ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Computational and Applied Mathematics 181 (2005) 24–45
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Accurate spectral solutions for the parabolic and elliptic partial differential equations by the ultraspherical tau method

E.H. Doha*, W.M. Abd-Elhameed

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

Received 27 October 2003

Abstract

We present a double ultraspherical spectral methods that allow the efficient approximate solution for the parabolic partial differential equations in a square subject to the most general inhomogeneous mixed boundary conditions. The differential equations with their boundary and initial conditions are reduced to systems of ordinary differential equations for the time-dependent expansion coefficients. These systems are greatly simplified by using tensor matrix algebra, and are solved by using the step-by-step method. Numerical applications of how to use these methods are described. Numerical results obtained compare favorably with those of the analytical solutions. Accurate double ultraspherical spectral approximations for Poisson's and Helmholtz's equations are also noted. Numerical experiments show that spectral approximation based on Chebyshev polynomials of the first kind is not always better than others based on ultraspherical polynomials.

MSC: 65M70; 65N35; 35C10; 42C10

Keywords: Parabolic and elliptic partial differential equations; Spectral methods; Orthogonal polynomials; Expansion coefficients

1. Introduction

The problem of approximating solutions of ordinary or partial differential equations by spectral methods, known as Galerkin approximation, involves the projection onto the span of some appropriate set

* Corresponding author. Tel.: +20 330 2329; fax: +20 572 7556.
E-mail address: eiddoha@frcu.eun.eg (E.H. Doha).

0377-0427/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2004.11.015

of basis functions, typically arising as the eigenfunctions of a singular Sturm–Liouville problem. The members of the basis may satisfy automatically the auxiliary conditions imposed on the problem, such as initial, boundary or more general conditions. Alternatively, these conditions may be imposed as constraints on the expansion coefficients, as in Lanczos τ-method, Lanczos [21].

It is well-known, Canuto et al. [3], that the eigenfunctions of certain singular-Sturm–Liouville problems allow the approximation of functions of $C^{\infty}[a, b]$ whose truncation error approaches zero faster than any negative power of the number of basis functions (retained modes) used in the approximation, as that number (order of truncation N) tends to ∞. This phenomenon is usually referred to as “spectral accuracy”, Gottlieb and Orszag [15].

It is of fundamental importance to know that the choice of the basis functions is responsible for the superior approximation properties of spectral methods when compared with finite difference and finite element methods.

Spectral methods provide a computational approach which achieved substantial popularity over the last three decades. They have gained new popularity in automatic computations for a wide class of physical problems in the fluid and heat flow. They have also proven highly efficient for time-dependent smooth flows in simple geometries, see Orszag [26] and Gottlieb and Orszag [15]. The principal advantage of spectral methods lies in their ability to achieve accurate results with substantially fewer degrees of freedom.

For spectral and pseudospectral methods, explicit expressions for the expansion coefficients of the solution are needed. Karageorghis [19], obtained an expression when the basis functions of expansion are shifted Chebyshev polynomials $T_n^*(x), x \in [0, 1]$. A corresponding formula for Legendre polynomials $P_n(x), x \in [-1, 1]$, is derived in [27]. Doha [7], has obtained a more general formula when the basis functions are the ultraspherical polynomials $C_n^{(\alpha)}(x), x \in [-1, 1], \alpha \in (-\frac{1}{2}, \infty)$; formulae for first and second kinds Chebyshev polynomials and Legendre polynomials $T_n(x), U_n(x)$ and $P_n(x)$ are given as special cases of $C_n^{(\alpha)}(x)$.

Spectral methods based on double Chebyshev polynomials for solving numerically partial differential equations have been used by many authors, among them, Dew and Scraton [4]; Doha [5,6]; Haidvogel and Zang [17] and Horner [18]. The existence and use of Fast Fourier Transform for Chebyshev polynomials to compute efficiently the matrix–vector product has made them more widely used than other sets of orthogonal polynomials, e.g., Legendre or ultraspherical polynomials. Streett et al. [29], show that in some cases it is faster to use the usual matrix–vector multiplications than to resort to other transform techniques.

For parabolic and hyperbolic partial differential equations employing an explicit finite-difference scheme in time and a spectral representation in space, Gottlieb and Orszag [15] have observed that the restriction on the time step for stability in Chebyshev methods is of $\mathrm{O}(1/N^2)$, where N is the number of retained modes in the representation. Tal-Ezer [31], shows that an improved stability condition of $\mathrm{O}(1/N)$ is obtained when Legendre polynomials are used.

In some other applications like the resolution of thin boundary layers, an expansion in Legendre polynomials may be appropriate, because such expansion gives an exceedingly good representation for functions that undergo rapid changes in narrow boundary layers, cf. Gottlieb and Orszag [15]. It also worthy to mention that Chebyshev- and Legendre-spectral methods are extremely sensitive to the proper formulation of boundary conditions. When proper boundary conditions are imposed so that the problem is well-posed, the methods yield very accurate results, when improper boundary conditions are applied, the methods are likely to be explosively unstable. An example is given by Gottlieb and Orszag [15], from which they conclude that while the Chebyshev-spectral method is unbounded and algebraically unstable, the Legendre-spectral is semi-bounded and stable. This motivated our interest in ultraspherical polynomials, because it includes Chebyshev and Legendre polynomials and some other polynomials as subclasses of it.

For solving high-order partial differential equations the ultraspherical coefficients of high-order partial derivatives of infinitely differentiable functions are required. Explicit expressions relating the ultraspherical coefficients of a general order partial derivatives of infinitely differentiable function in two and three variables in terms of the ultraspherical coefficient of the function itself are given in [10]. Similar expressions for the cases of double and triple Chebyshev and Legendre polynomials are considered in [8,9] respectively.

It is worthy to mention here that an accurate double Chebyshev and double Legendre spectral approximations for parabolic and elliptic partial differential equations have been developed in [11,12], respectively, but for the reasons previously discussed, we have to consider the ultra-spherical polynomials.

Our main aim in present paper is to reduce a parabolic partial differential equation in two space variables with its most general inhomogeneous mixed boundary and initial conditions to a system of ordinary differential equations in the spectral ultraspherical expansion coefficients, by using two alternative methods of solution; and to explain how tensor matrix algebra can be used to solve such system of differential equations.

We concentrate here on the spectral solutions of parabolic partial differential equations models, because we know that some problems in fluid dynamics governed by the Navier–Stokes equations, can be reduced in many cases and under certain physical conditions to such models, see for instance, Schlichting [28].

The paper is organized as follows. In Section 2, we state without proof a theorem from Doha [10]; this relates the expansion coefficients of the partial derivatives of a function of two variables in terms of its original ultraspherical expansion coefficients. In Section 3, we develop a spectral method based on expansion in doubly ultraspherical polynomials for solving parabolic partial differential equation in two space variables in a square subject to the most general inhomogeneous mixed boundary conditions. An alternative method of solution, based on the explicit formulae given in Section 2, is described in Section 4. In Section 5, two accurate double ultraspherical spectral approximation for solving Poisson's equation in a square are obtained directly from those given in Sections 3 and 4, without doing extra more analysis. Numerical results and comparisons are discussed in Section 6.

2. Explicit expressions for the ultraspherical coefficients of the derivatives

The ultraspherical polynomials associated with the real parameter ($\alpha > -\frac{1}{2}$) are a sequence of polynomials $\{C_n^{(\alpha)}(x), n = 0, 1, 2, \ldots\}$, each respectively of order n, satisfying the orthogonality relation

$$\int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} C_m^{(\alpha)}(x) C_n^{(\alpha)}(x)\, \mathrm{d}x = 0, \quad (m \neq n).$$

It is worthy to mention that many properties of ultraspherical polynomials may be found in [1,30], but for our present purposes it is convenient to standardize them so that $C_n^{(\alpha)}(1)=1$, $n=0, 1, 2, \ldots$. This is not the usual standardization, but has the desirable properties that $C_n^{(0)}(x)$ are identical with the Chebyshev polynomials of the first kind $T_n(x)$, $C_n^{(1/2)}(x)$ are the Legendre polynomials $P_n(x)$, and $C_n^{(1)}(x)$ are equal to $(1/(n+1))U_n(x)$, where $U_n(x)$ are the Chebyshev polynomials of the second kind. In this form the ultraspherical polynomials may be generated by using the recurrence relation

$$(n+2\alpha)C_{n+1}^{(\alpha)}(x)=2(n+\alpha)xC_n^{(\alpha)}(x)-nC_{n-1}^{(\alpha)}(x), \quad n=1,2,3,\ldots, \tag{1}$$

starting from $C_0^{(\alpha)}(x)=1$ and $C_1^{(\alpha)}(x)=x$, or obtained from the Rodrigue's formula

$$C_n^{(\alpha)}(x)=\left(-\frac{1}{2}\right)^n\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+\alpha+\frac{1}{2})}(1-x^2)^{(1/2)-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x^2)^{n+\alpha-(1/2)}].$$

Let $u(x,y)$ be a continuous function defined on the square $S[-1\leqslant x, y\leqslant 1]$, and let it has continuous and bounded partial derivatives of any order with respect to its variables x and y. Then it is possible to express

$$u(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y), \tag{2}$$

$$u^{(p,q)}(x,y)=\frac{\partial^{p+q}u(x,y)}{\partial x^p\partial y^q}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}^{(p,q)}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y), \tag{3}$$

where $a_{mn}^{(p,q)}$ denote the ultraspherical expansion coefficients of $u^{(p,q)}(x,y)$ and $a_{mn}^{(0,0)}=a_{mn}$.

Theorem.

$$\frac{(m+2\alpha-1)}{2m(m+\alpha-1)}a_{m-1,n}^{(p,q)}-\frac{(m+1)}{2(m+\alpha+1)(m+2\alpha)}a_{m+1,n}^{(p,q)}=a_{mn}^{(p-1,q)}, \quad m,p\geqslant 1, \tag{4}$$

$$\frac{(n+2\alpha-1)}{2n(n+\alpha-1)}a_{m,n-1}^{(p,q)}-\frac{(n+1)}{2(n+\alpha+1)(n+2\alpha)}a_{m,n+1}^{(p,q)}=a_{mn}^{(p,q-1)}, \quad n,q\geqslant 1, \tag{5}$$

$$\begin{aligned}a_{mn}^{(p,q)}={}&\frac{2^p(m+\alpha)\Gamma(m+2\alpha)}{(p-1)!m!}\sum_{i=1}^{\infty}\frac{(i+p-2)!\Gamma(m+i+p+\alpha-1)(m+2i+p-2)!}{(i-1)!\Gamma(m+i+\alpha)\Gamma(m+2i+p+2\alpha-2)}\\&\times a_{m+2i+p-2,n}^{(0,q)}, \quad p\geqslant 1,\end{aligned} \tag{6}$$

$$\begin{aligned}a_{mn}^{(p,q)}={}&\frac{2^q(n+\alpha)\Gamma(n+2\alpha)}{(q-1)!n!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\Gamma(n+j+q+\alpha-1)(n+2j+q-2)!}{(j-1)!\Gamma(n+j+\alpha)\Gamma(n+2j+q+2\alpha-2)}\\&\times a_{m,n+2j+q-2}^{(p,0)}, \quad q\geqslant 1,\end{aligned} \tag{7}$$

$$\begin{aligned}a_{mn}^{(p,q)}={}&\frac{2^{p+q}(m+\alpha)(n+\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{(p-1)!(q-1)!m!n!}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(j+q-2)!}{(i-1)!(j-1)!}\\&\times\frac{\Gamma(m+i+p+\alpha-1)\Gamma(n+j+q+\alpha-1)(m+2i+p-2)!(n+2j+q-2)!}{\Gamma(m+i+\alpha)\Gamma(n+j+\alpha)\Gamma(m+2i+p+2\alpha-2)\Gamma(n+2j+q+2\alpha-2)}\\&\times a_{m+2i+p-2,n+2j+q-2}, \quad p,q\geqslant 1.\end{aligned} \tag{8}$$

For proof of this theorem, see, Doha [10].

3. Statement of the problem and derivation of method of solution

In this section, we develop two alternative approximate methods based on an expansion in ultraspherical polynomials for solving numerically parabolic partial differential equations in two space variables, in addition to time variable, namely

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (-1 \leqslant x, y \leqslant 1), t \in [0, \infty), \tag{9}$$

subject to the most general inhomogeneous mixed boundary conditions

$$\left.\begin{array}{ll} \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial x} = \gamma_1(y,t), & x=-1 \\ \alpha_2 u + \beta_2 \dfrac{\partial u}{\partial x} = \gamma_2(y,t), & x=1 \end{array}\right\} -1 \leqslant y \leqslant 1, \tag{10}$$

$$\left.\begin{array}{ll} \alpha_3 u + \beta_3 \dfrac{\partial u}{\partial y} = \gamma_3(x,t), & y=-1 \\ \alpha_4 u + \beta_4 \dfrac{\partial u}{\partial y} = \gamma_4(x,t), & y=1 \end{array}\right\} -1 \leqslant x \leqslant 1 \tag{11}$$

and the initial condition

$$u(x, y, 0) = f(x, y), \quad x, y \in [-1, 1]. \tag{12}$$

It is assumed that the solution of the above problem can be expressed in a uniformly convergent double ultraspherical series expansion

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}(t) C_m^{(\alpha)}(x) C_n^{(\alpha)}(y). \tag{13}$$

Throughout this paper we assume that the function $f(x, y)$ satisfies the boundary conditions (10) and (11) to make sure that the solution of (9) is free of discontinuities. We also assume that $f(x, y)$ has a series expansion of the form

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y), \tag{14}$$

which is uniformly convergent in $-1 \leqslant x, y \leqslant 1$. It then follows that the solution of (9) has a double series expansion of form (13), and the solution is free of discontinuities. The case in which discontinuities are present at the vertices $(\pm 1, \pm 1)$, can often be treated by a method similar to that described in [20].

3.1. The first method of solution

If we satisfy the differential equation (9), we get

$$a_{mn}^{(2,0)}(t) + a_{mn}^{(0,2)}(t) = a'_{mn}(t), \tag{15}$$

where $a'_{mn}(t)$ denote the derivatives of $a_{mn}(t)$ with respect to t. Now application of relation (4) on (15) twice, gives

$$a_{mn}(t)+\sum_{i=0}^{\infty} A_{mi} a_{in}^{(0,2)}(t)=\sum_{i=0}^{\infty} A_{mi} a'_{in}(t), \quad m \geqslant 2, n \geqslant 0, \tag{16}$$

where

$$A_{mi}=\begin{cases}\dfrac{(m+2\alpha-1)(m+2\alpha-2)}{4m(m-1)(m+\alpha-1)(m+\alpha-2)}, & i=m-2,\\ \dfrac{-1}{2(m+\alpha-1)(m+\alpha-2)}, & i=m,\\ \dfrac{(m+1)(m+2)}{4(m+\alpha+1)(m+\alpha+2)(m+2\alpha)(m+2\alpha+1)}, & i=m+2,\\ 0 & \text{otherwise}.\end{cases} \tag{17}$$

Again application of relation (5) on (16), yields the following result:

$$\sum_{i=0}^{\infty} A_{mi} a_{in}(t)+\sum_{j=0}^{\infty} a_{mj}(t) B_{jn}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{mi} a'_{ij}(t) B_{jn}, \quad m, n \geqslant 2, \tag{18}$$

where $B_{ij}=A_{ji}$.

If we assume that $\gamma_i(y,t)$, $(i=1,2)$ and $\gamma_i(x,t)$, $(i=3,4)$, have the following ultraspherical expansions:

$$\gamma_i(y,t)=\sum_{n=0}^{\infty} \gamma_n^{(i)}(t) C_n^{(\alpha)}(y), \quad i=1,2, \tag{19}$$

$$\gamma_i(x,t)=\sum_{m=0}^{\infty} \gamma_m^{(i)}(t) C_m^{(\alpha)}(x), \quad i=3,4, \tag{20}$$

then the boundary conditions (10) and (11) give

$$\left.\begin{array}{l}\displaystyle\sum_{m=0}^{\infty}(-1)^m\left[\alpha_1-\frac{m(m+2\alpha)}{2\alpha+1}\beta_1\right] a_{mn}(t)=\gamma_n^{(1)}(t)\\ \displaystyle\sum_{m=0}^{\infty}\left[\alpha_2+\frac{m(m+2\alpha)}{2\alpha+1}\beta_2\right] a_{mn}(t)=\gamma_n^{(2)}(t)\end{array}\right], \quad n=0,1,2,\ldots, \tag{21}$$

$$\left.\begin{array}{l}\displaystyle\sum_{n=0}^{\infty}(-1)^n\left[\alpha_3-\frac{n(n+2\alpha)}{2\alpha+1}\beta_3\right] a_{mn}(t)=\gamma_m^{(3)}(t)\\ \displaystyle\sum_{n=0}^{\infty}\left[\alpha_4+\frac{n(n+2\alpha)}{2\alpha+1}\beta_4\right] a_{mn}(t)=\gamma_m^{(4)}(t)\end{array}\right], \quad m=0,1,2,\ldots\ . \tag{22}$$

It is not difficult to show that Eqs. (21) and (22), may be put in the form

$$\left.\begin{array}{l}\displaystyle a_{0n}(t)+\sum_{m=2}^{\infty}\mu_m a_{mn}(t)=g_n(t)\\ \displaystyle a_{1n}(t)+\sum_{m=2}^{\infty}v_m a_{mn}(t)=h_n(t)\end{array}\right], \quad n=0,1,2,\ldots, \tag{23}$$

$$\left.\begin{aligned} a_{m0}(t) + \sum_{n=2}^{\infty} U_n a_{mn}(t) = k_m(t) \\ a_{m1}(t) + \sum_{n=2}^{\infty} V_n a_{mn}(t) = \ell_m(t) \end{aligned}\right], \quad m = 0, 1, 2, \ldots, \tag{24}$$

where

$$\mu_m = \left\{(\alpha_1 - \beta_1)\left[\alpha_2 + \frac{m(m+2\alpha)}{2\alpha+1}\,\beta_2\right] + (-1)^m\,(\alpha_2+\beta_2)\left[\alpha_1 - \frac{m(m+2\alpha)}{2\alpha+1}\,\beta_1\right]\right\} \Big/ \delta_1,$$

$$\nu_m = \left\{\alpha_1\alpha_2(1-(-1)^m) + \frac{m(m+2\alpha)}{2\alpha+1}\,(\alpha_1\beta_2 + (-1)^m\alpha_2\beta_1)\right\} \Big/ \delta_1,$$

$$U_n = \left\{(\alpha_3 - \beta_3)\left[\alpha_4 + \frac{n(n+2\alpha)}{2\alpha+1}\,\beta_4\right] + (-1)^n\,(\alpha_4+\beta_4)\left[\alpha_3 - \frac{n(n+2\alpha)}{2\alpha+1}\,\beta_3\right]\right\} \Big/ \delta_2,$$

$$V_n = \left\{\alpha_3\alpha_4(1-(-1)^n) + \frac{n(n+2\alpha)}{2\alpha+1}\,(\alpha_3\beta_4 + (-1)^n\alpha_4\beta_3)\right\} \Big/ \delta_2,$$

$$g_n(t) = [(\alpha_2+\beta_2)\gamma_n^{(1)}(t) + (\alpha_1-\beta_1)\gamma_n^{(2)}(t)]/\delta_1,$$

$$h_n(t) = [\alpha_1\gamma_n^{(2)}(t) - \alpha_2\gamma_n^{(1)}(t)]/\delta_1,$$

$$k_m(t) = [(\alpha_4+\beta_4)\gamma_m^{(3)}(t) + (\alpha_3-\beta_3)\gamma_m^{(4)}(t)]/\delta_2,$$

$$\ell_m(t) = [\alpha_3\gamma_m^{(4)}(t) - \alpha_4\gamma_m^{(3)}(t)]/\delta_2,$$

$$\delta_1 = 2\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0, \quad \delta_2 = 2\alpha_3\alpha_4 + \alpha_3\beta_4 - \alpha_4\beta_3 \neq 0.$$

It is worthy to note here that the boundary conditions (23) and (24) are not all linearly independent. Clearly, four linear relations exist among them.

Eqs. (23) and (24), after some more manipulation, will have the forms

$$a_{00}(t) = g_0(t) - \sum_{m=2}^{\infty} \mu_m k_m(t) + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \mu_m U_n a_{mn}(t), \tag{25}$$

$$a_{01}(t) = g_1(t) - \sum_{m=2}^{\infty} \mu_m \ell_m(t) + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \mu_m V_n a_{mn}(t), \tag{26}$$

$$a_{10}(t) = h_0(t) - \sum_{m=2}^{\infty} \nu_m k_m(t) + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \nu_m U_n a_{mn}(t), \tag{27}$$

$$a_{11}(t) = h_1(t) - \sum_{m=2}^{\infty} \nu_m \ell_m(t) + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \nu_m V_n a_{mn}(t). \tag{28}$$

Elimination of $a_{0n}(t), a_{1n}(t), a_{m0}(t), a_{m1}(t)$ from the L.H.S of Eq. (18)—by making use of (23) and (24)—gives

$$A_{m0}g_n(t) + A_{m1}h_n(t) + B_{0n}k_m(t) + B_{1n}\ell_m(t) + \sum_{i=2}^{\infty}(A_{mi} - \mu_i A_{m0} - v_i A_{m1})a_{in}(t)$$
$$+ \sum_{j=2}^{\infty}(B_{jn} - U_j B_{0n} - V_j B_{1n})a_{mj}(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{mi}a'_{ij}(t)B_{jn}, \quad m, n \geqslant 2,$$

which may be written in the form

$$\sum_{i=2}^{\infty} C_{mi}a_{in}(t) + \sum_{j=2}^{\infty} a_{mj}(t)D_{jn} + b_{mn}(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{mi}a'_{ij}(t)B_{jn}, \quad m, n \geqslant 2, \tag{29}$$

where

$$C_{mi} = A_{mi} - \mu_i A_{m0} - v_i A_{m1}; \quad D_{jn} = B_{jn} - U_j B_{0n} - V_j B_{1n},$$
$$b_{mn}(t) = A_{m0}g_n(t) + A_{m1}h_n(t) + B_{0n}k_m(t) + B_{1n}\ell_m(t). \tag{30}$$

Eqs. (21)–(28) are all true for all values of t, hence we can differentiate with respect to t. The resulting equations can be used to eliminate $a'_{00}(t), a'_{01}(t), a'_{10}(t), a'_{11}(t), a'_{m0}(t), a'_{m1}(t), a'_{0n}(t)$, and $a'_{1n}(t)$ from the R.H.S. of (29) to give

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{mi}a'_{ij}(t)B_{jn} = d_{mn}(t) + \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} C_{mi}a'_{ij}(t)D_{jn}, \tag{31}$$

where

$$d_{mn}(t) = A_{m0}[B_{0n}g'_0(t) + B_{1n}g'_1(t)] + A_{m1}[B_{0n}h'_0(t) + B_{1n}h'_1(t)]$$
$$+ \sum_{i=2}^{\infty}\{C_{mi}[B_{0n}k'_i(t) + B_{1n}\ell'_i(t)] + B_{in}[A_{m0}g'_i(t) + A_{m1}h'_i(t)]\}. \tag{32}$$

Substitution from (31) into (29) yields

$$\sum_{i=2}^{\infty} C_{mi}a_{in}(t) + \sum_{j=2}^{\infty} a_{mj}(t)D_{jn} = e_{mn}(t) + \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} C_{mi}a'_{ij}(t)D_{jn}, \quad m, n \geqslant 2, \tag{33}$$

where

$$e_{mn}(t) = d_{mn}(t) - b_{mn}(t). \tag{34}$$

It is now necessary to assume that $a_{mn}(t)$ and $a'_{mn}(t)$ are negligible for $m > M$ and $n > N$ respectively, and as a result, Eq. (33) may be written in the finite matrix form as

$$\mathbf{CA}(t) + \mathbf{A}(t)\mathbf{D} = \mathbf{E}(\mathbf{t}) + \mathbf{CA}'(t)\mathbf{D}, \tag{35}$$

where

$$\mathbf{A}(t) = [a_{ij}(t), i = 2, 3, \ldots, M;\ j = 2, 3, \ldots, N], \quad \mathbf{C} = [C_{ij}, i,\ j = 2, 3, \ldots, M],$$
$$\mathbf{D} = [D_{ij}, i,\ j = 2, 3, \ldots, N], \quad \mathbf{E}(t) = [e_{ij}(t), i = 2, 3 \ldots, M;\ j = 2, 3, \ldots, N].$$

3.2. Solution of the system of differential equations (35)

Let $\mathbf{C} \otimes \mathbf{D} = [C_{ij}\ \ \mathbf{D}], i,\ j = 2, 3, \ldots, M$, be the tensor product of the two matrices $\mathbf{C}$ and $\mathbf{D}$; $\mathbf{C} \oplus \mathbf{D} = \mathbf{C} \otimes \mathbf{I}_{N-1} + \mathbf{I}_{M-1} \otimes \mathbf{D}$ be their tensor sum, where $\mathbf{I}_{M-1}$ and $\mathbf{I}_{N-1}$ are the identity matrices of order $M-1$ and $N-1$, respectively. Introducing the so-called block vectors:

$$\underline{a}(t) \equiv [\underline{a}_2(t), \underline{a}_3(t), \ldots, \underline{a}_N(t)]^{\mathrm{T}} \quad \text{and} \quad \underline{E}(t) \equiv [\underline{E}_2(t), \underline{E}_3(t), \ldots, \underline{E}_N(t)]^{\mathrm{T}},$$

where

$$\mathbf{A}(t) \equiv [\underline{a}_2(t)\, \underline{a}_3(t) \cdots\, \underline{a}_N(t)]^{\mathrm{T}}; \quad \mathbf{E}(t) \equiv [\underline{E}_2(t)\, \underline{E}_3(t) \cdots\, \underline{E}_N(t)]^{\mathrm{T}}$$

and

$$\operatorname{vec} \mathbf{A}(t) = \begin{bmatrix} \underline{a}_2(t) \\ \underline{a}_3(t) \\ \vdots \\ \underline{a}_N(t) \end{bmatrix}.$$

Utilizing the above definitions, one can reduce the system of Eqs. (35) to the following matrix differential form:

$$\mathbf{G}\underline{a}(t) = \underline{E}(t) + \mathbf{H}\underline{a}'(t), \tag{36}$$

where

$$\mathbf{G} = [\mathbf{C} \oplus \mathbf{D}^{\mathrm{T}}] \quad \text{and} \quad \mathbf{H} = [\mathbf{D}^{\mathrm{T}} \oplus \mathbf{C}].$$

An interested reader is referred to Graham [16] and Loan [23], for more details about the kronecker matrix algebra.

Eq. (36) represents a system of nonhomogeneous linear ordinary differential equations with constant coefficients which must be solved under the initial condition:

$$a_{mn}(0) = f_{mn},\ \ m = 0, 1, \ldots, M;\ \ n = 0, 1, \ldots, N,$$

where f_{mn} are the constants given by (14).

An analytical solution of Eq. (36) is given, explicitly, in [2]

$$\underline{a}(t) = \exp(\mathbf{H}^{-1}\mathbf{G}t) \left[\underline{a}(0) - \mathbf{H}^{-1} \int_0^t \exp(\mathbf{H}^{-1}\mathbf{G}s)\underline{E}(s)\, \mathrm{d}s\right]. \tag{37}$$

If $\underline{E}(t)$ is composed of exponential or oscillatory functions, a particular integral of (36) can be obtained by elementary means, and the complementary function by

$$\underline{a}(t) = \exp(\mathbf{H}^{-1}\mathbf{G}t)\, \underline{a}(0) = \exp(\mathbf{Q}t)\underline{a}(0), \tag{38}$$

but owing to the impracticability of evaluating the exponential of a large matrix this does not give feasible method for numerical solution. Let the eigenvalues of the matrix $\mathbf{Q}$ be $(-\lambda_i, i=1, 2, \ldots, k=(M-1)(N-1))$, and the corresponding eigenvectors $(\underline{x}_i, i=1, 2, \ldots, k)$. It will be assumed that λ_i are real and positive, a condition which normally guaranteed by physical considerations (the λ_i being approximations to the eigenvalues of the partial differential equation). If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are chosen so that

$$\underline{a}(0) = \sum_{i=1}^{k} \alpha_i \underline{x}_i,$$

solution (38) can be written in the form

$$\underline{a}(t) = \sum_{i=1}^{k} \alpha_i \mathrm{e}^{-\lambda_i t} \underline{x}_i. \tag{39}$$

Fox and Parker [14] suggest obtaining numerical solution in this form, and this is probably the best approach when k is small. For large values of k the determination of the eigenvalues and eigenvectors can make this an uneconomical method of solution.

Suppose $\phi(z)$ is a rational function which approximates e^{-z}, for real positive z. An approximate solution to the matrix differential equation can then be obtained in a step-by-step manner using the formula

$$\underline{a}(t + \Delta t) = \phi(-\Delta t \mathbf{Q}) \underline{a}(t). \tag{40}$$

This produces the approximate solution

$$\underline{a}(r\Delta t) = \sum_{i=1}^{k} \alpha_i [\phi(\lambda_i \Delta t)]^r \underline{x}_i \tag{41}$$

and one hopes that with a suitable choice of ϕ and Δt this will provide an adequate approximation to the correct solution given by Eq. (39).

The simple rational approximation

$$\phi(z) = \frac{1 - \frac{1}{2}z}{1 + \frac{1}{2}z} \tag{42}$$

has been used, but oscillations in the sign of some coefficients have been noticed. The reason for this oscillatory behaviour is not hard to find. Some of the λ_i are very large, so the $\lambda_i \Delta t$ is also large, and the value of $\phi(\lambda_i \Delta t)$ given by (42) is approximately -1. The corresponding terms in Eq. (41) produce the oscillations.

We can avoid altogether the oscillatory trouble by choosing a more satisfactory form for $\phi(z)$ so that $\phi(z)$ is small when z is large. It is merely necessary to ensure that the denominator of $\phi(z)$ is of higher degree than the numerator. Stability considerations also demand that $|\phi(z)| < 1$ for all positive z. There is a range of Padé approximations to e^{-z} satisfying these criteria, see for instance [13,32]. One may use Padé approximations of higher or lower order according to the degree of accuracy required, but for the sake of illustration we propose to use the expression

$$\phi(z) = \frac{1 - \frac{2}{5}z + \frac{1}{20}z^2}{1 + \frac{3}{5}z + \frac{3}{20}z^2 + \frac{1}{60}z^3}. \tag{43}$$

The error in this approximation, for small values of z, is given by

$$\phi(z) - \mathrm{e}^{-z} = \tfrac{1}{7200} z^6 + \mathrm{O}(z^7). \tag{44}$$

Now the error of the approximate solution (41) is given by

$$\sum_{i=1}^{k} \alpha_r E_r(\lambda_i \Delta t)\underline{x}_i,$$

where

$$E_r(z) = [\phi(z)]^r - \mathrm{e}^{-rz}$$

and in order to investigate this error it is necessary to consider the behaviour of $E_r(z)$. For small values of z, it follows from Eq. (44) that

$$E_r(z) = \tfrac{1}{7200} r z^6 \mathrm{e}^{-rz} + \mathrm{O}(z^7)$$

and the maximum value of $E_r(z)$ as r increases occurs when $r \approx 1/z$ and this approximately equals to $(1/7200e)z^5$. For large values of z, the maximum value of $E_r(z)$ occurs when $r = 1$, and $E_r(z)$ decreases rapidly as r increases.

The first step in a tabulation by the method suggested above involves the formation of the matrix $\phi(-\Delta t\mathbf{Q})$. If Eq. (43) is used for $\phi(z)$, this requires the determination of the matrix

$$(\mathbf{Q}^3 - \tfrac{3}{5}\Delta t\mathbf{Q}^2 + \tfrac{3}{20}(\Delta t)^2\mathbf{Q} - \tfrac{1}{60}(\Delta t)^3\mathbf{I})^{-1}(\mathbf{Q}^3 + \tfrac{2}{5}\Delta t\mathbf{Q}^2 + \tfrac{1}{20}(\Delta t)^2\mathbf{Q}). \tag{45}$$

If Δt is small, the matrix to be inverted approximates $\mathbf{Q}^3$, and in general this is less well conditioned than the matrix $\mathbf{Q}$. It is preferable therefore to write Eq. (45) in the alternative

$$\mathbf{I} + \Delta t\left\{\mathbf{Q} - \frac{\Delta t}{2}\mathbf{I} + \frac{(\Delta t)^2}{12}\left[\mathbf{Q} + \frac{(\Delta t)^2}{60}\left(\mathbf{Q} - \frac{\Delta t}{10}\mathbf{I}\right)^{-1}\right]^{-1}\right\}^{-1}, \tag{46}$$

which requires no more arithmetic, but in which the matrices to be inverted approximate to $\mathbf{Q}$.

To summarize the method, we first find the matrix $\mathbf{Q}$; then form the matrix $\phi(-\mathbf{Q}^{-1}h)$ by means of Eq. (46); then tabulate $\underline{a}(t)$ for the required values of t, using the step-by-step process (Eq. (40)), and finally find $a_{00}(t)$, $a_{01}(t)$, $a_{10}(t)$ and $a_{11}(t)$ for each t from Eqs. (25)–(28).

4. Alternative method of solution

An alternative method of solution based basically on the explicit expressions given by (6) and (7) is presented in this section. This method leads to a system similar to (36), but with some computational advantages. Direct substitution from (6) and (7) into (15) yields

$$a'_{mn}(t) = \frac{(m+\alpha)\Gamma(m+2\alpha)}{m!} \sum_{\substack{i=m+2\\(i-m)even}}^{\infty} (i-m)(i+m+2\alpha)\,\frac{i!}{\Gamma(i+2\alpha)}\,a_{in}(t)$$
$$+ \frac{(n+\alpha)\Gamma(n+2\alpha)}{n!} \sum_{\substack{j=n+2\\(j-n)even}}^{\infty} (j-n)(j+n+2\alpha)\,\frac{j!}{\Gamma(j+2\alpha)}\,a_{mj}(t),$$

which may be written in the equivalent form

$$a'_{mn}(t) = \sum_{i=0}^{\infty} K_{mi} a_{in}(t) + \sum_{j=0}^{\infty} a_{mj}(t) L_{jn}, \quad m, n \geqslant 0, \tag{47}$$

where

$$K_{mi} = \begin{cases} \dfrac{(m+\alpha)\Gamma(m+2\alpha) i!(i-m)(i+m+2\alpha)}{m!\Gamma(i+2\alpha)}, & i \geqslant m+2,\ (i-m)\text{even},\\ 0 & \text{otherwise} \end{cases}$$

and $L_{jn} = K_{nj}$.

It is necessary here to assume that the coefficients $a_{in}(t)$ and $a_{mj}(t)$ are negligible for $m \geqslant M+1$, $n \geqslant N+1$. The boundary conditions (21) and (22) are used to eliminate the coefficients $a_{i,N-1}(t)$, $a_{iN}(t)$, $a_{M-1,j}(t)$, $a_{Mj}(t)$. After straightforward but lengthy manipulation, we get

$$\left.\begin{aligned} a_{M-1,n}(t) + \sum_{m=0}^{M-2} \mu'_m a_{mn}(t) = g'_n(t),\\ a_{Mn}(t) + \sum_{m=0}^{M-2} v'_m a_{mn}(t) = h'_n(t), \end{aligned}\right], \quad n = 0, 1, \ldots, N, \tag{48}$$

$$\left.\begin{aligned} a_{m,N-1}(t) + \sum_{n=0}^{N-2} U'_n a_{mn}(t) = k'_m(t),\\ a_{mN}(t) + \sum_{n=0}^{N-2} V'_n a_{mn}(t) = \ell'_m(t), \end{aligned}\right], \quad m = 0, 1, \ldots, M, \tag{49}$$

where

$$\mu'_m = \left\{(-1)^M \left[\alpha_1 - \frac{M(M+2\alpha)}{2\alpha+1}\beta_1\right]\left[\alpha_2 + \frac{m(m+2\alpha)}{2\alpha+1}\beta_2\right] - (-1)^m \left[\alpha_1 - \frac{m(m+2\alpha)}{2\alpha+1}\beta_1\right]\left[\alpha_2 + \frac{M(M+2\alpha)}{2\alpha+1}\beta_2\right]\right\} \Big/ \delta'_1,$$

$$v'_m = \left\{(-1)^M \left[\alpha_1 - \frac{(M-1)(M+2\alpha-1)}{2\alpha+1}\beta_1\right]\left[\alpha_2 + \frac{m(m+2\alpha)}{2\alpha+1}\beta_2\right] + (-1)^m \left[\alpha_1 - \frac{m(m+2\alpha)}{2\alpha+1}\beta_1\right]\left[\alpha_2 + \frac{(M-1)(M+2\alpha-1)}{2\alpha+1}\beta_2\right]\right\} \Big/ \delta'_1,$$

$$U'_n = \left\{(-1)^N \left[\alpha_3 - \frac{N(N+2\alpha)}{2\alpha+1}\beta_3\right]\left[\alpha_4 + \frac{n(n+2\alpha)}{2\alpha+1}\beta_4\right] - (-1)^n \left[\alpha_3 - \frac{n(n+2\alpha)}{2\alpha+1}\beta_3\right]\left[\alpha_4 + \frac{N(N+2\alpha)}{2\alpha+1}\beta_4\right]\right\} \Big/ \delta'_2,$$

$$V'_n = \left\{(-1)^N \left[\alpha_3 - \frac{(N-1)(N+2\alpha-1)}{2\alpha+1}\beta_3\right]\left[\alpha_4 + \frac{n(n+2\alpha)}{2\alpha+1}\beta_4\right] + (-1)^n \left[\alpha_3 - \frac{n(n+2\alpha)}{2\alpha+1}\beta_3\right]\left[\alpha_4 + \frac{(N-1)(N+2\alpha-1)}{2\alpha+1}\beta_4\right]\right\} \Big/ \delta'_2,$$

$$g'_n(t) = \left\{(-1)^M \left[\alpha_1 - \frac{M(M+2\alpha)}{2\alpha+1}\beta_1\right]\gamma_n^{(2)}(t) - \left[\alpha_2 + \frac{M(M+2\alpha)}{2\alpha+1}\beta_2\right]\gamma_n^{(1)}(t)\right\} \Big/ \delta'_1,$$

$$h'_n(t) = \left\{\left[\alpha_2 + \frac{(M-1)(M+2\alpha-1)}{2\alpha+1}\beta_2\right]\gamma_n^{(1)}(t) + (-1)^M \left[\alpha_1 - \frac{(M-1)(M+2\alpha-1)}{2\alpha+1}\beta_1\right]\gamma_n^{(2)}(t)\right\} \Big/ \delta'_1,$$

$$k'_m(t) = \left\{(-1)^N \left[\alpha_3 - \frac{N(N+2\alpha)}{2\alpha+1}\beta_3\right]\gamma_m^{(4)}(t) - \left[\alpha_4 + \frac{N(N+2\alpha)}{2\alpha+1}\beta_4\right]\gamma_m^{(3)}(t)\right\} \Big/ \delta'_2,$$

$$\ell'_m(t) = \left\{\left[\alpha_4 + \frac{(N-1)(N+2\alpha-1)}{2\alpha+1}\beta_4\right]\gamma_m^{(3)}(t) + (-1)^N \left[\alpha_3 - \frac{(N-1)(N+2\alpha-1)}{2\alpha+1}\beta_3\right]\gamma_m^{(4)}(t)\right\} \Big/ \delta'_2,$$

$$\delta'_1 = (-1)^M \left[\left(\alpha_1 - \frac{(M-1)(M+2\alpha-1)}{2\alpha+1}\beta_1\right)\left(\alpha_2 + \frac{M(M+2\alpha)}{2\alpha+1}\beta_2\right) + \left(\alpha_1 - \frac{M(M+2\alpha)}{2\alpha+1}\beta_1\right)\left(\alpha_2 + \frac{(M-1)(M+2\alpha-1)}{2\alpha+1}\beta_2\right)\right] \neq 0,$$

$$\delta'_2 = (-1)^N \left[\left(\alpha_3 - \frac{(N-1)(N+2\alpha-1)}{2\alpha+1}\beta_3\right)\left(\alpha_4 + \frac{N(N+2\alpha)}{2\alpha+1}\beta_4\right) + \left(\alpha_3 - \frac{N(N+2\alpha)}{2\alpha+1}\beta_3\right)\left(\alpha_4 + \frac{(N-1)(N+2\alpha-1)}{2\alpha+1}\beta_4\right)\right] \neq 0.$$

Making use of Eqs. (48) and (49) to eliminate $a_{M-1,n}(t)$, $a_{Mn}(t)$, $a_{m,N-1}(t)$ and $a_{mN}(t)$ from the finite system of (47), give

$$\sum_{i=0}^{M-2} H_{mi}a_{in}(t) + \sum_{j=0}^{N-2} a_{mj}(t)T_{jn} + b'_{mn}(t) = a'_{mn}(t) \quad (0 \leqslant m \leqslant M-2;\ 0 \leqslant n \leqslant N-2), \tag{50}$$

where

$$H_{mi} = K_{mi} - \mu_i' K_{m,M-1} - v_i' K_{mM}; \quad T_{jn} = L_{jn} - U_j' L_{N-1,n} - V_j' L_{Nn},$$

$$b_{mn}'(t) = K_{m,M-1} g_n'(t) + K_{m,M} h_n'(t) + L_{N-1,n} k_m'(t) + L_{Nn} \ell_m'(t).$$

System (50) can be casted in the matrix form

$$\mathbf{HA}(t) + \mathbf{A}(t)\mathbf{T} + \mathbf{B}(t) = \mathbf{A}'(t), \tag{51}$$

where **A** is the matrix of the unknown coefficients of order $(M-1)(N-1)$; **H** and **T** are square matrices of orders $(M-1)$ and $(N-1)$, respectively.

This approach of truncating the exact infinite ultraspherical expansion for $u(x, y, t)$ by dropping the equations for the highest modes from Eq. (47) and determining them directly from the boundary conditions is amount to Lanczos τ-method.

The main advantages of this method can be seen by looking at the simpler form of matrices **H** and **T**. Note also that, although the method of this section computationally simpler than the first method, it is mathematically equivalent and will produce identical results.

5. Accurate double ultraspherical approximation for Poisson's equation

In the present section,we can obtain directly and without doing extra more analysis than that given in the previous two sections, the equations that give accurate double ultraspherical spectral approximations for solving numerically Poisson's equation, namely,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad x, y \in [-1, 1], \tag{52}$$

subject to the most general inhomogeneous mixed boundary conditions (10) and (11); noting here that γ_1 and γ_2 are functions of *y*, while γ_3 and γ_4 are functions of *x*, and all of them do not depend on *t*.

If the usual notation of Sections 3 and 4 kept unchanged, and returning to Eq. (29), replacing the coefficients $a_{ij}'(t)$ by the corresponding expansion coefficients f_{ij} of $f(x, y)$ that appear in (52), we get

$$\sum_{i=2}^{\infty} C_{mi} a_{in} + \sum_{j=2}^{\infty} a_{mj} D_{jn} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{mi} f_{ij} B_{jn} - b_{mn}, \quad m, n \geqslant 2,$$

or in the finite matrix form

$$\mathbf{CA} + \mathbf{AD} = \mathbf{S}, \quad 2 \leqslant m \leqslant M; \, 2 \leqslant n \leqslant N, \tag{53}$$

where **S** is a matrix of order $(M-1)(N-1)$ whose elements S_{mn} are given by

$$S_{mn} = \sum_{i=0}^{M} \sum_{j=0}^{N} A_{mi} f_{ij} B_{jn} - b_{mn}. \tag{54}$$

Note here that C_{mi}, D_{jn} and b_{mn} are as given by (30), and take into consideration that a_{mn}, g_n, h_n, k_m, and ℓ_m are all real constants which do not depend on the variable t.

It is not difficult to show that the system of linear algebraic equations (53) is equivalent to

$$\mathbf{G}\underline{a} = \underline{s}, \tag{55}$$

where $\mathbf{G}$ and $\underline{a}$ are as defined before.

Now returning to (50), we get

$$\sum_{i=0}^{M-2} H_{ni} a_{in} + \sum_{j=0}^{N-2} a_{mj} T_{jn} + b'_{mn} = f_{mn} \quad (0 \leqslant m \leqslant M-2;\ 0 \leqslant n \leqslant N-2),$$

or in the finite matrix form

$$\mathbf{HA} + \mathbf{AT} = \mathbf{W}. \tag{56}$$

This system of linear algebraic equations has the same methods of solution like that of the matrix equation (53).

It is worthy to mention here that the double ultraspherical expansion method can be extended to handle Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = f(x, y)$$

for constant λ subject to the most general boundary conditions as in the case of Poisson's equation.

6. Numerical results and comparisons

Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y, t) \in [-1, 1] \times [0, \infty), \tag{57}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \dfrac{\partial u}{\partial x} - 2u = \dfrac{\pi}{2}\, \mathrm{e}^{\frac{-\pi^2 t}{2}} \cos\left(\dfrac{\pi y}{2}\right), & x = -1, \\ \dfrac{\partial u}{\partial x} + 2u = -\dfrac{\pi}{2}\, \mathrm{e}^{\frac{-\pi^2 t}{2}} \cos\left(\dfrac{\pi y}{2}\right), & x = 1, \end{array}\right] , \quad -1 \leqslant y \leqslant 1,\ t > 0, \tag{58}$$

$$\left.\begin{array}{ll} \dfrac{\partial u}{\partial y} - 2u = \dfrac{\pi}{2}\, \mathrm{e}^{\frac{-\pi^2 t}{2}} \cos\left(\dfrac{\pi x}{2}\right), & y = -1, \\ \dfrac{\partial u}{\partial y} + 2u = -\dfrac{\pi}{2}\, \mathrm{e}^{\frac{-\pi^2 t}{2}} \cos\left(\dfrac{\pi x}{2}\right), & y = 1, \end{array}\right] , \quad -1 \leqslant x \leqslant 1,\ t > 0 \tag{59}$$

and the initial condition

$$u(x, y, 0) = \cos\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi y}{2}\right), \quad -1 \leqslant x, y \leqslant 1. \tag{60}$$

It is not difficult to show that the analytical solution of (57) subject to (58)–(60), is given by

$$u(x, y, t) = \mathrm{e}^{(-\pi^2 t/2)} \cos\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi y}{2}\right) \tag{61}$$

$$
\begin{aligned}
&= \frac{4\pi^{1-2\alpha}\, \mathrm{e}^{(-\pi^2 t/2)}}{\Gamma^2(\alpha+\frac{1}{2})} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2n+\alpha)(2m+\alpha)\Gamma(2n+2\alpha)\Gamma(2m+2\alpha)}{(2n)!(2m)!} \\
&\quad \times J_{2m+\alpha}\left(\frac{\pi}{2}\right) J_{2n+\alpha}\left(\frac{\pi}{2}\right) C_{2m}^{(\alpha)}(x)\, C_{2n}^{(\alpha)}(y), \qquad (62)
\end{aligned}
$$

where $J_i(z)$ denotes the Bessel function of the first kind, see [24].

It is obvious that the solution of (57) is symmetric about the axes of coordinates and therefore can be approximated by the series expansion

$$
u_{MN}(x,y,t) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{2m,2n}(t) C_{2m}^{(\alpha)}(x) C_{2n}^{(\alpha)}(y).
$$

If the first method of Section 3 is applied, then the boundary conditions (23) and (24) give

$$
a_{0n}(t) + \mu_2 a_{2n}(t) + \mu_4 a_{4n}(t) + \cdots = g_n(t), \quad n = 0, 2, 4, \ldots,
$$

$$
a_{m0}(t) + U_2 a_{m2}(t) + U_4 a_{m4}(t) + \cdots = k_m(t), \quad m = 0, 2, 4, \ldots,
$$

where

$$
\mu_{2m} = U_{2m} = 1 + \frac{2m(m+\alpha)}{2\alpha+1},
$$

$$
g_{2n}(t) = k_{2n}(t) = \mathrm{e}^{-\frac{\pi^2 t}{2}} \frac{(-1)^{n+1}(2n+\alpha)\Gamma(2n+2\alpha)}{2(\pi)^{\alpha-\frac{3}{2}}\Gamma(\alpha+\frac{1}{2})(2n)!} J_{2n+\alpha}\left(\frac{\pi}{2}\right).
$$

The matrices $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}(t)$ are calculated. After calculating, $\underline{E}(t) = \mathrm{vec}\,\mathbf{E}(t)$, $\underline{a}(0) = \mathrm{vec}\,\mathbf{A}(0)$, $\mathbf{G} = \mathbf{C} \oplus \mathbf{D}^{\mathrm{T}}$ and $\mathbf{H} = \mathbf{D}^{\mathrm{T}} \otimes \mathbf{C}$, then $\underline{a}(t)$ may be calculated from the step-by-step formula

$$
\underline{a}(t+\Delta t) = \mathrm{e}^{\mathbf{H}^{-1}\mathbf{G}\Delta t}\underline{a}(t) + (\mathrm{e}^{-(\pi^2\Delta t/2)}\mathbf{I} - \mathrm{e}^{\mathbf{H}^{-1}\mathbf{G}\Delta t})\left(\mathbf{G} + \frac{\pi^2}{2}\mathbf{H}\right)^{-1} \underline{E}(t), \qquad (63)
$$

where $\mathrm{e}^{\mathbf{H}^{-1}\mathbf{G}\Delta t}$ is approximated by formula (46).

If the alternative method of Section 4 is applied, then the boundary conditions (48) and (49) yield

$$
a_{2N,n} + v_0' a_{0n}(t) + v_2' a_{2n}(t) + \cdots + v_{2N-2,n}' a_{2N-2,n} = h_n'(t), \quad n = 0, 2, 4, \ldots, 2N,
$$

$$
a_{m,2N} + V_0' a_{m0}(t) + V_2' a_{m2}(t) + \cdots + V_{m,2N-2}' a_{m,2N-2} = \ell_m'(t), \quad m = 0, 2, 4, \ldots, 2M,
$$

where

$$
v_{2m}' = V_{2m}' = \frac{1 + 2m^2 + 2\alpha(m+1)}{1 + 2N^2 + 2\alpha(N+1)},
$$

$$
h_{2m}'(t) = \ell_{2m}'(t) = \frac{(-1)^{m+1}(2\alpha+1)(2m+\alpha)\Gamma(2m+2\alpha)}{4\Gamma(\alpha+\frac{1}{2})\pi^{\alpha-\frac{3}{2}}(1+2N^2+2\alpha(N+1))(2m)!} J_{2m+\alpha}\left(\frac{\pi}{2}\right).
$$

Table 1
Maximum pointwise error of $u - u_{MN}$ for $N = M = 4, 8, 12$

α	t	N	E	N	E	N	E
1			3.795×10^{-5}		4.142×10^{-9}		1.423×10^{-14}
$\frac{1}{2}$			8.120×10^{-5}		4.461×10^{-9}		4.407×10^{-14}
0	0.25	4	1.882×10^{-4}	8	1.475×10^{-8}	12	1.332×10^{-13}
-0.2			3.430×10^{-4}		2.503×10^{-8}		4.027×10^{-13}
-0.45			9.052×10^{-4}		6.597×10^{-8}		1.546×10^{-12}
1			1.087×10^{-5}		1.206×10^{-9}		5.100×10^{-15}
$\frac{1}{2}$			2.365×10^{-5}		1.299×10^{-9}		1.489×10^{-14}
0	0.5	4	5.491×10^{-4}	8	3.347×10^{-9}	12	4.179×10^{-14}
-0.2			1.051×10^{-4}		7.289×10^{-9}		1.753×10^{-13}
-0.45			2.636×10^{-4}		1.994×10^{-8}		1.309×10^{-12}
1			3.080×10^{-6}		3.512×10^{-10}		2.151×10^{-15}
$\frac{1}{2}$			6.892×10^{-6}		3.783×10^{-10}		5.699×10^{-15}
0	0.75	4	1.606×10^{-5}	8	9.773×10^{-10}	12	1.771×10^{-14}
-0.2			3.148×10^{-5}		2.122×10^{-9}		4.625×10^{-14}
-0.45			7.681×10^{-5}		5.809×10^{-9}		7.482×10^{-13}

The matrices $\mathbf{H}$, $\mathbf{T}$ and $\mathbf{B}(t)$ are calculated. Then $\underline{a}(t)$ may be calculated from the step-by-step formula

$$\underline{a}(t + \Delta t) = \mathrm{e}^{\mathbf{L}^{-1}\mathbf{G}\Delta t}\underline{a}(t) + (\mathrm{e}^{-(\pi^2 \Delta t/2)}\mathbf{I} - \mathrm{e}^{\mathbf{L}^{-1}\mathbf{G}\Delta t})\left(\mathbf{G} + \frac{\pi^2}{2}\mathbf{L}\right)^{-1}\underline{K}\mathrm{e}^{-(\pi^2 t/2)}, \tag{64}$$

where $\mathbf{G} = \mathbf{H}^{-1} \bigoplus \mathbf{H}^{-1}$, $\underline{K} = \mathrm{vec}(\mathbf{H}^{-1}\mathbf{B}\mathbf{T}^{-1})$, $\mathbf{L} = \mathbf{H}^{-1} \bigotimes \mathbf{H}^{-1}$.

Let E denotes the maximum pointwise error obtained by any of the two proposed methods. Then these values of E are tabulated for different values of α, M, $N(M = N)$ and t in Table 1.

For the sake of comparisons between the results obtained by using the first method and its alternative; we add alongside the theoretical values of the solution as calculated from (62). It is of interest to note here that the abbreviations FM, AM and T denote to the values of the solution calculated by using the first, alternative and the theoretical methods of solution for the example considered, respectively. The values of the solution obtained by the two methods at the data points, $A \equiv (0.2, 0.2)$, $B \equiv (0.2, 0.4)$ and $C \equiv (0.4, 0.4)$, for $t = 0.25, 0.5, 0.75$, can be compared with the theoretical solution for various values of the parameter α. In Table 2, the results are tabulated for the value $\alpha = 1$. For the sake of comparison with the results obtained in [12], we present the results for $\alpha = \frac{1}{2}$ in Table 3. Numerical results show that both methods compare favorably with the theoretical solution.

We present also a comparison for the model problem considered between our proposed ultraspherical expansion technique and a finite difference-scheme. The finite-difference scheme used is the Peaceman–Rashford method (See [25]), with the parameters $\Delta x = \Delta y = 0.05$ and $\Delta t = h = 0.0025$, and the calculation is continued for 100 time steps. The measure for error for the finite-difference method (EFDM) is the maximum pointwise error on the grid between the exact and calculated solution. The error at the grid point (0,0), are shown in Table 4 after a number of time steps (p). Note that the

Table 2
The case $\alpha = 1$

t	Data point	AM, FM	T
0.25	A	0.263404572	0.263404572
	B	0.224065313	0.224065312
	C	0.190601338	0.190601339
0.5	A	0.076767068	0.076767068
	B	0.065250717	0.065250719
	C	0.055505574	0.055505575
0.75	A	0.223380174	0.223380175
	B	0.190018526	0.190018526
	C	0.016163941	0.016163941

Table 3
The case $\alpha = \frac{1}{2}$

t	Data point	AM, FM	Doha & Al-Kholi	T
0.25	A	0.263404571	0.263404700	0.263404572
	B	0.224065312	0.224065500	0.224065313
	C	0.190601331	0.190601500	0.190601339
0.5	A	0.076767068	0.076706900	0.076767068
	B	0.065250717	0.065250800	0.065250716
	C	0.055505574	0.055505800	0.055505575
0.75	A	0.223380173	0.223380000	0.223380175
	B	0.190018526	0.190018000	0.190018526
	C	0.016163941	0.016163700	0.016163941

maximum pointwise error obtained by our proposed ultraspherical expansion technique is denoted by EUET.

Example 2. Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -32\pi^2 \sin(4\pi x)\sin(4\pi y),$$

subject to the boundary conditions

$$u \pm \frac{\partial u}{\partial x} = \pm 4\pi \sin(4\pi y), \quad x = \pm 1,$$

$$u \pm \frac{\partial u}{\partial y} = \pm 4\pi \sin(4\pi y), \quad x = \pm 1.$$

Table 4
Comparison between the maximum pointwise errors EFDM and EUET

α	p	t	EFDM	EUET
1				1.405×10^{-8}
$\frac{1}{2}$				2.513×10^{-8}
0	1	0.0025	9.6×10^{-4}	3.282×10^{-8}
-0.2				8.489×10^{-8}
-0.45				2.237×10^{-7}
1				1.337×10^{-8}
$\frac{1}{2}$				1.440×10^{-8}
0	5	0.0125	1.0×10^{-3}	3.124×10^{-8}
-0.2				8.081×10^{-8}
-0.45				2.211×10^{-7}
1				1.257×10^{-8}
$\frac{1}{2}$				1.354×10^{-8}
0	10	0.0250	8.0×10^{-4}	2.937×10^{-8}
-0.2				7.597×10^{-8}
-0.45				2.079×10^{-7}
1				1.111×10^{-8}
$\frac{1}{2}$				1.371×10^{-8}
0	20	0.0500	4.6×10^{-4}	2.289×10^{-8}
-0.2				6.715×10^{-8}
-0.45				1.838×10^{-7}
1				8.683×10^{-9}
$\frac{1}{2}$				9.353×10^{-9}
0	40	0.100	2.0×10^{-4}	1.021×10^{-8}
-0.2				3.624×10^{-8}
-0.45				9.551×10^{-8}
1				5.997×10^{-9}
$\frac{1}{2}$				6.460×10^{-9}
0	70	0.175	1.0×10^{-4}	1.401×10^{-8}
-0.2				3.624×10^{-8}
-0.45				9.551×10^{-8}
1				4.142×10^{-9}
$\frac{1}{2}$				4.461×10^{-9}
0	100	0.250	4.3×10^{-5}	1.475×10^{-8}
-0.2				2.503×10^{-8}
-0.45				6.597×10^{-8}

Table 5
Maximum pointwise error of $u - u_{MN}$ for $N = M = 16, 24, 32$

$M = N$	α	Error
16	1	6.773×10^{-2}
	$\frac{1}{2}$	7.758×10^{-2}
	0	9.616×10^{-2}
	-0.2	1.515×10^{-1}
	-0.45	1.809×10^{-1}
24	1	9.590×10^{-6}
	$\frac{1}{2}$	1.066×10^{-5}
	0	1.495×10^{-5}
	-0.2	1.737×10^{-5}
	-0.45	2.195×10^{-5}
32	1	2.356×10^{-10}
	$\frac{1}{2}$	5.187×10^{-10}
	0	7.891×10^{-10}
	-0.2	2.561×10^{-9}
	-0.45	9.657×10^{-9}

This problem has the analytical solution

$$\begin{aligned} u(x, y) &= \sin(4\pi x)\sin(4\pi y) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} C_i^{(\lambda)}(x) C_i^{(\lambda)}(y). \end{aligned}$$

The maximum pointwise error of the ultraspherical approximation for various choices of α, $M = N$ is illustrated in Table 5.

Example 3. Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

subject to the boundary conditions

$$u(x, \pm 1) = \cos\left(\frac{\pi x}{2}\right), \quad u(\pm 1, y) = 0.$$

This problem is symmetric about the axes of coordinates, and its analytical solution is given by

$$u(x, y) = \frac{\cos\left(\frac{\pi x}{2}\right)\cosh\left(\frac{\pi y}{2}\right)}{\cosh\left(\frac{\pi}{2}\right)}.$$

Table 6
Maximum pointwise error of $u - u_{MN}$ for $N = M = 4, 8, 12, 16$

$M = N$	α	Error
4	1	9.383×10^{-4}
	$\frac{1}{2}$	1.266×10^{-3}
	0	1.406×10^{-3}
	-0.2	3.112×10^{-3}
	-0.45	4.343×10^{-3}
8	1	8.911×10^{-8}
	$\frac{1}{2}$	1.298×10^{-7}
	0	1.811×10^{-7}
	-0.2	2.398×10^{-7}
	-0.45	9.165×10^{-7}
12	1	1.283×10^{-12}
	$\frac{1}{2}$	1.985×10^{-12}
	0	2.739×10^{-12}
	-0.2	3.432×10^{-12}
	-0.45	7.084×10^{-12}
16	1	8.326×10^{-16}
	$\frac{1}{2}$	6.106×10^{-15}
	0	9.015×10^{-14}
	-0.2	2.545×10^{-13}
	-0.45	1.509×10^{-12}

The maximum pointwise error of the ultraspherical approximation for various choices of α, $M = N$ is illustrated in Table 6.

From the numerical results presented in Tables 1–6, we see that the results corresponding to the value of the parameter $\alpha = 1$, are better than any the others. From this we conclude that the expansion based on Chebyshev polynomials of the first kind ($\alpha = 0$) is not always better than other ultraspherical series. This conclusion has been ascertained in [22].

References

[1] M. Abramowitz, I.A. Stegun, (Eds.), Handbook of mathematical functions, Applied Mathematical Series, vol. 55, National Bureau of Standards, New York, 1970.
[2] R. Bellman, K.L. Cooke, Differential-Difference Equations, Academic Press, New York, 1963.
[3] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer, Berlin, 1988.
[4] P.M. Dew, R.E. Scraton, Chebyshev methods for the numerical solution of parabolic partial differential equations in two and three space variables, J. Inst. Math. Appl. 16 (1975) 121–131.
[5] E.H. Doha, An accurate double Chebyshev approximation for Poisson's equation, Ann. Univ. Sci. Budapest. Sect. Comp. 10 (1990) 243–276.

[6] E.H. Doha, An accurate solution of parabolic equations by expansion in ultraspherical polynomials, Comput. Math. Appl. 19 (4) (1990) 75–88.
[7] E.H. Doha, The coefficients of differentiated expansions and derivatives of ultraspherical polynomials, Comput. Math. Appl. 21 (2–3) (1991) 115–122.
[8] E.H. Doha, The Chebyshev coefficients of general-order derivatives of an infinitely differentiable function in two or three variables, Ann. Univ. Sci. Budapest. Sect. Comp. 13 (1992) 83–91.
[9] E.H. Doha, On the coefficients of differentiated expansions of double and triple Legendre polynomials, Ann. Univ. Sci. Budapest. Sect. Comp. 15 (1995) 25–35.
[10] E.H. Doha, The coefficients of differentiated expansions of double and triple ultraspherical polynomials, Ann. Univ. Sci. Budapest. Sect. Comp. 19 (2000) 57–79.
[11] E.H. Doha, M.A. Helal, An accurate double Chebyshev spectral approximation for parabolic partial differential equations, J. Egypt. Math. Soc. 5 (1) (1997) 83–101.
[12] E.H. Doha, F.M.R. Al-Kholi, An efficient double Legendre spectral method for parabolic and elliptic partial differential equations, Internat. J. Comput. Math. 78 (2001) 413–432.
[13] G. Fairweather, Finite Element Galerkin Methods for Differential Equations, Marcel Dekker, Inc., New York, 1978.
[14] L. Fox, I.B. Parker, Chebyshev Polynomials in Numerical Analysis, Oxford University Press, Oxford, 1972.
[15] D. Gottlieb, S.A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 26, SIAM, Philadelphia, 1977.
[16] A. Graham, Kronecker Product and Matrix Calculus With Applications, Ellis Horwood Ltd., London, 1981.
[17] D.B. Haidvogel, T. Zang, The accurate solution of Poisson's equation by Chebyshev polynomials, J. Comput. Phys. 30 (1979) 167–180.
[18] T.S. Horner, A double Chebyshev series method for elliptic partial differential equations, in: J. Noye (Ed.), Numerical Solution of Partial Differential Equations, North-Holland, Amsterdam, 1982.
[19] A. Karageorghis, A note on the Chebyshev coefficients of the general order derivative of an infinitely differentiable function, J. Comput. Appl. Math. 21 (1988) 129–132.
[20] D. Knibb, R.E. Scraton, On the solution of parabolic partial differential equation in Chebyshev series, Comput. J. 14 (1971) 428–432.
[21] C. Lanczos, Applied Analysis, Pitman, London, 1957.
[22] W.A. Light, Are Chebyshev expansions really better, Bull. Inst. Math. Appl. 22 (1986) 180–181.
[23] C.V. Loan, Computational Frameworks for Fast Fourier Transform, SIAM, Philadelphia, USA, 1992.
[24] Y.L. Luke, The Special Functions and Their Approximations, vols. 1 and 2, Academic Press, New York, 1969.
[25] A.R. Mitchell, D.F. Griffiths, The Finite Difference Methods in Partial Differential Equations, Wiley, New York, 1980.
[26] S.A. Orszag, Accurate solution of the orr-Sommer field stability equation, J. Fluid Mech. 50 (1971) 689–703.
[27] T.N. Phillips, On the Legendre coefficients of a general order derivative of an infinitely differentiated function, IMA J. Numer. Anal. 8 (1988) 455–459.
[28] H. Schlichting, Boundary-Layer Theory, McGraw-Hill, New York, 1979.
[29] C.L. Streett, et al., Spectral multigrid methods with applications to transonic potential flow, J. Comput. Phys. 57 (1985) 43–76.
[30] Szegö, G., Orthogonal Polynomials, American Mathematical Society Colloquium Publications, vol. 23, 1985.
[31] H. Tal-Ezer, A pseudospectral Legendre method for hyperbolic equations with an improved stability condition, J. Comput. Phys. 67 (1986) 145–172.
[32] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.