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Energy conditions in $F(T,\Theta)$ gravity and compatibility with a stable de Sitter solution

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ABSTRACT

We study a new type of the modified teleparallel gravity of the form $F(T,\Theta)$ in which T, the torsion scalar, is coupled with Θ, the trace of the stress–energy tensor. In a perturbational approach, we study the stability of the solutions and as a special case we find a condition for stability of the de Sitter phase. Then we adopt a suitable form for $F(T,\Theta)$ that realizes a stable de Sitter solution so that the stability condition creates a specific constraint on the parametric space of the model. Finally, the energy conditions in the framework of $F(T,\Theta)$ gravity is investigated.

1. Introduction

Einstein's general relativity is a completely geometrical theory so that gravitation is described not as a force, but as a geometric deformation of flat Minkowski space–time. In this point of view, the gravitational field creates a curvature in space time so that its action on the particles is determined by allowing them to follow the geodesics of the space time. In this approach, trajectories are described by the geodesic equation not the force equation [1]. On the other hand, in 1928, Einstein in an attempt to build a unified gauge theory of gravitation and electrodynamics presented the other theory of gravity, the so-called teleparallel gravity [2]. In this theory torsion, the antisymmetric part of connection, is non-zero and torsion instead of curvature describes the gravitational interaction. In teleparallel gravity, tetrad (or vierbein) fields form the (pseudo) orthogonal bases for the tangent space at each point of flat space time. Similarly to the metric tensor in general relativity, here tetrad plays the role of the dynamical variables. Teleparallel gravity also uses the curvature-free Weitzenböck connection instead of Levi-Civita connection of general relativity to define covariant derivatives [3]. In spite of such fundamental conceptual differences between teleparallel theory and general relativity, it has been shown that teleparallel Lagrangian density only differs from Ricci scalar by a total divergence [4,5].

In general relativity, the dark energy puzzle can be addressed by introducing additional geometrical degree of freedom into the theory, that is called $F(R)$ modified gravity. In $F(R)$ gravity the late time acceleration of the universe is catched by dark geometry instead of dark energy [6]. The modification of gravity in teleparallel gravity is accomplished by supplementing an additional torsion term into Einstein–Hilbert Lagrangian [7]. The $F(T)$ gravity has interesting properties that the field equations are of second order, unlike $F(R)$ gravity which is of fourth order in the metric approach. In this context, $F(T)$ models have been extensively used in cosmology to explain the late time cosmic speed-up expansion without the need of dark energy [7,8].

In this Letter we construct a generalization of $F(T)$ modified gravity by considering coupling between torsion scalar T and trace of the stress–energy tensor Θ via a general function as $F(T,\Theta)$. Then we investigate stability of the de Sitter solution (when subjected to homogeneous perturbations) in this framework. In this sense, we obtain a stability condition for the de Sitter phase in the general $F(T,\Theta)$ theories. Then we propose a specific $F(T,\Theta)$ model and show that the stability condition can be expressed as a constraint equation between the parameters of the model. We also consider the constraints imposed by the energy conditions and investigate whether the parameters ranges of the proposed model are consistent with the stability conditions. We note that since homogeneous and isotropic perturbations can be considered as the route to determine the stability of different modified gravity theories (see for instance [9]), the full anisotropic analysis of the cosmological perturbations is not considered here.

The Letter is organized as follows: in Section 2 we explore the general features of the $F(T,\Theta)$ gravity theories by writing the

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corresponding modified Einstein equations. In Section 3 we introduce the evolution equations of perturbations appearing in this scenario once an FRW background is assumed. Section 4 is then devoted to the study of stability around the de Sitter solution. In Section 5 we present the energy conditions in $F(T,\Theta)$ gravity and compare the results with the obtained constraints from the stability conditions. We close the Letter by giving our conclusions in Section 6.

2. $F(T,\Theta)$ gravity

In this section, firstly a general $F(T,\Theta)$ function is considered for Lagrangian density of the action as follows

$$S=\int e\left(\frac{F(T,\Theta)}{2\kappa^2}+L_m\right)d^4x \quad (1)$$

where $\kappa^2=8\pi G$. $e=\sqrt{-g}$ is determinant of the vierbein e^i_μ and T is the torsion scalar. g is the determinant of the metric tensor and the metric of the space–time $g_{\mu\nu}$ is related to the vierbein by $g_{\mu\nu}=\eta_{ij}e^i_\mu e^j_\nu$. Here we use the Greek alphabet $(\mu,\nu,\rho,\ldots=0,1,2,3)$ to denote indices related to spacetime, and the Latin alphabet $(i,j,k,\ldots=0,1,2,3)$ to denote indices related to the tangent space. The ordinary matter part of the action is shown by L_m and the corresponding stress–energy tensor is

$$\Theta^\mu_i=-2\frac{\partial L_m}{\partial e^i_\mu}-2e^\mu_i L_m. \quad (2)$$

The connection that is used in general relativity, is the Levi-Civita connection

$$\hat{\Gamma}^\rho{}_{\mu\nu}=\frac{1}{2}g^{\rho\sigma}(\partial_\nu g_{\sigma\mu}+\partial_\mu g_{\sigma\nu}-\partial_\sigma g_{\mu\nu}). \quad (3)$$

This connection leads us to nonzero spacetime curvature but zero torsion [10]. In teleparallel gravity, tetrad fields give rise to a connection namely the Weitzenböck connection, instead of the Levi-Civita connection, which is defined by

$$\tilde{\Gamma}^\lambda{}_{\mu\nu}=e^\lambda_i\partial_\nu e^i_\mu=-e^i_\mu\partial_\nu e^\lambda_i. \quad (4)$$

One of the consequences of this definition is that the covariant derivative, D_μ, of the tetrad fields vanishes identically:

$$D_\mu e^i_\nu\equiv\partial_\mu e^i_\nu-\tilde{\Gamma}^\lambda{}_{\nu\mu}e^i_\lambda=0. \quad (5)$$

This equation leads us to a zero curvature but nonzero torsion [10]. We define the torsion and contortion by

$$T^\rho{}_{\mu\nu}=\tilde{\Gamma}^\rho{}_{\nu\mu}-\tilde{\Gamma}^\rho{}_{\mu\nu}, \quad (6)$$

$$K^\rho{}_{\mu\nu}=\tilde{\Gamma}^\rho{}_{\mu\nu}-\hat{\Gamma}^\rho{}_{\mu\nu}=\frac{1}{2}\left(T_\mu{}^\rho{}_\nu+T_\nu{}^\rho{}_\mu-T^\rho{}_{\mu\nu}\right) \quad (7)$$

respectively, that the contortion is expressed as the interrelation between Weitzenböck and Levi-Civita connections [4]. Now, one can define super-potential as follows

$$S_\sigma{}^{\mu\nu}\equiv\frac{1}{2}\left(K^{\mu\nu}{}_\sigma+\delta^\mu_\sigma T^{\xi\nu}{}_\xi-\delta^\nu_\sigma T^{\xi\mu}{}_\xi\right) \quad (8)$$

to obtain the torsion scalar

$$T\equiv S_\sigma{}^{\mu\nu}T^\sigma{}_{\mu\nu}=\frac{1}{4}T^{\xi\mu\nu}T_{\xi\mu\nu}+\frac{1}{2}T^{\xi\mu\nu}T_{\nu\mu\xi}-T_{\xi\mu}{}^\xi T^{\nu\mu}{}_\nu \quad (9)$$

which is used as the Lagrangian density in formulation of the teleparallel theories.

The generalized field equations are extracted by varying of the expression (1) with respect to the vierbein field e^i_ν as follows

$$\begin{aligned}&e^{-1}\partial_\rho\left(eS_i{}^{\mu\rho}\right)F_T+e^{-1}\partial_\rho\left(e\Xi_i{}^{\mu\rho}\right)F_\Theta+S_i{}^{\mu\rho}\partial_\rho T F_{TT}\\&\quad+\Xi_i{}^{\mu\rho}\partial_\rho\Theta F_{\Theta\Theta}+\frac{1}{4}e^\mu_i F-T^\sigma{}_{\nu i}S_\sigma{}^{\mu\nu}F_T-\Upsilon_i{}^\mu F_\Theta\\&=4\pi Ge^\sigma_i\Theta_\sigma{}^\mu\end{aligned} \quad (10)$$

where $\Xi_i{}^{\mu\rho}=\partial\Theta/\partial e^i_{\mu,\rho}$ and $\Upsilon_i{}^\mu=\frac{1}{4}\partial\Theta/\partial e^i{}_\mu$. Note that F_T and F_Θ (F_{TT} and $F_{\Theta\Theta}$) are the first (second) derivatives of the $F(T,\Theta)$ with respect to T and Θ, respectively.

Here, it is assumed that the Lagrangian of matter is only in terms of $e^i{}_\mu$, and so $\Xi_i{}^{\mu\rho}$ is zero. Since $\Theta=e^j_\alpha\Theta^\alpha_j$, one can say $\Upsilon_i{}^\mu$ is written as $\Upsilon_i{}^\mu=\frac{1}{4}\Theta_i{}^\mu+\Omega^\mu_i$ that $\Omega^\mu_i=\frac{1}{4}e^j_\alpha[\partial\Theta^\alpha_j/\partial e^i{}_\mu]$. So the field equations (10) can be rewritten as follows

$$\begin{aligned}&e^{-1}\partial_\rho\left(eS_i{}^{\mu\rho}\right)F_T+S_i{}^{\mu\rho}(\partial_\rho T)F_{TT}+\frac{1}{4}e^\mu_i F-e^\gamma_i T^\sigma{}_{\nu\gamma}S_\sigma{}^{\mu\nu}F_T\\&\quad-\Omega_i{}^\mu F_\Theta=4\pi G\Theta_i{}^\mu+\frac{1}{4}F_\Theta\Theta_i{}^\mu.\end{aligned} \quad (11)$$

On the other hand, Ω^μ_i by definition (2) takes the form

$$\Omega^\mu_i=\Theta^\mu_i+\frac{3}{2}e^\mu_i L_m-\frac{1}{2}e^j_\alpha\frac{\partial^2 L_m}{\partial e^i_\mu\partial e^j_\alpha}. \quad (12)$$

In this Letter, we consider a perfect fluid form for the stress–energy tensor of the matter as $\Theta^i_\mu=(\rho+p)u^iu_\mu-pe^i_\mu$, where ρ is the energy density, p is the pressure and u_μ describes the four-velocity. We also assume that the matter Lagrangian takes the form $L_m=-\rho$ [11] (see [12] for the case of $L_m=-p$). Thus, with these assumptions Ω^μ_i is rewritten as

$$\Omega^\mu_i=\Theta^\mu_i-\frac{3}{2}e^\mu_i\rho. \quad (13)$$

Now in a flat FRW background, $ds^2=dt^2-a^2(t)dX^2$ with scale factor $a(t)$, the field equations for $F(T,\Theta)$ gravity are given by

$$12H^2F_T+F=2\kappa^2\rho-\rho F_\Theta \quad (14)$$

and

$$48\dot{H}H^2F_{TT}-\left(12H^2+4\dot{H}\right)F_T-F=2\kappa^2p+(5p+6\rho)F_\Theta, \quad (15)$$

where a dot denotes derivative with respect to time. Torsion scalar as a function of the Hubble parameter $H=\frac{\dot{a}}{a}$ is expressed by

$$T=-6H^2. \quad (16)$$

Using Eqs. (14) and (15), one can obtain the modified Friedmann equations as follows

$$3H^2=\kappa^2(\rho+\rho_T+\rho_{(T,\Theta)}) \quad (17)$$

and

$$2\dot{H}+3H^2=-\kappa^2(p+p_T+p_{(T,\Theta)}) \quad (18)$$

where ρ_T and p_T are energy density and pressure contribution of torsion scalar, respectively. $\rho_{(T,\Theta)}$ and $p_{(T,\Theta)}$ are the energy density and pressure contribution of the coupling between torsion and stress–energy tensor, respectively. These quantities are defined as follows

$$\rho_T=\frac{1}{2F_T}(TF_T-F), \quad (19)$$

$$\rho_{(T,\Theta)}=-\frac{1}{2F_T}(\rho F_\Theta), \quad (20)$$

$$p_T=\frac{1}{2F_T}\left[F-TF_T-48\dot{H}H^2F_{TT}\right], \quad (21)$$

and

$$p_{(T,\Theta)} = \frac{1}{2F_T}(5p + 6\rho)F_\Theta. \tag{22}$$

From Eqs. (19)–(22), we can define gravitationally effective form of dark energy density $\rho_{DE} = \rho_T + \rho_{(T,\Theta)}$ and pressure $p_{DE} = p_T + p_{(T,\Theta)}$, so that equation of state parameter is defined as

$$\omega_{DE} = \frac{p_{DE}}{\rho_{DE}} = -1 + \frac{-48\dot{H}H^2 F_{TT} + 5(p+\rho)F_\Theta}{(TF_T - F) - \rho F_\Theta}. \tag{23}$$

In which follows, we want to rewrite the field equations (11) to a suitable form for our purpose in Section 5. To this end, we firstly multiply $g_{\mu\sigma}e^i_\nu$ on both sides of (11), so that the coefficient of the term F_T takes the following form

$$\begin{aligned} &e^{-1}e^i_\nu \partial_\rho\big(e e^\alpha_i S_\alpha{}^{\mu\rho}\big) - T^\lambda{}_{\rho\nu} S_\lambda{}^{\mu\rho} \\ &\quad = \partial_\rho S_\nu{}^{\mu\rho} - \tilde{\Gamma}^\alpha{}_{\nu\rho} S_\alpha{}^{\mu\rho} + \hat{\Gamma}^\tau{}_{\tau\rho} S_\nu{}^{\mu\rho} - T^\lambda{}_{\rho\nu} S_\lambda{}^{\mu\rho} \\ &\quad = -\nabla^\rho S_{\nu\rho}{}^\mu - K^\lambda{}_{\rho\nu} S^\mu{}_\lambda{}^\rho \end{aligned} \tag{24}$$

where we have used the following relation

$$K^{(\mu\nu)\sigma} = T^{\mu(\nu\sigma)} = S^{\mu(\nu\sigma)} = 0. \tag{25}$$

On the other hand, by Eq. (7), the Riemann tensor for the Levi-Civita connection is written in the following form

$$R^\rho{}_{\mu\lambda\nu} = \partial_\lambda \hat{\Gamma}^\rho{}_{\mu\nu} - \partial_\nu \hat{\Gamma}^\rho{}_{\mu\lambda} + \hat{\Gamma}^\rho{}_{\sigma\lambda}\hat{\Gamma}^\sigma{}_{\mu\nu} - \hat{\Gamma}^\rho{}_{\sigma\nu}\hat{\Gamma}^\sigma{}_{\mu\lambda} \tag{26}$$

then the corresponding Ricci tensor is written as

$$R_{\mu\nu} = \nabla_\nu K^\rho{}_{\mu\rho} - \nabla_\rho K^\rho{}_{\mu\nu} + K^\rho{}_{\sigma\nu} K^\sigma{}_{\mu\rho} - K^\rho{}_{\sigma\rho} K^\sigma{}_{\mu\nu}. \tag{27}$$

By using $K^\rho{}_{\mu\nu}$ given in Eq. (8) and the relations (25), and also by considering $S^\mu{}_{\rho\mu} = K^\mu{}_{\rho\mu} = T^\mu{}_{\mu\rho}$ one obtains [10,13,14]

$$\begin{aligned} &R_{\mu\nu} = -2\nabla^\rho S_{\nu\rho\mu} - g_{\mu\nu}\nabla^\rho T^\sigma{}_{\rho\sigma} - 2K_{\sigma\rho\nu} S^{\rho\sigma}{}_\mu, \\ &R = -T - 2\nabla^\mu T^\nu{}_{\mu\nu}, \end{aligned} \tag{28}$$

therefore, one reaches

$$G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = -2\big(\nabla^\rho S_{\nu\rho\mu} + K_{\sigma\rho\nu} S^{\rho\sigma}{}_\mu\big) \tag{29}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor. Finally, combining Eqs. (24) and (29), one can rewrite the field equations for $F(T,\Theta)$ gravity as follows

$$A_{\sigma\nu}F_T + B_{\sigma\nu}F_{TT} + \frac{1}{4}g_{\sigma\nu}F(T) - \Omega_{\sigma\nu}F_\Theta = \frac{1}{2}\Theta_{\sigma\nu} + \frac{1}{4}\Theta_{\sigma\nu}F_\Theta \tag{30}$$

where

$$\begin{aligned} A_{\sigma\nu} &= g_{\sigma\mu}e^i_\nu\big[e^{-1}\partial_\rho\big(e e^\alpha_i S^{\mu\rho}_\alpha\big) - e^\lambda_i T^\alpha{}_{\rho\lambda} S_\alpha{}^{\mu\rho}\big] \\ &= -\nabla^\rho S_{\nu\rho\sigma} - K_{\lambda\rho\nu} S^{\rho\lambda}{}_\sigma = \frac{1}{2}\big[G_{\nu\sigma} - (1/2)g_{\nu\sigma}T\big], \\ B_{\sigma\nu} &= S_{\nu\sigma}{}^\rho \nabla_\rho T. \end{aligned} \tag{31}$$

In upcoming sections, we use the trace of Eq. (30) as an independent relation to simplify the field equation. Since $A_\mu{}^\mu = -\frac{1}{2}(R + 2T)$, the mentioned trace can be expressed as

$$-\frac{1}{2}(R + 2T)F_T + BF_{TT} + F(T) + \Omega F_\Theta = \Theta + \frac{1}{4}\Theta F_\Theta \tag{32}$$

where $B = B_\mu{}^\mu$, $\Omega = \Omega_\mu{}^\mu$ and $\Theta = \Theta_\mu{}^\mu$ [10].

3. Perturbations of flat FRW solutions in $F(T,\Theta)$ gravity

Now we study the homogenous and isotropic perturbations around a particular cosmological solution for the model described by the action (1). First, we obtain the perturbed equations for the most general case. Then as a specific case, the de Sitter solution will be studied in which follows. Let us assume a general solution in the FRW cosmological background, which is given by a Hubble parameter $H = \bar{H}(t)$. This solution for a particular $F(T,\Theta)$ model satisfies Eq. (14). The matter fluid is assumed to be in the form of a perfect fluid with a constant equation of state $p = \omega\rho$, in which the matter energy density ρ satisfies the standard continuity equation

$$\dot{\rho} + 3H(1+\omega)\rho = 0. \tag{33}$$

Then the evolution of the matter energy density can be expressed in terms of particular solution $\bar{H}(t)$ and by solving the continuity equation (33) as follows

$$\hat{\rho}(t) = \rho_0 e^{-3(1+\omega)\int \bar{H}(t)\,dt}. \tag{34}$$

In order to investigate the perturbations around the solutions $H = \bar{H}(t)$ and energy density given by (34), one can consider small deviations from the Hubble parameter and the energy density evolution as [15]

$$H(t) = \bar{H}(t)\big[1 + \delta(t)\big], \qquad \rho(t) = \bar{\rho}(t)\big[1 + \delta_m(t)\big] \tag{35}$$

where $\delta(t)$ and $\delta_m(t)$ hold for the isotropic deviation of the background Hubble parameter and the matter density, respectively. In which follows we expand the $F(T,\Theta)$ function in the power of $\bar{T}$ and $\bar{\Theta}$ evaluated at the solution $H = \bar{H}(t)$ to study the behavior of the perturbations in linear regime:

$$F(T,\Theta) = \bar{F} + \bar{F}_T(T - \bar{T}) + \bar{F}_\Theta(\Theta - \bar{\Theta}) + \mathcal{O}^2 \tag{36}$$

where a bar indicates the value of $F(T,\Theta)$ function and its derivatives evaluated at $T = \bar{T}$ and $\Theta = \bar{\Theta}$. By substituting Eqs. (35) and (36) into the Friedmann equation (14), one can obtain an expression for the perturbations $\delta(t)$ in linear approximation as follows

$$\begin{aligned} &12\bar{H}^2\bar{F}_T\delta(t) + \big[\bar{F} + (\Theta - \bar{\Theta})\bar{F}_\Theta - \big(2\kappa^2 - \bar{F}_\Theta\big)\bar{\rho}_m\big] \\ &\quad = \big(2\kappa^2 - \bar{F}_\Theta\big)\bar{\rho}_m\delta_m(t). \end{aligned} \tag{37}$$

It seems that Eq. (37) is an algebraic equation for $\delta(t)$, but since Θ as trace of the stress–energy tensor itself is expressed in terms of H and $\dot{H}$, one should find a differential equation for $\delta(t)$. For this purpose, we firstly contract the field equations (11) by e^i_μ to find

$$\begin{aligned} \Theta = {} &\frac{4}{2 + F_\Theta}\big[e^{-1}F_T\big[\partial_\rho\big(eS_\mu{}^{\mu\rho}\big) - e\big(\partial_\rho e^i_\mu\big)S_i{}^{\mu\rho}\big] \\ &+ TF_T + S_\mu{}^{\mu\rho}\partial_\rho T F_{TT} + F - \Omega^\mu_\mu F_\Theta\big]. \end{aligned} \tag{38}$$

It is easy to show that

$$\begin{aligned} &\partial_\rho\big(eS_\mu{}^{\mu\rho}\big) = 3e\big(\dot{H} + H^2\big), \qquad \big(\partial_\rho e^i_\mu\big)S_i{}^{\mu\rho} = 3H^2, \\ &S_\mu{}^{\mu\rho}\partial_\rho T = 3H\dot{H}. \end{aligned} \tag{39}$$

Also by using expression (13) and $\Omega^\mu_\mu = e^i_\mu\Omega^\mu_i$, one obtains

$$\Omega^\mu_\mu = \Theta - 6\rho_m. \tag{40}$$

So, the expression for Θ can be deduced as follows

$$\Theta = \frac{12}{2 + 5F_\Theta}\bigg[\big(\dot{H} - 2H^2\big)F_T + 2\rho_m F_\Theta + H\dot{H}F_{TT} + \frac{1}{3}F\bigg]. \tag{41}$$

Now one can substitute Eqs. (35) and (36) into the expression for Θ and then Θ in Eq. (37) in order to get the corresponding differential equation for $\delta(t)$ as follows

$$C_2\dot{\delta}(t)+C_1\delta(t)+C_0=C_m\delta_m(t), \tag{42}$$

where $C_{0,1,2}$ and C_m depend explicitly on the $F(T,\Theta)$ and its derivatives evaluated in the background solution. These coefficients are explicitly written in the following

$$C_2=\frac{12\bar{H}}{2+\bar{F}_\Theta}\bar{F}_\Theta\bar{F}_T+\bar{H}^2\bar{F}_{TT}, \tag{43}$$

$$C_1=12\bar{H}^2\bar{F}_T+\frac{12\bar{F}_\Theta}{2+\bar{F}_\Theta}\left(\dot{\bar{H}}-8\bar{H}^2\right)\bar{F}_T+2\bar{H}\dot{\bar{H}}\bar{F}_{TT}, \tag{44}$$

$$C_0=\bar{F}-\left(2\kappa^2-\bar{F}_\Theta\right)\bar{\rho}_m-\bar{\Theta}\bar{F}_\Theta+\frac{12\bar{F}_\Theta}{2+\bar{F}_\Theta}\left[\bar{H}\dot{\bar{H}}\bar{F}_{TT}+\left(\dot{\bar{H}}-2\bar{H}^2\right)\bar{F}_T+2\bar{\rho}_m\bar{F}_\Theta-\bar{\Theta}\bar{F}_\Theta+\frac{1}{3}\bar{F}\right], \tag{45}$$

and

$$C_m=\bar{\rho}_m\left(2\kappa^2-\bar{F}_\Theta-\frac{24(\bar{F}_\Theta)^2}{2+\bar{F}_\Theta}\right). \tag{46}$$

In addition, there is a second perturbed equation obtained from the matter continuity equation (33) and perturbed expressions (35), as follows

$$\dot{\delta}_m(t)+3\bar{H}(t)\delta(t)=0. \tag{47}$$

In general relativity, the stability equation (42) is reduced to an algebraic relation between geometrical and the matter perturbations. For higher order theories of gravity, the evolution of the perturbations is in general determined by a coupled system of ordinary differential equations (42) and (47). Eq. (42) is a non-homogeneous and linear first order differential equation. To solve this differential equation, one firstly rewrites it in the standard form and then finds an integrating factor. Multiplying the standard equation by integrating factor, $\delta(t)$ can be obtained as

$$\delta(t)=\left[\int\frac{e^{\frac{C_1}{C_2}t}[C_m\delta_m(t)-C_0]}{C_2}\,dt\right]e^{-\frac{C_1}{C_2}t}. \tag{48}$$

Hence, for a particular FRW cosmological solution, the stability of the solution can be investigated in the context of $F(T,\Theta)$ gravity by analyzing and solving Eqs. (42) and (47). In the next section, we will illustrate the previous discussions by considering theories which include the de Sitter solution as the simplest cosmological solution.

4. The stability of the de Sitter solution

The de Sitter solution is one of the simplest cosmological solutions which can realize the late-time accelerated phase of the universe expansion as well as the inflationary epoch. On the other hand, the existence of a *stable de Sitter solution* helps the theory to be cosmologically viable. Therefore, we study the stability of the de Sitter solution,

$$H(t)=H_{(0)},\qquad a(t)=a_0e^{H_{(0)}t} \tag{49}$$

where $H_{(0)}$ is a constant. Since the de Sitter solution is a vacuum solution, the perturbations is affected only by the underlying gravitational theory. According to the differential equation for the perturbations, Eq. (42), now we have

$$C_2^{(0)}\dot{\delta}(t)+C_1^{(0)}\delta(t)+C_0^{(0)}=0 \tag{50}$$

with

$$C_2^{(0)}=F_\Theta^{(0)}\left(12F_T^{(0)}+F_{TT}^{(0)}\right)+2F_{TT}^{(0)},$$
$$C_1^{(0)}=12F_T^{(0)}\left(2-7F_\Theta^{(0)}\right),$$
$$C_0^{(0)}=\frac{F^{(0)}}{H_{(0)}^2}\left(6+F_\Theta^{(0)}\right)-24F^{(0)}F_\Theta^{(0)} \tag{51}$$

where the notation (0) indicates the value of each function evaluated in the de Sitter phase $T=T_{(0)}$ and $\Theta=\Theta_{(0)}$. The general solution of Eq. (50) demonstrates the dynamical behavior of the gravitational perturbations as follows

$$\delta(t)=-\frac{C_0^{(0)}}{C_1^{(0)}}+\alpha e^{-\frac{C_1^{(0)}}{C_2^{(0)}}t} \tag{52}$$

where α is an arbitrary integration constant. As we see, the growth of the gravitational perturbations tends to the constant value $-\frac{C_0^{(0)}}{C_1^{(0)}}$ with the stability condition:

$$\frac{C_1^{(0)}}{C_2^{(0)}}=\frac{12F_T^{(0)}(2-7F_\Theta^{(0)})}{12F_\Theta^{(0)}F_T^{(0)}+2F_{TT}^{(0)}+F_\Theta^{(0)}F_{TT}^{(0)}}>0. \tag{53}$$

Therefore, the stability of the de Sitter solution depends on the values of the function $F(T,\Theta)$ and its derivatives evaluated at $T_{(0)}$ and $\Theta_{(0)}$.

In order to illustrate the previous calculations, let us consider the function (see for instance [15])

$$F(T,\Theta)=k_1T+k_2T^m\Theta^n. \tag{54}$$

Here k_1 and k_2 are positive or negative coupling constants. Now one can easily solve Eq. (14) to find

$$\Theta_{(0)}^n=\frac{k_1}{(1-2m)k_2}T_{(0)}^{1-m}. \tag{55}$$

Then by using Eqs. (16) and (41) (in the linear regime), we obtain

$$\Theta_{(0)}=\frac{k_1}{2(1-2m)}T_{(0)}\left[12(1-m)-5n\right]. \tag{56}$$

Now, the combination of the last two equations gives the following de Sitter solution

$$H_{(0)}=\left[(-6)^{m+n-1}\left(\frac{k_1}{1-2m}\right)^{n-1}\times\left(6[1-m]-\frac{5}{2}n\right)^n k_2\right]^{\frac{1}{2(1-m-n)}}. \tag{57}$$

As a specific example, we consider the case with $m=\frac{2}{3}$ and $n=1$. Thus the de Sitter solution takes the following form

$$H_{(0)}=\left(\frac{-2}{9k_2^3}\right)^{\frac{1}{4}}. \tag{58}$$

As we see, in this model there is a determinate district for k_2 and it is that for a universe with a well-defined de Sitter expansion, k_2 must be negative. Now, the stability condition (53) by using (56) gives

$$\frac{36(1+k_2T_0^{\frac{2}{3}})(2-7k_2T_0^{\frac{2}{3}})}{36k_2T_0^{\frac{2}{3}}(1+k_2T_0^{\frac{2}{3}})-2k_2T_0^{-\frac{1}{3}}-k_2^2T_0^{\frac{1}{3}}}>0. \tag{59}$$

Substituting (16) and (58) into (59), the stability condition is rewritten as

$$\frac{-576}{144+\sqrt{2}(-k_2)^{\frac{7}{6}}+\sqrt{2}(-k_2)^{\frac{5}{6}}}>0.$$

It is clear that this inequality is incorrect. In other words, in the model with $m=\frac{3}{2}$ and $n=1$ the perturbation grows exponentially and the de Sitter solution becomes unstable. So, it is not cosmologically a viable model. The other example is a model with $m=\frac{6}{7}$ and $n=1$ (we note that these choices are restricted from the energy conditions viewpoint as we see later). In this case, the de Sitter solution takes the form $H_{(0)}=(-32k_2^7)^{-\frac{1}{12}}$. Again for a universe with a well-defined de Sitter expansion, k_2 must be negative. Here the stability condition (53) by using (56) and (58) takes the following form

$$\frac{210}{0.63(-k_2^7)^{\frac{1}{6}}-25.6}>0. \tag{60}$$

Thus the stability condition reduces to

$$k_2<-0.76. \tag{61}$$

Note that for the models with $n=1$, the stability condition has no dependence on the parameter k_1. Generally, to recover the teleparallel equivalent of the general relativity, k_1 should be positive.

In the upcoming section, we discuss the energy conditions in the general $F(T,\Theta)$ theories, then we specify a kind of the $F(T,\Theta)$ function in the spirit of (54) and finally we investigate whether the energy conditions can be satisfied in the context of the constraint (61) or not.

5. Energy conditions

The Raychaudhuri equation is origin of the strong and null energy conditions together with the requirement that gravity is attractive for the space time manifold that is endowed by a metric $g_{\mu\nu}$. For a congruence of timelike geodesics with tangent vector field u^μ, Raychaudhuri equation as the temporal variation of expansion θ [16] is defined as follows

$$\frac{d\theta}{d\tau}=-\frac{1}{3}\theta^2-\sigma_{\mu\nu}\sigma^{\mu\nu}+\omega_{\mu\nu}\omega^{\mu\nu}-R_{\mu\nu}u^\mu u^\nu \tag{62}$$

where θ is expansion parameter, $\sigma^{\mu\nu}$ and $\omega_{\mu\nu}$ are, respectively, shear and rotation associated with the congruence defined by the vector field u^μ. In the case of null vector field n^μ the temporal version of the expansion is given by

$$\frac{d\theta}{d\tau}=-\frac{1}{2}\theta^2-\sigma_{\mu\nu}\sigma^{\mu\nu}+\omega_{\mu\nu}\omega^{\mu\nu}-R_{\mu\nu}n^\mu n^\nu. \tag{63}$$

Note that the Raychaudhuri equation is a purely geometric equation and hence, it is not restricted to a specific theory of gravitation. Since the shear tensor is purely spatial $\sigma_{\mu\nu}\sigma^{\mu\nu}>0$, thus, for any hypersurface of orthogonal congruence ($\omega_{\mu\nu}=0$), the conditions for gravity to be attractive, become

$$\textbf{SEC}:\quad R_{\mu\nu}u^\mu u^\nu\geqslant 0, \tag{64}$$

$$\textbf{NEC}:\quad R_{\mu\nu}n^\mu n^\nu\geqslant 0. \tag{65}$$

By using the field equations of any gravitational theory, one can relate the Ricci tensor to the energy–momentum tensor $\Theta_{\mu\nu}$. Thus, the combination of the field equations and Raychaudhuri equation sets a series of physical conditions for the energy–momentum tensor. By employing (64) and (65) in the general relativity framework, one can restrict the energy–momentum tensor as follows

$$R_{\mu\nu}u^\mu u^\nu=\left(\Theta_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\Theta\right)u^\mu u^\nu\geqslant 0 \tag{66}$$

and

$$R_{\mu\nu}n^\mu n^\nu=\Theta_{\mu\nu}n^\mu n^\nu\geqslant 0 \tag{67}$$

where for a perfect fluid with density ρ and pressure p, this expression reduces to the well-known forms of the SEC and NEC in general relativity:

$$\rho+3p\geqslant 0\quad\text{and}\quad\rho+p\geqslant 0. \tag{68}$$

Note that in the inequalities (66) and (67) we have set $\kappa^2=1$. In which follows we continue to use this convention.

5.1. The energy condition in $F(T,\Theta)$ gravity

The Raychaudhuri equation together with attractor character of the gravitational interaction have led us to Eqs. (66) and (67). These relations hold for any theory of gravity. In which follows, we apply this approach to drive the strong energy condition (SEC) and null energy condition (NEC) in $F(T,\Theta)$ gravity. First we rewrite the field equations (30) as follows

$$G_{\mu\nu}=\frac{1}{F_T}\left(\Theta_{\mu\nu}-2B_{\mu\nu}F_{TT}+\frac{1}{2}[TF_T-F]g_{\mu\nu}+2F_\Theta\left[\Omega_{\mu\nu}+\frac{1}{4}\Theta_{\mu\nu}\right]\right). \tag{69}$$

From this equation and the trace of the field equations, Eq. (32), we have

$$R_{\mu\nu}=\mathcal{T}_{\mu\nu}-\frac{1}{2}\mathcal{T}g_{\mu\nu} \tag{70}$$

where

$$\mathcal{T}_{\mu\nu}=\frac{1}{F_T}\left[\Theta_{\mu\nu}-2B_{\mu\nu}F_{TT}+2F_\Theta\left(\Omega_{\mu\nu}+\frac{1}{4}\Theta_{\mu\nu}\right)\right], \tag{71}$$

$$\mathcal{T}=\frac{1}{F_T}\left[\Theta+TF_T-F-2BF_{TT}-2F_\Theta\left(\Omega-\frac{1}{4}\Theta\right)\right]. \tag{72}$$

Now in an FRW background, from Eqs. (6) and (9) along with Eq. (31), we obtain

$$A_{00}=3H^2,\qquad A_{ij}=-a^2(3H^2+\dot{H})\delta_{ij}, \tag{73}$$

$$B_{ij}=12a^2H^2\dot{H}\delta_{ij},\qquad B=-36H^2\dot{H}. \tag{74}$$

For a perfect fluid of density ρ and pressure p, $\Theta_{\mu\nu}=e_i^\alpha g_{\alpha\nu}\Theta_\mu^i$, and by taking $u_\mu=(1,0,0,0)$ and $n_\mu=(1,a,0,0)$, we obtain the $\mathcal{T}_{\mu\nu}$ and its trace $\mathcal{T}$ as follows

$$\mathcal{T}_{00}=\frac{1}{F_T}\left(1-\frac{F_\Theta}{4}\right)\rho,$$

$$\mathcal{T}_{ij}=\frac{a^2}{F_T}\left(p-24H^2\dot{H}F_{TT}+\frac{1}{2}(5p+6\rho)F_\Theta\right)\delta_{ij} \tag{75}$$

and

$$\mathcal{T}=\frac{1}{F_T}\left[\rho-3p+TF_T-F+72H^2\dot{H}F_{TT}+\frac{1}{2}F_\Theta(21\rho+9p)\right]. \tag{76}$$

Here we can use Eqs. (64) and (65) together with Eqs. (75) and (76), for a general $F(T,\Theta)$ gravity, to achieve the strong and null energy conditions respectively as

$$\textbf{SEC}: \quad \frac{1}{2F_T}\bigg(\rho + 3p + F - TF_T - 72H^2\dot{H}F_{TT} - \frac{1}{2}(23\rho + 9p)F_\Theta\bigg) \geqslant 0, \tag{77}$$

$$\textbf{NEC}: \quad \frac{1}{F_T}\bigg(\rho + p - 24H^2\dot{H}F_{TT} + \frac{3}{2}(\rho + p)F_\Theta\bigg) \geqslant 0. \tag{78}$$

As one may expect, the energy conditions in the general relativity framework *i.e.* Eq. (68), can be recovered as a particular case of SEC and NEC in the context of $F(T,\Theta)$ gravity if we set $F(T,\Theta) = T$.

By defining an effective energy–momentum tensor in the context of $F(T,\Theta)$ gravity, SEC and NEC can also be re-casted in the form of that of GR ($\rho^{eff} + 3p^{eff} \geqslant 0$ and $\rho^{eff} + p^{eff} \geqslant 0$, respectively). In this respect we can drive weak energy condition (WEC) and dominant energy condition (DEC). The effective energy–momentum tensor in the framework of $F(T,\Theta)$ gravity is defined as follows (similarly to $F(T)$ gravity in Ref. [14])

$$\Theta^{eff}_{\mu\nu} = \frac{1}{F_T}\bigg[\Theta_{\mu\nu} - 2B_{\mu\nu}F_{TT} + \frac{1}{2}(TF_T - F)g_{\mu\nu} + 2F_\Theta\bigg(\Omega_{\mu\nu} + \frac{1}{4}\Theta_{\mu\nu}\bigg)\bigg] \tag{79}$$

ρ^{eff} and p^{eff} can be derived via the effective energy–momentum tensor by the following definitions

$$\rho^{eff} = g^{00}\Theta^{eff}_{00}, \qquad p^{eff} = -\frac{1}{3}g^{ij}\Theta^{eff}_{ij}. \tag{80}$$

Thus, using the effective energy–momentum tensor approach, the weak energy condition (WEC) in $F(T,\Theta)$ gravity ($\rho^{eff} \geqslant 0$) is written as

$$\textbf{WEC}: \quad \frac{1}{F_T}\bigg[\rho + \frac{1}{2}(TF_T - F) - \frac{1}{2}\rho F_\Theta\bigg] \geqslant 0. \tag{81}$$

Finally one can write the dominant energy condition ($\rho^{eff} > |p^{eff}| \geqslant 0$) as follows

$$\textbf{DEC}: \quad \frac{1}{F_T}\bigg[\rho - p + (TF_T - F) + 24H^2\dot{H}F_{TT} - \frac{1}{2}(7\rho + 5p)F_\Theta\bigg] \geqslant 0. \tag{82}$$

In the next section, we test one of the $F(T,\Theta)$ models in the context of the energy conditions we derived. In this way we obtain a constraint on the parametric space of the model.

5.2. Constraining $F(T,\Theta)$ models from energy conditions

We firstly list the energy conditions in terms of the phenomenological parameter of deceleration $q = -\frac{\ddot{a}}{a}H^{-2} = -(1 + \frac{\dot{H}}{H^2})$. The positivity of the Newtonian gravitational constant requires also the constraint $F_T > 0$. With these notifications, the energy conditions are rewritten as follows

$$\textbf{WEC}: \quad 2\rho_0 + T_0F_{T_0} - F_0 - \rho_0F_{\Theta_0} \geqslant 0, \tag{83}$$

$$\textbf{NEC}: \quad \rho_0 + p_0 + 24H_0^4(1 + q_0)F_{T_0T_0} + \frac{3}{2}(\rho_0 + p_0)F_{\Theta_0} \geqslant 0, \tag{84}$$

$$\textbf{SEC}: \quad \rho_0 + 3p_0 + F_0 - T_0F_{T_0} + 72H_0^4(1 + q_0)F_{T_0T_0} - \frac{1}{2}(23\rho_0 + 9p_0)F_{\Theta_0} \geqslant 0, \tag{85}$$

$$\textbf{DEC}: \quad \rho_0 - p_0 + T_0F_{T_0} - F_0 - 24H_0^4(1 + q_0)F_{T_0T_0} - \frac{1}{2}(7\rho_0 + 5p_0)F_{\Theta_0} \geqslant 0. \tag{86}$$

We note that all the above conditions depend on the present value of pressure p_0, so for simplicity we assume $p_0 = 0$.

Then we should adopt a specific function for $F(T,\Theta)$ to obtain the constraints on the parametric space of the considered model from the point of view of the energy conditions. On the other hand, we know that in order for a theoretical model to be cosmologically viable, it should satisfy at least the weak energy condition. This leads us to the mentioned constraints on parametric space of the model. Here we again consider $F(T,\Theta) = k_1T + k_2T^m\Theta^n$ as our background gravitational model. The weak energy condition together with Eqs. (55) and (56) is satisfied by

$$\frac{m-1}{1-2m}k_1 \leqslant \Omega_{m_0}\bigg(1 + \frac{n}{12m + 5n - 12}\bigg). \tag{87}$$

By restricting the parameter m values, one can constrain the parameter k_1. Also to recover the teleparallel equivalent of the general relativity, k_1 should be positive. Now one can obtain three ranges for m as $\frac{1}{2} < m < 1$, $m > 1$ and $m < \frac{1}{2}$ in which the constraint (87) is rewritten as follows:

1. The case with $m < \frac{1}{2}$ and $1 < m$:

$$k_1 \geqslant \frac{1-2m}{m-1}\Omega_{m_0}\bigg(1 + \frac{n}{12m + 5n - 12}\bigg). \tag{88}$$

Here by considering the condition for recovery of general relativity, that is, $k_1 > 0$, we are led to the other constraint on the parameter n and m as

$$5 \leqslant \frac{5n}{2(1-m)} < 6.$$

2. The case with $\frac{1}{2} < m < 1$:

$$k_1 \leqslant \frac{1-2m}{m-1}\Omega_{m_0}\bigg(1 + \frac{n}{12m + 5n - 12}\bigg). \tag{89}$$

Now the mentioned condition, $k_1 > 0$, leads us to the following constraint:

$$n < 2(1-m), \qquad n > 2.4(1-m).$$

So, in the model with $m = \frac{2}{3}$ and $n = 1$ which is considered in Section 4, the weak energy condition can be realized with condition $k_1 \leqslant 2\Omega_{m_0}$, but the de Sitter solution is unstable in this case. Nevertheless, we could find the suitable values for m and n for which the weak energy condition can be realized as well as the stable de Sitter solution. This can be done if we set $m = \frac{6}{7}$ and $n = 1$ for instance. In this case the weak energy condition holds if $k_1 \leqslant \frac{150}{23}\Omega_{m_0}$ and the de Sitter phase is stable.

Also one can investigate consistency of the null energy condition in the de Sitter phase. Note that one should set the value of $q_0 = -1$ for the de Sitter phase, so that the coefficient of the term $F_{T_0T_0}$ in Eq. (85) vanishes. The NEC in the de Sitter phase imposes a constraint on the parameters m and n as follows

$$\frac{n}{12(m-1) + 5n} \leqslant \frac{1}{3}. \tag{90}$$

On the other hand, the constraints of WEC on the m and n (along with the positivity of k_1) which have already been mentioned, can be used to obtain a more restricted ranges of the parameters m and n. For example, in the second case in which $\frac{1}{2} < m < 1$, the NEC in the de Sitter solution imposes the following constraints

$$n < 2(1-m), \qquad n > 6(1-m). \tag{91}$$

So, in the model with $m = \frac{2}{3}$ and $n = 1$, in spite of the realization of the WEC with condition $k_1 \leqslant 2\Omega_{m_0}$, the null energy condition cannot be satisfied. While in the model with $m = \frac{6}{7}$ and $n = 1$ in the de Sitter phase, both of the WEC (with $k_1 \leqslant \frac{150}{23}\Omega_{m_0}$) and NEC are realized as well as the stable de Sitter solution. Thus, the later model is cosmologically viable.

6. Cosmography and crossing the phantom divide

In this section, we check our $F(T,\Theta)$ scenario by cosmographical considerations and compare the results with the case of $F(R,\Theta)$. Also we check the evolution of the equation of state parameter in this framework. The equation of state parameter of the dark energy ω_{DE} is given by Eq. (23) for a general $F(T,\Theta)$. Albeit, to obtain an explicit expression for $\omega_{DE}(z)$, we need the exact solution of the field equations to determine $H(z)$. Generally this is not an easy task and usually for modified gravity theories it is better to use the *cosmography* approach to obtain an explicit relation between ω_{DE} and redshift z. The paradigm of cosmography follows the cosmological principle and proposes that the scale factor can be expanded in terms of a Taylor series around the present time t_0, yielding [17]

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) - \frac{q_0}{2}H_0^2(t-t_0)^2 + \frac{j_0}{3!}H_0^3(t-t_0)^3 + \frac{s_0}{4!}H_0^4(t-t_0)^4 + \frac{l_0}{5!}H_0^5(t-t_0)^5 + \mathcal{O}(t-t_0)^6. \qquad (92)$$

The coefficients of the power series in this expansion are known as the cosmographic parameters: H, q, j, s and l which are named Hubble, deceleration, jerk and snap parameters, respectively. However, so far, no universal name is attributed to the cosmographic parameter l. A similar expansion can be applied also for the derivatives of the scale factor. In which follows, we expand each of the cosmographic parameters around the present day value [18]. Then, we substitute the obtained expressions in Eqs. (16) and (54) with $m = \frac{6}{7}$ and $n = 1$. By substituting the results in Eq. (23) and using the relation $\dot{H} = -H^2(1+q)$ [19], we obtain ω_{DE} as a function of redshift for a specific model of $F(T,\Theta)$ gravity with $m = \frac{6}{7}$ and $n = 1$. To assist the present theories, we compare the results of cosmographic analysis for this type of $F(T,\Theta)$ with the corresponding $F(R,\Theta)$ gravity of the type $F(R,\Theta) = k_1R + k_2R^m\Theta^n$ with $m = \frac{6}{7}$ and $n = 1$. Using the relation $\omega_{DE} = -1 - \frac{\dot{\rho}_{DE}}{3H\rho_{DE}}$ and the definition of ρ_{DE} in $F(R,\Theta)$ theories (see for instance [20]) as

$$\rho_{DE} = F_\Theta \rho^{(M)} + 3H^2(1-F_R) - 3H\dot{F}_R - \frac{1}{2}(F - RF_R), \qquad (93)$$

we obtain the expression for ω_{DE} for general $F(R,\Theta)$ theories. Now the curvature scalar R and its derivatives can be rewritten in terms of cosmographic parameters [18]. So we can express the EoS parameter of the dark energy of $F(R,\Theta) = k_1R + k_2R^{\frac{6}{7}}\Theta$ theory as a function of redshift. The expressions for ω_{DE} in $F(T,\Theta)$ and $F(R,\Theta)$ models can be obtained by simple algebra in terms of redshift and present day values of the cosmographic parameters. These relations are very lengthy and for the sake of economy we don't present them here. In Fig. 1, the EoS parameters of $F(T,\Theta) = k_1T + k_2T^{\frac{6}{7}}\Theta$ and $F(R,\Theta) = k_1R + k_2R^{\frac{6}{7}}\Theta$ models versus the redshift z are plotted. In this figure the present day values of cosmographic parameters H_0, q_0, j_0, s_0 and l_0 are taken from Ref. [18]. Based on the observational data, crossing of the phantom divide ($\omega_{DE} = -1$) occurs at $z \simeq 0.68$. Also this crossing proceeds from quintessence phase ($\omega_{DE} > -1$) towards the phantom phase ($\omega_{DE} < -1$). As our analysis shows, in both $F(T,\Theta) = k_1T + k_2T^{\frac{6}{7}}\Theta$ and $F(R,\Theta) = k_1R + k_2R^{\frac{6}{7}}\Theta$, crossing the phantom divide line occurs at $z \simeq 0.68$. However, for $F(T,\Theta)$ the equation of state parameter proceeds in observationally viable direction.

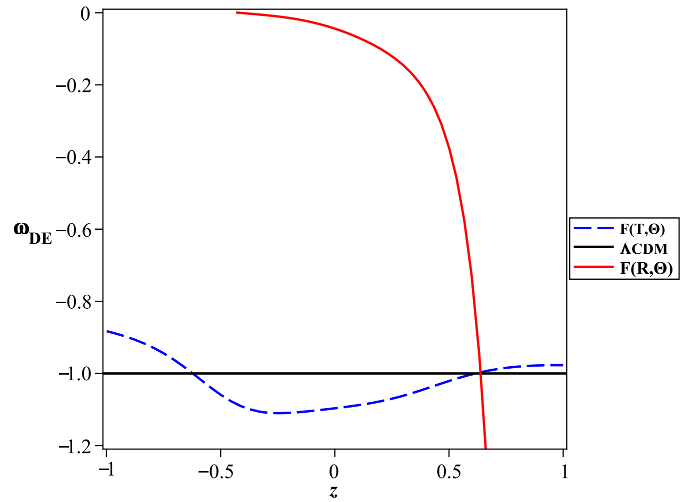

Fig. 1. The EoS parameter versus the redshift for two model $F(T,\Theta) = k_1T + k_2T^m\Theta^n$ and $F(R,\Theta) = k_1R + k_2R^m\Theta^n$ with $m = \frac{6}{7}$ and $n = 1$.

7. Conclusion

In this work we discussed the cosmological viability of an alternative gravitational theory, namely, the modified $F(T,\Theta)$ gravity, where T is the Torsion scalar and Θ is the trace of the energy–momentum tensor. The viability of the model is based on the existence of a stable de Sitter solution and the realization of all the energy conditions or at least some of them. In a perturbational approach, we have obtained a differential equation for $\delta(t)$. As a special case, we analyzed the differential equation for the de Sitter solution and we obtained a condition for the stability of this solution. Then we focused on the case where the algebraic function $F(T,\Theta)$ is cast into $F(T,\Theta) = k_1T + k_2T^m\Theta^n$, where k_1, k_2, m and n are input parameters. We firstly adopted the case with $m = \frac{2}{3}$ and $n = 1$ and we have shown that the perturbations in the model grow with time exponentially. Then we considered the other case with $m = \frac{6}{7}$ and $n = 1$. This model realizes a stable de Sitter phase with the condition $k_2 < -0.76$. Note that for simplicity we have adopted the value $n = 1$, because by this choice one gets rid of the dependence of the stability condition on the parameter k_1. Finally we investigated the energy conditions in the $F(T,\Theta)$ models. We focused on the fact that WEC is the main condition for the cosmological viability of the theory to obtain a constraint on the parameters m, n and k_1. Then by assuming that the parameter k_1 should be positive to recover the teleparallel equivalent of the general relativity, we achieved the more restricted parametric space for m and n. In the next step, the adopted values for m and n (in the stability discussion) are applied. We have shown that the case with $m = \frac{2}{3}$, $n = 1$ and the other case with $m = \frac{6}{7}$ and $n = 1$ can realize the WEC along with $k_1 < 2\Omega_{m_0}$ and $k_1 \leqslant \frac{150}{23}\Omega_{m_0}$, respectively. We considered the cosmological viability of the model from the point of view of the NEC. Since the purpose of our study was the comparison of the energy conditions with the stability of the de Sitter phase, we considered NEC at $q = -1$ (in the de Sitter solution). Here we obtained the more complete constraint on the m and n, so that it entails both WEC and NEC. As we saw, the case with $m = \frac{6}{7}$ and $n = 1$ realizes NEC too and is cosmologically a viable gravitational theory. In the last step, we treated the cosmography of the model presented in this Letter and compared its observational outcome with $F(R,\Theta)$ model with the same parameter. As an important result, crossing of the phantom divide by equation of state parameter in $F(T,\Theta)$ gravity proceeds in observationally viable direction.

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