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INVERTING GRAPHS OF RECTANGULAR MATRICES

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This paper addresses the question of determining the class of rectangular matrices having a given graph as a row or column graph. We also determine equivalent conditions on a given pair of graphs in order for them to be the row and column graphs of some rectangular matrix. In connection with these graph inversion problems we discuss the concept of minimal inverses. This concept turns out to have two different forms in the case of one-graph inversion. For the two-graph case we present a method of determining when an inverse is minimal. Finally we apply the two-graph theorem to a class of energy related matrices.

1. Introduction

Matrices and related constructs from linear algebra have been widely used by graph theorists in order to study various properties of graphs and, in particular, to define invariants associated with graphs (see [1] and [2] for reasonably up to date and extensive discussions of the application of algebraic methods to graph theory). In recent years, on the other hand, there has been a growing use of graph theory to provide insights into structural problems in matrix theory and to help solve such problems as the sign solvability problem and the sign stability problem (see [4], [10], [11], [16], [17], [18], and [22] among others). A systematic use of various graphs to help study structural problems for rectangular matrices has recently been initiated by the present authors (see [6] and [7]).

In this paper we consider the problem of identifying Boolean matrices A having a given row graph RG(A), or column graph, CG(A). Specifically, in the row graph the rows of A correspond to the points of RG(A), and two rows are adjacent in RG(A) if and only if they have a nonzero entry in the same column of A. The column graph is defined similarly.

The basic concept that we use is the fact that the columns of A correspond to cliques of RG(A). The main result of Section 2, the one-graph inversion theorem, has several applications to problems that involve edge coverings by cliques (see Roberts[21]). In particular, Theorem 1 is used in [12] to characterize competition graphs (see Roberts [20]), in [13] to characterize upper bound graphs (see McMorris and Zaslavsky [19]), and in [14] to characterize consanguinity graphs (see Florence [3]). In Section 2 we also investigate the concept of minimal inverses.

We consider the two-graph inversion problem in Section 3. In particular, we characterize the pairs of graphs for which there exists a matrix whose row and column graphs are precisely the given graphs. Theorem 2 is used in [15] to characterize the pairs of graphs that are the competition graph and common enemy graph of the same food web, in [14] to characterize the pair of graphs that are the upper bound graph and consanguinity graph of the same digraph, and in [8] to characterize 2-step graphs.

One application of our results is to the class of physical flows networks which have been identified by H.J. Greenberg [9] as often being an important component of energy related linear programming problems. We show in Section 4 exactly how the two-graph inversion theorem applies to such problems.

Throughout this paper we deal only with matrices having the property that each row and column has at least one nonzero element. We will call such a matrix a *regular matrix*.

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2. One-graph inversion

Let $G=(V,E)$ be a graph with p points, $p=|V|$. Observe that if a matrix A can be found such that $\mathrm{RG}(A)=G$, then the matrix A^{T} has the property that $\mathrm{CG}(A^{\mathrm{T}})=G$. Consequently we can and shall consider only matrices A such that $\mathrm{RG}(A)=G$.

In order to discover the family of regular Boolean matrices such that $\mathrm{RG}(A)=G$ for any A in the family, we require the following definitions. Recall that a k-clique, $k\geq 1$, of a graph is a complete subgraph on k-points. Given a graph $G=(V,E)$, a finite set S of cliques of G will be called a *clique cover* if every point and edge of G belongs to at least one clique in S. We will use the notation $\langle X\rangle$ to denote the subgraph of G generated by the set X when $X\subset V$.

With the above definition as background we can now formulate our main one-graph inversion theorem.

Theorem 1. *Given the graph $G=(V,E)$ with $p=|V|$, the regular boolean matrix A has the property that* $\mathrm{RG}(A)=G$ *iff A has p rows and columns of A correspond to a clique cover of G.*

Proof. Suppose first that $\mathrm{RG}(A)=G$. Then the rows of A correspond to the points of G. Let c_k be a column of A with nonzero entries in rows $i_1, i_2, \ldots, i_p$. Then $\langle i_1, i_2, \ldots, i_p \rangle$ is a p-clique of RG. Thus, each column of A corresponds to a clique of $\mathrm{RG}(A) = G$. Also, the regularity of A assures that each point of G belongs to one of these cliques. Finally, observe that if $[x, y]$ is an edge of G, then some column of A must contain a nonzero element in rows x and y, so $[x, y]$ belongs to at least one clique corresponding to a column of A.

For the converse let $S = \{C_1, \ldots, C_\sigma\}$ be a clique cover of G. Construct the boolean matrix A with p rows and σ columns as follows. Let the points of G be labelled $r_1, r_2, \ldots, r_p$. Let column C_k, $1 \le k \le \sigma$, have ones in rows $r_{k1}, \ldots, r_{kp}$ where $C_k = \langle r_{k1}, \ldots, r_{kp} \rangle$ and zeros in all the remaining rows. Then each row and column of A has at least one nonzero element and A is regular. We must show $\mathrm{RG}(A) = G$. Obviously $|V| = p$, so the points of G are in 1-1 correspondence with the rows of A. It remains to show that $[r_i, r_j]$ belongs to G iff at least one column of A has ones in rows r_i and r_j. But the line $[r_i, r_j]$ belongs to G iff it belongs to at least one clique in S, hence iff at least one column of A satisfies the conditions. This proves the theorem.

The following corollary establishes existence of the inversion problem.

Corollary 1. *Let $G = (V, E)$ be a graph with $p = |V|$, $n = |E|$. Then there exists a $p \times (n + p_0)$ matrix A such that $\mathrm{RG}(A) = G$, where p_0 is the number of isolated points in G.*

Proof. The set of edges and isolated points form a clique cover S of G. Using S, the construction in the proof of the theorem produces a matrix A satisfying the conditions of the corollary.

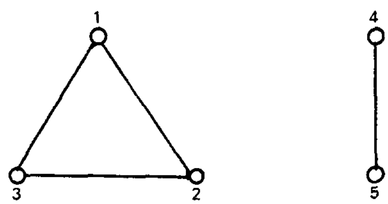

Fig. 1.

Let us consider as a simple example the graph of Fig. 1. For this graph, the matrices all have the property that $\mathrm{RG}(A_i) = G$.

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus it is clear that the inverse of G is not uniquely defined. Also note that repetitions of columns are allowed corresponding to a clique occurring more than once in a clique cover. The clique covers associated with these matrices are:

$$S_1=\{\langle 1,2\rangle,\langle 1,3\rangle,\langle 2,3\rangle,\langle 4,5\rangle\}; \qquad S_2=\{\langle 1,2,3\rangle,\langle 4,5\rangle\},$$
$$S_3=\{\langle 1,2,3\rangle,\langle 1,2,3\rangle,\langle 4,5\rangle\}.$$

Before we discuss minimality it will be useful for our work on two-graph inversion to introduce an additional graph. Let S be a clique cover of the graph G. We associate with S a *clique cover graph*, $Q(S)$, in the following way. Let $S=\{C_1,\dots,C_\sigma\}$. Then $Q(S)$ is a graph on the points $1,2,\dots,\sigma$ and the line $[i,j]$ belongs to $Q(S)$ iff C_i and C_j contain at least one common point. Hence, we have the following reformulation of Theorem 1 using the graph $Q(S)$.

Theorem 1′. *Given the graph $G=(V,E)$ with $p=|V|$, the regular boolean matrix A has the property that* $\mathrm{RG}(A)=G$ *iff A has p rows and there exists a clique cover S of G such that* $\mathrm{CG}(A)$ *is isomorphic to* $Q(S)$.

It will be convenient to set
$\mathscr{R}(G)=\{A:A$ is a regular Boolean matrix and $\mathrm{RG}(A)=G\}$,
$\mathscr{C}(G)=\{A:A$ is a regular Boolean matrix and $\mathrm{CG}(A)=G\}$.

Given a matrix $A\in\mathscr{R}(G)$, we next consider the problem of finding $A'\in\mathscr{R}(G)$ that has fewer nonzeros than A or fewer columns than A. In other words we may desire to find, in some well-defined sense, a minimal inverse of G.

Let S be a clique cover of G. If $A\in\mathscr{R}(G)$, A is said to be *generated* by the clique cover S if $\mathrm{CG}(A')=Q(S)$. We say that S is *minimal* or a *spanning set of cliques* for G if the removal of any clique from the set S violates the cover condition. The matrix A will be called a *minimal row inverse of G* if A is generated by S where S is minimal. Similarly we can define a *minimal column inverse of G*. The following result is an immediate consequence of Theorem 1.

Corollary 2. *A is a minimal row inverse of G iff the deletion of any column of A changes the graph of* $\mathrm{RG}(A)$.

We note that for the graph illustrated above in Fig. 1 the clique covers S_1 and S_2 are minimal. Therefore the matrices A_1 and A_2 are minimal row inverses of G.

Let us denote by $\mathscr{R}_0(G)$ the set of minimal row inverses of G and by $\mathscr{C}_0(G)$ the set of minimal column inverses of G (defined in a similar way). It is clear that the elements of $\mathscr{R}_0(G)$ have fewer columns, in some sense than the elements in $\mathscr{R}(G)-\mathscr{R}_0(G)$. However, the graph in Fig. 1 shows that not every element of $\mathscr{R}_0(G)$ has fewer columns than every element of $\mathscr{R}(G)-\mathscr{R}_0(G)$. Indeed, $A_1\in\mathscr{R}_0(P)$ has more columns than $A_3\in\mathscr{R}(G)-\mathscr{R}_0(G)$.

Using Corollary 2, we get the following result.

Corollary 3. *For every $A \in \mathscr{R}(G)$ columns can be deleted so that the resulting matrix is in $\mathscr{R}_0(G)$.*

To find a row inverse of G with the minimum number of columns, we must find a clique cover with the smallest number of cliques. This number is denoted by $k(G)$, the clique cover number of G (see [14]). We next deduce the following result.

Corollary 4. *Given the graph $G=(V,E)$ with $p=|V|$ and $\varepsilon=k(G)$, there exists a regular $p\times\varepsilon$ boolean matrix A such that* $\mathrm{RG}(A)=G$ *and if A' is any other $p\times\varepsilon'$ matrix with* $\mathrm{RG}(A')=G$, *then $\varepsilon\le\varepsilon'$.*

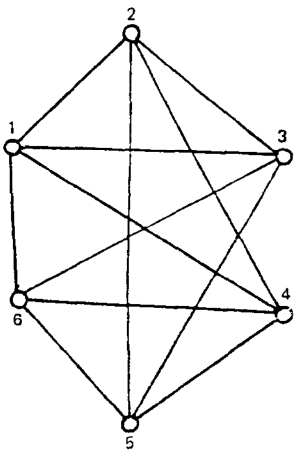

Fig. 2.

The matrix A in Corollary 4 need not be unique as is illustrated for the graph in Fig. 2. For this graph it is easy to verify that $k(G)=4$. Furthermore, $S_1=\{\langle 1,2,3\rangle,\langle 1,4,6\rangle,\langle 2,4,5\rangle,\langle 3,5,6\rangle\}$ and $S_2=\{\langle 1,2,4\rangle,\langle 1,3,6\rangle,\langle 2,3,5\rangle,\langle 3,5,6\rangle\}$ are clique covers which generate the following matrices:

$$A_1=\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\0&1&1&0\\0&0&1&1\\0&1&0&1\end{bmatrix},\qquad A_2=\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\1&0&0&1\\0&0&1&1\\0&1&0&1\end{bmatrix}$$

The following result is an immediate corollary of Theorem 1.

Corollary 5. *Let G be a graph. The set $\mathscr{R}_0(g)$ consists of a unique matrix A iff every maximal clique of G is either a 1-clique or a 2-clique.*

The set of graphs defined in the corollary includes some rather important subsets.

For example every bipartite graph satisfies the conditions and so does every graph that is a cycle of length ≥ 4. Also among the graphs in the set defined in the lemma are those row graphs associated with 'physical flows network' matrices (PFN), which will be discussed in Section 4 (see [9] for applications of these matrices).

For the purpose of minimizing the number of nonzeros in a row inverse, the concept of minimality can be modified. Consider the graph in Fig. 3. For this graph, $k(G)=3$, *and* $S_1=\{\langle 1,2,5\rangle, \langle 2,4,5\rangle, \langle 2,3,4\rangle\}$ and $S_2=\{\langle 1,2,5\rangle, \langle 4,5\rangle, \langle 2,3,4\rangle\}$ are minimal clique covers, and

$$A_1=\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\\0&1&1\\1&1&0\end{bmatrix}, \qquad A_2=\begin{bmatrix}1&0&0\\1&0&1\\0&0&1\\0&1&1\\1&1&0\end{bmatrix}$$

are the corresponding minimal row inverses. A_2 has fewer nonzeros than A_1.

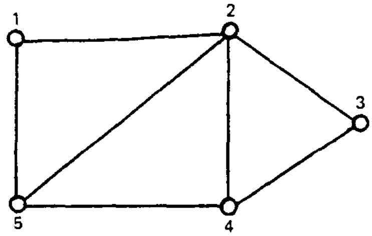

Fig. 3.

The above example leads to the following definition. Let S be a clique cover of G. We say that S is a *subminimal clique cover* for G if it is not possible to remove a clique or replace any one clique with a proper subclique while retaining the covering property. The matrix A will be called a *subminimal row inverse* of G if A is generated by S where S is subminimal. If we let $\mathscr{R}_0'(G)$ be the set of subminimal row inverses of G, then clearly $\mathscr{R}_0'(G) \subsetneq \mathscr{R}_0(g)$ and the above example shows that this containment may be proper.

Suppose $A, A' \in \mathscr{R}(G)$ with dimensions $m\times n$ and $m \times n'$, respectively. The relation $A'<A$ is defined to mean $n'\leq n$, $a'_{ij}\leq a_{ij}$ (for $i=1,\ldots,m; j=1,\ldots,n'$) and $A'\neq A$. Then we have the following corollary of Theorem 1.

Corollary 6. *A is a subminimal row inverse of G iff* $\mathrm{RG}(A)=G$ *and $A'<A$ implies* $\mathrm{RG}(A')\neq G$.

3. Two-graph inversion

In this section we consider the following problem. Given graphs G_1 and G_2, when can a regular matrix A be constructed having the property that $\mathrm{RG}(A)=G_1$

and $\mathrm{CG}(A)=G_2$? If such a matrix A exists, we say that (G_1, G_2) is invertible. The first point to be made is that for a regular matrix A, the graphs $\mathrm{CG}(A)$ and $\mathrm{RG}(A)$ have the same number of components [7]. It follows that we cannot always invert the pair (G_1, G_2); however, we can provide equivalent conditions for existence.

Theorem 2. *Given two graphs G_1 and G_2, the following are equivalent:*
(i) *(G_1, G_2) is invertible.*
(ii) *G_1 is isomorphic to a clique cover graph of G_2.*
(iii) *G_2 is isomorphic to a clique cover graph of G_1.*

Proof. We show first that (i) implies (iii). If G_1 and G_2 are invertible, there exists a matrix A such that $\mathrm{RG}(A)=G_1$ and $\mathrm{CG}(A)=G_2$ so that we have $A \in \mathcal{R}(G_1)$ and $A \in \mathcal{C}(G_2)$. By the one graph inversion theorem $A \in \mathcal{R}(G_1)$ implies that $\mathrm{CG}(A)=Q(S)$ where S is the clique cover of G_1. It follows that $\mathrm{CG}(A)=G_2$ is isomorphic to a clique cover graph of G_1. Thus (i) implies (iii). Next, we show that (iii) implies (i). Suppose G_2 is isomorphic to a clique cover graph of G_1. Then G_2 is isomorphic to $Q(S)$ where S is a clique cover of G_1. Let $A \in \mathcal{R}(G_1)$ be determined by S, then $\mathrm{RG}(A)=G_1$ and $\mathrm{CG}(A)=Q(S)=G_2$ by Theorem 1′. Hence G_1 and G_2 are invertible. The equivalence of (i) and (ii) follows by symmetry.

We have already pointed out a special case where G_1 and G_2 are not invertible. One might conjecture that if G_1 and G_2 are both connected, then they are invertible. But this is not true as we shall show, by considering the pair of graphs illustrated in Fig. 4. Since G_1 has only two points and $k(G_2)=4$, G_1 can't be isomorphic to a clique cover graph of G_2.

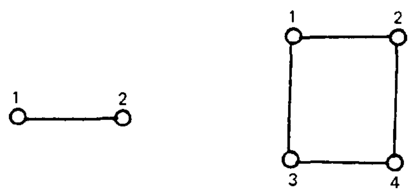

Fig. 4.

Now we will show how Theorem 2 can be used to invert a pair of graphs. Consider the pair of graphs in Fig. 5. If we choose the clique cover $S=\{\langle 1,2,3\rangle, \langle 1,3\rangle, \langle 2,4\rangle\}$ of G_1, then G_2 is isomorphic to the clique cover graph of S. Using the method in the proof of Theorem 1 we can then construct

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly $\mathrm{RG}(A)=G_1$ and $\mathrm{CG}(A)=G_2$.

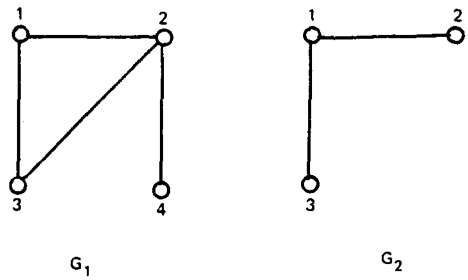

Fig. 5.

Next, consider the concept of minimality for two-graph inversion. The pair of graphs,

$$G_1 = G_2 = \overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\circ},$$

has two solutions (maintaining equivalence classes of rearrangements):

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since $A_1 \leq A_2$, we may regard A_1 as minimal. Note, however that A_1 is neither a minimal row inverse of G_1 nor a minimal column inverse of G_2. This is because row 1 is a singleton, and column 2 is a singleton, respectively. Thus, minimality cannot be equivalent to minimal clique covers as in the case of one-graph inversion. We call the inverse of (G_1, G_2), A, *minimal* if there is no other inverse, A', for which $A' < A$ is regular. In other words, A is minimal if no nonzero element can be replaced by a zero and still satisfy $\mathrm{RG}(A) = G_1$ and $\mathrm{CG}(A) = G_2$.

Minimality can be tested in an algebraic fashion as follows. Given an entry (i, j) rearrange the rows and columns of A using permatation matrices and P and Q to the form:

$$PAQ = \left[\begin{array}{c|c|c} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \\ \hline 0 \ \cdots \ 0 & (i,j) & 1 \ \cdots \ 1 \\ \hline & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & K_{ij} \end{array}\right]$$

We call the matrix K_{ij} the *connective* matrix of entry (i, j). If row i or column j has only one nonzero entry, K_{ij} is the empty set in which case we will call it trivial.

Theorem 3. *An inverse, $A = (G_1, G_2)^{-1}$, is minimal iff the connective of every nonzero entry is not regular or trivial.*

Proof. Suppose A is minimal. Assume, for contradiction, that $a_{ij}=1$ and K_{ij} is regular. Let A' be the same as A except $a'_{ij}=0$. We will show A' is an inverse of (G_1, G_2), thus contradicting minimality of A. The potential effect on row adjacency when complementing element (i,j) (i.e., $a_{ij} \leftarrow 1-a_{ij}$), is the elimination of edges between i and the rows of K_{ij}. However, since K_{ij} is regular, every row of K_{ij} has a nonzero, so the edges between i and the rows of K_{ij} remain. Hence, $\mathrm{RG}(A)=A$, and a similar argument shows that $\mathrm{CG}(A') \doteq \mathrm{CG}(A)$, so that A' is an inverse of (G_1, G_2).

Now suppose K_{ij} is not regular or trivial when $a_{ij}=1$. If A is not minimal, then there is a regular boolean matrix $A'<A$ that is an inverse and such that A' differs from A only in the i,j entry with $a'_{ij}=0$ and $a'_{ij}=1$. Now if K_{ij} is trivial, then A' would not be regular, a contradiction, so we can assume that K_{ij} is not regular. Now suppose row r is a null row. From the construction of K_{ij} it is easy to see that $[i,r]$ is in $\mathrm{RG}(A)$, but $[i,r]$ is not $\mathrm{RG}(A')$. A similar argument shows that $\mathrm{CG}(A) \neq \mathrm{CG}(A')$ if K_{ij} has a null column. Thus we have completed the proof.

The notion of connective makes it easy to test for minimality, particularly if the matrix A is sparse. Of peripheral interest is the general result: An element may be complemented without affecting row and column adjacency iff its connective is regular. We may thus look at the zeros of A to see if they can be made ones without introducing a new adjacency. We call A *maximal* if $A<A'$ implies $A' \neq (G_1, G_2)^{-1}$. The proof of the following result is similar to the proof of Theorem 3.

Theorem 4. *An inverse $A=(G_1, G_2)^{-1}$, is maximal iff the connective of every zero element is not regular.*

4. Application to PFN

It is of interest to apply our results in Sections 2 and 3 to the PFN-matrix case. The matrices are defined as follows [9].

The rows can be partitioned into 2 nonempty classes $R1$ and $R2$ and the columns into 3 nonempty classes $C1$, $C2$ and $C3$ such that:

(1) $a_{ij}=0$ for $i \in R1$ and $j \in C3$ and for $i \in R2$ and $j \in C1$.

(2) Each column in $C1$ and $C3$ has exactly 1 nonzero.

(3) Columns in $C2$ have two nonzeros, one in $R1$ and one in $R2$.

If A is a PFN-matrix, $\mathrm{RG}(A)$ is bipartite on $R1$ and $R2$, and adjacency in the row graph is completely determined by columns in $C2$. Using Corollary 5 it is thus simple to characterize $\mathscr{R}_0(G)$ when G is allegedly from a PFN-matrix.

We will now derive a characterization of a physical flows matrix A in terms of $\mathrm{RG}(A)$ and $\mathrm{CG}(A)$. First the definition implies that there exists permutation matrices P and Q such that if A is a PFN-matrix

$$PAQ=\begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{21} & A_{22} \end{bmatrix} \tag{1}$$

where each of the nonzero blocks in (1) has exactly one nonzero element in each column.

From (1) we can deduce that if A is a PFN-matrix with row graph RG(A), then CG(A) is isomorphic to one of the following clique cover graphs of RG(A). By Corollary 5, RG(A) has a unique minimum clique cover S_0 consisting of the maximal cliques of RG(A). Let $R1=\{r_{11},\dots,r_{1p}\}$ and $R3=\{r_{21},\dots,r_{2q}\}$. If $\langle r_{1i}\rangle$ is a 1-clique of RG(A), let $\langle r_{1i}\rangle^*$ denote a finite repetition of $\langle r_{1i}\rangle$, i.e. $\langle r_{1i}\rangle^*=\{r_{1i},\dots,r_{1i}\}$ where $\langle r_{1i}\rangle^*$ may be empty. Similarly define $\langle r_{2j}\rangle^*$. Finally, if $\langle r_{1i},r_{2j}\rangle$ is a 2-clique, let $\langle r_{1i},r_{2j}\rangle^*$ denote a finite repetition of $\langle r_{1i},r_{2j}\rangle$. Then

$$\mathrm{CG}(A)=Q\left(S_0\cup\left\{\bigcup_{i=1}^{p}\langle r_{1i}\rangle^*\right\}\cup\left\{\bigcup_{j=1}^{q}\langle r_{2j}\rangle^*\right\}\cup\left\{\bigcup_{\substack{1\le i\le p\\ 1\le j\le q}}\langle r_{1i},r_{2j}\rangle^*\right\}\right). \tag{2}$$

Let us present an example to clarify (2). The matrix

$$A=\left[\begin{array}{ccccc|cccccc|cccc} 1&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\ 0&1&0&1&1&0&1&0&1&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&1&0&0&1&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&1&1 \end{array}\right]$$

is a PFN-matrix. For this matrix, $R1=\{r_1,r_2,r_3\}$ and $R1=\{r_4,r_5,r_6\}$. Then,

$$S_0=\{\langle r_1\rangle,\langle r_2\rangle,\langle r_3\rangle,\langle r_1,r_4\rangle,\langle r_2,r_5\rangle,\langle r_3,r_6\rangle,\langle r_2,r_4\rangle,\langle r_4\rangle,\langle r_5\rangle,\langle r_6\rangle\}.$$

The nonempty finite repetitions are as follows:

$$\langle r_2\rangle^*=\{\langle r_2\rangle,\langle r_2\rangle\},\qquad \langle r_1,r_4\rangle^*=\langle r_1,r_4\rangle,$$

$$\langle r_2,r_5\rangle^*=\langle r_2,r_5\rangle,\qquad \langle r_6\rangle^*=\langle r_6\rangle.$$

Then CG(A) is isomorphic to the clique cover graph defined by the clique cover

$$S=S_0\cup\{\langle r_2\rangle,\langle r_2\rangle,\langle r_1,r_4\rangle,\langle r_2,r_5\rangle,\langle r_6\rangle\},$$

which has the form (2).

It is clear that if G_1 is bipartite and if G_2 is given in the form (2) relative to G_1, then G_1 and G_2 are invertible and the inverse matrix A with RG(A) $=G_1$ and CG(A) $=G_2$ is a PFN-matrix. Thus we can state

Theorem 5. *The matrix A is a PFN-matrix iff* RG(A) *is bipartite and* CG(A) *is given by* (2) *where S_0 is the set of maximal cliques of* RG(A) *and $\langle r_{1i}\rangle^*$, $\langle r_{2j}\rangle^*$, and $\langle r_{1i},r_{2j}\rangle^*$ denote finite repetitions of $\langle r_{1i}\rangle$, $\langle r_{2j}\rangle$, and $\langle r_{1i},r_{2j}\rangle$, respectively.*

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