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Orbits in symmetric spaces ☆

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Abstract

We characterize those elements in fully symmetric spaces on the interval $(0, 1)$ or on the semi-axis $(0, \infty)$ whose orbits are the norm-closed convex hull of their extreme points.

Keywords: Symmetric function spaces; Orbits; Closed convex hull of extreme points

1. Introduction

The following semigroups of bounded linear operators play a fundamental role in the interpolation theory of linear operators for the couple (L_1, L_∞) of Lebesgue measurable functions on intervals $(0, 1)$ and $(0, \infty)$. The semigroup of absolute contractions, or admissible operators (see e.g. [10, II.3.4])

$$\Sigma := \{T : L_1 + L_\infty \to L_1 + L_\infty:\ \max(\|T\|_{L_1 \to L_1}, \|T\|_{L_\infty \to L_\infty}) \leqslant 1\},$$

the semigroup of substochastic operators (see e.g. [2, p. 107])

$$\Sigma_+ := \{0 \leqslant T \in \Sigma\}$$

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and, in the case of the interval (0, 1), the semigroup of doubly stochastic operators

$$\Sigma' := \left\{ 0 \leqslant T \in \Sigma_+ : \int_0^1 (Tx)(s)\,ds = \int_0^1 x(s)\,ds,\ \forall x \geqslant 0,\ T1 = 1 \right\}$$

(see e.g. [13]). If $x \in L_1 + L_\infty$ (respectively, $0 \leqslant x \in L_1 + L_\infty$ or $0 \leqslant x \in L_1(0,1)$) we denote by $\Omega(x)$ (respectively $\Omega_+(x)$ and $\Omega'(x)$) the *orbit* of x with respect to the semigroups Σ (respectively, Σ_+, and Σ'). A Banach function space E (on $(0,1)$ or $(0,\infty)$, see [2, pp. 2–3]) is called an exact interpolation space if every $T \in \Sigma$ maps E into itself and $\|T\|_{E \to E} \leqslant 1$, or alternatively, if $\Omega(x) \subset E$ and $\|y\|_E \leqslant \|x\|_E$ for every $x \in E$ and every $y \in \Omega(x)$. The class of exact interpolation spaces admits an equivalent description in terms of (sub)majorization in the sense of Hardy, Littlewood and Polya. Recall, that if $x, y \in L_1 + L_\infty$, then y is said to be a *submajorized* by x in the sense of Hardy, Littlewood and Polya, written $y \prec\prec x$ if and only if

$$\int_0^t y^*(s)\,ds \leqslant \int_0^t x^*(s)\,ds, \quad t \geqslant 0.$$

Here, x^* denotes the non-increasing right-continuous rearrangement of x given by

$$x^*(t) = \inf\bigl\{s \geqslant 0:\ m\bigl(\{|x| \geqslant s\}\bigr) \leqslant t\bigr\}$$

and m is Lebesgue measure. If $0 \leqslant x, y \in L_1$, then we say that y is *majorized* by x (written $y \prec x$) if and only if $y \prec\prec x$ and $\|y\|_1 = \|x\|_1$. A Banach function space E is said to be *fully symmetric* if and only if $x \in E$, $y \in L_1 + L_\infty$ $y \prec\prec x \Rightarrow y \in E$ and $\|y\|_E \leqslant \|x\|_E$. The classical Calderon–Mityagin theorem (see [4,10,2]) gives an alternative description of the sets $\Omega(x)$, $x \in L_1 + L_\infty$ and $\Omega_+(x)$, $0 \leqslant x \in L_1 + L_\infty$ as follows

$$\Omega(x) = \{y \in L_1 + L_\infty:\ y \prec\prec x\}, \qquad \Omega_+(x) = \{0 \leqslant y \in L_1 + L_\infty:\ y \prec\prec x\}$$

and (in the case of the interval (0, 1) and $0 \leqslant x \in L_1(0,1)$)

$$\Omega'(x) = \{0 \leqslant y \in L_1:\ y \prec x\},$$

which shows, in particular, that the classes of exact interpolation spaces and fully symmetric spaces coincide.

Let fully symmetric Banach function space E be fixed. The principal aim of the paper is to give conditions for a given $0 \leqslant x \in E$ which are necessary and sufficient for each of the sets $\Omega_+(x)$, $\Omega'(x)$ to be the norm closure of the convex hull of their extreme points.

If $E = L_1(0,1)$, then it has been shown by Ryff (see [13]) that if $0 \leqslant x \in E$, then the orbit $\Omega'(x)$ is weakly compact and so, due to the Krein–Milman theorem, the orbit $\Omega'(x)$ is the weak (and hence norm)-closed convex hull of its extreme points. It follows from the results of [7] that the set $\Omega'(x)$ is weakly compact in any separable symmetric space E. Hence, $\Omega'(x)$ is the weak (and hence norm)-closed convex hull of its extreme points in any separable symmetric space E.

If a fully symmetric space E is not separable, then it is not the case in general that orbits are weakly compact. A trivial example yields the orbit $\Omega(\chi_{[0,1]})$ in fully symmetric non-separable space $L_\infty(0,1)$. Indeed, it is obvious that this orbit coincides with the unit ball of $L_\infty(0,1)$ and the latter is not weakly compact since the space $L_\infty(0,1)$ is non-reflexive. Nonetheless, it is an interesting question to give necessary and sufficient conditions that the orbit of a given element should be the norm-closed convex hull of its extreme points. This question was considered by Braverman and Mekler (see [3]) in the case of the interval (0, 1) and orbits $\Omega(x)$. They showed that for every fully symmetric space E on (0, 1) satisfying the condition

$$\lim_{\tau\to\infty}\frac{1}{\tau}\|\sigma_\tau\|_{E\to E}=0 \tag{1}$$

that $\Omega(x)$ is indeed the norm-closed convex hull of the set of its extreme points, for every $x\in E$ (see [3, Theorem 3.1]). Here σ_τ denotes the usual dilation operator (see the following section for definition and properties). They showed as well that the converse assertion is valid in case that E is a Marcinkiewicz space on (0, 1). As explained below, this converse assertion, however, fails for arbitrary fully symmetric spaces.

We show (Theorem 21) that if E is a fully symmetric space on (0, 1) and if $0\leqslant x\in E$, then $\Omega'(x)$ is the norm-closed convex hull of its extreme points if and only if

$$\varphi(x):=\lim_{\tau\to\infty}\frac{1}{\tau}\left\|\sigma_\tau(x^*)\right\|_E=0. \tag{2}$$

As shown in Corollary 27 this implies the result of Braverman and Mekler. In Appendix A, we demonstrate that the conditions (1) and (2) are distinct in the class of Orlicz spaces. If E is an Orlicz space, then it is the case that (2) holds, and so by Theorems 21 and 22 for every $0\leqslant x\in E$, the sets $\Omega'(x)$, $\Omega_+(x)$ and $\Omega(x)$ are the norm-closed convex hulls of its extreme points. However, there are non-separable Orlicz spaces E which fail condition (1).

In Appendix A, we also introduce the notion of symmetric and fully symmetric functionals. The latter are a "commutative" counterpart of Dixmier traces appeared in non-commutative geometry (see e.g. [5]). Symmetric and fully symmetric functionals are extensively studied in [8,9] (see also [5] and references therein). Note, however, that our terminology differs from that used in just cited articles. A subclass of Marcinkiewicz spaces admitting symmetric functionals which fail to be fully symmetric is described in [9]. It follows from our results that any symmetric functional on a fully symmetric space satisfying (2) is automatically fully symmetric. In particular, this implies that an Orlicz space does not possess any singular symmetric functionals (see Proposition 34). This latter result strengthens the result of [8, Theorem 3.1] that an Orlicz space does not possess any singular fully symmetric functionals.

Results similar to Theorems 21 and 22 hold also for fully symmetric spaces E on the semi-axis (see Theorems 23–26).

The main results of this article are contained in Section 4. In the following section we present some definitions from the theory of symmetric spaces, as some of our results hold in a slightly more general setting than that of fully symmetric spaces. For more details on the latter theory we refer to [10,11,2]. Section 3 treats various properties of the functional φ and its modifications needed in Section 4. We would like to emphasize the difference between geometric properties of the orbit $\Omega(x)$ and those of $\Omega'(x)$ and $\Omega_+(x)$. This is especially noticeable in the description of the respective sets of their extreme points. The extreme points of the sets $\Omega(x)$ and $\Omega'(x)$, $x\geqslant 0$ are well known (see [14,6]) and are given by

$$\operatorname{extr}\big(\Omega(x)\big)=\{y\in L_1+L_\infty:\ y^*=x^*\},\qquad \operatorname{extr}\big(\Omega'(x)\big)=\{0\leqslant y\in L_1:\ y^*=x^*\},$$

whereas the description of extreme points of the set $\Omega_+(x)$, $x \geqslant 0$ given by

$$\operatorname{extr}\big(\Omega_+(x)\big) = \{0 \leqslant y \in L_1 + L_\infty\colon\ y^* = x^*\chi_{[0,\beta]} \text{ for some } \beta \leqslant \infty\}$$

when $x^*(\infty) := \lim_{t\to\infty} x^*(t) = 0$, and by

$$\operatorname{extr}\big(\Omega_+(x)\big) = \{0 \leqslant y \in L_1 + L_\infty\colon\ y^* = x^*\chi_{[0,\beta]} \text{ for some } \beta \leqslant \infty \text{ and } y\chi_{\{y<y^*(\infty)\}} = 0\}$$

when $x^*(\infty) > 0$, is somewhat less known, so we present in Appendix A a careful exposition of the latter equalities.

2. Preliminaries

Let L_0 be a space of Lebesgue measurable functions either on (0, 1) or $(0,\infty)$ finite almost everywhere (with identification m-a.e.). Here m is a Lebesgue measure. Define S_0 as the subset of L_0 which consists of all functions x such that $m(\{|x| > s\})$ is finite for some s.

Let E be a Banach space of real-valued Lebesgue measurable functions either on (0, 1) or $(0,\infty)$ (with identification m-a.e.). E is said to be *ideal lattice* if $x \in E$ and $|y| \leqslant |x|$ implies that $y \in E$ and $\|y\|_E \leqslant \|x\|_E$.

The ideal lattice $E \subseteq S_0$ is said to be *symmetric space* if for every $x \in E$ and every y the assumption $y^* = x^*$ implies that $y \in E$ and $\|y\|_E = \|x\|_E$.

If $E = E(0,1)$ is a symmetric space on (0, 1), then

$$L_\infty \subseteq E \subseteq L_1.$$

If $E = E(0,\infty)$ is a symmetric space on $(0,\infty)$, then

$$L_1 \cap L_\infty \subseteq E \subseteq L_1 + L_\infty.$$

Symmetric space E is said to be *fully symmetric* if and only if $x \in E$, $y \in L_1 + L_\infty$ $y \prec\prec x$ $\Rightarrow y \in E$ and $\|y\|_E \leqslant \|x\|_E$.

We now gather some additional terminology from the theory of symmetric spaces that will be needed in the sequel.

Suppose E is a symmetric space. Following [3], E will be called *strictly symmetric* if and only if whenever $x, y \in E$ and $y \prec\prec x$ then $\|y\|_E \leqslant \|x\|_E$.

It is clear that if E is fully symmetric then E is strictly symmetric, but the converse assertion is not valid.

The norm $\|\cdot\|_E$ is called Fatou norm if, for every sequence $x_n \uparrow x \in E$, it follows that $\|x_n\|_E \uparrow \|x\|_E$. This is equivalent to the assertion that the unit ball of E is closed with respect to almost everywhere convergence.

It is well known that if the norm on E is a Fatou norm then E is strictly symmetric.

If $\tau > 0$, the dilation operator σ_τ is defined by setting $(\sigma_\tau(x))(s) = x(\frac{s}{\tau})$, $s > 0$ in the case of the semi-axis. In the case of the interval (0, 1) the operator σ_τ is defined by

$$(\sigma_\tau x)(s) = \begin{cases} x(s/\tau), & s \leqslant \min\{1,\tau\}, \\ 0, & \tau < s \leqslant 1. \end{cases}$$

The operators σ_τ ($\tau \geqslant 1$) satisfy semigroup property $\sigma_{\tau_1}\sigma_{\tau_2} = \sigma_{\tau_1\tau_2}$. If E is a symmetric space and if $\tau > 0$, then the dilation operator σ_τ is a bounded operator on E and

$$\|\sigma_\tau\|_{E\to E} \leqslant \max\{1, \tau\}.$$

We will need also the notion of a partial averaging operator (see [3]).

Let $\mathcal{A} = \{A_k\}$ be a (finite or infinite) sequence of disjoint sets of finite measure and denote by $\mathfrak{A}$ the collection of all such sequences. Denote by A_∞ the complement of $\bigcup_k A_k$. The partial averaging operator is defined by

$$P(x|\mathcal{A}) = \sum_k \frac{1}{m(A_k)}\left(\int_{A_k} x(s)\,ds\right)\chi_{A_k} + x\chi_{A_\infty}.$$

Note, that we do not require A_∞ to have a finite measure.

Every partial averaging operator is a contraction both in L_1 and L_∞. Hence, $P(\cdot|\mathcal{A})$ is also a contraction in E. In case of the interval $(0,1)$, $P(\cdot|\mathcal{A})$ is a doubly stochastic operator in the sense of [13].

Since $P(\cdot|\mathcal{A}) \in \Sigma$, then $P(x|\mathcal{A}) \in \Omega(x)$ (respectively, $P(x|\mathcal{A}) \in \Omega'(x)$ if $x \in L_1$) for every $\mathcal{A} \in \mathfrak{A}$. As will be seen, elements of the form $P(x|\mathcal{A})$ play a central role.

The following properties of rearrangements can be found in [10]. If $x, y \in L_1 + L_\infty$, then

$$(x+y)^* \prec\prec x^* + y^* \tag{3}$$

and

$$(x^* - y^*) \prec\prec (x-y)^*. \tag{4}$$

Let us recall some classical examples of fully symmetric spaces.

Let ψ be a concave increasing continuous function. The Marcinkiewicz space M_ψ is the linear space of those functions $x \in S_0$, for which

$$\|x\|_{M_\psi} = \sup_t \frac{1}{\psi(t)} \int_0^t x^*(s)\,ds < \infty.$$

Equipped with the norm $\|x\|_{M_\psi}$, M_ψ is a fully symmetric space with Fatou norm.

Let $M(t)$ be a convex function on $[0,\infty)$ such that $M(t) > 0$ for all $t > 0$ and such that

$$0 = M(0) = \lim_{t\to 0} \frac{M(t)}{t} = \lim_{t\to\infty} \frac{t}{M(t)}. \tag{5}$$

Denote by L_M the Orlicz space on $[0,\infty)$ (see e.g. [11,10]) endowed with the norm

$$\|x\|_{L_M} = \inf\left\{\lambda\colon\ \lambda > 0,\ \int_0^\infty M\big(|x(t)|/\lambda\big)\,dt \leqslant 1\right\}.$$

Equipped with the norm $\|x\|_{L_M}$, L_M is a fully symmetric space with Fatou norm.

For further properties of Marcinkiewicz and Orlicz spaces, we refer to [10–12].

For $0 \leqslant x \in L_1 + L_\infty$, we set

$$Q_+(x) = \overline{\mathrm{Conv}}\big(\mathrm{extr}\big(\Omega_+(x)\big)\big).$$

For $0 \leqslant x \in L_1$, we set

$$Q'(x) = \overline{\mathrm{Conv}}\big(\mathrm{extr}\big(\Omega'(x)\big)\big).$$

For $0 \leqslant x \in L_1 + L_\infty$, we set

$$Q'(x) = \overline{\mathrm{Conv}}\{y^* = x^*,\ y\chi_{\{y<y^*(\infty)\}} = 0\}.$$

Here, $\overline{\mathrm{Conv}}$ means the norm-closed convex hull. See Appendix A for the precise description of the extreme points.

3. The dilation functional and its properties

The following assertion is widely used in the literature. However, no direct reference is available.

Lemma 1. *If $0 \leqslant x, y \in L_1 + L_\infty$, then*

$$x^* + y^* \prec\prec 2\sigma_{\frac{1}{2}}\big((x+y)^*\big). \tag{6}$$

Proof. Fix $\varepsilon > 0$. It follows from [10, II.2.1],

$$\int_0^t x^*(s)\,ds \leqslant \varepsilon + \int_{e_1} x(s)\,ds, \qquad \int_0^t y^*(s)\,ds \leqslant \varepsilon + \int_{e_2} y(s)\,ds$$

for some e_1 and e_2 with $m(e_i) = t$. However,

$$\int_{e_1} x(s)\,ds + \int_{e_2} y(s)\,ds \leqslant \int_{e_1\cup e_2} (x+y)(s)\,ds$$

$$\leqslant \sup_{m(e)=2t} \int_e (x+y)(s)\,ds = \int_0^{2t} (x+y)^*(s)\,ds.$$

Note, that $\int_0^{2t} u(s)\,ds = \int_0^t (2\sigma_{\frac{1}{2}} u)(s)\,ds$. □

Lemma 2. *If $x, y \in L_1 + L_\infty$ and $y \prec\prec x$, then,*

$$\big(\sigma_\tau(y)\big)^* \leqslant \sigma_\tau(y^*) \prec\prec \sigma_\tau(x^*).$$

Proof. Set $d_y(s) = m(t\colon |y(t)| > s)$. In the case of the semi-axis, $d_{\sigma_\tau y} = \tau d_y = d_{\sigma_\tau(y^*)}$. In the case of the interval $(0,1)$, $d_{\sigma_\tau y} \leqslant \tau d_y$ and $d_{\sigma_\tau(y^*)} = \min\{1, \tau d_y\}$. Hence, $d_{\sigma_\tau y} \leqslant d_{\sigma_\tau(y^*)}$ and so $(\sigma_\tau(y))^* \leqslant \sigma_\tau(y^*)$. Finally,

$$\int_0^t \sigma_\tau(y^*)(s)\,ds = \tau \int_0^{\frac{t}{\tau}} y^*(s)\,ds \leqslant \tau \int_0^{\frac{t}{\tau}} x^*(s)\,ds = \int_0^t \sigma_\tau(x^*)(s)\,ds. \qquad \square$$

The next lemma introduces the dilation functional φ on E, which is a priori non-linear. The behavior of the functional φ on the positive part E_+ of E provides the key to our main question.

Lemma 3. *For every $x \in E$ the following limit exists and is finite.*

$$\varphi(x) = \lim_{s\to\infty} \frac{1}{s} \left\| \sigma_s(x^*) \right\|_E, \quad x \in E. \tag{7}$$

If, in addition, $E = E(0,\infty)$, then the following limits exist and are finite.

$$\varphi_{fin}(x) = \lim_{s\to\infty} \frac{1}{s} \left\| \sigma_s(x^*)\chi_{[0,1]} \right\|_E, \quad x \in E, \tag{8}$$

$$\varphi_{cut}(x) = \lim_{s\to\infty} \frac{1}{s} \left\| \sigma_s(x^*)\chi_{[0,s]} \right\|_E, \quad x \in E. \tag{9}$$

The following properties hold.

(i) *If E is symmetric, then $\varphi(y) \leqslant \varphi(x)$ provided that $x, y \in E$ satisfy $y^* \leqslant x^*$.*
(ii) *If E is symmetric, then $\varphi(x) \leqslant \|x\|_E$ for every $x \in E$.*
(iii) *If E is strictly symmetric, then $\varphi(y) \leqslant \varphi(x)$ provided that $x, y \in E$ satisfy $y \prec\prec x$.*
(iv) *If E is symmetric, then $\varphi(\sigma_\tau(x^*)) = \tau\varphi(x)$, $\tau > 0$.*
(v) *If E is strictly symmetric, then φ is norm-continuous.*
(vi) *If E is strictly symmetric, then φ is convex.*

If, in addition, $E = E(0,\infty)$, then φ_{fin} also satisfies (i)–(vi), *while φ_{cut} satisfies* (i), (ii), (iii), (v) *and* (vi). *If, in addition, $E \not\subseteq L_1$, then φ_{cut} also satisfies* (iv).

Proof. We prove that the function $s \to \frac{1}{s}\|\sigma_s x^*\|_E$ is decreasing. Let $s_2 > s_1$. We have $s_2 = s_3 s_1$ and $s_3 > 1$. Therefore,

$$\frac{1}{s_2} \left\| \sigma_{s_3}\big(\sigma_{s_1}(x^*)\big) \right\|_E \leqslant \frac{\|\sigma_{s_3}\|_{E\to E}}{s_2} \left\| \sigma_{s_1}(x^*) \right\|_E \leqslant \frac{1}{s_1} \left\| \sigma_{s_1}(x^*) \right\|_E,$$

since $\|\sigma_{s_3}\|_{E\to E} \leqslant s_3$. It follows immediately that the limit in (7) exists.

(i) Trivial.

(ii) This follows from the fact that $\|\sigma_s(x^*)\|_E \leqslant s\|x\|_E$.

(iii) Since $y \prec\prec x$, it follows that $\sigma_s(y^*) \prec\prec \sigma_s(x^*)$. Since E is strictly symmetric, it follows that $\|\sigma_s(y^*)\|_E \leqslant \|\sigma_s(x^*)\|_E$. Therefore,

$$\varphi(y) = \lim_{s\to\infty} \frac{1}{s} \left\| \sigma_s(y^*) \right\|_E \leqslant \lim_{s\to\infty} \frac{1}{s} \left\| \sigma_s(x^*) \right\|_E = \varphi(x).$$

(iv) Applying the semigroup property of the dilation operators σ_τ,

$$\lim_{s\to\infty}\frac{1}{s}\big\|\sigma_s\big(\sigma_\tau(x^*)\big)\big\|_E=\tau\lim_{\tau\to\infty}\frac{1}{s\tau}\big\|\sigma_{s\tau}(x^*)\big\|_E=\tau\varphi(x).$$

(v) By triangle inequality,

$$\big|\big\|\sigma_s(x^*)\big\|_E-\big\|\sigma_s(y^*)\big\|_E\big|\leqslant\big\|\sigma_s(x^*-y^*)\big\|_E.$$

Using (4) and Lemma 2 one can obtain $\sigma_s(x^*-y^*)\prec\prec\sigma_s((x-y)^*)$. Since E is strictly symmetric,

$$\big|\big\|\sigma_s(x^*)\big\|_E-\big\|\sigma_s(y^*)\big\|_E\big|\leqslant\big\|\sigma_s\big((x-y)^*\big)\big\|_E.$$

Now, one can divide by s and let $s\to\infty$. Therefore,

$$\big|\varphi(x)-\varphi(y)\big|\leqslant\varphi(x-y)\leqslant\|x-y\|_E.$$

(vi) It follows from (3) and Lemma 2 that $\sigma_s((x+y)^*)\prec\prec\sigma_s(x^*)+\sigma_s(y^*)$. Therefore,

$$\varphi(x+y)=\lim_{s\to\infty}\frac{1}{s}\big\|\sigma_s\big((x+y)^*\big)\big\|_E\leqslant\lim_{s\to\infty}\frac{1}{s}\big(\big\|\sigma_s(x^*)\big\|_E+\big\|\sigma_s(y^*)\big\|_E\big)=\varphi(x)+\varphi(y).$$

Existence and properties (i)–(vi) of φ_{fin} can be proved in a similar way. Existence and properties (i), (ii), (iii), (iv), (vi) of φ_{cut} can be proved in a similar way. Let us prove (iv) for φ_{cut}.

(iv) Assume $E\not\subset L_1$. By Lemma 4 below, $\varphi(x^*\chi_{[\tau^{-1},1]})=\varphi_{cut}(x^*\chi_{[\tau^{-1},1]})=0$. Hence,

$$\varphi(x^*\chi_{[0,\tau^{-1}]})\leqslant\varphi(x^*\chi_{[0,1]})\leqslant\varphi(x^*\chi_{[0,\tau^{-1}]})+\varphi(x^*\chi_{[\tau^{-1},1]})=\varphi(x^*\chi_{[0,\tau^{-1}]}).$$

Therefore,

$$\varphi_{cut}\big(\sigma_\tau(x^*)\big)=\varphi\big(\sigma_\tau(x^*\chi_{[0,\tau^{-1}]})\big)=\tau\varphi(x^*\chi_{[0,\tau^{-1}]})=\tau\varphi_{cut}(x).\qquad\square$$

Lemma 4. *If $E=E(0,1)$ be a symmetric space on $(0,1)$ and $x\in L_\infty$, then $\varphi(x)=0$. If $E=E(0,\infty)$ be a symmetric space on $(0,\infty)$ and $x\in L_\infty\cap E$, then $\varphi_{fin}(x)=0$. If $E=E(0,\infty)\nsubseteq L_1$ and $x\in E\cap L_\infty$, then $\varphi_{cut}(x)=0$. In particular, the functional φ vanishes on every separable space $E=E(0,1)$.*

Proof. Clearly, $\varphi(x)=\varphi(x^*\chi_{[0,1]})\leqslant\|x\|_\infty\varphi(\chi_{[0,1]})$ in the first case. Similarly, $\varphi_{fin}(x)\leqslant\|x\|_\infty\varphi_{fin}(\chi_{[0,1]})$ ($\varphi_{cut}(x)\leqslant\|x\|_\infty\varphi_{cut}(\chi_{[0,1]})$) in the second (third) case. It is clear that $\varphi(\chi_{[0,1]})=0$ ($\varphi_{fin}(\chi_{[0,1]})=0$) in the first (second) case. Also, $E\not\subset L_1$ implies that $\|\chi_{[0,n]}\|_E=o(n)$ and, therefore, $\varphi_{cut}(\chi_{[0,1]})=0$. The assertion follows immediately. $\square$

Lemma 5. *Let E be a strictly symmetric space. For functions $0\leqslant x_1,\ldots,x_k\in E$ and numbers $\lambda_1,\ldots,\lambda_k\geqslant 0$*

$$\varphi\left(\sum_{i=1}^k\lambda_i x_i\right)=\varphi\left(\sum_{i=1}^k\lambda_i x_i^*\right).$$

If $E = E(0,\infty)$, then the same is valid for φ_{fin}. If, in addition, $E \nsubseteq L_1$, then the same is valid for φ_{cut}.

Proof. Applying the inequality (6) n times, we have for positive functions $x_1, \dots, x_{2^n}$

$$\left(x_1^* + \dots + x_{2^n}^*\right) \prec\prec 2^n \sigma_{2^{-n}}(x_1 + \dots + x_{2^n}).$$

Therefore, by Lemma 3(iii),

$$\varphi\left(x_1^* + \dots + x_{2^n}^*\right) \leqslant \varphi\left(2^n \sigma_{2^{-n}}(x_1 + \dots + x_{2^n})\right).$$

By Lemma 3(iv), $2^k \varphi(\sigma_{2^{-k}}(z^*)) = \varphi(z^*)$. Therefore,

$$\varphi\left(x_1^* + \dots + x_{2^n}^*\right) \leqslant \varphi(x_1 + \dots + x_{2^n}).$$

Converse inequality follows trivially from (3) and Lemma 3(iii).

The assertion of the lemma follows now from Lemma 3(v). □

Note, that y and z in the proposition below are arbitrary, that is, y, z do not necessary belong to $Q_+(x)$.

Proposition 6. *Let E be a symmetric space equipped with a Fatou norm. If $x \geqslant 0 \in E$, then in each of the following cases there exists a decomposition $x = y + z$, such that $y, z \geqslant 0$ and such that the following assertions hold.*

(i) *If $E = E(0,1)$, then $\varphi(x) = \varphi(y) = \varphi(z)$.*
(ii) *If $E = E(0,\infty)$ and $\varphi_{cut}(x) = 0$, then $\varphi(x) = \varphi(y) = \varphi(z)$.*
(iii) *If $E = E(0,\infty)$, then $\varphi_{fin}(x) = \varphi_{fin}(y) = \varphi_{fin}(z)$.*
(iv) *If $E = E(0,\infty)$, then $\varphi_{cut}(x) = \varphi_{cut}(y) = \varphi_{cut}(z)$.*

Proof. We will prove only the first assertion. The proofs of the third and fourth assertions are exactly the same. The proof of the second assertion requires replacement of the interval $[\frac{1}{m}, \frac{1}{n}]$ with the interval $[n, m]$.

We may assume that $x = x^*$. Fix $n \in N$. The sequence $\sigma_n(x\chi_{[\frac{1}{m}, \frac{1}{n}]})$ converges to $\sigma_n(x\chi_{[0, \frac{1}{n}]})$ almost everywhere when $m \to \infty$.

By the definition of Fatou norm,

$$\left\|\sigma_n(x\chi_{[\frac{1}{m}, \frac{1}{n}]})\right\|_E \to_m \left\|\sigma_n(x\chi_{[0, \frac{1}{n}]})\right\|_E.$$

For each $n \in N$, one can select $f(n) > n$, such that

$$\left\|\sigma_n(x\chi_{[\frac{1}{f(n)}, \frac{1}{n}]})\right\|_E \geqslant \left(1 - \frac{1}{n}\right)\left\|\sigma_n(x\chi_{[0, \frac{1}{n}]})\right\|_E.$$

Fix some n_0 and set $n_k = f^k(n_0)$, $k \in N$. Here, $f^k = f \circ \dots \circ f$ (k times). Define

$$y=\sum_{k=0}^{\infty}x\chi_{[\frac{1}{n_{2k+1}},\frac{1}{n_{2k}}]},$$

$$z=\sum_{k=1}^{\infty}x\chi_{[\frac{1}{n_{2k}},\frac{1}{n_{2k-1}}]}.$$

It is clear, that

$$\frac{1}{n_{2k}}\left\|\sigma_{n_{2k}}(y^*)\right\|_E\geqslant\frac{1}{n_{2k}}\left\|\sigma_{n_{2k}}(y)\right\|_E\geqslant\frac{1}{n_{2k}}\left\|\sigma_{n_{2k}}(x\chi_{[\frac{1}{n_{2k+1}},\frac{1}{n_{2k}}]})\right\|_E. \tag{10}$$

By definition of n_k,

$$\frac{1}{n_{2k}}\left\|\sigma_{n_{2k}}(x\chi_{[\frac{1}{n_{2k+1}},\frac{1}{n_{2k}}]})\right\|_E\geqslant\frac{1}{n_{2k}}\left(1-\frac{1}{n_{2k}}\right)\left\|\sigma_{n_{2k}}(x\chi_{[0,\frac{1}{n_{2k}}]})\right\|_E. \tag{11}$$

It follows from (10) and (11) that

$$\frac{1}{n_{2k}}\left\|\sigma_{n_{2k}}(y^*)\right\|_E\geqslant\left(1-\frac{1}{n_{2k}}\right)\frac{1}{n_{2k}}\left\|\sigma_{n_{2k}}(x\chi_{[0,\frac{1}{n_{2k}}]})\right\|_E\geqslant\left(1-\frac{1}{n_{2k}}\right)\varphi(x\chi_{[0,\frac{1}{n_{2k}}]}). \tag{12}$$

By Lemma 4, $\varphi(x\chi_{[\frac{1}{n_{2k}},1]})=0$. Since φ is convex, then

$$\varphi(x\chi_{[0,\frac{1}{n_{2k}}]})\leqslant\varphi(x)\leqslant\varphi(x\chi_{[0,\frac{1}{n_{2k}}]})+\varphi(x\chi_{[\frac{1}{n_{2k}},1]})=\varphi(x\chi_{[0,\frac{1}{n_{2k}}]}). \tag{13}$$

It follows from (12) and (13) that

$$\frac{1}{n_{2k}}\left\|\sigma_{n_{2k}}(y^*)\right\|_E\geqslant\left(1-\frac{1}{n_{2k}}\right)\varphi(x).$$

Passing to the limit, we obtain $\varphi(y)\geqslant\varphi(x)$. The converse inequality is obvious.

Hence, $\varphi(y)=\varphi(x)=\varphi(z)$, and this completes the proof of the proposition. □

Lemma 7. *If space E is strictly symmetric, then $\varphi(y)=\varphi(x)$ for every $y\in Q'(x)$. If, in addition, $E=E(0,\infty)$, then $\varphi_{fin}(y)=\varphi_{fin}(x)$ for every $y\in Q'(x)$. If $E\nsubseteq L_1$, then $\varphi_{cut}(y)=\varphi_{cut}(x)$ for every $y\in Q'(x)$.*

Proof. Let

$$z=\sum_{i=1}^{s}\lambda_i x_i,$$

where $\lambda_i\geqslant 0$, $\sum_{i=1}^{s}\lambda_i=1$, $x_i\geqslant 0$ and $x_i^*=x$. By Lemma 5, we obtain $\varphi(z)=\varphi(x)$. However, $y\in Q'(x)$ can be approximated by such z. Since φ is continuous in strictly symmetric spaces, the lemma follows readily.

The proofs are the same in cases of φ_{fin} and φ_{cut}. □

If A is a convex set, then function $\theta: A \to R$ is called additive homogeneous if and only if

$$\theta(\alpha y_1 + \beta y_2) = \alpha\theta(y_1) + \beta\theta(y_2), \quad y_1, y_2 \in A, \ \alpha, \beta \in R_+.$$

Proposition 8. *Let E be a strictly symmetric space and $x \in E$. Then the following assertions hold.*

(i) *If $E = E(0,1)$, then φ is additive homogeneous on $Q_+(x)$.*
(ii) *If $E = E(0,\infty)$, then φ_{fin} is additive homogeneous on $Q_+(x)$.*
(iii) *If $E \nsubseteq L_1$, then φ_{cut} is additive homogeneous on $Q_+(x)$.*

Proof. We will only prove the first assertion. The proofs of the other two assertions are exactly the same.

Let $y \in \mathrm{Conv}(\mathrm{extr}(\Omega_+(x)))$, so that

$$y = \sum_{i=1}^{m} \lambda_i x_i,$$

where $\lambda_i \geqslant 0$, $\sum_{i=1}^m \lambda_i = 1$, $x_i \geqslant 0$ and $x_i^* = x^*\chi_{[0,\beta_i]}$. Denote $z = \sum_{i=1}^m \lambda_i x^*\chi_{[0,\beta_i]}$ and $u = \sum_{\beta_i > 0} \lambda_i x^* \chi_{[0,1]}$. By Lemma 5, $\varphi(y) = \varphi(z)$.

Since $|z-u| \in L_\infty$, then $\varphi(|u-z|) = 0$ by Lemma 4. By the triangle inequality,

$$\varphi(u) \leqslant \varphi(z) + \varphi(|u-z|) = \varphi(z) \leqslant \varphi(u) + \varphi(|u-z|) = \varphi(u).$$

Hence, $\varphi(y) = \varphi(u) = (\sum_{\beta_i>0} \lambda_i)\varphi(x)$. It is clear that the last expression is additive homogeneous on the set $\mathrm{Conv}(\mathrm{extr}(\Omega_+(x)))$. By Lemma 3, the functional φ is continuous on $Q_+(x)$. Hence, it is additive homogeneous on the set $Q_+(x)$. □

Proposition 9. *Let $E = E(0,\infty)$ be a symmetric space on semi-axis equipped with a Fatou norm. Suppose that $E \nsubseteq L_1$ and $x \in E$. If $\Omega_+(x) = Q_+(x)$, then φ is additive homogeneous on $\Omega_+(x)$.*

Proof. It follows from Proposition 8 that φ_{cut} is additive homogeneous on $Q_+(x)$. By assumption, $\Omega_+(x) = Q_+(x)$. Hence, φ_{cut} is additive homogeneous on $\Omega_+(x)$. It follows now from Proposition 6(iv) that $\varphi_{cut}(x) = 0$. This assertion and Lemma 2 imply that $\varphi(x^*\chi_{[0,\beta]}) = 0$ for every finite β.

Let $y \in \mathrm{Conv}(\mathrm{extr}(\Omega_+(x)))$. Hence,

$$y = \sum_{i=1}^{m} \lambda_i x_i,$$

where $\lambda_i \geqslant 0$, $\sum_{i=1}^m \lambda_i = 1$, $x_i \geqslant 0$ and $x_i^* = x^*\chi_{[0,\beta_i]}$. By convexity of φ,

$$\varphi(y) \leqslant \varphi\biggl(\sum_{\beta_i \in [0,\infty)} \lambda_i x_i\biggr) + \varphi\biggl(\sum_{\beta_i = \infty} \lambda_i x_i\biggr).$$

However,

$$0 \leqslant \varphi\biggl(\sum_{\beta_i \in [0,\infty)} \lambda_i x_i\biggr) \leqslant \sum_{\beta_i \in [0,\infty)} \lambda_i \varphi(x^* \chi_{[0,\beta_i]}) = 0.$$

It then follows that

$$\varphi(y) \leqslant \varphi\biggl(\sum_{\beta_i=\infty} \lambda_i x_i\biggr).$$

The converse inequality is obvious. By Lemma 5,

$$\varphi(y) = \varphi\biggl(\sum_{\beta_i=\infty} \lambda_i x_i\biggr) = \varphi\biggl(\sum_{\beta_i=\infty} \lambda_i x_i^*\biggr) = \biggl(\sum_{\beta_i=\infty} \lambda_i\biggr)\varphi(x).$$

Clearly, the last expression is additive homogeneous on $\mathrm{Conv}(\mathrm{extr}(\Omega_+(x)))$. Hence, the functional φ is additive homogeneous on $Q_+(x) = \Omega_+(x)$. $\square$

Lemma 10. *Let $E = E(0,\infty)$ be a strictly symmetric space on $(0,\infty)$ and $x \in E$. Suppose, that $E \nsubseteq L_1$. If $P(x|\mathcal{A}) \in Q'(x)$ for every $\mathcal{A}$, then $\varphi_{cut}(x) = 0$.*

Proof. Suppose that $x = x^*$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A}) \in E \cap L_\infty$. By the assumption, $y \in Q'(x)$. By Lemmas 7 and 4, $\varphi_{cut}(x) = \varphi_{cut}(y) = 0$. $\square$

Lemma 11. *Let E and x be as in Lemma 10. If $L_\infty \subseteq E$, then $\varphi(x) = 0$.*

Proof. Due to the choice of E, we have $1 \in E$. However, $\sigma_\tau(1) = 1$ implies $\varphi(1) = 0$. Thus, for every $z \in E \cap L_\infty$, we have $\varphi(z) = 0$. However, for every $x \in E$, we have $\varphi(x^*\chi_{[0,1]}) = 0$ due to Lemma 10. Hence,

$$0 \leqslant \varphi(x) = \varphi(x^*) \leqslant \varphi(x^*\chi_{[0,1]}) + \varphi(x^*\chi_{[1,\infty)}) = 0 + 0 = 0. \qquad \square$$

Lemma 12. *Let E and x be as in Lemma 10. If $y \in E \cap L_\infty$ and if*

$$\omega(y) := \limsup_{t\to\infty} \frac{\int_0^t y^*(s)\,ds}{\int_0^t x^*(s)\,ds},$$

then $\varphi(y) = \omega(y)\varphi(x)$. In particular, if in addition $\varphi(x) > 0$, then $\omega(y) < \infty$.

Proof. Fix $\varepsilon > 0$. There exists $T > 0$, such that for every $t > T$,

$$\int_0^t y^*(s) \leqslant \bigl(\omega(y) + \varepsilon\bigr) \int_0^t x^*(s)\,ds.$$

It then follows that $y \prec\prec (\omega(y)+\varepsilon)(x^*+C\chi_{[0,T]})$ for some constant C. By Lemma 3(iii), $\varphi(y) \leqslant (\omega(y)+\varepsilon)\varphi(x^*+C\chi_{[0,T]})$. By Lemma 4, $\varphi(C\chi_{[0,T]})=0$ and, therefore, $\varphi(x^*+C\chi_{[0,T]})=\varphi(x)$. Hence $\varphi(y)\leqslant \omega(y)\varphi(x)$.

Now, fix $\omega<\omega(y)$. There exists a sequence $t_k\to\infty$, such that

$$\int_0^{t_k} y^*(s)\,ds \geqslant \omega\int_0^{t_k} x^*(s)\,ds.$$

Without loss of generality, $t_0=0$. Set $u=P(x^*|\mathcal{A})$, where $\mathcal{A}=\{[t_k,t_{k+1})\}$. It then follows that $\omega u \prec\prec y$ and $\omega\varphi(u)\leqslant\varphi(y)$. However, $u\in Q'(x)$ and $\varphi(u)=\varphi(x)$ due to Lemma 7. Hence $\omega(y)\varphi(x)\leqslant\varphi(y)$. □

Proposition 13. *Let $E=E(0,\infty)$ be a symmetric space on the semi-axis and let $x\in E$. If $\varphi(x)=0$, then $x\chi_A\in Q'(x)$ for every Lebesgue measurable subset $A\subseteq(0,\infty)$.*

Proof. Let $[0,\infty)=B\cup C$, where B,C are disjoint sets such that $m(B)=m(A)$ and $m(C)=\infty$. Fix a partition $C=\bigcup_{i=1}^{n+1}C_i$, where $m(C_i)=m(R_+\setminus A)$, $1\leqslant i\leqslant n$. Let $\gamma: B\to A$ and $\gamma_i: C_i\to R_+\setminus A$, $1\leqslant i\leqslant n$ be measure-preserving transformations. Define functions x_n^i, $1\leqslant i\leqslant n$ by the following construction. Set $x_n^i\chi_B=x\circ\gamma$, $x_n^i|_{C_i}=x\circ\gamma_i$ and $x_n^i|_{C_j}=0$ if $i\neq j$. Clearly, $x_n^i\sim x$ and

$$\left\|(x\chi_A)\circ\gamma-\frac{1}{n}\sum_{i=1}^n x_n^i\right\|_E=\frac{1}{n}\left\|\sigma_n(x\chi_{[0,\infty)\setminus A})\right\|_E\leqslant\frac{1}{n}\left\|\sigma_n(x^*)\right\|_E\to 0.$$

Hence, $(x\chi_A)\circ\gamma\in Q'(x)$. Thus, $x\chi_A\in Q'(x)$. □

Corollary 14. *Let $E=E(0,\infty)$ be a symmetric space on semi-axis. If $\varphi(x)=0$, then $y\chi_A\in Q'(x)$ for every $y\in Q'(x)$.*

Proof. It follows from assumption and Lemma 7 that $\varphi(y)=\varphi(x)=0$. Lemma 13 implies that $y\chi_A\in Q'(y)$. Since $y\chi_A\in Q'(y)$ and $y\in Q'(x)$, then Lemma 16 implies $y\chi_A\in Q'(x)$. □

An assertion somewhat similar to the lemma below is contained in [3, Lemma 1.3].

Lemma 15. *Assume that $x\in E$ satisfies conditions of Proposition 13. If $y\in Q'(x)$ and $0\leqslant z\leqslant y$, then $z\in Q'(x)$.*

Proof. Define sets e_n^i, $i=1,\dots,n$ by

$$e_n^i=\left\{t:\ \frac{i-1}{n}y(t)\leqslant z(t)\leqslant\frac{i}{n}y(t)\right\}.$$

Define functions y_n^k, $k=1,\dots,n$ as $y_n^k=y\sum_{k<(i+n)/2}\chi_{e_n^i}$. By Corollary 14, $y_n^k\in Q'(x)$. Put

$$s_n=\frac{1}{n}\sum_{k=1}^n y_n^k\in Q'(x).$$

Clearly,

$$\left|s_n(t) - \big(y(t) + z(t)\big)/2\right| \leqslant \frac{2y(t)}{n}, \quad \forall t \in e_n^i.$$

Hence, $s_n \to (y+z)/2$ by norm. Therefore, $(y+z)/2 \in Q'(x)$. We can repeat this procedure n times and obtain $2^{-n}((2^n-1)z + y) \in Q'(x)$. Therefore, $z \in Q'(x)$. □

The following assertion seems to be known. We include the details of the proof for lack of a convenient reference.

Lemma 16. *Let E be a symmetric space either on $(0,1)$ or $(0,\infty)$ and $x \in E$. If $y \in Q'(z)$ and $z \in Q'(x)$, then $y \in Q'(x)$.*

Proof. Without loss of generality, $y = y^*$, $z = z^*$ and $x = x^*$. Let $y \in Q'(z)$. Hence, for every $\varepsilon > 0$, one can find $n \in N$, $\lambda_i \in R_+$ and measurable functions $z_i \sim z$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \lambda_i = 1$ and

$$\left\| y - \sum_{i=1}^n \lambda_i z_i \right\|_E \leqslant \varepsilon.$$

One can find measure-preserving transformations γ_i, such that

$$\|z_i - z \circ \gamma_i\|_{L_1 \cap L_\infty} \leqslant \varepsilon.$$

Hence,

$$\left\| y - \sum_{i=1}^n \lambda_i z \circ \gamma_i \right\|_E \leqslant 2\varepsilon.$$

However, $z \in Q'(x)$. Consequently, arguing in a similar way, one can find $m \in N$, $\mu_j \in R_+$ and measure preserving transformations δ_j, $1 \leqslant j \leqslant n$ such that $\sum_{j=1}^m \mu_j = 1$ and

$$\left\| z - \sum_{j=1}^m \mu_j x \circ \delta_j \right\|_E \leqslant 2\varepsilon.$$

Therefore,

$$\left\| y - \sum_{i=1}^n \sum_{j=1}^m \lambda_i \mu_j x \circ \gamma_i \circ \delta_j \right\| \leqslant 4\varepsilon$$

and this suffices to complete the proof. □

Remark 17. The collection of sets $\{Q(x),\ x \in E\}$ also satisfies the transitivity property expressed in Lemma 16. We do not know whether this is the case for the collection $\{Q_+(x),\ x \in E\}$.

4. Main results

The implication (ii) $\Rightarrow$ (i) in the following theorem is almost verbatim repetition of the argument given in [3, Lemma 3.1] for the case of finite measure. For convenience of the reader, we present here a proof of the most important case.

Theorem 18. (a) *Let E be a fully symmetric space and $x \in E$. If $E = E(0,1)$ or $E = E(0,\infty)$ and $E \nsubseteq L_1$, then the following conditions are equivalent.*

(i) $P(x|\mathcal{A}) \in Q'(x)$ *for every* $\mathcal{A} \in \mathfrak{A}$.
(ii) $\varphi(x) = 0$.

(b) *If $E = E(0,\infty)$ and $E \subseteq L_1$, then the following conditions are equivalent.*

(i) $P(x|\mathcal{A}) \in Q'(x)$ *for every* $\mathcal{A} \in \mathfrak{A}$.
(ii) $\varphi_{fin}(x) = 0$.

Proof. (a) (i) $\Rightarrow$ (ii) Let $E = E(0,1)$ and $x = x^*$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A})$. By assumption, $y \in Q'(x)$. By Lemmas 7 and 4, $\varphi(x) = \varphi(y) = 0$.

Let $E = E(0,\infty)$ and $L_\infty \subseteq E \nsubseteq L_1$. The assertion is proved in Lemma 11.

Let $E = E(0,\infty)$ and $L_\infty \nsubseteq E \nsubseteq L_1$. Suppose that $x = x^*$ and $\varphi(x) > 0$. Set $\mathcal{B} = \{[0,1]\}$, $\psi' = P(x|\mathcal{B})$ and $\psi(t) = \int_0^t \psi'(s)\,ds$. By Lemma 7, $\varphi(\psi') = \varphi(x)$.

Let $y \in E \cap L_\infty$. It follows from Lemma 12, that $\omega(y) < \infty$. Therefore, $y \in M_\psi$. Hence, $E \cap L_\infty \subseteq M_\psi$. Since E is fully symmetric and $\psi' \in E \cap L_\infty$, then $M_\psi \subseteq E \cap L_\infty$. Therefore, $E \cap L_\infty = M_\psi$.

If $u = 2\sigma_{\frac{1}{2}}\psi'$, then $\varphi(u) = \varphi(\psi')$ by Lemma 3(v). Hence $\omega(u)\varphi(x) = \varphi(x)$ and $\omega(u) = 1$. However,

$$\omega(u) = \limsup_{t\to\infty} \frac{\int_0^t 2x(2s)\,ds}{\int_0^t x(s)\,ds} = \limsup_{t\to\infty} \frac{\psi(2t)}{\psi(t)}.$$

Thus,

$$\lim_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1. \tag{14}$$

Let G be the set defined by

$$G = \left\{ y \in E\colon\ \exists C\ \sup_{t\geqslant 1} \frac{y^*(t)}{\psi'(Ct)} < \infty \right\}.$$

Note, that our set G differs from the one introduced in [10]. If $y_1, y_2 \in G$, then $y_i^*(t) \leqslant C_i\psi(Ct)$ for $t \geqslant \frac{1}{2}$. It then follows

$$(y_1 + y_2)^*(t) \leqslant y_1^*\left(\frac{t}{2}\right) + y_2^*\left(\frac{t}{2}\right) \leqslant (C_1 + C_2)\psi'\left(\frac{C}{2}t\right).$$

In particular, G is a linear set and $\mathrm{Conv}(\{y^* = x^*\}) \subseteq G$. If the condition (14) holds, then there exists a sequence t_k, such that $t_0 = 0$, $t_1 = 1$ and for every k

$$\frac{\psi(t_{k+1}) - \psi(t_k)}{t_{k+1} - t_k} \geqslant \frac{2}{3}\frac{\psi(\frac{1}{2}t_{k+1})}{t_{k+1}}.$$

Set $\mathcal{A} = \{[t_k, t_{k+1}]\}$ and $z = P(x|\mathcal{A})$. It follows from the construction given in [10] that $\|(z - y)\chi_{[\frac{1}{2}t_k, t_k]}\|_{M_\psi} \geqslant \frac{1}{4}$ for every $y \in G$ and every sufficiently large k. However, $\|(y - z)\chi_{[\frac{1}{2}t_k, t_k]}\|_{L_\infty} \to 0$. Since $M_\psi = E \cap L_\infty$, then $\|(z - y)\chi_{[\frac{1}{2}t_k, t_k]}\|_E \geqslant \frac{1}{4}$ for sufficiently large k. In particular, $\|y - z\|_E \geqslant \frac{1}{4}$. Hence, $\mathrm{dist}_E(z, G) \geqslant \frac{1}{4}$ and $\mathrm{dist}_E(z, Q'(x)) \geqslant \frac{1}{4}$. This contradicts the assumption that $P(x|\mathcal{A}) \in Q'(x)$.

(a) (ii) $\Rightarrow$ (i) Let $E = E(0, 1)$ or $E = E(0, \infty) \nsubseteq L_1$. We will prove the assertion for the case when $\mathcal{A} = \{[0, 1]\}$. The general proof is similar. Without loss of generality, x decreases on $[0, 1]$. Define functions x_n^i, $i = 0, \ldots, n-1$ such that (i) $x_n^i = x$ outside $(0, 1)$ and (ii) $x_n^i(t) = x((t + \frac{i}{n}) (\mathrm{mod}\ 1))$ if $t \in (0, 1)$. Set $x_n(t) = x(t - \frac{i}{n})$ if $\frac{i}{n} \leqslant t \leqslant \frac{i+1}{n}$, $0 \leqslant i \leqslant n-1$ and $x_n(t) = 0$ if $t \geqslant 1$. Clearly, $x_n^i \sim x$ and $(x_n)^* \leqslant \sigma_n(x^*)$.

We will show that

$$\int_0^1 x(s)\,ds - \frac{1}{n}\sum_{i=0}^{n-1} x\left(\left(t + \frac{i}{n}\right)(\mathrm{mod}\ 1)\right) \leqslant \int_0^{\frac{1}{n}} x(s)\,ds$$

and

$$\int_0^1 x(s)\,ds - \frac{1}{n}\sum_{i=0}^{n-1} x\left(\left(t + \frac{i}{n}\right)(\mathrm{mod}\ 1)\right) \geqslant -\frac{1}{n}x_n(t).$$

We will prove only the first inequality. The proof of the second one is identical.

Without loss of generality, $t \in [0, \frac{1}{n}]$. Clearly,

$$\frac{1}{n}x\left(t + \frac{i}{n}\right) \geqslant \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} x(s)\,ds$$

for $i = 0, \ldots, n-2$. Hence,

$$\int_0^1 x(s)\,ds - \frac{1}{n}\sum_{i=0}^{n-1} x\left(t + \frac{i}{n}\right) = \int_0^{\frac{1}{n}} x(s)\,ds - \frac{1}{n}x\left(t + \frac{n-1}{n}\right) - \sum_{i=0}^{n-2}\left(\frac{1}{n}x\left(t + \frac{i}{n}\right) - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} x(s)\,ds\right) \leqslant \int_0^{\frac{1}{n}} x(s)\,ds.$$

Therefore,

$$\left|\int_0^1 x(s)\,ds - \frac{1}{n}\sum_{i=0}^{n-1} x_n^i(t)\right| \leqslant \frac{1}{n} z_n(t), \quad t \in [0,1],$$

where $z_n = x_n + (\int_0^1 x_n(s)\,ds)\chi_{[0,1]}$. Obviously, $z_n \prec\prec 2x_n \leqslant 2\sigma_n(x^*)$ and, therefore, $\|z_n\|_E \leqslant 2\|\sigma_n(x^*)\|_E$.

It then follows that

$$\left\|P(x|\mathcal{A}) - \frac{1}{n}\sum_{i=0}^{n-1} x_n^i\right\|_E \leqslant \frac{2}{n}\|\sigma_n(x^*)\|_E \to 0.$$

(b) (i) ⇒ (ii) Let $E = E(0,\infty)$ and $E \subset L_1$. Set $A = \{s\colon x(s) \geqslant 1\}$ and $\mathcal{A} = \{A\}$. Set $y = P(x|\mathcal{A}) \in E \cap L_\infty$. Lemma 4 implies that $\varphi(y) = 0$. By the assumption, $y \in Q'(x)$. By Lemma 7, $\varphi(x) = \varphi(y) = 0$.

(b) (ii) ⇒ (i) The assertion follows from Theorem 23. □

The following proposition is the core technical result of the article. In case of the interval (0, 1) it may be found in [3, Lemma 3.2]. However, our proof is more general, simpler and shorter.

We consider the functions of the form

$$x = \sum_{i\in Z} x_i \chi_{[a_{i-1},a_i]}, \qquad y = \sum_{i\in Z} y_i \chi_{[a_{i-1},a_i]}, \tag{15}$$

where $\{a_i\}_{i\in Z}$ is an increasing sequence (possibly finite or one-sidedly infinite).

Proposition 19. *Let $y = y^*$ and $x = x^*$ be the functions of the form* (15) *either on* $(0,1)$ *or on* $(0,\infty)$. *If $y \prec\prec x$, then there exists a countable collection $\{\Delta_k\}_{k\in\mathcal{K}}$ of disjoint sets, where $\Delta_k = I_k \cup J_k$ with intervals I_k and J_k of finite measure, such that*

- (i) *the functions x and y are constant on the intervals I_k and J_k and the interval I_k lies to the left of J_k, $k \in \mathcal{K}$.*
- (ii) $y|_{\Delta_k} \prec x|_{\Delta_k}$, $k \in \mathcal{K}$.
- (iii) $y(t) \leqslant x(t)$ *if* $t \notin \bigcup_{k\in\mathcal{K}} \Delta_k$.

If, in addition, x and y are functions on $(0,1)$ *and* $\int_0^1 y(s)\,ds = \int_0^1 x(s)\,ds$, *then $y(t) = x(t)$ if $t \notin \bigcup_{k\in\mathcal{K}} \Delta_k$.*

Proof. There exists a subsequence $\{a_{m_i}\}_{i\in\mathcal{I}}$ (possibly finite or one-sidedly infinite) such that $\{x < y\} = \bigcup_{\in\mathcal{I}} [a_{m_i - 1}, a_{m_i}]$. Since $y \prec\prec x$, we have

$$\int_0^t (x-y)_+(s)\,ds - \int_0^t (y-x)_+(s)\,ds = \int_0^t x(s)\,ds - \int_0^t y(s)\,ds \geqslant 0.$$

For each $i \in \mathcal{I}$, denote by b_i the minimal $t > 0$, such that

$$\int_0^t (x-y)_+(s)\,ds = \int_0^{a_{m_i}} (y-x)_+(s)\,ds.$$

Clearly, for every $i \in \mathcal{I}$,

$$\int_0^{a_{m_i-1}} (x-y)_+(s)\,ds = \int_0^{a_{m_i}} (x-y)_+(s)\,ds \geqslant \int_0^{a_{m_i}} (y-x)_+(s)\,ds.$$

Hence, $b_i \leqslant a_{m_i-1}$. For each $i \in \mathcal{I}$, the set $[b_{i-1}, b_i] \cap \{x > y\}$ is a finite union $\bigcup_{j=1}^{n_i} I_i^j$ of disjoint intervals on which each of x and y is finite. By the definition of b_i, we have

$$\int_{a_{m_i-1}}^{a_{m_i}} (y-x)_+(s)\,ds = \int_{b_{i-1}}^{b_i} (x-y)_+(s)\,ds = \sum_{j=1}^{n_i} \int_{I_i^j} (x-y)_+(s)\,ds.$$

Set $\mathcal{K} = \{(i,j)\colon\ 1 \leqslant j \leqslant n_i,\ i \in \mathcal{I}\}$. If $k = (i,j) \in \mathcal{K}$, set $I_k = I_i^j$ and

$$J_k = J_i^j = \big[a_{m_i-1} + (y_{m_i} - x_{m_i})^{-1} c_i^{j-1}, a_{m_i-1} + (y_{m_i} - x_{m_i})^{-1} c_i^j\big],$$

where

$$c_i^j = \sum_{l=1}^{j} \int_{I_i^l} (x-y)_+(s)\,ds, \quad i \in \mathcal{I},\ 0 \leqslant j \leqslant n_i.$$

Using the fact that x and y are constant on the interval $[a_{m_i-1}, a_{m_i}]$, we obtain $J_k \subset [a_{m_i-1}, a_{m_i}]$ and $\bigcup_{j=1}^{n_i} J_i^j = [a_{m_i-1}, a_{m_i}]$.

(i) Both x and y are constant on I_k and J_k, $k \in \mathcal{K}$. Since $b_i \leqslant a_{m_i-1}$ for each $i \in \mathcal{I}$, then I_k lies to the left of J_k for $k \in \mathcal{K}$.

It then follows from (i), that

$$\int_{I_k} (x-y)_+(s)\,ds = \int_{J_k} (y-x)_+(s)\,ds, \quad k \in \mathcal{K}. \tag{16}$$

(ii) Since $x|_{I_k} \geqslant y|_{I_k}$ and $x|_{J_k} \leqslant y|_{J_k}$ for all $k \in \mathcal{K}$, then the assertion follows directly from (i) and (16).

(iii) The set $\{y > x\} = \bigcup_{i \in \mathcal{I}} \bigcup_{j=1}^{n_i} J_i^j \subseteq \bigcup_{k \in \mathcal{K}} \Delta_k$.

The last assertion is immediate. □

Corollary 20. *Let E be a fully symmetric space either on the interval* $(0,1)$ *or on the semi-axis. If x, y and $\mathcal{B}=\{\Delta_k\}_{k\in\mathcal{K}}$ are as in Proposition* 19 *and $y(t)=x(t)$ if $t\notin\bigcup_k\Delta_k$, then y can be arbitrary well approximated in the norm of E by convex combinations of functions of the form $P(x|\mathcal{A})$, $\mathcal{A}\in\mathfrak{A}$.*

Proof. Set $\lambda_k=(y|_{I_k}-y|_{J_k})/(x|_{I_k}-x|_{J_k})$, $k\in\mathcal{K}$. Since $y|_{\Delta_k}\prec x|_{\Delta_k}$, it is not difficult to verify that $\lambda_k\in[0,1]$, $k\in\mathcal{K}$. Further, a simple calculation shows that $y=(1-\lambda_k)P(x|\mathcal{B})+\lambda_k x$ on Δ_k, $k\in\mathcal{K}$.

As is well known, every $[0,1]$-valued sequence can be uniformly approximated by convex combinations of $\{0,1\}$-valued sequences.

Fix $\varepsilon>0$. There exists $\mu\in l_\infty(\mathcal{K})$ with $\mu=\sum_{i=1}^n\theta_i\chi_{D_i}$ for some $n\in N$, $0\leqslant\theta_i\in R$ and $D_i\subseteq\mathcal{K}$ such that $\sum_{i=1}^n\theta_i=1$ and $\|\lambda-\mu\|_\infty\leqslant\varepsilon$. Set $z=(1-\mu_k)P(x|\mathcal{B})+\mu_k x$ on Δ_k, $k\in\mathcal{K}$ and $z=x$ outside $\bigcup_{k\in\mathcal{K}}\Delta_k$. It is clear that $|y-z|\chi_{\Delta_k}=|\lambda_k-\mu_k||x-P(x|\mathcal{B})|\chi_{\Delta_k}$, $k\in\mathcal{K}$ and $|y-z|=\sum_{k\in\mathcal{K}}|y-z|\chi_{\Delta_k}\leqslant 2\varepsilon(x+P(x|\mathcal{B}))$. Therefore, $\|y-z\|_E\leqslant 2\varepsilon\|x\|_E$.

Set $F_i=\bigcup_{k\in D_i}\Delta_k$ and $\mathcal{A}_i=\{\Delta_k\}_{k\notin D_i}\in\mathfrak{A}$, $1\leqslant i\leqslant n$. It is then clear that

$$z=\sum_{i=1}^n\theta_i\big((1-\chi_{F_i})P(x|\mathcal{B})+\chi_{F_i}x\big)=\sum_{i=1}^n\theta_i P(x|\mathcal{A}_i). \qquad\Box$$

4.1. The case that $E\subseteq L_1$

Theorem 21. *Let $E=E(0,1)$ be a fully symmetric space on the interval* $(0,1)$. *If $x\in E$, then the following statements are equivalent.*

(i) $\Omega'(x)=Q'(x)$.
(ii) $\varphi(x)=0$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $Q'(x)=\Omega'(x)$. Set $\mathcal{A}=\{[0,1]\}$ and $y=P(x|\mathcal{A})$. Clearly, $y\in\Omega'(x)=Q'(x)$. Lemma 7 implies that $\varphi(x)=\varphi(y)$. Lemma 4 implies $\varphi(y)=0$. The assertion is proved.

(ii) $\Rightarrow$ (i) Let $x=x^*$ and $0\leqslant y\in\Omega'(x)$. In this case, $y=y^*\circ\gamma$ for some measure-preserving transformation γ (see [15] or [2, Theorem 7.5, p. 82]). Without loss of generality, we may assume that $y=y^*$. Fix $\varepsilon>0$. Set

$$s_n(\varepsilon)=\inf\big\{s:\ y(s)\leqslant y(1)+n\varepsilon\big\},\quad n\in N.$$

Let $\mathcal{A}_\varepsilon$ be the partition, determined by the points $s_n(\varepsilon)$, $n\in N$. Set $u=P(y|\mathcal{A}_\varepsilon)$ and $z=P(x|\mathcal{A}_\varepsilon)$. The functions u and z satisfy the condition $u\prec z$ and are of the form given in (15).

By Lemma 3(iii), $\varphi(z)\leqslant\varphi(x)=0$. By Theorem 18, $P(z|\mathcal{A})\in Q'(z)$ for every $\mathcal{A}\in\mathfrak{A}$. It follows now from Corollary 20 that $u\in Q'(z)$. However, $z\in Q'(x)$ by Theorem 18. Therefore, by Lemma 16, $u\in Q'(x)$. However, $\|y-u\|_{L_\infty}\leqslant\varepsilon$. Since ε is arbitrary, $y\in Q'(x)$. $\Box$

Theorem 22. *Let $E=E(0,1)$ be a fully symmetric space on the interval* $(0,1)$. *If $x\in E$ and $\varphi(x)=0$, then $\Omega_+(x)=Q_+(x)$. If, in addition, the norm on E is a Fatou norm, then converse assertion also holds.*

Proof. Suppose that $\varphi(x) = 0$ and let $y \in \Omega_+(x)$. Hence, there exists $s_0 \in [0,1]$, such that $\int_0^{s_0} x^*(s)\,ds = \int_0^1 y^*(s)\,ds$. Set $z = x^*\chi_{[0,s_0]}$. By Theorem 21, $y \in Q'(z)$. Hence, $y \in Q'(z) \subseteq Q_+(x)$.

By Proposition 6, there exist $0 \leqslant y, z \in E$, such that $x = y + z$ and $\varphi(x) = \varphi(y) = \varphi(z)$. By Proposition 8, $\varphi(x) = \varphi(y) + \varphi(z)$. Consequently, $\varphi(x) = 0$. □

Now, consider the case that $E = E(0,\infty)$.

Theorem 23. *Let $E = E(0,\infty)$ be a fully symmetric space on semi-axis. If $E \subseteq L_1$ and $x \in E$, then the following assertions are equivalent.*

(i) $\Omega'(x) = Q'(x)$.
(ii) $\varphi_{fin}(x) = 0$.

Proof. (i) ⇒ (ii) Let $x = x^*$ and suppose that $Q'(x) = \Omega'(x)$. Set $\mathcal{A} = \{[0,1]\}$ and $y = P(x|\mathcal{A})$. Clearly, $y \in \Omega'(x) = Q'(x)$. Lemma 7 implies that $\varphi_{fin}(x) = \varphi_{fin}(y)$. Lemma 4 implies $\varphi_{fin}(y) = 0$. The assertion is proved.

(ii) ⇒ (i) Let $x = x^*$ and $0 \leqslant y \in \Omega'(x)$. It follows from [10, Lemma II.2.1] that for every fixed $\varepsilon > 0$ there exists a measure-preserving transformation γ such that $\|y - y^* \circ \gamma\|_E \leqslant \varepsilon$. Without loss of generality, we may assume that $y = y^*$. For every $S > 0$,

$$\frac{1}{\tau}\big\|(\sigma_\tau x)\chi_{[0,S]}\big\|_E \leqslant \frac{S}{\tau}\big\|(\sigma_\tau x)\chi_{[0,1]}\big\|_E \to 0.$$

(a) Suppose first that $\operatorname{supp}(x) = \operatorname{supp}(y) = (0,\infty)$. Fix $\varepsilon > 0$. There exists T, such that

$$\|x\chi_{[T,\infty)}\|_{L_1\cap L_\infty} \leqslant \varepsilon, \qquad \|y\chi_{[T,\infty)}\|_{L_1\cap L_\infty} \leqslant \varepsilon.$$

Clearly, $\int_0^T x(s)\,ds < \int_0^\infty x(s)\,ds$. Hence, there exists $S \geqslant T$, such that $\int_0^S y(s)\,ds = \int_0^T x(s)\,ds$. By Theorem 21, $y\chi_{[0,S]} \in Q'(x\chi_{[0,T]})$. Hence, $y \in Q'(x) + y\chi_{(S,\infty)} - Q'(x\chi_{(T,\infty)})$ and, therefore, $\operatorname{dist}(y, Q'(x)) \leqslant 2\varepsilon$. Since ε is arbitrary, $y \in Q'(x)$.

(b) Suppose now that $m(\operatorname{supp}(x)) < \infty$ or $m(\operatorname{supp}(y)) < \infty$. Fix $z = z^* \in L_1 \cap L_\infty$ with infinite support. It is clear that $(y + \varepsilon z) \in \Omega'(x + \varepsilon z)$, $\varepsilon > 0$. By assumption and Lemma 4, $\varphi_{fin}(x + \varepsilon z) = 0$. Hence, using (a) preceding, it follows that $(y + \varepsilon z) \in Q'(x + \varepsilon z) \subset Q'(x) + \varepsilon Q'(z)$. Hence, $\operatorname{dist}(y, Q'(x)) \leqslant \varepsilon$ for every $\varepsilon > 0$ and, therefore, $y \in Q'(x)$. □

Theorem 24. *Let $E = E(0,\infty)$ be a fully symmetric space on $(0,\infty)$ such that $E \subseteq L_1$. If $0 \leqslant x \in E$ and $\varphi_{fin}(x) = 0$, then $\Omega_+(x) = Q_+(x)$. If, in addition, the norm on E is a Fatou norm, then converse assertion also holds.*

Proof. Let $\varphi_{fin}(x) = 0$ and $y \in \Omega_+(x)$. As in Theorem 23, we may assume $y = y^*$. Fix $\varepsilon > 0$. There exists $T > 0$ such that

$$\|x\chi_{[T,\infty)}\|_{L_1\cap L_\infty} \leqslant \varepsilon, \qquad \|y\chi_{[T,\infty)}\|_{L_1\cap L_\infty} \leqslant \varepsilon.$$

Select $S \leqslant T$ such that

$$\int_0^S x^*(s)\,ds = \int_0^T y^*(s)\,ds.$$

Clearly, $y\chi_{[0,T]} \in \Omega'(x^*\chi_{[0,S]})$. By Theorem 21, $y\chi_{[0,T]} \in Q'(x^*\chi_{[0,S]}) \subseteq Q_+(x)$. Hence, $y \in Q_+(x)$.

By Proposition 6, there exist $0 \leqslant y, z \in E$, such that $x = y + z$ and $\varphi_{fin}(x) = \varphi_{fin}(y) = \varphi_{fin}(z)$. By Proposition 8, $\varphi_{fin}(x) = \varphi_{fin}(y) + \varphi_{fin}(z)$. Consequently, $\varphi_{fin}(x) = 0$. □

4.2. The case that $E \nsubseteq L_1$

Theorem 25. *Let $E = E(0,\infty)$ be a fully symmetric space on the semi-axis and let $x \in E$. If $\varphi(x) = 0$, then $\Omega_+(x) = Q'(x)$.*

Proof. Let us assume first that $y = y^* \in \Omega_+(x)$. Fix $\varepsilon > 0$. Set $t_n(\varepsilon) = 1 + n\varepsilon$,

$$s_n(\varepsilon) = \inf\{s\colon\ y(s) \leqslant y(1) + n\varepsilon\},$$
$$s_{-n}(\varepsilon) = \sup\{s\colon\ y(s) \geqslant y(1) - n\varepsilon\}.$$

Let $\mathcal{A}_\varepsilon$ be the partition, determined by the points $s_{\pm n}(\varepsilon)$, $t_n(\varepsilon)$. Set $u = P(y|\mathcal{A}_\varepsilon)$ and $z = P(x|\mathcal{A}_\varepsilon)$. The functions u and z satisfy the conditions $u \prec\prec z$ and (15). Set

$$v = u\sum_{k\in\mathcal{K}} \chi_{\Delta_k} + z\chi_{(0,\infty)\setminus\bigcup_{k\in\mathcal{K}}\Delta_k},$$

where the collection $\{\Delta_k\}_{k\in\mathcal{K}}$ is given by Proposition 19.

By Lemma 3(iii), $\varphi(z) \leqslant \varphi(x) = 0$. By Theorem 18, $P(z|\mathcal{A}) \in Q'(z)$ for every $\mathcal{A} \in \mathfrak{A}$. It follows now from Corollary 20 that $v \in Q'(z)$. Since $u \leqslant v$, it follows from Lemma 15 that $u \in Q'(z)$. Theorem 18 implies that $z \in Q'(x)$. By Lemma 16, $u \in Q'(x)$. However,

$$\operatorname{dist}\bigl(y, Q'(x)\bigr) \leqslant \|y - u\|_E \leqslant \bigl\|y - P(y|\mathcal{A}_\varepsilon)\bigr\|_{L_1\cap L_\infty} \leqslant \varepsilon\bigl(1 + y(1)\bigr).$$

Since ε is arbitrary, $y \in Q'(x)$.

Let now $y \in \Omega_+(x)$ be arbitrary. By [10, Lemma II.2.1 and Theorem II.2.1], for every fixed $\varepsilon > 0$, there exist $y_1 \in E$, $y_2 \in E$, $y = y_1 + y_2$ and measure-preserving transformation γ such that $0 \leqslant y_1 \leqslant y^* \circ \gamma$ and $\|y_2\|_E \leqslant \varepsilon$. Since we already proved that $y^* \in Q'(x)$, the assertion follows immediately. □

Theorem 26. *Let $E = E(0,\infty)$ be a fully symmetric space on semi-axis. Suppose that $E \nsubseteq L_1$ and $x \in E$. If $\varphi(x) = 0$, then the set $\Omega_+(x)$ is the norm-closed convex hull of its extreme points. If, in addition, the norm on E is a Fatou norm, then converse assertion also holds.*

Proof. Suppose that $\varphi(x) = 0$. Applying Theorem 25 and noting the embedding $Q'(x) \subseteq Q_+(x)$ yields the assertion.

Conversely, by Proposition 6(iv), there exist $0 \leqslant y_1, z_1 \in E$, such that $x = y_1 + z_1$ and $\varphi_{cut}(x) = \varphi_{cut}(y_1) = \varphi_{cut}(z_1)$. By assumption, $\Omega_+(x) = Q_+(x)$ and so $y_1, z_1 \in Q_+(x)$. By Proposition 8, $\varphi_{cut}(x) = \varphi_{cut}(y_1) + \varphi_{cut}(z_1)$. Consequently, $\varphi_{cut}(x) = 0$. By Proposition 6(ii), there exist $0 \leqslant y_2, z_2 \in E$, such that $x = y_2 + z_2$ and $\varphi(x) = \varphi(y_2) = \varphi(z_2)$. Again, by the assumption, we have $y_2, z_2 \in Q_+(x)$ and therefore, by Proposition 9, we have $\varphi(x) = \varphi(y_2) + \varphi(z_2)$. Consequently, $\varphi(x) = 0$. □

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Appendix A

A.1. An application to the case of orbits $\Omega(x)$

The following consequence of Theorem 22 is essentially due to Braverman and Mekler [3].

Corollary 27. *If $\varphi(x) = 0$, then $\Omega(x)$ is the norm-closed convex hull of its extreme points.*

Proof. Let $x = x^*$ and $y \in \Omega(x)$. Clearly, $y = u \cdot |y|$, where $|u| = 1$ a.e. and $|y| \in \Omega_+(x)$. Fix $\varepsilon > 0$. By Theorem 22, there exist $n \in N$, scalars $\lambda_{n,i}, \beta_{n,i} \in [0, 1]$ and functions $x_{n,i} \sim x\chi_{[0,\beta_{n,i}]}$, such that $\sum_{i=1}^{n} \lambda_{n,i} = 1$ and

$$\left\| |y| - \sum_{i=1}^{n} \lambda_{n,i} x_{n,i} \right\|_E \leqslant \varepsilon.$$

There exist measure-preserving transformations $\gamma_{n,i}$, $1 \leqslant i \leqslant n$ (see [15]), such that $x_{n,i} = (x^*\chi_{[0,\beta_{n,i}]}) \circ \gamma_{n,i}$. Set $x^1_{n,i} = u \cdot x \circ \gamma_{n,i}$ and $x^2_{n,i} = u \cdot (x\chi_{[0,\beta_{n,i}]} - x\chi_{[\beta_{n,i},1]}) \circ \gamma_{n,i}$, $1 \leqslant i \leqslant n$. It is clear that $x_{n,i} \sim x$, $1 \leqslant i \leqslant n$, and

$$\left\| y - \frac{1}{2} \sum_{i=1}^{n} \lambda_{n,i} x^1_{n,i} - \frac{1}{2} \sum_{i=1}^{n} \lambda_{n,i} x^2_{n,i} \right\|_E \leqslant \varepsilon. \qquad \square$$

A.2. Extreme points of the orbit $\Omega_+(x)$

The following theorem is due to Ryff (see [14]).

Theorem 28. *If $0 \leqslant x \in L_1(0, 1)$, then $y \in \operatorname{extr}(\Omega'(x))$ if and only if $y^* = x^*$.*

Corollary 29. *If $0 \leqslant x \in L_1(0, 1)$, then $y \in \operatorname{extr}(\Omega_+(x))$ if and only if $y^* = x^*\chi_{[0,\beta]}$ for some $\beta \geqslant 0$.*

Proof. Indeed, if $\int_0^\beta x^*(s)\,ds = \int_0^1 y^*(s)\,ds$, then $y \in \Omega'(x^*\chi_{[0,\beta]})$. Therefore, if $y \in \operatorname{extr}(\Omega_+(x))$, then obviously $y \in \operatorname{extr}(\Omega'(x^*\chi_{[0,\beta]}))$ and the assertion follows immediately from Theorem 28.

If $y^* = x^*\chi_{[0,\beta]}$ and $y = \frac{1}{2}(u_1 + u_2)$ with $u_i \in \Omega_+(x)$, then $\int_0^t u_i^*(s)\,ds = \int_0^t x^*(s)\,ds$ for $t \in [0, \beta]$ and $\operatorname{supp}(u_i) = \operatorname{supp}(y)$. Therefore, $(u_1 + u_2)^* = u_1^* + u_2^*$. It follows now from [10, (II.2.19)] that $u_1 = u_2$. □

Lemma 30. *If* $0 \leqslant x \in L_1 + L_\infty$ *and* $y \in \operatorname{extr}(\Omega_+(x))$, *then* $y\chi_{\{y<y^*(\infty)\}} = 0$.

Proof. Assume, the contrary. Thus, the Lebesgue measure of the set $A = \{y \in (0, \lambda y^*(\infty))\}$ does not vanish for some $\lambda \in (0, 1)$. Let $0 \leqslant \varepsilon$ be such that $(1 + \varepsilon)\lambda < 1$. Set $y_1 = (1 + \varepsilon)y\chi_A + y\chi_{(0,\infty)\setminus A}$ and $y_2 = (1 - \varepsilon)y\chi_A + y\chi_{(0,\infty)\setminus A}$. Clearly, $y_i^* = y^*$ and, therefore, $y_i \in \Omega_+(x)$, for $i = 1, 2$. Hence, $y = \frac{1}{2}(y_1 + y_2) \notin \operatorname{extr}(\Omega_+(x))$. □

Corollary 31. *Let* $0 \leqslant x \in L_1 + L_\infty$ *and* $y \in \operatorname{extr}(\Omega_+(x))$. *It then follows that*

(1) *If* $x^*(\infty) = 0$, *then* $y^* = x^*\chi_{[0,\beta]}$ *for some* $\beta \in [0, \infty]$.
(2) *If* $x^*(\infty) > 0$, *then either* $y^* = x^*\chi_{[0,\beta]}$ *for some* $\beta \in [0, \infty)$ *or* $y^* = x^*$ *and* $y\chi_{\{y<y^*(\infty)\}} = 0$.

Conversely, functions as above belong to the set $\operatorname{extr}(\Omega_+(x))$.

Proof. If y belongs to $\operatorname{extr}(\Omega_+(x))$, then so does y^* (see [14] and [6]). Fix $t_1 > 0$ and find $t_2 \leqslant t_1$ such that $\int_0^{t_2} x^*(s)\,ds = \int_0^{t_1} y^*(s)\,ds$. Clearly, $y^*\chi_{[0,t_1]} \prec x^*\chi_{[0,t_2]}$ and $y^*\chi_{[t_1,\infty)} \prec\prec x^*\chi_{[t_2,\infty)}$. If $y^*\chi_{[0,t_1]} = \frac{1}{2}(u_1 + u_2)$ with $u_1, u_2 \in \Omega'(x^*\chi_{[0,t_2]})$, then set $y_i = u_i\chi_{[0,t_1]} + y^*\chi_{[t_1,\infty)}$. We claim $y_i \prec\prec x$. Indeed, if $e \in (0, \infty)$ and $m(e) < \infty$, then $e = e_1 \cup e_2$ with $e_1 \subset [0, t_1]$ and $e_2 \subset [t_1, \infty)$. Therefore,

$$\int_e y_i(s)\,ds = \int_{e_1} u_i(s)\,ds + \int_{e_2} y^*(s)\,ds \leqslant \int_0^{m(e_1)} u_i^*(s)\,ds + \int_{t_1}^{t_1+m(e_2)} y^*(s)\,ds$$
$$\leqslant \int_0^{\min\{t_2,m(e_1)\}} x^*(s)\,ds + \int_{t_2}^{t_2+m(e_2)} x^*(s)\,ds \leqslant \int_0^{m(e)} x^*(s)\,ds.$$

Hence, $y_i \in \Omega_+(x)$ and $y = \frac{1}{2}(y_1 + y_2)$. Thus, $y \notin \operatorname{extr}(\Omega_+(x))$. Therefore, $y^*\chi_{[0,t_1]} \in \operatorname{extr}(\Omega'(x^*\chi_{[0,t_2]}))$. By Theorem 28, $y^* = x^*$ on $[0, t_2]$. The assertion follows now from Lemma 30.

The converse assertion is easy. □

Corollary 32. *If* $x \in L_1(0, \infty)$, *then* $0 \leqslant y \in \operatorname{extr}(\Omega'(x))$ *if and only if* $y^* = x^*$.

The proof is identical to that of Corollary 31.

A.3. Marcinkiewicz spaces with trivial functional φ

It follows from Lemma 3 and the definition of Marcinkiewicz space, that $\varphi = 0$ if and only if $\varphi(\psi') = 0$. It is now easy to derive, that in case of the interval $(0, 1)$ this is equivalent to the condition

$$\liminf_{t\to 0}\frac{\psi(2t)}{\psi(t)} > 1.$$

In case of the semi-axis, the condition

$$\liminf_{t\to \infty}\frac{\psi(2t)}{\psi(t)} > 1$$

needs to be added.

A.4. A comparison of conditions (1) and (2) in Orlicz spaces

Let M be a convex function satisfying (5) and let L_M be the corresponding Orlicz space on $(0,1)$. The following proposition shows that L_M always satisfies condition (2).

Proposition 33. *We have $\varphi(x) = 0$ for every $x \in L_M$.*

Proof. Using the description of relatively weakly compact subsets in L_M given in [1] (see also [12, p. 144]) we see that for every $0 \leqslant y \in L_M$

$$n\int_0^{\frac{1}{n}} M\left(\frac{1}{n}y(s)\right)ds \to 0.$$

We are going to prove that $\frac{1}{n}\|\sigma_n x\|_{L_M} \to 0$ for every $x \in L_M$. Assume the contrary. Let $\|\sigma_n x\|_{L_M} \geqslant n\alpha$ for some $0 \leqslant x \in L_M$, some $\alpha > 0$ and for arbitrary large $n \geqslant 1$. By the definition of the norm $\|\cdot\|_{L_M}$, we have

$$\int_0^1 M\left(\frac{1}{n\alpha}\sigma_n x(s)\right)ds \geqslant 1.$$

Hence,

$$n\int_0^{\frac{1}{n}} M\left(\frac{1}{n}y(s)\right)ds \geqslant 1$$

with $y = \alpha^{-1}x \in L_M$. A contradiction. □

However, there exists an Orlicz space L_M which fails to satisfy condition (1).

For the definition of Boyd indices $1 \leqslant p_E \leqslant q_E \leqslant \infty$ of a fully symmetric space E, we refer the reader to [11, 2.b.1 and p. 132]. It is clear, that the condition (1) holds for a fully symmetric space E if and only if $p_E > 1$. However (see e.g. [11]) Orlicz space L_M is separable if and only if $q_{L_M} < \infty$. It is well known that there exists a non-separable Orlicz space L_M with $p_{L_M} = 1$.

A.5. An application to symmetric functionals

Let E be a fully symmetric space. A positive functional $f \in E^*$ is said to be symmetric (respectively, fully symmetric) if $f(y) = f(x)$ (respectively, $f(y) \leqslant f(x)$) for all $0 \leqslant x, y \in E$ such that $y^* = x^*$ (respectively, $y \prec\prec x$). We refer to [8,5] and references therein for the exposition of the theory of singular fully symmetric functionals and their applications. Recently, symmetric functionals which fail to be fully symmetric were constructed in [9] on some Marcinkiewicz spaces. However, for Orlicz spaces situation is different. The following proposition shows that a symmetric functional on an Orlicz space on the interval (0, 1) is necessary fully symmetric.

Proposition 34. *Any symmetric functional on L_M is fully symmetric.*

Proof. Let $\omega \in E^*$ be symmetric. It is clear, that $\omega(x^*\chi_{[0,\beta]}) \leqslant \omega(x)$ for $x \geqslant 0$. Therefore, $\omega(y) \leqslant \omega(x)$ for $y \in \overline{\mathrm{Conv}}\{y^* = x^*\chi_{[0,\beta]}\}$. Since ω is continuous, we have $\omega(y) \leqslant \omega(x)$ for $y \in Q_+(x)$. By Theorem 22 and Proposition 33, we have $Q_+(x) = \Omega_+(x)$, and so ω is a fully symmetric functional on L_M. □

Corollary 35. *Any singular symmetric functional on L_M vanishes.*

Proof. Indeed, there are no fully symmetric singular functionals on L_M (see [8, Theorem 3.1]). □

We also formulate the following hypothesis: If E is a fully symmetric space, then functional φ vanishes if and only if there are no singular symmetric functionals on E.

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