha) \to \Diamond(M, <, \alpha_3) \vee \Diamond(M, <, \alpha_2)$ (4.2.1);
(2) TLR $\vdash \alpha_3(a) \to \alpha_1(a)$ (a tautology);
(3) TLR $\vdash \Diamond(M, <, \alpha_3) \to \Diamond(M, <, \alpha_1)$ ((2), 4.3).

Now, the desired condition follows from (1) and (3). □

4.5. Lemma. $\mathrm{TLR} \vdash A \to \Diamond\langle A\rangle \Rightarrow \mathrm{TLR} \vdash \neg A$.

Proof. Suppose $\mathrm{TLR} \vdash A \to \Diamond\langle A\rangle$, i.e., $\mathrm{TLR} \vdash \neg(A \wedge \neg\Diamond\langle A\rangle)$; then, by 4.2.6, $\mathrm{TLR} \vdash \neg\Diamond\langle A \wedge \neg\Diamond\langle A\rangle\rangle$, whence, by 4.2.2, $\mathrm{TLR} \vdash \neg\Diamond\langle A\rangle$; consequently (since $\mathrm{TLR} \vdash A \to \Diamond\langle A\rangle$), $\mathrm{TLR} \vdash \neg A$. □

4.6. Lemma. $\mathrm{TLR} \vdash \Diamond\langle\Diamond(Tr)\rangle \to \Diamond(Tr)$.

Proof.

(1) $\mathrm{TLR} \vdash \Diamond(Tr) \to \top \wedge \Diamond(Tr)$ (a tautology);

(2) $\mathrm{TLR} \vdash \Diamond\langle\Diamond(Tr)\rangle \to \Diamond\langle\top \wedge \Diamond(Tr)\rangle$ ((1), 4.3).

In fact $\Diamond\langle\top \wedge \Diamond(Tr)\rangle = \Diamond(M, <, \alpha)$, where $m = \{a\}$, $<$ is empty and $\alpha(a) = \top \wedge \Diamond(Tr)$. Then, by 4.2.4c,

(3) $\mathrm{TLR} \vdash \Diamond\langle\top \wedge \Diamond(Tr)\rangle \to \Diamond((M, <, \alpha) +_a (Tr))$.

Clearly $(Tr) \subseteq ((M, <, \alpha) +_a (Tr))$. Therefore, by 4.2.4a,

(4) $\mathrm{TLR} \vdash \Diamond((M, <, \alpha) +_a (Tr)) \to \Diamond(Tr)$.

$\mathrm{TLR} \vdash \Diamond\langle\Diamond(Tr)\rangle \to \Diamond(Tr)$ follows from (2), (3), (4) by propositional logic. □

4.7. Lemma. $\mathrm{TLR} \vdash \neg\Diamond(Tr) \wedge \Diamond(M, <, \alpha_1) \to \Diamond(M, <, \alpha_2)$, *if there is* $a \in M$ *such that* $\alpha_1 =_a \alpha_2$ *and* $\alpha_2(a) = \alpha_1(a) \wedge \neg\Diamond(Tr)$.

Proof. Assume the above conditions.

Let $\alpha_3 =_a \alpha_1$ and $\alpha_3(a) = \Diamond(Tr)$. We have:

(1) $\mathrm{TLR} \vdash \Diamond(M, <, \alpha_1) \to \Diamond(M, <, \alpha_2) \vee \Diamond(M, <, \alpha_3)$ (4.2.1);

(2) $\mathrm{TLR} \vdash \Diamond(M, <, \alpha_3) \to \Diamond\langle\Diamond(Tr)\rangle$ (4.2.4a);

(3) $\mathrm{TLR} \vdash \Diamond\langle\Diamond(Tr)\rangle \to \Diamond(Tr)$ (4.6).

The desired formula follows from (1), (2), (3) by propositional logic. □

4.8. Lemma. $\mathrm{TLR} \vdash \Diamond(M, <, \alpha_1) \to \Diamond(M, <, \alpha_2)$, *if* $d = \mathrm{ROOT}[M, <]$, $\alpha_1 =_d \alpha_2$ *and* $\alpha_2(d) = \alpha_1(d) \wedge \neg\Diamond(M, <, \alpha_1)$.

Proof. Assume the above conditions. Let α_3 and α_4 be evaluators of M such that $\alpha_3 =_d \alpha_1 =_d \alpha_4$, $\alpha_3(d) = \alpha_1(d) \wedge \neg(\alpha_1(d) \wedge \Diamond(M, <, \alpha_1))$ and $\alpha_4(d) = \alpha_1(d) \wedge \Diamond(M, <, \alpha_1)$. We have:

(1) $\mathrm{TLR} \vdash \Diamond(M, <, \alpha_1) \to \Diamond(M, <, \alpha_3) \vee \Diamond(M, <, \alpha_4)$ (4.2.1);

(2) $\mathrm{TLR} \vdash \alpha_3(d) \to \alpha_2(d)$ (a tautology);

(3) $\mathrm{TLR} \vdash \Diamond(M, <, \alpha_3) \to \Diamond(M, <, \alpha_2)$ ((2), 4.3);

(4) $\text{TLR} \vdash \Diamond(M, <, \alpha_4) \rightarrow \Diamond\langle \alpha_4(d) \rangle$ (4.2.4a);

(5) $\text{TLR} \vdash \alpha_4(d) \rightarrow \Diamond(M, <, \alpha_1)$ (a tautology);

(6) $\text{TLR} \vdash \Diamond\langle \alpha_4(d) \rangle \rightarrow \Diamond\langle \Diamond(M, <, \alpha_1) \rangle$ ((5), 4.3);

(7) $\text{TLR} \vdash \Diamond(M, <, \alpha_1) \wedge \neg\Diamond(M, <, \alpha_2) \rightarrow \Diamond\langle \Diamond(M, <, \alpha_1) \rangle$
(from (1), (3), (4), (6) by propositional logic);

(8) $\text{TLR} \vdash \Diamond(M, <, \alpha_1) \wedge \neg\Diamond(M, <, \alpha_2) \rightarrow \Diamond\langle \Diamond(M, <, \alpha_1) \wedge \neg\Diamond(M, <, \alpha_2)$
((7), 4.7).

Now the desired condition follows from (8) by 4.5. □

5. TLR-models

5.1. Definition. A 1-*model* is a triple $\langle W, G, \vDash \rangle$, where:
W is a nonempty set (of 'possible worlds');
G is a relation included in $W \times \{$*≈-equivalence classes of trees of elements of* $W\}$.
$\vDash$ is a ('forcing') relation included in $W \times \{$*propositional letters*$\}$, which is extended to complex TLR-formulas in the following unique way: for any $w \in W$,
(a) $w \vDash$ commutes with Boolean connectives;
(b) $w \vDash \Diamond(M, <, \alpha)$ iff there is an evaluator $\beta : M \rightarrow W$ such that $w\, G\, (M, <, \beta)$ and for each $a \in M$, $\beta(a) \vDash \alpha(a)$. □

A 1-model $\langle W, G, \vDash \rangle$ is said to be a *countermodel* for a TLR-formula A, if *not* $\omega \vDash A$ for some $w \in W$.

5.2. Definition. A *finite* TLR-*model* is a 1-model $\langle W, G, \vDash \rangle$ with finite W and the following properties (for all $w, u, v \in W$ and all trees $[Tr]$, $[Tr_1]$, $[Tr_2]$ of elements of W):
1. *not* $w\, G\, \langle w \rangle$;
2. $w\, G\, \langle u, v \rangle$, $v\, G\, (Tr) \Rightarrow u\, G\, (Tr)$;
3. $w\, G\, (Tr_1) \Rightarrow w\, G\, (Tr_2)$, if one of the following holds:
 (a) $(Tr_2) \subseteq (Tr_1)$;
 (b) (Tr_2) is a duplicate of (Tr_1);
 (c) $[Tr_1] = [M_1, <_1, \alpha_1]$, there is $d \in M_1$ with $\alpha_1(d)\, G\, (Tr)$ for some tree $[Tr]$ of elements of W such that $(Tr_2) = (Tr_1) +_d (Tr)$.

5.3. Theorem. $\text{TLR} \vdash A$ *iff there is no finite* TLR-*countermodel for* A.

Proof. (⇒) follows immediately from 6.3, 7.5 and 8.1. Therefore we prove here only (⇐). Let us fix a TLR-formula A.

5.3.1. Let Sb be the smallest set of TLR-formulas such that:
(a) any subformula of A is contained in Sb;
(b) if $\Diamond(M, <, \alpha) \in Sb$ and $(M', <', \alpha') \subseteq (M, <, \alpha)$, then $\Diamond(M', <', \alpha') \in Sb$;
(c) if $B \in Sb$ and B is not a negation, then $\neg B \in Sb$.

Note that Sb is finite.
We *define* a 1-model $\langle W, G, \vDash\rangle$ by 5.3.2, 5.3.3 and 5.3.4:

5.3.2. W is the set of all maximal TLR-consistent subsets of Sb, i.e., $w \in W$ iff;
(a) $w \subseteq Sb$ (thus, every $w \in W$ is finite);
(b) for any B such that $B, \neg B \in Sb$, we have $B \in w$ or $\neg B \in w$;
(c) TLR $\nvdash \neg \bigwedge w$ (recall 2.2.3).

5.3.3. $w\, G\, (M, <, \alpha)$ iff $[M, <, \alpha]$ is a tree of elements of w such that:
(1) there is $\Diamond(Tr') \in w$ such that $\neg\Diamond(Tr') \in \text{root}(M, <, \alpha)$;
(2) if $\neg\Diamond(Tr') \in w$, then $\neg\Diamond(Tr') \in \text{root}(M, <, \alpha)$;
(3) if $\neg\Diamond(Tr') \in \alpha(a)$ and $a < b$, then $\neg\Diamond(Tr') \in \alpha(b)$;
(4) for no trees $[M_1, <_1, \alpha_1]$, $[M_2, <_2, \alpha_2]$ of TLR-formulas and no function $h: M_2 \to M$ do we have:
(a) $\neg\Diamond(M_1, <_1, \alpha_1) \in w$;
(b) $[M_2, <_2, \alpha_2]$ is an initial part (recall 1.7.1) of $[M_1, <_1, \alpha_1]$;
(c) for all $a <_2 b$, $ha \preccurlyeq hb$;
(d) for all $a \in M_2$, $\alpha_2(a) \in \alpha(ha)$;
(e) for all $a \in M_2$ and all $a <_1 b \notin M_2$, $\Diamond(M_1, <_1, \alpha_1)_b \in \alpha(ha)$ (recall 1.9.3).

5.3.4. For any propositional letter p, $w \vDash p$ iff $p \in w$.

5.3.5. Lemma. $\langle W, G, \vDash\rangle$ *is a finite* TLR-*model.*

Proof. The finiteness of W is evident. The property 5.2.1 is guaranteed by 5.3.3.1. The property 5.2.2 easily follows from 5.3.3.3, and the property 5.2.3a can also be checked immediately.

Checking of the property 5.2.3b. Suppose $[Tr_1] = [M_1, <_1, a_1]$ and $[Tr_2] = [M_2, <_2, \alpha_2]$ are trees of elements of W, and (Tr_2) is a duplicate of (Tr_1), i.e., for some $d \in M_1$, $(Tr_2) = (Tr_1) +_d (Tr_1)_d$. We may suppose that for some tree $[Tr] = [M, <, \alpha]$ of elements of w such that $M \cap M_1 = \emptyset$ and $[Tr_1]_d \approx [Tr]$, the following conditions hold:

(1) $\quad M_2 = M_1 \cup M;$

(2) $\quad <_2 = <_1 \cup < \cup \{(a, b){:}\ a \preccurlyeq_1 d \text{ and } b \in M\};$

(3) $\quad$ for all $a \in M_1$, $\alpha_2(a) = \alpha_1(a)$ and for all $a \in M$, $\alpha_2(a) = \alpha(a)$.

And suppose that *not* $w\, G\, (Tr_2)$. We want to show that *not* $w\, G\, (Tr_1)$. It is easy to see that if the reason for *not* $w\, G\, (Tr_2)$ is that one of the conditions 5.3.3.1–3 is not satisfied, then the same conditions fail for $w\, G\, (Tr_1)$ and so we will have *not* $w\, G\, (Tr_1)$. Now suppose that the reason for *not* $w\, G\, (Tr_2)$ is that the condition

5.3.3.4 is not satisfied, i.e., there are trees $[M_3, <_3, \alpha_3]$, $[M_4, <_4, \alpha_4]$ of TLR-formulas and a function $h: M_4 \to M_2$ such that:

(4) $\neg\Diamond(M_3, <_3, \alpha_3) \in w$;

(5) $[M_4, <_4, \alpha_4]$ is an initial part of $[M_3, <_3, \alpha_3]$;

(6) for all $a <_4 b$, $ha \leqslant_2 hb$;

(7) for all $a \in M_4$, $\alpha_4(a) \in \alpha_2(ha)$;

(8) for all $a \in M_4$ and all $a <_3 b \notin M_4$, $\Diamond(M_3, <_3, \alpha_3)_b \in \alpha_2(ha)$.

Let us fix an isomorphism i for which $[Tr_1]_d \approx_i [Tr]$ (recall 1.8).

We define a function $h': M_4 \to M_1$ as follows: for each $a \in M_4$, $h'a = ha$, if $ha \in M_1$, and $h'a = c$, if ($d \leqslant_1 c$ and) $ha = ic \in M$. It is easy to check that we have:

(9) for all $a, b \in M_4$, $ha \leqslant_2 hb \Rightarrow h'a \leqslant_1 h'b$;

(10) for all $a \in M_4$, $\alpha_2(ha) = \alpha_1(h'a)$.

Now, (4)–(8) together with (9) and (10) imply that *not* $w\,G\,(M_1, <_1, \alpha_1)$ because 5.3.3.4 fails.

Checking of the property 5.2.3c. Assume $(Tr_2) = (Tr_1) +_d (Tr)$, where $[Tr] = [M, <, \alpha]$, $[Tr_1] = [M_1, <_1, \alpha_1]$, $[Tr_2] = [M_2, <_2, \alpha_2]$ are trees of elements of W, $d \in M_1$ and $\alpha_1(d)\,G\,(Tr)$. We may suppose that $M_1 \cap M = \emptyset$ and $[Tr_2] = [Tr_1] +_d [Tr]$, i.e., the above conditions (1)–(3) are satisfied.

And suppose *not* $w\,G\,(Tr_2)$. We want to show that *not* $w\,G\,(Tr_1)$.

Suppose the reason for *not* $w\,G\,(Tr_2)$ is that one of the conditions 5.3.3.1 or 5.3.3.2 is not satisfied. Then the same condition will fail for $w\,G\,(Tr_1)$ because $\mathrm{root}(Tr_1) = \mathrm{root}(Tr_2)$.

Suppose the reason for *not* $w\,G\,(Tr_2)$ is that the condition 5.3.3.3 is not satisfied, i.e., for some tree $[Tr']$ of TLR-formulas and some $\alpha <_2 b$, $\neg\Diamond(Tr') \in \alpha_2(a)$ and $\Diamond(Tr') \in \alpha_2(b)$. If both a, b belong to M, then the condition 5.3.3.3 fails for $\alpha_1(d)\,G\,(Tr)$. This is impossible because $\alpha_1(d)\,G\,(Tr)$. Therefore only the following two cases are possible: (1) both a, b belong to M_1 and (2) $a \leqslant_1 d$ and $b \in M$. In the case 1 condition 5.3.3.3 fails for $w\,G\,(Tr_1)$ (and so *not* $w\,G\,(Tr_1)$). Suppose now that the case 2 takes place and $w\,G\,(Tr_1)$. Then, by 5.3.3.3, $\neg\Diamond(Tr') \in \alpha_1(d)$. But then, since $\alpha_1(d)\,G\,(Tr)$, by 5.3.3.2, $\neg\Diamond(Tr') \in \mathrm{root}(Tr)$, whence, by 5.3.3.3, $\neg\Diamond(Tr') \in \alpha(b)$, i.e. (since $\alpha(b) = a_2(b)$), $\neg\Diamond(Tr') \in \alpha_2(b)$. We have obtained a contradiction. Thus, in both the cases 1 and 2, *not* $w\,G\,(Tr_1)$.

Finally, suppose the reason for *not* $w\,G\,(Tr_2)$ is that the condition 5.3.3.4 is not satisfied. It means that there are trees $[M_3, <_3, \alpha_3]$, $[M_4, <_4, \alpha_4]$ of TLR-formulas and a function $h: M_4 \to M_2$ satisfying the above conditions (4)–(8). In fact $M_4 = M_5 \cup M_6$ for some $M_5 \cap M_6 = \emptyset$, where for each $a \in M_5$, $ha \in M_1$ and for each $a \in M_6$, $ha \in M$.

First we want to show that

(11) $\Diamond(Tr_3)_c \in \alpha_1(d)$ for all $c \in M_6$.

Indeed, deny this: suppose $c \in M_6$ and $\Diamond(Tr_3)_c \notin \alpha_1(d)$, i.e. (by 5.3.1b, 5.3.2b), $\neg\Diamond(Tr_3)_c \in \alpha_1(d)$. Let $[Tr_7] = [M_7, <_7, \alpha_7] = [Tr_4]_c$. Let h' be the restriction of h to M_7. Observe that the values of h' belong to M. Now, one can check that the conditions 5.3.3.4a–e are satisfied if we take $\alpha_1(d)$ for w, $[Tr]$ for $[Tr]$, $[Tr_3]_c$ for $[M_1, <_1, \alpha_1]$, $[Tr_7]$ for $[M_2, <_2, \alpha_2]$ and h' for h (this checking is routine and we skip it). It means that *not* $\alpha_1(d)\ G\ (Tr)$, a contradiction.

Let h'', $<_5$ and α_5 be the restrictions of h, $<_4$ and α_4 to M_5, respectively. Observe that the values of h'' belong to M_1.

Suppose $w\ G\ (Tr_1)$. We claim that then the conditions 5.3.3.4a–e are satisfied if we take w for w, $[Tr_1]$ for $[Tr]$, $[Tr_3]$ for $[M_1, <_1, \alpha_1]$, $[M_5, <_5, \alpha_5]$ for $[m_2, <_2, \alpha_2]$ and h'' for h (and thus, *not* $w\ G\ (Tr_1)$, a contradiction).

Indeed, the condition 5.3.3.4(a) is just (4), and (b), (c), (d) we get from (5), (6), (7) almost automatically. Let us now check (e). Consider any $a <_3 b$ with $a \in M_5$ and $b \notin M_5$. We want to show that $\Diamond(M_3, <_3, \alpha_3)_b \in \alpha_1(h''a)$. If $b \notin M_6$, then $b \notin M_4$ and, by (8), $\Diamond(M_3, <_3, \alpha_3)_b \in \alpha_2(ha)$, whence (since $a \in M_5$, i.e. $ha \in M_1$) $\Diamond(M_3, <_3, \alpha_3)_b \in \alpha_1(ha) = \alpha_1(h''a)$. And if $b \in M_6$, then, by (11), $\Diamond(Tr_3)_b \in \alpha_1(d)$. It is easy to verify that $h''a \preccurlyeq_1 d$; therefore our supposition $w\ G\ (Tr_1)$ implies by 5.3.3.3 that $\Diamond(Tr_3)_b \in \alpha_1(h''a)$, as needed. □

5.3.6. Lemma. *For any $B \in Sb$ and any $w \in W$, $w \vDash B$ iff $B \in w$.*

Proof. By induction on the complexity of B. The only nonstraightforward case is $B = \Diamond(Tr)$, where $[Tr] = [M, <, \alpha]$.

($\Rightarrow$): Suppose $w \vDash \Diamond(Tr)$, i.e., there is an evaluator $\beta: M \to W$ such that $w\ G\ (M, <, \beta)$ and for each $a \in M$, $\beta(a) \vDash \alpha(a)$. Then, by the induction hypothesis, for each $a \in M$, $\alpha(a) \in \beta(a)$. Now, suppose $\neg\Diamond(Tr) \in w$. Define a function $h: M \to M$ by setting $ha = a$ for all $a \in M$. Then the conditions 5.3.3.4a–e are satisfied if we take w for w, $[Tr]$ for $[Tr]$, $[M, <, \beta]$ for $[M_1, <_1, \alpha_1]$, $[M, <, \beta]$ for $[M_2, <_2, \alpha_2]$ and h for h. It means that *not* $w\ G\ (M, <, \beta)$, a contradiction.

($\Leftarrow$): Suppose $\Diamond(Tr) \in w$. Let $d = \mathrm{ROOT}[M, <]$. Define an evaluator δ of M by: $\delta =_d \alpha$ (recall 1.5) and $\delta(d) = \alpha(d) \wedge \neg\Diamond(Tr)$. By 4.8 we have:

$$(*) \qquad \mathrm{TLR} \vdash \Diamond(Tr) \to (M, <, \delta).$$

Let H be the set of all evaluators $\beta: M \to W$ such that for each $a \in M$, $\alpha(a) \in \beta(a)$ and, if $a = d$, $\neg\Diamond(Tr) \in \beta(a)$ as well.

Let for each $\beta \in H$, β^+ be the evaluator of M such that for each $a \in M$, $\beta^+(a) = \bigwedge \beta(a)$ (the conjunction of all the elements of $\beta(a)$).

And let γ be the evaluator of M such that for any $a \in M$, $\gamma(a) = \bigvee \{\beta^+(a) : \beta \in H\}$.

Taking 5.3.2b into account, it is easy to see that for each $a \in M$, $\delta(a)$ implies $\gamma(a)$ by propositional logic and so $\mathrm{TLR} \vdash \delta(a) \to \gamma(a)$. It follows from this and (*) by 4.3 that

$$\mathrm{TLR} \vdash \Diamond(Tr) \to \Diamond(M, <, \gamma).$$

From this by 4.4 we obtain TLR$\vdash \Diamond(Tr) \to \bigvee\{\Diamond(M, \prec, \beta^+): \beta \in H\}$, whence it follows:

5.3.6.1. Lemma. TLR$\vdash \bigwedge w \to \bigvee \{\Diamond(M, \prec, \beta^+): \beta \in H\}$.

5.3.6.2. Lemma. *For any $\beta \in H$, if* not $w\,G\,(M, \prec, \beta)$, *then*

TLR$\vdash \bigwedge w \to \neg\Diamond(M, \prec, \beta^+)$.

Proof. Suppose $\beta \in H$ and *not* $w\,G\,(M, \prec, \beta)$. Since $\Diamond(Tr) \in w$ and $\neg\Diamond(Tr) \in \beta(d)$, the reason for *not* $w\,G\,(M, \prec, \beta)$ cannot be that the conditions 5.3.3.1 is not satisfied. Therefore only the following three cases (i)–(iii) are possible:

(i) The reason for *not* $w\,G\,(M, \prec, \beta)$ is that the condition 5.3.3.2 is not satisfied, i.e., there is $\neg\Diamond(Tr') \in w$ with $\Diamond(Tr') \in \beta(d)$. Thus we have:

(1) TLR$\vdash \bigwedge w \to \neg\Diamond(Tr')$;

(2) TLR$\vdash \beta^+(d) \to \Diamond(Tr')$;

(3) TLR$\vdash \Diamond(M, \prec, \beta^+) \to \Diamond\langle\beta^+(d)\rangle$ (4.2.4a);

(4) TLR$\vdash \Diamond\langle\beta^+(d)\rangle \to \Diamond\langle\Diamond(Tr')\rangle$ ((2), 4.3);

(5) TLR$\vdash \Diamond(M, \prec, \beta^+) \to \Diamond(Tr')$ ((3), (4), 4.6);

(6) TLR$\vdash \bigwedge w \to \neg\Diamond(M, \prec, \beta^+)$ ((1), (5)).

(ii) The reason for *not* $w\,G\,(M, \prec, \beta)$ is that the condition 5.3.3.3 is not satisfied, i.e., there are $a \prec b$ and $\neg\Diamond(Tr') \in \beta(a)$ with $\Diamond(Tr') \in \beta(b)$. Thus we have:

(1) TLR$\vdash \beta^+(a) \to \neg\Diamond(Tr')$;

(2) TLR$\vdash \beta^+(b) \to \Diamond(Tr')$;

(3) TLR$\vdash \Diamond(M, \prec, \beta^+) \to \Diamond\langle\beta^+(a), \beta^+(b)\rangle$ (4.2.4a);

(4) TLR$\vdash \Diamond(M, \prec, \beta^+) \to \Diamond\langle\neg\Diamond(Tr'), \Diamond(Tr')\rangle$ ((1), (2), (3), 4.3);

(5) TLR$\vdash \Diamond\langle\neg\Diamond(Tr'), \Diamond(Tr')\rangle \to \Diamond\langle\neg\Diamond(Tr') \wedge \Diamond(Tr')\rangle$ (4.2.3);

(6) TLR$\vdash \neg(\neg\Diamond(Tr') \wedge \Diamond(Tr'))$ (a tautology);

(7) TLR$\vdash \neg\Diamond\langle\neg\Diamond(Tr') \wedge \Diamond(Tr')\rangle$ ((6), 4.2.6);

(8) TLR$\vdash \neg\Diamond(M, \prec, \beta^+)$ ((4), (5), (7)), i.e.,

TLR$\vdash \bigwedge w \to \neg\Diamond(M, \prec, \beta^+)$.

(iii) The reason for *not* $w\,G\,(M, \prec, \beta)$ is that the condition 5.3.3.4 is not satisfied, i.e., there are trees $[M_1, \prec_1, \alpha_1]$, $[M_2, \prec_2, \alpha_2]$ of TLR-formulas and a function $h: M_2 \to M$ such that:

(a) $\neg\Diamond(M_1, \prec_1, \alpha_1) \in w$;

(b) $[M_2, \prec_2, \alpha_2]$ is an initial part of $[M_1, \prec_1, \alpha_1]$;

(c) for all $a \prec_2 b$, $ha \preccurlyeq hb$;

(d) for all $a \in M_2$, $\alpha_2(a) \in \beta(ha)$;

(e) for all $a \in M_2$ and all $a \prec_1 b \notin M_2$, $\Diamond(M_1, \prec_1, \alpha_1)_b \in \beta(ha)$.

Thus we have:

(1) $\mathrm{TLR} \vdash \bigwedge w \rightarrow \neg \Diamond(M_1, <_1, \alpha_1)$ (from (a));

(2) $\mathrm{TLR} \vdash \beta^+(ha) \rightarrow \alpha_2(a)$ for any $a \in M_2$ (from (d));

(3) $\mathrm{TLR} \vdash \beta^+(ha) \rightarrow \Diamond(M_1, <_1, \alpha_1)_b$
for any $a \in M_2$ and any $a <_1 b \notin M_2$ (from (e)).

Let $[Tr_3] = [M_3, <_3, \alpha_3]$, where $M_3 = \{ha: a \in M_2\}$, and $<_3$ and α_3 are the restrictions of $<$ and β^+ to M_3. It is easy to verify that $[Tr_3]$ is a tree of TLR-formulas and, clearly, $[Tr_3] \subseteq [M, <, \beta^+]$.

Let α_2^+ be the evaluator of M_2 such that for each $a \in M_2$, $\alpha_2^+(a) = \beta^+(ha)$.

The implication in (4) below can be deduced in TLR using several times the axioms 4.2.4a and 4.2.4b:

(4) $\mathrm{TLR} \vdash \Diamond(Tr_3) \rightarrow \Diamond(M_2, <_2, \alpha_2^+)$.

Let α_2^{++} be the evaluator of M_2 such that for each $a \in M_2$, $\alpha_2^{++}(a) = \alpha_2(a) \wedge \bigwedge\{\Diamond(M_1, <_1, \alpha_1)_b: a <_1 b \notin M_2\}$. Then we have:

(5) $\mathrm{TLR} \vdash \Diamond(M_2, <_2, \alpha_2^+) \rightarrow \Diamond(M_2, <_2, \alpha_2^{++})$ ((2), (3), 4.3).

The implication in (6) below can be derived in TLR using several times the axioms 4.2.4c, 4.2.4a and Lemma 4.3:

(6) $\mathrm{TLR} \vdash \Diamond(M_2, <_2, \alpha_2^{++}) \rightarrow \Diamond(M_1, <_1, \alpha_1)$.

Now, (1), (4), (5) and (6) imply that $\mathrm{TLR} \vdash \bigwedge w \rightarrow \neg\Diamond(Tr_3)$. Since $(Tr_3) \subseteq (M, <, \beta^+)$, the axiom 4.2.4a gives $\mathrm{TLR} \vdash \Diamond(M, <, \beta^+) \rightarrow \Diamond(Tr_3)$. Consequently, $\mathrm{TLR} \vdash \bigwedge w \rightarrow \neg\Diamond(M, <, \beta^+)$. 5.3.6.2 is thus proved. □

Since w is TLR-consistent, 5.3.6.1–2 imply that there is $\beta \in H$ with $w\, G\, (M, <, \beta)$. Recalling the definition of H, it means that for each $a \in M$, $\alpha(a) \in \beta(a)$, whence, by the induction hypothesis, $\beta(a) \vDash \alpha(a)$. It means that $w \vDash \Diamond(Tr)$. 5.3.6 is thus proved. □

We can now complete the proof of Theorem 5.3($\Leftarrow$). If $\mathrm{TLR} \nvdash A$, then obviously there is a maximal TLR-consistent subset of Sb (i.e., an element of W) w such that $A \notin w$. Then, by 5.3.6, *not* $w \vDash A$; taking 5.3.5 into account, it means that $\langle W, G, \vDash \rangle$ is a finite TLR-countermodel for A. □

6. Visser models

6.1. Definition. A *2-model* is a tuple $\langle V, R, S, \Vdash \rangle$, where:
V is a nonempty set (of 'possible worlds');
R and S are binary relations on V;
$\Vdash$ is a ('forcing') relation included in $V \times \{propositional\ letters\}$, which is

extended to complex TLR-formulas in the following unique way: for any $w \in V$,

(a) $w \Vdash$ commutes with Boolean connectives;

(b) $w \Vdash \Diamond(M, <, \alpha)$ iff there is an evaluator $\beta : M \to V$ such that for all $a \in M$, we have $\beta(a) \Vdash \alpha(a)$, $w\, R\, \beta(a)$ and, for all $a < b$, $\beta(a)\, S\, \beta(b)$.

A 2-model $\langle V, R, S, \Vdash\rangle$ is said to be a *countermodel* for a TLR-formula A, if *not* $w \Vdash A$ for some $w \in V$.

6.2. Definition. A *finite Visser model*[3] is a 2-model $\langle V, R, S, \Vdash\rangle$ with finite V and the following properties:

1. R is transitive and irreflexive;
2. S is transitive and reflexive;
3. $R \subseteq S$;
4. $w\, S\, u\, R\, v \Rightarrow w\, R\, v$ (all w, u, v).

And a finite Visser model $\langle V, R, D, \Vdash\rangle$ is said to be *strengthened* iff it has the following additional property:

5. $w\, R\, u\, S\, v \Rightarrow w\, R\, v$ (all w, u, v).

6.3. Lemma. *If there is a finite* TLR-*countermodel for* A, *then there is a finite strengthened Visser countermodel for* A.

Proof. The proof partially uses some technical ideas developed by Visser in [17].

Assume that $\langle W, G, \Vdash\rangle$ is a finite TLR-countermodel for A.

6.3.1. Let X be the smallest set of $\approx$-equivalence classes of trees of elements of W such that:

(a) if $\Diamond(M, <, \beta)$ is a subformula of A and α is an evaluator of M in W, then $(M, <, \alpha) \in X$ (we suppose here that there is at least one such subformula);

(b) if $(M, <, \alpha) \in X$ and $(M', <', \alpha') \subseteq (M, <, \alpha)$, then $(M', <', \alpha') \in X$.

6.3.2. Let us define a 2-model $\langle V, R, S, \Vdash\rangle$ as follows:

V is the set of all (finite) nonempty sequences $\langle (Tr_1), \ldots, (Tr_n)\rangle$ of elements of X, where for any $1 \leq i < n$, $\mathrm{root}(Tr_i)\; G\; (Tr_{i+1})$ or $Tr_{i+1} \subset Tr_i$ (recall 1.9.2, 1.6). Note that V is nonempty because for each $w \in W$, $\langle\!\langle w \rangle\!\rangle \in V$.

$\langle (Tr_1), \ldots, (Tr_n)\rangle R w$ iff $w = \langle (Tr_1), \ldots, (Tr_n), (Tr_{n+1}), \ldots, (Tr_m)\rangle$ for some $m > n$ and there is $n \leq k < m$ such that $\mathrm{root}(Tr_k)\; G\; (Tr_{k+1})$.

$w\, S\, u$ iff u is an end-extension of w.

For any propositional letter p, $w \Vdash p$ iff $w^\circ \vDash p$, where the notation w° is explained below in 6.3.3.2.

6.3.3. Notation. 1. If $w = \langle (Tr_1), \ldots, (Tr_n)\rangle \in V$, we use $\mathrm{Last}(w)$ to denote (Tr_n).

[3] The tuples we call Visser models, have in fact only common frames (i.e., the part $\langle V, R, S\rangle$) with the models for ILM studied by Visser [17]. The forcing relations are, of course, defined in different ways, as the languages of TLR and ILM are different.

2. If $w \in V$, we use w° to denote root Last(w).

3. If α is an evaluator of a set M in V, then α° denotes the evaluator of M in W such that for each $\alpha \in M$, $\alpha^\circ(a) = (\alpha(a))^\circ$.

The following lemma follows easily from 5.2.3a and 5.2.3c:

6.3.4. Lemma. *Suppose* $\langle (Tr_1), \dots, (Tr_n) \rangle \in V$, $1 \leq i \leq N$ *and* $\text{root}(Tr_i)\, G\, (Tr)$. *Then for each* $1 \leq j \leq i$, $\text{root}(Tr_j)\, G\, (Tr)$.

6.3.5. Lemma. $\langle V, R, S, \Vdash \rangle$ *is a finite strengthened Visser model.*

Proof. The properties 6.2.1–5 immediately follow from the definitions of R and S. We only need to verify that V is finite. First notice that since W is finite, X is finite. Therefore, if V is infinite, there are arbitrarily long elements of V. In other words, for arbitrary large n, there are elements of V of the form

$$\langle (Tr_1), \dots, (Tr_m), \dots, (Tr_{m+n}), \dots, (Tr_k) \rangle$$

with $(Tr_m) = (Tr_{m+n})$. We may suppose that n is sufficiently large, namely, that there are no $(Tr'_1) \subset \cdots \subset (Tr'_n)$ in X. Then it follows from the definition of V that there is j with $m < j < m+n$ such that $\text{root}(Tr_j)\, G\, (Tr_{j+1})$ and $(Tr_{m+n}) \subseteq (Tr_{j+1})$. Then, by 5.2.3a, $\text{root}(Tr_j)\, G\, (Tr_{m+n})$ and, by 6.3.4, $\text{root}(Tr_m)\, G\, (Tr_{m+n})$, i.e. (as $(Tr_m) = (Tr_{m+n})$), $\text{root}(Tr_m)\, G\, (Tr_m)$, whence, by 5.2.3a, $\text{root}(Tr_m)$ $G\, \langle \text{root}(Tr_m) \rangle$. But, according to 5.2.1, this is impossible. □

6.3.6. Lemma. *Suppose* $w\, S\, u$ *and* not $\text{Last}(u) \subseteq \text{Last}(w)$. *Then* $w\, R\, u$.

Proof. Assume the conditions of the lemma. Assume $w = \langle (Tr_1), \dots, (Tr_n) \rangle$. Then u has the form $\langle (Tr_1), \dots, (Tr_n), \dots, (Tr_m) \rangle$, where *not* $(Tr_m) \subseteq (Tr_n)$. It follows then from the definition of V that there is $n \leq i < m$ with $\text{root}(Tr_i)\, G\, (Tr_{i+1})$. It means by the definition of R that $w\, R\, u$. □

6.3.7. Lemma. *If* $w\, R\, u$, *then* $w^\circ\, G\, \text{Last}(u)$.

Proof. Suppose $w = \langle (Tr_1), \dots, (Tr_n) \rangle$ and $w\, R\, u$. It follows from the definitions of V and R that u has the form $\langle (Tr_1), \dots, (Tr_n), \dots, (Tr_m) \rangle$, where for some $n \leq i < m$, $\text{root}(Tr_i)\, G\, (Tr_{i+1})$ and for each $i+1 \leq j < m$, $(Tr_{j+1}) \subset (Tr_j)$. Namely, $(Tr_m) \subseteq (Tr_{i+1})$. And since $\text{root}(Tr_i)\, G\, (Tr_{i+1})$, we have $\text{root}(Tr_i)\, G\, (Tr_m)$ by 5.2.3a; consequently, by 6.3.4, $\text{root}(Tr_n)\, G\, (Tr_m)$, i.e., $w^\circ\, G\, \text{Last}(u)$. □

6.3.8. Lemma. *Suppose* $u \in W$ *and* $[M, <, \alpha]$ *is a tree of elements of the set* X *such that*:

(a) $u\, G\, \text{root}(M, <, \alpha)$ *and*

(b) *for all* $a < b$, $\alpha(b) \subseteq \alpha(a)$.

Let β *be the following evaluator of* M *in* W:

$$\text{for all } \alpha \in M, \quad \beta(a) = \text{root}\ \alpha(a).$$

Then there is a tree $[M', <', \beta']$ *of elements of* W *such that*:

(c) $u\, G\, (M', <', \beta')$;

(d) $[M, <, \beta] \subseteq [M', <', \beta']$;

(e) *for each* $a \in M$, $\alpha(a) \subseteq (M', <', \beta)_a$.

Proof. By induction on the cardinality of M. Assume the conditions of the lemma.

Suppose M consists of a single element b. Choose $[M', <', \beta']$ such that $(M', <', \beta') = \alpha(b)$ $(= \text{root}(M, <, \alpha))$ and $\text{ROOT}[M', <'] = b$. Then we trivially have (e), and the condition (c) is just (a). Clearly (d) is also satisfied.

Suppose now that M consists of ≥ 2 elements. Choose then a pair $b, c \in M$ such that the depth of c in $[M, <, \alpha]$ is 0 and b is the immediate predecessor of c in $[M, <, \alpha]$. Let M^- be $M - \{c\}$, and $<^-, \alpha^-, \beta^-$ be the restrictions of $<, \alpha, \beta$ to M^-. The conditions (a) and (b) clearly continue to hold when $M^-, <^-, \alpha^-$ stand for $M, <, \alpha$. Hence, by the induction hypothesis, there is a tree $[M^*, <^*, \beta^*]$ of elements of W such that:

(c^-) $v\, G\, (M^*, <^*, \beta^*)$;

(d^-) $[M^-, <^-, \beta^-] \subseteq [M^*, <^*, \beta^*]$;

(e^-) for each $a \in M^-$, $\alpha^-(a) \subseteq (M^*, <^*, \beta)_a$.

We may suppose that $c \notin M^*$. Choose then $[M^{**}, <^{**}, \beta^{**}]$ such that $(M^{**}, <^{**}, \beta^{**}) = \alpha(c)$, $M^{**} \cap M^* = \emptyset$ and $\text{ROOT}[M^{**}, <^{**}] = c$. And let

$$[M', <', \beta'] = [M^*, <^*, \beta^*] +_b [M^{**}, <^{**}, \beta^{**}].$$

We now want to show that (c), (d) and (e) are satisfied.

(c): According to (b), $\alpha(c) \subseteq \alpha(b) = \alpha^-(b)$; consequently, by ($e^-$), $\alpha(c) \subseteq (M^*, <^*, \beta^*)_b$, i.e.,

$$(M^{**}, <^{**}, \beta^{**}) \subseteq (M^*, <^*, \beta^*)_b.$$

Therefore, it is clear that

$$(1) \qquad (M', <', \beta') = (M^*, <^*, \beta^*) +_b (M^{**}, <^{**}, \beta^{**}) \subseteq (M^*, <^*, \beta^*) +_b (M^*, <^*, \beta^*)_b.$$

But $(M^*, <^*, \beta^*) +_b (M^*, <^*, \beta^*)_b$ is a duplicate of $(M^*, <^*, \beta^*)$ and hence, by (c^-) and 5.2.3b,

$$(2) \qquad u\, G\, (M^*, <^*, \beta^*) +_b (M^*, <^*, \beta^*)_b.$$

Now, (c) follows from (1) and (2) by 5.2.3a.

(d): Taking (d^-) into account, it suffices to show that

$$(3) \qquad \beta(c) = \beta'(c),$$

$$(4) \qquad \text{for all } a \in M, \quad c < a \Leftrightarrow c <' a, \quad \text{and}$$

$$(5) \qquad \text{for all } a \in M, \quad a < c \Leftrightarrow a <' c.$$

Observe that $\beta'(c) = \beta^{**}(c)$. And since $c = \text{ROOT}[M^{**}, <^{**}]$, we have $\beta^{**}(c) = \text{root}(M^{**}, <^{**}, \beta^{**})$. But $(M^{**}, <^{**}, \beta^{**}) = \alpha(c)$ and hence

$$\text{root}(M^{**}, <^{**}, \beta^{**}) = \text{root } \alpha(c).$$

On the other hand, by the definition of β, root $\alpha(c) = \beta(c)$. Consequently, (3) holds.

The ($\Rightarrow$) direction of (4) trivially holds because, according to our choice of c, there is no $a \in M$ with $c \prec a$. And the ($\Leftarrow$) part also holds because $\{a: c \prec' a\} = (M^{**} - \{c\})$ and $(M^{**} - \{c\}) \cap M = \emptyset$.

Let us now check (5). Consider an arbitrary $a \in M$. Since b is the immediate predecessor of c in $[m, \prec]$, we have $a \prec c \Leftrightarrow a \preccurlyeq^- b$; on the other hand, by (d^-), $a \preccurlyeq^- b \Leftrightarrow a \preccurlyeq^* b$; finally, since $[M', \prec', \beta'] = [M^*, \prec^*, \beta^*] +_b [M^{**}, \prec^{**}, \beta^{**}]$ and $c = \mathrm{ROOT}[M^{**}, \prec^{**}, \beta^{**}]$, we have $a \preccurlyeq^* b \Leftrightarrow a \prec' c$. These three equivalences give $a \prec c \Leftrightarrow a \prec' c$.

(e): Consider an arbitrary $a \in M$. If $a = c$, then $\alpha(a)$ is just $(M', \prec', \beta')_a$. Suppose now $a \neq c$, i.e., $a \in M^-$. Then, by (e^-), $\alpha(a) \subseteq (M^*, \prec^*, \beta^*)_a$. But we have $[M^*, \prec^*, \beta^*] \subseteq [M', \prec', \beta']$ and this clearly implies that $(M^*, \prec^*, \beta^*)_a \subseteq (M', \prec', \beta')_a$. Consequently, $\alpha(a) \subseteq (M', \prec', \beta')_a$ □

6.3.9. Definition. Let $[Tr] = [M, \prec, \alpha]$ be a tree of elements of V such that for all $a \prec b$, $\alpha(a)\, S\, \alpha(b)$. Then Rank($Tr$) is the number of the pairs $a, b \in M$ such that a is the immediate predecessor of b in $[M, \prec]$ and *not* Last($\alpha(b)$) $\subseteq$ Last($\alpha(a)$).

6.3.10. Lemma. *Suppose $w \in V$ and $[Tr] = [M, \prec, a]$ is a tree of elements of V such that*:

(1) *for each* $a \in M$, $w\, R\, \alpha(a)$;

(2) *for all* $a \prec b$, $\alpha(a)\, S\, \alpha(b)$.

Then $w^\circ\, G\, (M, \prec, \alpha^\circ)$.

Proof. By induction on Rank(Tr). Assume the conditions of the lemma.

Suppose Rank(Tr) = 0. It means that

(3) for all $a \prec b$, $\mathrm{Last}(\alpha(b)) \subseteq \mathrm{Last}(\alpha(a))$.

From (1) we have $w\, R\, \mathrm{root}(Tr)$, whence, by 6.3.7,

(4) $w^\circ\, G\, \mathrm{Last}(\mathrm{root}(Tr))$.

Let λ be the evaluator of M such that

$$\lambda(a) = \mathrm{Last}(\alpha(a)) \qquad (\text{all } a \in M).$$

Then (4) implies

(5) $w^\circ\, G\, \mathrm{root}(M, \prec, \lambda)$

and (3) implies

(6) for all $a \prec b$, $\lambda(b) \subseteq \lambda(a)$.

But (5) and (6) mean that the conditions (a) and (b) of Lemma 6.3.8 are satisfied, when w° stands for u and λ stands for α. Then there is a tree $[M', \prec', \beta']$ of elements of W such that the conditions (c) and (d) of Lemma 6.3.8 are also satisfied. These two conditions imply by 5.2.3a that $w^\circ = u\, G\, (M, \prec, \beta)$. It means that $w^\circ\, G\,(M, <, \alpha^\circ)$, because $\beta = \alpha^\circ$.

Suppose now $\mathrm{Rank}(Tr) > 0$. Let us fix any pair $c, d \in M$ such that c is the immediate predecessor of d in $[M, <]$ and *not* $\mathrm{Last}(\alpha(d)) \subseteq \mathrm{Last}(\alpha(c))$. Then, by (2) and 6.3.6,

(7) $\alpha(c)\, R\, \alpha(d)$.

Let $[Tr_1] = [M_1, <_1, \alpha_1] = [Tr]_d$. It follows from (7), (2) and 6.2.5 (taking 6.3.5 into account) that the condition (1) (as well as (2)) is satisfied when we put $\alpha(c)$ for w and $[Tr_1]$ for $[Tr]$; notice also that $\mathrm{Rank}(Tr_1) < \mathrm{Rank}(Tr)$. Consequently, by the induction hypothesis,

(8) $\alpha^\circ(c)\, G\,(M_1, <_1, \alpha_1^\circ)$.

Let $[Tr_2] = [M_2, <_2, \alpha_2]$, where $M_2 = M - \{d\preccurlyeq\}$ (recall 1.2.5) and $<_2$ and α_2 are the restrictions of $<$ and α to M_2. Clearly $\mathrm{Rank}(Tr_2) < \mathrm{Rank}(Tr)$, and (Tr_2) (if we put it for (Tr)) satisfies the conditions of Lemma 6.3.10. Consequently, by the induction hypothesis,

(9) $w^\circ\, G\,(M_2, <_2, \alpha_2^\circ)$.

Now, (9) and (8) imply by 5.2.3c that $w^\circ\, G\,(M, <, \alpha^\circ)$. □

6.3.11. Lemma. *For any subformula B of A and any $w \in V$, $w \Vdash B$ iff $w^\circ \vDash B$.*

Proof. By induction on the complexity of B. The only nonstraightforward case is $B = \Diamond(Tr)$, where $[Tr] = [M, <, \alpha]$. Assume the conditions of the lemma.

($\Rightarrow$): Suppose $w \Vdash \Diamond(Tr)$. It means that for some evaluator $\beta : M \to V$, we have:

(a) for all $a \in M$, $w\, R\, \beta(a)$ and, for all $a < b$, $\beta(a)\, S\, \beta(b)$;

(b) for all $a \in M$, $\beta(a) \Vdash \alpha(a)$.

By 6.3.10, (a) implies that $w^\circ\, G\,(M, <, \beta^\circ)$, and by the induction hypothesis, (b) implies that for each $a \in M$, $\beta^\circ(a) \vDash \alpha(a)$. It means that $w^\circ \vDash \Diamond(Tr)$.

($\Leftarrow$): Suppose $w^\circ \vDash \Diamond(Tr)$. It means that for some evaluator $\beta : M \to W$, we have:

(a) $w^\circ\, G\,(M, <, \beta)$;

(b) for each $a \in M$, $\beta(a) \vDash \alpha(a)$.

Assume $w = \langle (Tr_1), \ldots, (Tr_n) \rangle$. Let us define an evaluator $\gamma : M \to V$ by induction on the height of a in $[M, <]$ (recall 1.2.3):

(1) $\gamma(\mathrm{ROOT}[M, <]) = \langle (Tr_1), \ldots, (Tr_n), (M, <, \beta) \rangle$;

(2) suppose the height of a in $[M, <]$ is $m + 1$, b is the immediate predecessor of a in $[M, <]$ (so the height of b is m) and $\gamma(b) = \langle (Tr_1), \ldots, (Tr_n), \ldots, (Tr_{n+1+m}) \rangle$. Then $\gamma(a) = \langle (Tr_1), \ldots, (Tr_n), \ldots, (Tr_{n+1+m}), (M, <, \beta)_a \rangle$.

It follows from (a) that for each $a \in M$, $w\, R\, \gamma(a)$; it is obvious also that for all $a < b$, $\gamma(a)\, S\, \gamma(b)$. On the other hand, notice that for each $a \in M$, $\gamma^\circ(a) = \beta(a)$, whence, by the induction hypothesis, (b) implies $\gamma(a) \Vdash \alpha(a)$. This all means that $w \Vdash \Diamond(Tr)$. □

We can now complete the proof of Lemma 6.3. Since $\langle W, G, \vDash\rangle$ is a countermodel for A, we have *not* $w \vDash A$ for some $w \in W$. Clearly $\langle\!\langle w\rangle\!\rangle \in V$ and, by 6.3.11, *not* $\langle\!\langle w\rangle\!\rangle \Vdash A$. In view of 6.3.5, it means that $\langle V, R, S, \Vdash\rangle$ is a finite strengthened Visser countermodel for A. □

6.4. Theorem. (a) TLR $\vdash A$ *iff there is no finite Visser countermodel for* A.
(b) TLR $\vdash A$ *iff there is no finite strengthened Visser countermodel for* A.

Proof. By 5.3($\Leftarrow$), 6.3, 7.5, 8.1. □

6.5. Corollary. TLR *is decidable.*

7. The arithmetical soundness of TLR

7.1. Notation. 1. If α is an evaluator of a set M in the set of arithmetical sentences, then $\bar{\alpha}$ denotes the evaluator of M that assigns to each $a \in M$ the theory PA + $\alpha(a)$.

2. If D denotes the $\approx$-equivalence class of a tree $[M, <, \alpha]$ of arithmetical sentences, then CnD denotes a natural formalization (in the language of PA) of the assertion "$(M, <, \bar{\alpha})$ is Π_1-consistent".

3. For an arithmetical formula F, $\mathrm{Pr}(\ulcorner F\urcorner)$ abbreviates $\neg\mathrm{Cn}\langle\neg F\rangle$.

4. If α is an evaluator of a set M in the set of TLR-formulas and f is a function: {TLR-formulas} $\to$ {arithmetical sentences}, then $f\alpha$ denotes the evaluator of M that assigns to each $a \in M$ the arithmetical sentence $f(\alpha(a))$.

7.2. Remark. It follows easily from 2.5 that $\mathrm{Cn}(Tr) \in \Pi_1$ for any tree $[Tr]$ of arithmetical sentences. Note also that $\mathrm{Cn}\langle F\rangle$ (resp. $\mathrm{Pr}(\ulcorner F\urcorner)$) expresses that F is a sentence consistent with PA (resp. provable in PA).

In view of 3.7, Cn can also be regarded as a formalization of the predicate of tolerance over PA.

7.3. Definition. A *realization* f is a function that assigns to each propositional letter p a sentence fp of PA. f is extended to complex TLR-formulas in the following unique way:

(a) f commutes with Boolean connectives;

(b) $f\Diamond(M, <, \alpha) = \mathrm{Cn}(M, <, f\alpha)$.

7.4. Definition. A TLR-formula A is said to be PA-*valid* iff PA $\vdash fA$ for every realization f.

7.5. Lemma. *If* TLR $\vdash A$, *then* A *is* PA-*valid.*

Proof. Modus Ponens clearly preserves PA-validity, and the rule 4.2.6 corresponds to the well-known principle $\mathrm{PA} \vdash F \Rightarrow \mathrm{PA} \vdash \mathrm{Pr}(\ulcorner F \urcorner)$, so it preserves PA-validity, too. All tautologies are clearly PA-valid. As for the axiom 4.2.2, it is in fact the well-known Lob's axiom which is PA-valid (cf. [3]). So we need now to check only the axioms 4.2.1, 4.2.3, 4.2.4a–c.

In the following A is an arithmetical sentence and $[M, <, \alpha]$, $[M_1, <_1, \alpha_1]$, $[M_2, <_2, \alpha_2]$ are trees of arithmetical sentences.

Work in PA.

Axiom 4.2.1. Assume $d \in M$, $\alpha_1 =_d \alpha_2 =_d \alpha$, $\alpha_1(d) = \alpha(d) \wedge \neg A$ and $\alpha_2(d) = A$. Suppose that both $(M, <, \bar{\alpha}_1)$ and $(M, <, \bar{\alpha}_2)$ are Π_1-inconsistent. We want to show that then $(m, <, \bar{\alpha})$ is Π_1-inconsistent. According to 2.5, Π_1-inconsistency of $(M, <, \bar{\alpha}_1)$ and $(M, <, \bar{\alpha}_2)$ means that there are Π_1-sentences E_a, F_a: $a \in M$ such that:

(1) $\mathrm{PA} \vdash \neg E_{\mathrm{ROOT}[M, <]}$ and $\mathrm{PA} \vdash \neg F_{\mathrm{ROOT}[M, <]}$;

(2) for each $d \neq a \in M$, $\mathrm{PA} + \alpha(a) \vdash \bigwedge \{E_b : a < b\} \rightarrow E_a$ and $\mathrm{PA} + \alpha(a) \vdash \bigwedge \{F_b : a < b\} \rightarrow F_a$;

(3) $\mathrm{PA} + \alpha(d) \wedge \neg A \vdash \bigwedge \{E_a : d < a\} \rightarrow E_d$ and $\mathrm{PA} + A \vdash \bigwedge \{F_a : d < a\} \rightarrow F_d$.

We may suppose here that

(4) for each $a \in M$, $\bigwedge \{E_b : a < b\}$ (resp. $\bigwedge \{F_b : a < b\}$) is a conjunct of E_α (resp. of F_a).

Let for each $a \leqslant d$, $H_b = F_b \vee E_b$, and for each $a \in M$ with *not* $a \leqslant d$, $H_b = F_b \wedge E_b$. Note that for each $a \in M$, $H_b \in \Pi_1$. It can be easily verified that (1)–(4) imply the following:

(5) $\mathrm{PA} \vdash \neg H_{\mathrm{ROOT}[M, <]}$;

(6) for each $a \in M$, $\mathrm{PA} + \alpha(a) \vdash \bigwedge \{H_b : a < b\} \rightarrow H_a$.

(5) and (6) imply by 2.5 that $(M, <, \bar{\alpha})$ is Π_1-consistent.

Axiom 4.2.3. Suppose $\mathrm{PA} + (A \wedge \mathrm{Cn}(Tr))$ is inconsistent. It means that $\mathrm{PA} + A \vdash \neg \mathrm{Cn}(Tr)$; and since $\mathrm{Cn}(Tr) \in \Pi_1$, we have that the Π_1-avalanche on $\langle \mathrm{PA} + A, \mathrm{PA} + \mathrm{Cn}(Tr) \rangle$ is inconsistent, i.e., $\langle \mathrm{PA} + A, \mathrm{PA} + \mathrm{Cn}(Tr) \rangle$ is Π_1-inconsistent.

Axiom 4.2.4a. Suppose $(M_2, <_2, \alpha_2) \subseteq (M_1, <_1, \alpha_1)$. Let Av_1 and Av_2 be the Π_1-avalanches on $(M_1, <_1, \bar{\alpha}_1)$ and $(M_2, <_2, \bar{\alpha}_2)$, respectively. It is easy to see that consistency of Av_1 implies consistency of Av_2; in other words, Π_1-consistency of $(M_1, <_1, \bar{\alpha}_1)$ implies Π_1-consistency of $(M_2, <_2, \bar{\alpha}_2)$.

Axiom 4.2.4b. Suppose $(M_2, <_2, \alpha_2)$ is a duplicate of $(M_1, <_1, \alpha_1)$. Let Av_1 and Av_2 be the Π_1-avalanches on $(M_1, <_1, \bar{\alpha}_1)$ and $(M_2, <_2, \bar{\alpha}_2)$, respectively. It is easy to see that $Av_2 = Av_1$. Consequently, if $(M_1, <_1, \bar{\alpha}_1)$ is Π_1-consistent, i.e., Av_1 is consistent, then Av_2 is consistent, i.e., $(M_2, <_2, \bar{\alpha}_2)$ is Π_1-consistent.

Axiom 4.2.4c. Assume $d \in M_1$, $\alpha_1(d) = A \wedge \mathrm{Cn}(M, <, \alpha)$, $M_1 \cap M = \emptyset$, $M_2 = M_1 \cup M$, for each $a \in M_1$, $\alpha_2(a) = \alpha_1(a)$, for each $a \in M$, $\alpha_2(a) = \alpha(a)$, and $<_2 = <_1 \cup < \cup \{(a, b)\colon\ a \leqslant_1 d$ and $b \in M\}$. Suppose $(M_2, <_2, \bar{\alpha}_2)$ is Π_1-inconsistent. We want to show that then $(M_1, <_1, \bar{\alpha}_1)$ is Π_1-inconsistent. By 2.5, Π_1-inconsistency of $(M_2, <_2, \bar{\alpha}_2)$ means that there are Π_1-sentences $F_a : a \in M$ such that:

(1) $\mathrm{PA} \vdash \neg F_{\mathrm{ROOT}[M_2, <_2]}$;

(2) for each $a \in M_2$, $\mathrm{PA} + \alpha_2(a) \vdash \bigwedge \{F_b\colon a <_2 b\} \to F_a$.

Observe that since $(M, <, \alpha) \subseteq (M_2, <_2, \alpha_2)$, by (2) and 2.5, we have:

(3) for each $a \in M$, $\mathrm{PA} \vdash \mathrm{Cn}(M, <, \alpha)_a \to \neg \mathrm{Pr}(\ulcorner \neg F_a \urcorner)$.

Clearly we may suppose that each $F_b \in \Pi_1!$ By formalizing in PA the argument that a false $\Pi_1!$-sentence cannot be consistent with PA, (3) implies:

(4) for each $a \in M$, $\mathrm{PA} \vdash \mathrm{Cn}(M, <, \alpha)_a \to F_a$.

Taking into account that the axiom 4.2.4a is PA-valid, we have:

$$\mathrm{PA} + \mathrm{Cn}(m, <, \alpha) \vdash \bigwedge \{\mathrm{Cn}(M, <, \alpha)_a\colon a \in M\};$$

therefore, by (4),

(5) $\mathrm{PA} + \mathrm{Cn}(M, <, \alpha) \vdash \bigwedge \{F_a\colon a \in M\}$.

Since $\alpha_1(d) = A \wedge \mathrm{Cn}(M, <, \alpha)$, $\{F_a\colon d <_2 a\} = \{F_a\colon a \in M\} \cup \{F_a\colon d <_1 a\}$ and $\alpha_2(d) = \alpha_1(d)$, we have that (2) and (5) imply:

(6) $\mathrm{PA} + \alpha_1(d) \vdash \bigwedge \{F_a\colon d <_1 a\} \to F_d$.

Let us consider any $a \in M_1$. We have $\alpha_2(a) = \alpha_1(a)$. If $a <_1 d$, then $\{F_b\colon a <_2 b\} = \{F_b\colon a <_1 b\} \cup \{F_b\colon b \in M\}$; we may suppose (take (2) into account) that $\mathrm{PA} \vdash F_d \to \bigwedge \{F_b\colon b \in M\}$; it means by (2) that

(∗) $\mathrm{PA} + \alpha_1(a) \vdash \{F_b\colon a <_1 b\} \to F_a$.

If *not* $a \leqslant_1 d$, then $\{F_b\colon a <_2 b\} = \{F_b\colon a <_1 b\}$, and (∗) follows from (1) at once. Finally, if $a = d$, (∗) is just (6). Thus we have:

(7) for each $a \in M_1$, $\mathrm{PA} + \alpha_1(a) \vdash \bigwedge \{F_b\colon a <_1 b\} \to F_a$.

Now, (7) and (1) imply by 2.5 that $(m_1, <_1, \bar{\alpha}_1)$ is Π_1-inconsistent. □

8. The arithmetical completeness of TLR

8.1. Lemma. *If there is a finite Visser countermodel for A, then A is not* PA-*valid.*

Proof. Let us fix a finite Visser countermodel $\langle V', R', S', \Vdash' \rangle$ for A. Without loss of generality we may assume that $V' = \{1, \ldots, e\}$ for some $e \geq 1$, $1\, R\, w$ for each $1 < w \leq e$, and *not* $1 \Vdash A$.

8.1.1. Let us define $\langle V, R, S, \Vdash \rangle$ by:

$V = V' \cup \{0\}$;

R and S are the extensions of R' and S' to V by setting $0\, R\, w$ for all $w \in V'$ and $0\, S\, w$ for all $w \in V$;

$\Vdash$ is the extension of $\Vdash'$ to V by setting for every propositional variable p, $0 \Vdash p \Leftrightarrow 1 \Vdash' p$.

Note that $\langle V, R, S, \Vdash \rangle$ is a finite Visser model and for all $w \in V'$ and every TLR-formula B, $w \Vdash B$ iff $w \Vdash' B$.

8.1.2. Notation. We write $\vdash_x F$ to express that x is the Gödel number of a PA-proof of the formula F.

The Diagonal Lemma (cf. [3]) enables us to construct for each $w \in V$ an arithmetical formula L_w expressing that w is the limit of the function h: $\omega \rightarrow V$, defined as follows:

8.1.3. Definition. $h(0) = 0$ and $h(x+1)$ is determined by:

1. $h(x+1) = w$, if $h(x)\, R\, w$ and $\vdash_x \neg L_w$;
2. otherwise, $h(x+1) = u$, if $h(x)\, S\, u \neq h(x)$ and there are $y < x$ and F such that:
   (a) $F \in \Pi_1!$ and F has a $<x$ regular counterwitness (recall 2.7);
   (b) $h(y)\, R\, h(x)$ and $h(y)\, R\, u$;
   (c) $\vdash_y L_u \rightarrow F$;
   (d) there are no u': $h(x)\, S\, u'$ (possibly $u' = h(x)$), y': $y' < y$, F' satisfying the conditions (a)–(c), when u', y', F' stand for u, y, F;
3. otherwise, $h(x+1) = h(x)$.

Note that the function h is primitive recursive.

8.1.4. Lemma. (PA $\vdash$:) *If* $x \leq y$, *then* $h(x)\, S\, h(y)$.

Proof. Immediately from 6.2.2–3 and the definition of h. □

8.1.5. Lemma. (PA $\vdash$:) *There is* z *such that for all* $z \leq z' \leq z''$, not $h(z')\, R\, h(z'')$.

Proof. Argue in PA. Suppose, for a contradiction, that for any z there are $z \leq z' \leq z''$ with $h(z')\, R\, h(z'')$; since, according to 8.1.4, $h(z)\, S\, h(z')$, we have by 6.2.4 that $h(z)\, R\, h(z'')$. Thus, for any z there is $t \geq z$ with $h(z)\, R\, h(t)$. It means that there is an infinite sequence $w_1 R w_2 R \cdots$. But this is impossible because V is finite and R is transitive and irreflexive. □

8.1.6. Lemma. (PA $\vdash$:) *Suppose* w *is the limit of the function* h, $w\, R\, u$, $w\, R\, v$ *and* $u\, S\, v$. *Then* PA + L_v *is* Π_1*-conservative over* PA + L_u.

Proof. Argue in PA. Assume the conditions of the lemma. We may suppose that $v \neq u$. Let F be any $\Pi_1!$-sentence provable in PA + L_v. Since w is the limit of h

and every provable formula has arbitrary long proofs, we have $\mathrm{PA}\vdash_y L_v \to F$ for some y with $h(y)=w$. Clearly PA proves that y is the Gödel number of a PA-proof of $L_v \to F$ and (as h is primitive recursive) $h(y)=w$.

Now argue in $\mathrm{PA}+L_u$: Suppose F is false, i.e., there exists a regular counterwitness z for F. Since u is the limit of h, there is x with $x>y, z$ and $h(x)=h(x+1)=u$. Then, according to 8.1.3.2, the only reason for $h(x+1)=u$ ($\neq v$) can be the following: there is a false Π_1-sentence F' (with a $<x$ regular counterwitness) such that $\vdash_{<y} L_u \to F'$. But 'we' (i.e., $\mathrm{PA}+L_u$) know that this does not hold. Consequently, F is true. □

8.1.7. Lemma. *For all $w, u \in V$:*

(a) $\mathrm{PA}\vdash \bigvee \{L_w : w \in V\}$.

(b) *If* $w \neq u$, *then* $\mathrm{PA}\vdash \neg(L_w \wedge L_u)$.

(c) *Suppose $[M, <, \alpha]$ is a tree of elements of V such that for each $a \in M$, $w\,R\,\alpha(a)$ and for all $a<b$, $\alpha(a)\,S\,\alpha(b)$. Let β be the evaluator of M such that for each $a \in M$, $\beta(\alpha)=L_{\alpha(a)}$. Then* $\mathrm{PA}\vdash L_w \to \mathrm{Cn}(M, <, \beta)$.

(d) *Suppose* not $w\,S\,u$. *Then* $\mathrm{PA}\vdash \neg\mathrm{Cn}\langle L_w, L_u\rangle$.

(e) *Suppose* $w \neq 0$ *and* not $w\,R\,u$. *Then* $\mathrm{PA}\vdash \mathrm{Pr}(\ulcorner \neg L_u \urcorner)$.

(f) L_0 *is true.*

Proof. In (a)–(e) we argue in PA.

(a): Let us fix the number z from 8.1.5. Then for each $x \geq z$, the transfer from $h(x)$ to $h(x+1)$ is determined by 8.1.3.2 or 8.1.3.3. We claim that the case 8.1.3.2 (when $x \geq z$) can take place at most z times (whence automatically follows that h has a limit and this limit is, of course, one of the elements of V).

Indeed, deny this claim. Let then $x_1 < \cdots < x_{z+1}$ be exactly the first $z+1$ numbers more or equal to z such that for each $1 \leq i \leq z+1$, the transfer from $h(x_i)$ to $h(x_i+1)$ is determined by 8.1.3.2. Let for each $1 \leq i \leq z+1$, y_i be the number y from 8.1.3.2 (putting $x=x_i$). The irreflexivity of R implies $z>y_1$; and, taking into account the reflexivity of S, it is easy to see that for each $1 \leq i \leq z$, $y_i > y_{i+1}$. Thus we have obtained a contradiction: $z>y_1>\cdots>y_{z+1}$.

(b): The function h clearly cannot have two limits.

(c): Assume the conditions of 8.1.7c. Suppose that w is the limit of h and $(M, <, \bar\beta)$ is Π_1-inconsistent, i.e., by 2.5, there are Π_1-sentences F_a: $a \in M$ such that $\mathrm{PA}\vdash \neg F_{\mathrm{ROOT}[M,<]}$ and for each $a \in M$, $\bar\beta(a) \vdash \bigwedge \{F_b : a<b\} \to F_a$. Let us show by induction on the depth in $[M, <]$ that for each $a \in M$, $\bar\beta(a)\vdash F_a$. The case when the depth of a in $[M, <]$ is 0, is trivial. Suppose now that the depth of a in $[M, <]$ is $n+1$. Let us consider any b with $a<b$. The depth of b in $[M, <]$ is $\leq n$ and, by the induction hypothesis, $\bar\beta(b)\vdash F_b$; but since $a<b$, we have $\alpha(a)\,S\,\alpha(b)$, which means by 8.1.6 that $\bar\beta(b)$ is Π_1-conservative over $\bar\beta(a)$; consequently, $\bar\beta(a)\vdash F_b$. Thus, $\bar\beta(a)\vdash \bigwedge \{F_b : a<b\}$ and, since $\bar\beta(a)\vdash \bigwedge \{F_b : a< b\} \to F_a$, we have $\bar\beta(a)\vdash F_a$ (for every $a \in M$). Namely, $\mathrm{root}(M, <, \bar\beta)\vdash F_{\mathrm{ROOT}[M,<]}$, and since $\mathrm{PA}\vdash \neg F_{\mathrm{ROOT}[M,<]}$, we have $\mathrm{PA}\vdash \neg\mathrm{root}(M, <, \beta)$. It follows then easily from this by 8.1.3.1 (taking into account that $w\,R\,\text{root}(M, <, \alpha)$) that w cannot be the limit of h. We have obtained a contradiction.

(d): Assume the conditions of 8.1.7d. Taking 8.1.4 into account, it is easy to see that $PA + L_u$ proves (the formalization of) the assertion "for all x, $h(x) \neq w$", which is clearly in Π_1; on the other hand, $PA + L_w$ clearly proves the negation of this statement. It means that $\langle PA + L_w, PA + L_u \rangle$ is Π_1-inconsistent.

(e): Assume the conditions of 8.1.7e. Suppose w is the limit of h. If $u = w$, then (since $w \neq 0$) there is x such that $h(x) \neq w$ and $h(x+1) = w$; the transfer from $h(x)$ to $h(x+1)$ is determined by 8.1.3.1 or 8.1.3.2; in both cases $PA \vdash \neg L_w$ (in the case 8.1.3.2 because L_w implies in PA a false Π_1!-sentence).

Suppose now $u \neq w$. Let us fix a number z with $h(z) = w$. Since h is primitive recursive, PA proves that $h(z) = w$. Now argue in $PA + L_u$: Since u is the limit of h, there is a number x with $x \geq z$ such that $h(x) \neq u$ and $h(x+1) = u$. Since *not* $h(z)\,R\,u$, by 8.1.4 and 5.2.4 we have:

(*) for each $z \leq y \leq x$, *not* $h(y)\,R\,u$.

It means that the transfer from $h(x)$ to $h(x+1)$ can be determined only by 8.1.3.2. Then (*) implies that the number y from 8.1.3.2 is less than z. That is, there is a false Π_1!-formula F such that for some $y < z$, $\vdash_y L_u \rightarrow F$; but 'we' (i.e., $PA + L_u$) know that this doesn't hold.

Thus, arguing in $PA + L_u$, we have obtained a contradiction. It means that $PA \vdash \neg L_u$.

(f): 8.1.7a implies that one of the L_w: $w \in V$, is true; if $w \neq 0$, then, by 8.1.7e (since R is irreflexive), $PA \vdash L_w \rightarrow \Pr(\ulcorner \neg L_w \urcorner)$ and, therefore, L_w is false. □

8.1.8. Let us define a realization f by setting for each propositional letter p, $fp = \bigvee \{L_w\colon w \Vdash p\}$.

8.1.9. Lemma. *Suppose $0 \neq w \in V$. Then for any modal formula B:*

(a) *if $w \Vdash B$, then* $PA \vdash L_w \rightarrow fB$;

(b) *if* not $w \Vdash B$, *then* $PA \vdash L_w \rightarrow \neg fB$.

Proof. Induction on the complexity of B. Assume $0 \neq w \in V$.

Suppose B is a propositional letter p. If $w \Vdash p$, then L_w is a disjunct of fp and therefore $PA \vdash L_w \rightarrow fp$. And if *not* $w \Vdash p$, then L_w is not a disjunct of fp and, by 8.1.7b, $PA \vdash L_w \rightarrow \neg fp$.

The truth-functional cases are straightforward.

Suppose now $B = \Diamond(Tr)$, where $[Tr] = [M, <, \alpha]$.

(a) Suppose $w \Vdash \Diamond(Tr)$, i.e., there is an evaluator $\beta\colon M \rightarrow V$ such that for each $a \in M$, we have: $\beta(a) \Vdash \alpha(a)$, $w\,R\,\beta(a)$ (so, by 8.1.1, $\beta(a) \neq 0$) and, for all $a < b$, $\beta(a)\,S\,\beta(b)$. Let γ be the evaluator of M such that for each $a \in M$, $\gamma(a) = L_{\beta(a)}$.

Argue in $PA + L_w$. By the induction hypothesis, for each $a \in M$, $PA \vdash \gamma(a) \rightarrow f\alpha(a)$, i.e., the theory $\bar{\gamma}(a)$ contains the theory $\overline{f\alpha}(a)$. It is easy to see therefore, that if $(M, <, \bar{\gamma})$ is Π_1-consistent, then $(M, <, \overline{f\alpha})$ is Π_1-consistent, too. But by

8.1.7c, $(M, <, \bar{\gamma})$ is Π_1-consistent; consequently, $(M, <, \overline{f\alpha})$ is Π_1-consistent, i.e., $f\Diamond(Tr)$ holds.

(b) Suppose *not* $w \Vdash \Diamond(Tr)$. Let ε be the evaluator of M such that for each $a \in M$, $\varepsilon(a) = \bigvee \{L_u\colon w\,R\,u \text{ and } u \Vdash \alpha(a)\}$. Notice that L_0 cannot be a disjunct of any $\varepsilon(a)$: $a \in M$, because *not* $w\,R\,0$.

Argue in $\mathrm{PA} + L_w$. It follows from 8.1.7a, 8.1.7e and the induction hypothesis that for each $\alpha \in M$, $\mathrm{PA} \vdash f\alpha(a) \to \varepsilon(a)$; it means that the theory $\overline{f\alpha}(a)$ contains the theory $\bar{\varepsilon}(a)$.

Suppose $f\Diamond(Tr)$ is true, i.e., $(M, <, \overline{f\alpha})$ is Π_1-consistent; then, since $\overline{f\alpha}(a)$ contains $\bar{\varepsilon}(a)$ for all $a \in M$, $(M, <, \bar{\varepsilon})$ is Π_1-consistent, too. By 4.4 (taking into account the arithmetical soundness of TLR), this is possible only if $(M, <, \bar{\varepsilon}_1)$ is Π_1-consistent for some evaluator ε_1 of M such that for each $a \in M$, $\varepsilon_1(a)$ is a disjunct of $\varepsilon(a)$, i.e., $\varepsilon_1(a) = L_u$ for some u with $w\,R\,u$ *and* $u \Vdash \alpha(a)$. Let ε_2 be the evaluator of M that assigns to each $a \in M$ the very u for which $\varepsilon_1(a) = L_u$. Observe that since *not* $w \Vdash \Diamond(Tr)$, we have *not* $\varepsilon_2(a)\,S\,\varepsilon_2(b)$ for some $a < b$, whence, by 8.1.7d, $\langle \bar{\varepsilon}_1(a), \bar{\varepsilon}_1(b)\rangle$ is Π_1-inconsistent, whence, by axiom 4.2.4a (taking into account the arithmetical soundness of TLR), $(M, <, \bar{\varepsilon}_1)$ is Π_1-inconsistent, a contradiction. We conclude that $(m, <, \overline{f\alpha})$ is Π_1-inconsistent, i.e., $\neg f\Diamond(Tr)$. □

We can now complete the proof of Lemma 8.1. Since *not* $1 \Vdash A$, by 8.1.9, $\mathrm{PA} \vdash L_1 \to \neg fA$. By 8.1.7c, since $0\,R\,1$, $\mathrm{PA} \vdash L_0 \to \mathrm{Cn}\langle L_1\rangle$; since L_0 is true, it follows that $\mathrm{PA} \nvdash \neg L_1$. Consequently, $\mathrm{PA} \nvdash fA$. □

The following theorem is an immediate consequence of 7.5, 6.4a($\Leftarrow$) and 8.1:

8.2. Theorem. $\mathrm{TLR} \vdash A$ *iff* A *is* PA-*valid.*

8.3. Remark. Our function h defined in 8.1.3 is similar to the Berarducci [2] function F. But the advantage of h is that it can be immediately employed to prove the completeness of ILM and TLR as the logics of Π_1-conservativity and Π_1-consistency for a wider class of 'sufficiently rich' base theories instead of PA, including finitely axiomatized ones.

9. Logic TLRω

Logic TLRω is an extension of TLR in the same language.
The *AXIOMS* of TLRω are:

theorems of TLR;
$A \to \Diamond\langle A\rangle$ (for every TLR-formula A).

The *rule* of inference of TLRω is Modus Ponens.

9.1. Notation. For any TLR-formula A, A^+ denotes the conjunction of all the formulas of the form

$$(\bigwedge \{\alpha(a)\colon a \in M'\} \wedge \bigwedge \{\Diamond(M, <, \alpha)_a\colon a \in (M - M')\})$$
$$\rightarrow \Diamond\langle \bigwedge \{\alpha(a)\colon a \in M'\} \wedge \bigwedge \{\Diamond(M, <, \alpha)_a\colon a \in (M - M')\}\rangle,$$

where $\Diamond(M, <, \alpha)$ is a subformula of A and, for the corresponding $<'$, $[M', <']$ is an initial part of $[M, <]$.

9.2. Definition. A modal formula A is said to be *ω-valid* iff fA is true for every realization f.

9.3. Lemma. *If there is a finite Visser countermodel for $A^+ \rightarrow A$, then A is not ω-valid.*

Proof. Assume the condition of the lemma. We may suppose that the model $\langle V, R, S, \Vdash\rangle$, defined in 8.1.1, is (at the same time) a countermodel for $A^+ \rightarrow A$ with *not* $1 \Vdash A^+ \rightarrow A$. Accept also all the other definitions from the proof of Lemma 8.1. Clearly Lemmas 8.1.7 and 8.1.9 continue to hold.

9.3.1. Lemma. *For any subformula B of A, $0 \Vdash B$ iff $1 \Vdash B$.*

Proof. Induction on the complexity of B. The only nonstraightforward case is $B = \Diamond(Tr)$ ($\Rightarrow$).

Suppose $0 \Vdash \Diamond(Tr)$, where $[Tr] = [M, <, \alpha]$. Then there is an evaluator $\beta\colon M \rightarrow V$ such that for each $a \in M$, $0\, R\, \beta(a)$ (i.e., $\beta(a) = 1$ or $1\, R\, \beta(a)$), $\beta(a) \Vdash \alpha(a)$, and for all $a < b$, $\beta(a)\, S\, \beta(b)$. Observe that for each $w \in V$, if $0 \neq w \neq 1$, then, since $1\, R\, w$, we have *not* $w\, S\, 1$ by 6.2.1 and 6.2.4. It follows then that one of the following two cases takes place:

Case 1: there is no $a \in M$ with $\beta(a) = 1$;

Case 2: there is an initial part $[M', <']$ of $[m, <]$ such that for each $a \in M'$, $\beta(a) = 1$ and for each $a \in (M - M')$, $1\, R\, \beta(a)$.

In the case 1, $1\, R\, \beta(a)$ for each $a \in M$, and clearly $1 \Vdash \Diamond(Tr)$.

Now suppose that the case 2 takes place. Observe that then we have

(1) $1 \Vdash C$, where

$$C = \bigwedge \{\alpha(a)\colon a \in M'\} \wedge \bigwedge \{\Diamond(M, <, \alpha)_a\colon a \in (M - M')\}.$$

Let us observe now that $1 \Vdash A^+$ because *not* $1 \Vdash A^+ \rightarrow A$. It follows from this by (1) that

(2) $1 \Vdash \Diamond\langle C\rangle$.

Let β_1 be the evaluator of M' such that for each $a \in M'$, $\beta_1(a) = C$. Using several times the axiom 4.2.4b, we have $\mathrm{TLR} \vdash \Diamond\langle C\rangle \rightarrow \Diamond(M', <', \beta_1)$, whence, by (2),

(3) $1 \Vdash \Diamond(M', <', \beta_1)$.

Let β_2 be the evaluator of M' such that for each $a \in M'$, $\beta_2(a) = \alpha(a) \wedge \bigwedge \{\Diamond(M, <, \alpha)_b : b \in (M - M')$ and b is an immediate successor of a in $[M, <]\}$. Clearly for each $a \in M'$, $\mathrm{TLR} \vdash C \rightarrow \beta_2(a)$, i.e., $\mathrm{TLR} \vdash \beta_1(a) \rightarrow \beta_2(a)$. Therefore, by 4.3 and (3),

(4) $\quad 1 \Vdash \Diamond(M', <', \beta_2)$.

Let $(a_1, b_1), \ldots, (a_n, b_n)$ be an enumeration of all the pairs (a, b) such that $a \in M'$, $b \in (M - M')$ and b is an immediate successor of a in $[M, <]$. (Note that $i \neq j$ doesn't imply $a_i \neq a_j$). We define the trees $[M_0, <_0, \alpha_0], \ldots, [M_n, <_n, \alpha_n]$ of TLR-formulas by induction: $[M_0, <_0, \alpha_0] = [M', <', \beta_2]$ and $[M_{i+1}, <_{i+1}, \alpha_{i+1}] = [M_i, <_i, \alpha_i] +_{a_{i+1}} [M, <, \alpha]_{b_{i+1}}$. By the axiom 4.2.4c, for any $0 \leq i < n$ we have:

$$\mathrm{TLR} \vdash \Diamond(M_i, <_i, \alpha_i) \rightarrow \Diamond(M_{i+1}, <_{i+1}, \alpha_{i+1});$$

consequently, by (4),

(5) $\quad 1 \Vdash \Diamond(M_n, <_n, \alpha_n)$.

Let us now observe that $[M_n, <_n] = [M, <]$. Observe also that for each $a \in M$, $\alpha(a)$ is a conjunct of $\alpha_n(a)$ (namely, for each $a \in (M - M')$, $\alpha_n(\alpha) = \alpha(a)$ and for each $a \in M'$, $\alpha_n(a)$ equals to $\beta_2(a)$ and the latter contains $\alpha(a)$ as a conjunct). Therefore, by 4.3, $\mathrm{TLR} \vdash \Diamond(M_n, <_n, \alpha_n) \rightarrow \Diamond(M, <, \alpha)$, whence, by (5), $1 \Vdash \Diamond(M, <, \alpha)$. 9.3.1 is thus proved. □

9.3.2. Lemma. *For any subformula B of A,*

(a) *if* $0 \Vdash B$, *then* $\mathrm{PA} \vdash L_0 \rightarrow fB$;

(b) *if* not $0 \Vdash B$, *then* $\mathrm{PA} \vdash L_0 \rightarrow \neg fB$.

Proof. Induction on the complexity of B. The only case when the reasoning differs from that given in the proof of Lemma 8.1.9, is $B = \Diamond(Tr)$ (b).

Suppose *not* $0 \Vdash \Diamond(Tr)$, where $[Tr] = [M, <, \alpha]$. Let us first verify the following proposition:

(*) Suppose an evaluator $\beta : M \rightarrow V$ is such that for each $a \in M$, $\beta(a) \Vdash \alpha(a)$. Then there are $a < b$ with *not* $\beta(a)\, S\, \beta(b)$.

Indeed, assume the conditions of (*), and suppose, for a contradiction, that for all $a < b$, $\beta(a)\, S\, \beta(b)$. Let then β' be the evaluator of M such that for each $a \in M$, $\beta'(a) = 1$, if $\beta(a) = 0$, and $\beta'(a) = \beta(a)$ otherwise. Then $0\, R\, \beta'(a)$ for all $a \in M$. By 9.3.1, for each $a \in M$, $\beta'(a) \Vdash \alpha(a)$; on the other hand, it can be easily verified that for each $a < b$, $\beta'(a)\, S\, \beta'(b)$ (take into account that, by 5.2.1, and 5.2.4, $w \neq 0$ implies *not* $w\, S\, 0$). It means that $0 \Vdash \Diamond(Tr)$, a contradiction.

Let ε be the evaluator of m such that for each $a \in M$,

$$\varepsilon(a) = \bigvee \{L_u : u \Vdash \alpha(a)\}.$$

Argue in PA. It follows from 8.1.7a, 8.1.9 and the induction hypothesis that for each $a \in M$, $\mathrm{PA} \vdash f\alpha(a) \rightarrow \varepsilon(a)$; it means that the theory $\overline{f\alpha}(a)$ contains the theory $\bar{\varepsilon}(a)$.

Suppose $f\Diamond(Tr)$ *is* true, i.e., $(M, <, \overline{f\alpha})$ is Π_1-consistent; then, since $\overline{f\alpha}(a)$ contains $\bar{\varepsilon}(a)$ for all $a \in M$, $(M, <, \bar{\varepsilon})$ is Π_1-consistent, too. By 4.4 (taking into account the arithmetical soundness of TLR), this is possible only if $(M, <, \bar{\varepsilon}_1)$ is Π_1-consistent for some evaluator ε_1 of M such that for each $a \in M$, $\varepsilon_1(a)$ is a disjunct of $\varepsilon(a)$, i.e., $\varepsilon_1(a) = L_u$ for some u with $u \Vdash \alpha(a)$. Let ε_2 be the evaluator of M that assigns to each $a \in M$ the very u for which $\varepsilon_1(a) = L_u$. By (*), we have *not* $\varepsilon_2(a)\, S\, \varepsilon_2(b)$ for some $a < b$, whence, by 8.1.7d, $\langle \bar{\varepsilon}_1(a), \bar{\varepsilon}_1(b) \rangle$ is Π_1-inconsistent, whence, by the axiom 4.2.4a (in view of the arithmetical soundness of TLR), $(M, <, \bar{\varepsilon}_1)$ is Π_1-inconsistent, a contradiction. We conclude that $(M, <, \overline{f\alpha})$ is Π_1-inconsistent, i.e., $\neg f\Diamond(Tr)$. □

We can now complete the proof of Lemma 9.3. Since *not* $1 \Vdash A^+ \to A$, we have *not* $1 \Vdash A$, whence, by 9.3.1, *not* $0 \Vdash A$, whence, by 9.3.2, $\text{PA} \vdash L_0 \to \neg fA$; and since L_0 is true, fA is false. □

9.4. Theorem. TLR$\omega \vdash A$ *iff* A *is* ω*-valid.*

Proof. ($\Rightarrow$): All theorems of TLR are ω-valid because they are PA-valid (7.5); the axiom $A \to \Diamond\langle A \rangle$ is obviously ω-valid, too, and Modus Ponens clearly preserves ω-validity.

($\Leftarrow$): By 6.4a($\Leftarrow$), 9.3 and the evident fact that TLR$\omega \nvdash A$ implies TLR $\nvdash A^+ \to A$. □

9.5. Remark. As we noted in the above paragraph, TLR$\omega \nvdash A$ implies TLR $\nvdash A^+ \to A$. On the other hand, by 6.4a($\Leftarrow$), 9.3 and 9.4($\Rightarrow$), TLR $\nvdash A^+ \to A$ implies TLR$\omega \nvdash A$. Thus, TLR$\omega \vdash A \Leftrightarrow$ TLR $\vdash A^+ \to A$ and, since TLR is decidable (6.5), TLRω is decidable, too.

10. TLR and provability logic with propositional quantifiers

The language of provability logic contains, besides the symbols used in classical logic, the unary modal operator □. Formulas of this language are considered as schemata of arithmetical formulas, where □A is understood as a formalization of the assertion "A is provable (say, in PA)". Under this approach there arise two natural classes of modal formulas:

(1) class P of the modal formulas that are schemata of PA-provable arithmetical formulas, and

(2) class T of the modal formulas that are schemata of true arithmetical formulas.

And the main task is to characterize these two classes — first and foremost, to determine their arithmetical complexities.

The answer on this main question depends on what language is taken as the basic one to which the modal operator □ is added.

If the basic language is that of propositional logic (without quantifiers), everything is 'smooth': as it was shown by Solovay [14], both sets P and T are decidable. But if the language of predicate logic is taken as the basic one, the situation deteriorates at once: Vardanyan [16] showed that in this case the set P is not r.e., and before that Artemov [1] showed that the set T is not even arithmetical.

A different approach is taking the basic language to be a propositional language with quantifiers over propositions. There are several natural ways of doing this, and we do not know whether any of them leads to undecidability of P or T. Moreover, up to now there are not known any results concerning decidability of more or less considerable fragments of provability logic (i.e., the sets P and T) with propositional quantifiers, when the range of the latters is not restricted to some very specific class of arithmetical formulas. Theorem 10.6 below can be regarded as the first positive result of this kind.

Studying provability logic with propositional quantifiers, some restrictions or conditions are necessary to be taken. E.g., the formula $\Box(\forall p\,(\Box p \rightarrow p))$ hardly can have any reasonable interpretation, if '$\forall p$' is understpod as 'for any arithmetical formula p', because in this case the expression '$\forall p\,(\Box p \rightarrow p)$' has no natural translation into arithmetic. This difficulty will be avoided, if the propositional variables range over arithmetical formulas of restricted complexity. Below we define a language, the quantifiers of which are interpreted as quantifiers over Σ_1-sentences — the most interesting class of arithmetical formulas.

In *language* L, besides $\top$ and $\bot$, we have two sorts of atomic formulas:

(1) *propositional letters*: $p_1, p_2, \ldots$; as we see, the set of propositional letters of L coincides with that of the language of TLR;

(2) *propositional variables* that we denote by $x, y, z, x_1, x_2, \ldots, y_1, y_2, \ldots$. We suppose that the set of propositional variables of L coincides with the set of individual variables of PA.

10.1. Definition. The set of *formulas of* L (*L-formulas*) is defined as the smallest one such that:

(1) propositional letters, propositional variables, $\top$ and $\bot$ are L-formulas;

(2) Boolean combinations of L-formulas are L-formulas;

(3) if A is an L-formula, then $\Box A$ is an L-formula;

(4) if A is an L-formula and x is a propositional variable, then $\forall x\, A$ is an L-formula.

10.2. Notation. For any arithmetical formula E, $\mathrm{Pr}[E]$ is an r.e. arithmetical formula with exactly the same free variables that naturally expresses the PA-provability of the result of substituting for each variable free in E the numeral for the value of that variable (cf. [3]).

Recall also that $\exists y\, \Theta(y, x)$ is a formalization of the predicate "x is the Gödel number of a true Σ_1!-sentence" (2.7).

10.3. Definition. A *realization,* as in case of the lagnuage of TLR, is a function f that assigns to each propositional letter p_i a sentence fp_i of PA. f is extended to complex L-formulas in the following unique way:

(a) for any propositional variable x, $fx = \exists y\, \Theta(y, x)$;

(b) f commutes with Boolean connectives;

(c) $f(\forall x\, A) = \forall x\, (fA)$;

(d) $f(\Box A) = \Pr[fA]$.

We say that an L-formula is *pure,* if it doesn't contain propositional letters (but it may contain propositional variables). Notice that if A is a pure L-formula, then fA doesn't depend on the choice of the realization f.

10.4. Problems. 1. What are the arithmetical complexities of the sets of closed L-formulas

$$\mathrm{P} = \{A\colon \mathrm{PA} \vdash fA \text{ for every realization } f\} \quad \text{and}$$
$$\mathrm{T} = \{A\colon fA \text{ is true for every realization } f\}?$$

2. What are the complexities of the above sets restricted to pure formulas?

We are now going to define a fragment L′ of L and show, that formulas of L′ are 'expressible' in the language of TLR; it implies the decidability of the restrictions of P and T to L′-formulas.

10.5. Definition. The set of *L′-formulas* is defined as the smallest one such that:

(1) propositional letters, $\top$ and $\bot$ are L′-formulas;

(2) Boolean combinations of L′-formulas are L′-formulas;

(3) if $\boldsymbol{A} = A_1, \ldots, A_n$ are L′-formulas, $\boldsymbol{x} = x_1, \ldots, x_m$, $\boldsymbol{y} = y_1, \ldots, y_k$ are propositional variables (possibly $n, m, k = 0$) and $Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y})$ is a Boolean combination of $\boldsymbol{A}$, $\boldsymbol{x}$, $\boldsymbol{y}$, then $\exists \boldsymbol{x}\, \forall \boldsymbol{y}\, \Box Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y})$ is an L′-formula.

2.5 shows that, roughly speaking, in L′ can be expressed everything expressible in the language of TLR. E.g., $\Diamond(p_1, p_2)$ can be expressed by $\neg\exists x\, \Box((p_2 \to \neg x) \wedge (p_1 \to x))$. The following theorem establishes that such an 'expressibility' holds in the opposite direction as well:

10.6. Theorem. *There is an effective mapping * that assigns to every* L′*-formula A a* TLR*-formula A^* containing exactly the same propositional letters such that for every realization f,*

$$\mathrm{PA} \vdash fA \leftrightarrow fA^*.$$

To prove this theorem, we need two Lemmas 10.7 and 10.8.

10.7. Lemma. *Let z, $\boldsymbol{x} = x_1, \ldots, x_m$, $\boldsymbol{y} = y_1, \ldots, y_k$ be propositional variables (possibly m, $k = 0$), $\boldsymbol{A} = A_1, \ldots, A_n$ be* L′*-formulas (possibly $n = 0$), and $Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z)$ be a Boolean combination of $\boldsymbol{A}$, $\boldsymbol{x}$, $\boldsymbol{y}$, z. Then for any realization f,*

$$\mathrm{PA} \vdash f\, \exists \boldsymbol{x}\, \forall \boldsymbol{y}\, \forall z\, \Box Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z) \leftrightarrow f\, \exists \boldsymbol{x}\, \forall \boldsymbol{y}\, \Box(Bl)(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, \top) \wedge Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, \bot)).$$

Proof. It is enough to show that $\mathrm{PA} \vdash F \leftrightarrow E$, where

$$F = f\, \forall z\, \Box Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z) \quad \text{and} \quad E = f\Box(Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, \top) \wedge Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, \bot)).$$

Since $\top, \bot \in \Sigma_1$, the fact $\mathrm{PA} \vdash F \rightarrow E$ is evident. Now, it is easy to see that

$$\mathrm{PA} \vdash E \rightarrow f\Box((z \rightarrow Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z)) \wedge (\neg z \in Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z))),$$

whence

$$\mathrm{PA} \vdash E \rightarrow f\Box((z \vee \neg z) \rightarrow Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z)),$$

whence

$$\mathrm{PA} \vdash E \rightarrow f(\Box(z \vee \neg z) \rightarrow \Box Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z)),$$

whence

$$\mathrm{PA} \vdash E \rightarrow f(\forall z\, \Box(z \vee \neg z) \rightarrow \forall z\, \Box Bl(\boldsymbol{A}, \boldsymbol{x}, \boldsymbol{y}, z)),$$

i.e.

$$\mathrm{PA} \vdash E \rightarrow (f\, \forall z\, \Box(z \vee \neg z) \rightarrow F).$$

But $\mathrm{PA} \vdash f\, \forall z\, \Box(z \vee \neg z)$. Consequently, $\mathrm{PA} \vdash E \rightarrow F$. $\Box$

10.8. Lemma. *Suppose $N = \{1, \ldots, n\}$ and M is the set of all linearly ordered subsets of N (i.e., sequences $\langle k_1, \ldots, k_m \rangle$ (possibly $m = 0$) such that for each $1 \leqslant i \leqslant m$, $k_i \in N$ and if $j \neq i$, then $k_i \neq k_j$). For all $a, b \in M$, let $a < b$ iff b is a proper end-extension of a* (2.1.8). *Notice that $[M, <]$ is a tree and* ROOT$[M, <]$ *is the empty sequence $\langle\ \rangle$. For each $a \in M$, let a° be the very subset of N that is linearly ordered in a. Let α be an evaluator of M that assigns to each $a \in M$ some superarithmetical theory, such that if $a^\circ = b^\circ$, then $\alpha(a) = \alpha(b)$. Then* PA *proves the equivalence of the assertions* (i) *and* (ii):

(i) *$(M, <, \alpha)$ is Π_1-inconsistent;*

(ii) *there are $x_1, \ldots, x_n \in \Sigma_1!$ such that for each $a \in M$, we have:*

(1) $$\alpha(a) \vdash \neg(\bigwedge\{x_i\colon i \in a^\circ\} \wedge \bigwedge\{\neg x_i; i \in (N - a^\circ)\}).$$

Proof. Argue in PA.

(ii)$\Rightarrow$(i): Assume (ii). Taking 2.5 into account, it is enough to show that for each $a \in M$,

(2) $$\alpha(a) \vdash \bigwedge\{\neg\bigwedge\{x_i\colon i \in b^\circ\}\colon a < b\} \rightarrow \neg\bigwedge\{x_i\colon i \in a^\circ\}.$$

Consider an arbitrary $a \in M$.

If each $1 \leqslant i \leqslant n$ is in a°, then (2) immediately follows from (1).

Otherwise, let us consider any $1 \leqslant i \leqslant n$ with $i \notin a^\circ$. Let b be the result of adding i as the last element to the sequence a. We have $a < b$ and

(*) $\quad \mathrm{PA} \vdash x_i \wedge \bigwedge \{x_j : j \in a^\circ\} \rightarrow \bigwedge \{x_j : j \in b^\circ\}.$

Thus, for any $i \in (N - a^\circ)$ there is b with $a < b$ such that (*) is satisfied. It means that we have

$$\mathrm{PA} \vdash \bigvee \{x_i : i \in (N - a^\circ)\} \wedge \bigwedge \{x_i : i \in a^\circ\} \rightarrow \bigvee \{\bigwedge \{x_i : i \in b^\circ\} : a < b\},$$

i.e., by contraposition,

$$\mathrm{PA} \vdash \bigwedge \{\neg \bigwedge \{x_i : i \in b^\circ\} : a < b\} \rightarrow \neg \bigwedge \{x_i : i \in a^\circ\} \vee \bigwedge \{\neg x_i : i \in (N - a^\circ)\}.$$

Now, (2) easily follows from this and (1) by propositional logic.

(i)$\Rightarrow$(ii): Assume (i). It means by 2.5 that for each $a \in M$ there is a Σ_1-sentence F_a such that $\mathrm{PA} \vdash F_{\langle\,\rangle}$ and

$$\alpha(a) \vdash \bigwedge \{\neg F_b : a < b\} \rightarrow \neg F_\alpha.$$

For each $a \in M$, let F_a^+ be a Σ_1!-sentence that is PA-equivalent to $\bigvee \{F_b : b^\circ = \alpha^\circ\}$. Taking into account that $a^\circ = b^\circ$ implies $\alpha(a) = \alpha(b)$, it is easy to verify that for each $a \in M$,

(3) $\quad \alpha(a) \vdash \bigwedge \{\neg F_b^+ : a^\circ \subset b^\circ\} \rightarrow \neg F_a^+$

(recall: $a^\circ \subset b^\circ$ means that $a^\circ \subseteq b^\circ$ and $a^\circ \neq b^\circ$).

We may suppose that the sentences F_a^+ are chosen in such a way that if $a^\circ \neq b^\circ$, then $F_a^+ \neq F_b^+$ and so PA proves that F_a^+ and F_b^+ cannot have common regular witnesses (recall 2.7).

Let us define a p.r. function $g : \omega \rightarrow \{a^\circ : a \in M\}$ as follows: $g(0) = \emptyset$ and

$$g(k+1) = \begin{cases} a^\circ, & \text{if } a \in M,\ g(k) \subset a^\circ \text{ and } k \text{ is a regular witness for } F_a^+; \\ g(k), & \text{if such an } a^\circ \text{ doesn't exist.} \end{cases}$$

Let now for each $1 \leqslant i \leqslant n$, x_i be a Σ_1!-sentence expressing that $i \in g(k)$ for some k. We want to show that then (1) holds.

First of all let us observe that (PA proves that) the function g has a limit; let *Lim* denote this limit. It is easy to see that each x_i is PA-equivalent to the assertion that $i \in \mathit{Lim}$. It follows that for each $a \in M$,

(4) $\quad \bigwedge \{x_i : i \in a^\circ\} \wedge \bigwedge \{\neg x_i : i \in (M - a^\circ)\}$ is PA-equivalent to $\mathit{Lim} = a^\circ$;

on the other hand, it is also easy to verify that

(5) $\quad \mathit{Lim} = a^\circ$ implies in PA $\bigwedge \{\neg F_b^+ : a^\circ \subset b^\circ\} \wedge F_a^+$.

Now, (1) follows from (3), (4) and (5). □

Proof of Theorem 10.6. We define A^* by induction on the complexity of A.

If A is a propositional letter, $\top$ or $\bot$, set $A^*=A$; if $A=(B\to C)$, set $A^*=(B^*\to C^*)$; similarly for the other Boolean connectives.

Suppose now $A=\exists\boldsymbol{x}\,\forall\boldsymbol{y}\,\Box Bl(\boldsymbol{B},\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{B}=B_1,\ldots,B_k$ are L′-formulas, $\boldsymbol{x}=x_1,\ldots,x_n$, $\boldsymbol{y}=y_1,\ldots,y_m$ are propositional variables (possibly $k,n,m=0$), and $Bl(\boldsymbol{B},\boldsymbol{x},\boldsymbol{y})$ is a Boolean combination of $\boldsymbol{B},\boldsymbol{x},\boldsymbol{y}$.

First we use Lemma 10.7 m times and obtain an L′-formula $\exists\boldsymbol{x}\,\Box Bl_1(\boldsymbol{B},\boldsymbol{x})$, where Bl_1 is a Boolean combination of $\boldsymbol{B}$, $\boldsymbol{x}$ and for any realization f,

(1) $\mathrm{PA}\vdash fA\leftrightarrow f\,\exists\boldsymbol{x}\,\Box Bl_1(\boldsymbol{B},\boldsymbol{x})$.

Let $N=\{1,\ldots,n\}$, $K=\{1,\ldots,k\}$, S be the set of all subsets of N and U be the set of all subsets of K. For each $s\in S$ and $u\in U$, let

$$\hat{s}=\bigwedge\{x_i\colon i\in s\}\wedge\bigwedge\{\neg x_i\colon i\in(N-s)\}\quad\text{and}$$
$$\hat{u}=\bigwedge\{B_i\colon i\in u\}\wedge\bigwedge\{\neg B_i\colon i\in(K-u)\}.$$

Now, by propositional logic, there is a Boolean combination Bl_2 of $\boldsymbol{B}$, $\boldsymbol{x}$ that is a conjunction, each conjunct of which is $\hat{u}\to\neg\hat{s}$ for some $s\in S$ and $u\in U$, such that $Bl_2(\boldsymbol{B},\boldsymbol{x})$ is tautologically equivalent to $Bl_1(\boldsymbol{B},\boldsymbol{x})$.

Let for each $s\in S$, R_s be the disjunction of all $\hat{u}$ such that $\hat{u}\to\neg\hat{s}$ is a conjunct of $Bl_2(\boldsymbol{B},\boldsymbol{x})$. Then $\bigwedge\{R_s\to\neg\hat{s}\colon s\in S\}$ is tautologically equivalent to $Bl_2(\boldsymbol{B},\boldsymbol{x})$. Thus for any realization f,

(2) $\mathrm{PA}\vdash f\,\exists\boldsymbol{x}\,Bl_1(\boldsymbol{B},\boldsymbol{x})\leftrightarrow f\exists\boldsymbol{x}\,\Box(\bigwedge\{R_s\to\neg\hat{s}\colon s\in S\})$.

For each $a\in M$, let a° be the very subset of N that is linearly ordered in a.

In the following three paragraphs we define a tree $[M,\prec,\alpha]$ of TLR-formulas

Let M be the set of all linearly ordered subsets of N.

For all $a,b\in M$, let $a\prec b$ iff b is a proper end-extension of a.

By the induction hypothesis, B_i^* is already defined for each B_i $(1\leq i\leq k)$. Each R_s $(s\in S)$ is a Boolean combination of $B_1,\ldots,B_k$, and, since $*$ commutes with Boolean connectives, R_s^* is also defined. Taking this remark into account, let α be the evaluator of M which assigns to each $a\in M$ the TLR-formula R_s^*, where $s=a^\circ$.

Now, we define A^* by setting $A^*=\neg\Diamond(M,\prec,\alpha)$.

We want to show that for every realization f,

(3) $\mathrm{PA}\vdash f\,\exists\boldsymbol{x}\,\Box(\bigwedge\{R_s\to\neg\hat{s}\colon s\in S\})\leftrightarrow fA^*$.

Let us fix a realization f and argue in PA. $f\,\exists\boldsymbol{x}\,\Box(\bigwedge\{R_s\to\neg\hat{s}\colon s\in S\})$ means that there are Σ_1!-sentences $\boldsymbol{x}=x_1,\ldots,x_n$ such that for each $s\in S$, $\mathrm{PA}\vdash fR_s\to\neg\hat{s}$; on the other hand, the induction hypothesis implies that $\mathrm{PA}\vdash fR_s\leftrightarrow fR_s^*$; it means that for each $s\in S$, $\mathrm{PA}+fR_s^*\vdash\neg\hat{s}$, and this, by Lemma 10.8, is equivalent to the assertion that $(M,\prec,\overline{f\alpha})$ is Π_1-inconsistent (recall notations 7.1.1 and 7.1.4). Thus, $f\,\exists\boldsymbol{x}\,\Box(\bigwedge\{R_s\to\neg\hat{s}\colon s\in S\})$ iff $(M,\prec,\overline{f\alpha})$ is Π_1-inconsistent. But "$(M,\prec,\overline{f\alpha})$ is Π_1-inconsistent" means nothing else but that fA^* is true. Thus, (3) holds.

It follows from (1), (2) and (3) that for every realization f, $\mathrm{PA} \vdash fA \leftrightarrow fA^*$. Finally, let us observe that A^* contains exactly the same propositional letters as A and that the mapping $*$ is effective. This completes the proof of Theorem 10.6. □

Appendix A

Notes on some logics between GL *and* TLR

For background information, we define the three logics that are between GL and TLR in Figure 2, see Introduction.

The axioms of *logic* HGL^- (the □, Σ_1-fragment of the logic of arithmetical hierarchy, [5]) are given by the following schemata:

0. tautologies;
1. $\Box(A \to B) \to (\Box A \to \Box B)$;
2. $\Box(\Box A \to A) \to \Box A$;
3. $\Sigma_1 A \wedge \Sigma_1 B \to \Sigma_1(A \wedge B) \wedge \Sigma_1(A \vee B)$;
4. $\Sigma_1 A \wedge \Box(A \leftrightarrow B) \to \Sigma_1 B$;
5. $\Sigma_1 A \to \Box \Sigma_1 A$;
6. $\Sigma_1 \bot$;
7. $\Sigma_1 \Box A$;
8. $\Sigma_1 \Sigma_1 A$;
9. $\Sigma_1 A \to \Box(A \to \Box A)$.

The rules of inference are Modus Ponens and $\vdash A \Rightarrow \vdash \Box A$.

The axiom schemata of Ignatiev's [10] *logic* ELH (the logic of Σ_1-interpolability) are:

0. tautologies;
1. $\Box(A \to B) \to (\Box A \to \Box B)$;
2. $\Box(\Box A \to A) \to \Box A$;
3. $A \twoheadrightarrow B \to (A \wedge C) \twoheadrightarrow B$;
4. $A \twoheadrightarrow B \wedge C \twoheadrightarrow B \to (A \vee C) \twoheadrightarrow B$;
5. $A \twoheadrightarrow B \to A \twoheadrightarrow (B \vee C)$;
6. $A \twoheadrightarrow B \wedge A \twoheadrightarrow C \to A \twoheadrightarrow (B \wedge C)$;
7. $\Box(A \to B) \wedge B \twoheadrightarrow C \to A \twoheadrightarrow C$;
8. $A \twoheadrightarrow B \wedge \Box(B \to C) \to A \twoheadrightarrow C$;
9. $\bot \twoheadrightarrow \bot$;
10. $\top \twoheadrightarrow \top$;
11. $A \twoheadrightarrow B \to \Box(A \twoheadrightarrow B)$;
12. $(A \twoheadrightarrow B) \twoheadrightarrow (A \twoheadrightarrow B)$;
13. $(A \twoheadrightarrow B) \to \Box(A \to B)$;
14. $(A \twoheadrightarrow B) \to \Box(A \to \Box B)$.

The rules of inference are Modus Ponens and $\vdash A \Rightarrow \vdash \Box A$.

The axiom schemata of *logic* TOL (the logic of linear tolerance, [6]) are:

0. tautologies;
1. $\Diamond(\boldsymbol{C}, A, \boldsymbol{D}) \to \Diamond(\boldsymbol{C}, A \wedge \neg B, \boldsymbol{D}) \vee \Diamond(\boldsymbol{C}, B, \boldsymbol{D})$;
2. $\Diamond(A) \to \Diamond(A \wedge \neg\Diamond(A))$;
3. $\Diamond(\boldsymbol{C}, A, \boldsymbol{D}) \to \Diamond(\boldsymbol{C}, \boldsymbol{D})$;
4. $\Diamond(\boldsymbol{C}, A, \boldsymbol{D}) \to \Diamond(\boldsymbol{C}, A, A, \boldsymbol{D})$;
5. $\Diamond(A, \Diamond(\boldsymbol{C})) \to \Diamond(A \wedge \Diamond(\boldsymbol{C}))$;
6. $\Diamond(\boldsymbol{C}, \Diamond(\boldsymbol{D})) \to \Diamond(\boldsymbol{C}, \boldsymbol{D})$.

($\boldsymbol{A}$ abbreviates $A_1, \ldots, A_n$ for any $n \geq 0$; if $n = 0$, $\Diamond(\boldsymbol{A})$ is identified with $\top$.)

The rules of inference are Modus Ponens and $\neg A \vdash \neg\Diamond A$.

We see that TOL has a simpler axiomatization than ELH. This is one more argument showing that the non-generalized, binary relation of weak interpretability is not quite natural.

In view of the interpretations of the logics TLR, TOL, ELH and GL and the corresponding arithmetical completeness theorems, we can say that TOL, ELH and GL are the 'linear', the 'binary' and the 'unary' fragments of TLR, respectively.

In particular, define a translation * from the languages of the logics GL, HGL^-, ELH and TOL into the language of TLR by:

* commutes with Boolean connectives;
$(\Box A)^* = \neg\Diamond\langle\neg A\rangle$;
$(\Sigma_1 A)^* = \neg\Diamond\langle A, \neg A\rangle$;
$(A \twoheadrightarrow B)^* = \neg\Diamond\langle A, \neg B\rangle$;
$\Diamond(\boldsymbol{A})^* = \Diamond\langle\boldsymbol{A}\rangle$.

Then, if L is one of these four logics and A is a formula of the language of L, we have:

$$L \vdash A \text{ iff } \text{TLR} \vdash A^*.$$

Modulo the arithmetical completeness theorem, the following proposition implies that tolerance, in general, cannot be modal-logically defined in terms of linear tolerance and hence, TLR is an essential extension of TOL:

Proposition. *Let* $Tr = [M, <, \alpha]$, where:

$M = \{1, 2, 3\}$;
$< = \{(1, 2), (1, 3)\}$;
$a(i) = p_i$ (*all* $i \in M$).

Then for any formula A of the language of TOL,

$$\text{not TLR} \vdash A^* \leftrightarrow \Diamond(Tr).$$

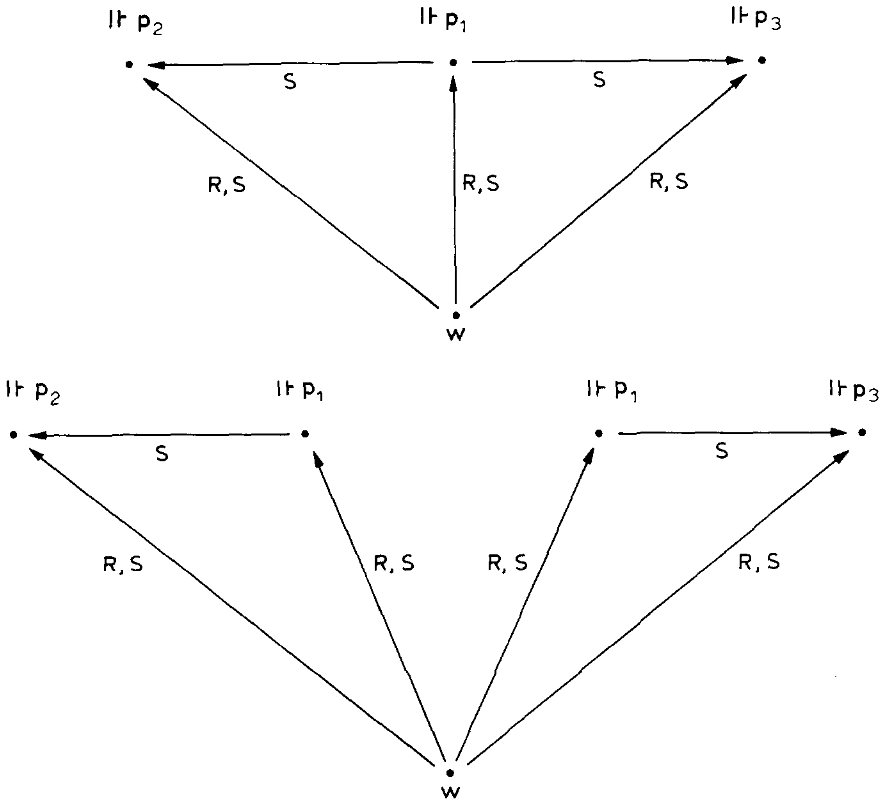

Fig. 3.

Indeed, consider the two strengthened Visser models displayed in Fig. 3, where each world forces only the indicated propositional letters (the S-arrows are also supposed to be reflexive).

It is easy to see that both models force precisely the same formulas of the language of TLR at the world w whereas we have $w \Vdash \Diamond(Tr)$ in the first model and *not* $w \Vdash \Diamond(Tr)$ in the other one. In view of the soundness of TLR with respect to Visser models, it follows that for no formula A of the language of TOL do we have $\mathrm{TLR} \vdash A^* \leftrightarrow \Diamond(Tr)$.

A similar method can be used to prove that each logic between GL and TLR in Figure 2 is an essential extension of its predecessors.

Appendix B

Proof of Theorem 3.6b

In the following T and S are superarithmetical theories. By a 'translation' we mean a translation from the language of PA into the language of PA. A 'cointerpretation of S in T' means a translation that satisfies 3.3b. PC is pure predicate calculus. Notation and terminology not explained here are standard (cf. [7], [11]).

The following Lemmas B.1 and B.2 can be called variants of Lindström's [11] Lemmas 2 and 4, respectively:

B.1. Lemma. *There is a formula $\sigma(x)$ such that:*
(i) *$\sigma(x)$ binumerates T (i.e., the set of axioms of T) in S;*
(ii) *if $S \vdash \mathrm{Pr}_\sigma(\ulcorner E \urcorner)$, then $S \vdash \mathrm{Pr}_{T\restriction m}(\ulcorner E \urcorner)$ for some m (all E).*

B.2. Lemma. *Suppose $\alpha(x)$ binumerates T in $S + \mathrm{Con}_\alpha$. There is then a translation t such that for all E,*

$$\text{if } S + \mathrm{Con}_\alpha \vdash tE, \text{ then } \quad S + \mathrm{Con}_\alpha \vdash \mathrm{Pr}_\alpha(\ulcorner E \urcorner).$$

B.3. Lemma. *There is a translation t such that for all E,*

$$\text{if } S \vdash tE, \text{ then } \quad S \vdash \mathrm{Pr}_{T\restriction m}(\ulcorner E \urcorner) \text{ for some } m.$$

Proof. Let us fix the formula $\sigma(x)$ from Lemma B.1. According to (i), $\sigma(x)$ binumerates T in S and hence in $S + \mathrm{Con}_\sigma$. Then, by Lemma B.2, there is a translation t such that for all E,

(*) if $S + \mathrm{Con}_\sigma \vdash tE$, then $S + \mathrm{Con}_\sigma \vdash \mathrm{Pr}_\sigma(\ulcorner E \urcorner)$.

Suppose $S \vdash tE$. Then $S + \mathrm{Con}_\sigma \vdash tE$ and, by (*), $S + \mathrm{Con}_\sigma \vdash \mathrm{Pr}_\sigma(\ulcorner E \urcorner)$; on the other hand, we clearly have $S + \neg\mathrm{Con}_\sigma \vdash \mathrm{Pr}_\sigma(\ulcorner E \urcorner)$; consequently, $S \vdash \mathrm{Pr}_\sigma(\ulcorner E \urcorner)$. Then, by B.1(ii), $S \vdash \mathrm{Pr}_{T\restriction m}(\ulcorner E \urcorner)$ for some m. □

B.4. Lemma. *Let E be a sentence and t be a translation with the relativizing formula $\delta(x)$ (see 3.1). Then*

$$\mathrm{PA} \vdash \mathrm{Pr}_\emptyset(\ulcorner E \urcorner) \rightarrow \mathrm{Pr}_\emptyset(\ulcorner \exists x\, \delta(x) \rightarrow tE) \urcorner).$$

Proof. Standard. Argue in PA.

Suppose Prf is a proof of E in pure predicate calculus, and let $x_1, \ldots, x_n$ be all the variables occurring free in Prf. Let then

$$\Delta = \delta(x_1) \wedge \cdots \wedge \delta(x_n).$$

By induction on Prf, one can easily verify that $\mathrm{PC} \vdash \Delta \rightarrow tE$ and hence (as E is closed) $\mathrm{PC} \vdash \exists\Delta \rightarrow tE$ ($\exists\Delta$ denotes the existential quantifiers closure of Δ). On the other hand, $\mathrm{PC} \vdash \exists x\, \delta(x) \rightarrow \exists\Delta$. Consequently, $\mathrm{PC} \vdash \exists x\, \delta(x) \rightarrow tE$. □

B.5. Theorem. *The following are equivalent:*
(i) *S is cointerpretable in T.*
(ii) *If $S \vdash \mathrm{Pr}_{T\restriction m}(\ulcorner E \urcorner)$ for some m, then $T \vdash E$ (all E).*
(iii) *S is Σ_1-conservative over T.*

Proof. (i)$\Rightarrow$(ii): Suppose t (with the relativizing formula $\delta(x)$) is a cointerpretation of S in T, and $S \vdash \mathrm{Pr}_{T \restriction m}(\ulcorner E \urcorner)$, i.e., $S \vdash \mathrm{Pr}_{\emptyset}(\ulcorner \bigwedge T \restriction m \rightarrow E \urcorner)$. Then, by B.4, $S \vdash \mathrm{Pr}_{\emptyset}(\ulcorner \exists x\, \delta(x) \rightarrow t(\bigwedge T \restriction m \rightarrow E)\urcorner)$ and, since S is essentially reflexive, $S \vdash \exists x\, \delta(x) \rightarrow t(\bigwedge T \restriction m \rightarrow E)$, whence $S \vdash t(\exists x\, (x = x) \rightarrow (\bigwedge T \restriction m \rightarrow E))$, whence (as t is a cointerpretation of S in T) $T \vdash \exists x\, (x = x) \rightarrow (\bigwedge T \restriction m \rightarrow E)$, whence $T \vdash E$.

(ii)$\Rightarrow$(i): Assume (ii). Let t be the translation from B.3. Suppose $S \vdash tE$. Then, by B.3, $S \vdash \mathrm{Pr}_{T \restriction m}(\ulcorner E \urcorner)$ for some m, whence, by (ii), $T \vdash E$. It means that t is a cointerpretation of S in T.

(ii)$\Rightarrow$(iii): Assume (ii). Suppose E is a Σ_1-sentence and $S \vdash E$. Then $S \vdash \mathrm{Pr}_{T \restriction m}(\ulcorner E \urcorner)$ for some 'sufficiently large' m, whence, by (ii), $T \vdash E$.

(iii)$\Rightarrow$(ii): Suppose S is Σ_1-conservative over T and $S \vdash \mathrm{Pr}_{T \restriction m}(\ulcorner E \urcorner)$. Since $\mathrm{Pr}_{T \restriction m}(\ulcorner E \urcorner)$ is a Σ_1-sentence, it follows that $T \vdash \mathrm{Pr}_{T \restriction m}(\ulcorner E \urcorner)$ and, T being essentially reflexive, $T \vdash E$.

The theorem is proved. □

Observe that the above proof can be formalized in PA, and Theorem 3.6b is thus the (i)$\Leftrightarrow$(iii) part of B.5.

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