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FINITE-DIMENSIONAL, IRREDUCIBLE REPRESENTATIONS OF SOME CROSSED PRODUCTS AND GROUP RINGS

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1. Introduction

Let A be a finitely-generated algebra over a field k. For simplicity, we will assume throughout the paper that k is algebraically closed. The set Spec A of prime ideals can be given the Jacobson topology, in which the closed sets have the form

$$V(I)=\{P\in \operatorname{Spec} A \mid P\supset I\},$$

and the subset $\operatorname{Spec}_n A$ of maximal ideals which are kernels of irreducible representations of degree n is a locally closed subspace. M. Artin has proved that $\operatorname{Spec}_n A$ is homeomorphic to an open subscheme of a variety, which has a sheaf of Azumaya algebras corresponding to the degree n representations of A [2]. Let d_n denote the dimension of $\operatorname{Spec}_n A$. In the same paper, Artin asked whether anything can be said in general about the asymptotic behavior of the sequence $\{d_n\}$. The purpose of this paper is to present some examples of algebras for which $\operatorname{Spec}_n A$ can be explicitly described, and we will find that the dimension sequence $\{d_n\}$ can behave somewhat wildly.

Let G be a finite group of size n and let S be the algebra

$$k[y_1,y_1^{-1},\dots,y_m,y_m^{-1}].$$

Given a faithful action of G as k-automorphisms of S and a 2-cocycle $f: G\times G\to S^*$, we may form the crossed product $S*G$. It is generated as an S-module by elements $\{\bar g : g\in G\}$, and multiplication is defined by the rule

$$s\bar g=\bar g s^g,\qquad \bar g_1\bar g_2=\overline{g_1g_2}f(g_1,g_2).$$

In Section 2, we determine $\operatorname{Spec}_n S*G$ as the open subscheme of Spec S^G which is the complement of the branch locus with respect to the cover by Spec S. More

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precisely, the algebra $S*G$ has S^G as its center, and so defines a sheaf of algebras over Spec S^G. At branch points, $S*G$ has irreducible representations of degree $<n$, while on the complement, $S*G$ restricts to a sheaf of rank n^2 Azumaya algebras, whose stalks have the degree n irreducible representations as their residue rings. The irreducible representations of degree $<n$ are determined, but the topological structure of the corresponding set of primes is not.

The examples of Section 3 are all skew group rings $T*G$ (crossed products with trivial 2-cocycle), whose base ring T is the group ring of a free abelian group F, with a not necessarily finite group G acting on F. Thus we could just as well regard these examples as group rings of semi-direct products of F by G. In all cases, we show that under any finite-dimensional, irreducible representation, the elements of G act with finite order on the image of T. This reduces us to crossed products of the form described in Section 2, and we are able to describe $\mathrm{Spec}_n T*G$ for all n.

The first example, for any positive integer m, is a noetherian algebra A_m with irreducible representations only in degree r^m, for all r prime to the characteristic of k. The spaces $\mathrm{Spec}_{r^m} A_m$ which are non-empty have dimension $2m+1$. For $m>1$, the resulting generating function $\sum d_n t^n$ is not rational, answering a question in [2, p. 532].

The next two examples B and C are skew group rings whose group G is the infinite dihedral group. We are reduced to studying crossed products by various dihedral groups, and have to analyze the irreducible representations arising from the branch locus of the center. We do this completely for B, and are able to find the dimension sequences for both B and C, provided k does not have characteristic 2. The algebra C has the sequence $1,3,3,5,5,\ldots$ and in characteristic 0, the sequence for B is $1,3,1,3,\ldots$.

The last family of examples E_m are easy to analyze, but have the most complicated dimension sequence. The algebra E_1 has the sequence $0,3,4,5,6,\ldots$, while for E_2, the number d_n is 0 for n prime, and otherwise

$$d_n = p + (n/p) + 2,$$

for p the smallest prime divisor of n. The sequences for the other E_m behave similarly and suggest that there is no reasonable asymptotic behavior for $\{d_n\}$ in general.

In Section 4 we show that for any simple Lie algebra besides sl_2 in characteristic 0, the polynomial ring over its enveloping algebras has a generating function $\sum d_n t^n$ which is not rational. This follows immediately from the fact that the set of degrees of finite-dimensional, irreducible representations has density 0.

2. The space of irreducible representations of a crossed product

As noted in the introduction, for any finitely-generated k-algebra A, the space $\mathrm{Spec}_n A$ can be given the structure of a scheme X_n, along with a sheaf $\mathscr{A}_n$ of

Azumaya algebras of rank n^2 over the structure sheaf $\mathcal{O}_n$. In addition, there is a map of A into the global sections $\Gamma(X_n, \mathscr{A}_n)$ such that the irreducible, n-dimensional representations of A arise, up to equivalence, by mapping A to the residue rings of the stalks of $\mathscr{A}_n$ at the points $x \in X_n$:

$$A \to \mathscr{A}_n \underset{\mathcal{O}_n}{\otimes} k(x). \tag{1}$$

This is proved in [2, p. 556–557].

The algebras discussed in this section provide a good example of the geometric situation above. Let $S = k[y_1, y_1^{-1}, \dots, y_m, y_m^{-1}]$ and let G be a group of k-automorphisms of S of size n, with a 2-cocycle $f\colon G \times G \to S^*$. We may form the crossed product $A = S * G$, as described in the introduction, and its center will be $R = S^G$. The irreducible representations of A all have degree $\leq n$, and $\operatorname{Spec}_n A$ admits a precise description. The space $\operatorname{Spec} S$ is a branched cover of $\operatorname{Spec} R$ of degree n. Let X be the open subset of $\operatorname{Spec} R$ complementary to the branch locus. Then for any maximal ideal $\mathbf{m} \in X$, we will find that $A/\mathbf{m}A \cong M_n(k)$, but this is not the case for $\mathbf{m} \notin X$. This identifies $\operatorname{Spec}_n A$ set-theoretically with X, but more is true. Let R' be a localization of R for which $\operatorname{Spec} R' \subset X$. Then $A' = A \otimes_R R'$ is an Azumaya algebra of rank n^2 over R', and letting $S' = S \otimes_R R'$, we find that $A' \otimes_{R'} S' = M_n(S')$. In particular, the subspace of $\operatorname{Spec}_n A$ lying over $\operatorname{Spec} R'$ is homeomorphic to $\operatorname{Spec} R'$. Patching together these homeomorphisms, we have $\operatorname{Spec}_n A$ homeomorphic to X. This situation is most easily described via sheaves. The algebra A induces a sheaf $\tilde{A}$ of algebras over $\operatorname{Spec} R$, and the restriction of $\tilde{A}$ to X is a sheaf of Azumaya algebras of rank n^2 over the structure sheaf $\mathcal{O}$ of X. We may take $\tilde{A}|_X$ to be the $\mathscr{A}_n$ of the preceding paragraph. Then A maps to the global sections

$$\Gamma(X, \tilde{A}|_X)$$

by the canonical restriction map, since $A = \Gamma(\operatorname{Spec}(A), \tilde{A})$. Also, for any $x = \mathbf{m} \in X$, the map (1) is simply the map $A \to A/\mathbf{m}A = M_n(k)$. The fact that, locally, the algebras A' are split by S', with $A' \otimes_{R'} S' = M_n(S')$, may be rephrased to say that the restriction of $\tilde{S}$ to X is a sheaf of commutative algebras such that

$$\tilde{A}|_X \underset{\mathcal{O}}{\otimes} \tilde{S}|_X = M_n(S)|_X.$$

In case the open subset X is affine, the picture is more easily described, and sheaves are unnecessary. An example in which this occurs is the group ring of the infinite dihedral group.

Example. Assume that k does not have characteristic 2, and let $S = k[y, y^{-1}]$, with G, the group of two elements generated by x, acting on S via $y^x = y^{-1}$. The crossed product $A = S * G$ is generated over k by x, x^{-1}, y, y^{-1} with the relation

$$x^{-1} y x = y^{-1},$$

and the center of A is $R = k[y + y^{-1}]$. Write $\hat{y}$ for $y + y^{-1}$, and let $r = \hat{y}^2 - 4$.

Proposition 2.1. *The closed set* $V(r)$ *of* Spec R *is the branch locus with respect to the double cover by* Spec S. *The complementary open set* Spec R_r *is homeomorphic to* $\mathrm{Spec}_2 A$, *and* A_r *is an Azumaya algebra of rank* 4 *over* R_r *which is split by* S_r.

Proof. Any point on the line Spec R with coordinate c is covered in Spec S by the solutions to the equation $t^2-ct+1=0$. Thus the branch points are $c=2, -2$, which is precisely $V(r)$.

Any maximal ideal of A intersects R in a maximal ideal. Let us determine the maximal ideals of A lying over the branch points. Let $\mathbf{m}=(\hat{y}-2)$ and let I be a prime of A containing $\mathbf{m}$. Then $y^{-1}=2-y$ in $A/\mathbf{m}A$, so that

$$y^2-2y+1=y(y-2)+1=-yy^{-1}+1=0.$$

Thus I contains $(y-1)^2$. But

$$(y-1)x=x(y^{-1}-1)=x(1-y)$$

in $A/\mathbf{m}A$, so that I contains $(y-1)A(y-1)$, and $y-1\in I$. The only primes in $A/\mathbf{m}A$ are therefore $(y-1,x-1)$ and $(y-1,x+1)$, and these give rise to one-dimensional irreducible representations. A similar situation holds for $\mathbf{m}=(\hat{y}+2)$.

Therefore $\mathrm{Spec}_2 A$ and $\mathrm{Spec}_2 A_r$ are the same. Let us use primes in place of subscripted r's. We prove that A' is Azumaya by showing that for any $\mathbf{m}=(\hat{y}-c)$ in Spec R', the algebra $A'/\mathbf{m}A'$ is isomorphic to $M_2(k)$. Let a, a^{-1} be the two roots of $t^2-ct+1=0$. Then the desired map is

$$x\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad y\mapsto\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}.$$

The image of $y-y^{-1}$ is

$$\begin{pmatrix}a-a^{-1}&0\\0&a^{-1}-a\end{pmatrix}$$

and since $a\neq a^{-1}$ (we chose $\mathbf{m}$ unramified), the image of the map contains

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

This matrix and x produce the four matrix units, so the map is surjective. Injectivity follows because $A'/\mathbf{m}A'$ is spanned by four elements.

To prove the final statement, let $k[z,z^{-1}]$ be another copy of S, with $z+z^{-1}=\hat{y}$. We can map $A'\otimes_{R'}S'$ into $M_2(S')$ via

$$x\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad y\mapsto\begin{pmatrix}z&\\&z^{-1}\end{pmatrix},\qquad z\mapsto\begin{pmatrix}z&\\&z\end{pmatrix}.$$

The kernel intersects S' in (0), and since $A'\otimes S'$ is Azumaya over S', the map is injective. The image of $y-y^{-1}$ is

$$\begin{pmatrix} z-z^{-1} & \\ & z^{-1}-z \end{pmatrix},$$

and we can prove surjectivity just as above if we can invert $z-z^{-1}$ in S'. But

$$(z-z^{-1})^2 = z^2+z^{-2}-2=(z+z^{-1})^2-4=r,$$

so $z-z^{-1}$ is invertible.

We now return to the general situation, with S, G, A and R as before, and prove the facts described. Recall that X is the open subscheme of $\operatorname{Spec} R$ complementary to the branch locus, and let Y be its inverse image in $\operatorname{Spec} S$.

Proposition 2.2. *The spaces X and Y are non-empty and the induced map $g: Y\to X$ is a finite étale map of non-singular varieties.*

Proof. Since S is a normal domain, so is R, and their spectra are irreducible. Hence, unramified points exist provided the fraction fields $K(R)\subset K(S)$ form a separable extension [7, p. 117]. But in fact, the extension is Galois since $K(R)=K(S)^G$. It suffices to check the second statement locally, so let $\operatorname{Spec} R'$ be an affine open in X and let $S'=S\otimes_R R'$, so that $\operatorname{Spec} S'=g^{-1}(\operatorname{Spec} R')$. Since S' is normal, the map $g':\operatorname{Spec} S'\to\operatorname{Spec} R'$ is étale by [7, p. 120 and 1, VI.4.5]. Non-singularity of X follows because Y is non-singular and g is étale [7, p. 120].

Theorem 2.3. *Let R' be a localization of R such that* $\operatorname{Spec} R'\subset X$. *Then $A'=A\otimes_R R'$ is Azumaya of rank n^2 over R'. Moreover, $S'=S\otimes_R R'$ splits A', with*

$$A'\otimes_{R'} S'=M_n(S').$$

Proof. By 2.2, S' is faithfully flat over R', so A' is Azumaya if $A'\otimes_{R'}S'$ is [5, p. 104]. Thus we need only prove the second statement. Let us write $S'=k[z_1,z_1^{-1},\dots,z_m,z_m^{-1}]'$ for the copy of S' on the right of the tensor product, and let

$$\{v(g): g\in G\}$$

be a basis for the free S'-module V of rank n. We define a map of $A'\otimes S'$ to $M_n(S')$ by describing an $A'\otimes S'$-action on V. Let

$$v(g)\cdot\bar{h}=v(gh)f(g,h)$$

and

$$v(g)\cdot y_i=v(g)(z_i)^{g^{-1}}.$$

Recalling that the z_i's commute with $\bar{h}$, we can check that

$$v(g)y_i\bar{h}=v(gh)f(g,h)z_i^{g^{-1}}=v(g)\bar{h}y_i^h.$$

In addition,

$$\nu(g)\cdot(\bar{h}_1\bar{h}_2)=(\nu(g)\bar{h}_1)\cdot\bar{h}_2,$$

as one checks via the 2-cocycle condition, so the action is well-defined.

Let $\mathbf{m}=(z_i-a_i)$ be a maximal ideal of S'. We claim that the map $A'\otimes S'\to M_n(S')$ defined above induces an isomorphism

$$(A'\otimes S')/\mathbf{m}(A'\otimes S')\to M_n(S'/\mathbf{m})=M_n(k).$$

Surjectivity is equivalent to irreducibility of the induced module. Since the images of g act transitively on the basis, it suffices to show that the image of $\{y_i\}$ allow one to obtain a basis vector from any non-zero vector. The matrices involved are

$$\begin{bmatrix} a_i^{g_1} & & & & \\ & a_i^{g_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & a_i^{g_n} \end{bmatrix}$$

for $i=1,\dots,m$ and some enumeration $\{g_i\}$ of G, and the fact that $\mathbf{m}$ is not a ramification point in Spec S means precisely that the G-conjugates of $(a_1,\dots,a_m)$ are distinct. Hence, for any two basis vectors, one of the matrices has distinct corresponding eigenvalues, and this permits us to reduce any non-zero vector to a basis vector. Injectivity of the map follows because $A'\otimes S'/\mathbf{m}(A'\otimes S')$ has a spanning set of n^2-elements.

Therefore $A'\otimes S'$ is Azumaya, and the map into $M_n(S')$ is injective, since it is injective on the center S'. All that remains is to show surjectivity, which may be checked at each maximal ideal $\mathbf{m}$ of S' after localizing. Then we can use Nakayama's Lemma to pass to the quotient of $S'_{\mathbf{m}}$ by $\mathbf{m}S'_{\mathbf{m}}$, and we are reduced to the special case above, in which we have already checked surjectivity.

Corollary 2.4. *The open subscheme $X\subset \operatorname{Spec} R$ is homeomorphic to* $\operatorname{Spec}_n A$, *and the sheaf $\tilde{A}$ restricts to a sheaf of rank n^2 Azumaya algebras on X, which is split by the étale cover Y.*

Proof. This is just a reformulation of 2.3, except for the first statement. We know that X is homeomorphic to the subset of maximal ideals of $\operatorname{Spec}_n A$ whose intersection with R lies in X, by 2.3. Thus it suffices to check that for $\mathbf{m}\notin X$, the maximal ideals of $A/\mathbf{m}A$ correspond to irreducible representations of degree $<n$.

Let $M_1,\dots,M_t$ be the maximal ideals of S lying over $\mathbf{m}$. Since $\mathbf{m}$ is a branch point, $t<n$. The ideal $M=M_1\cdots M_t$ is contained in the radical of $S/\mathbf{m}S$, so for some r we have

$$(M_1\cdots M_t)^r\subset \mathbf{m}S.$$

But any element g of G permutes the ideals $\{M_i\}$, so that

$$M_1 \cdots M_t g = g M_1 \cdots M_t.$$

Therefore, given a prime ideal I of A containing $\mathbf{m}A$, we obtain

$$M_1 \cdots M_t A (M_1 \cdots M_t)^{r-1} \subset I.$$

This forces $M_1 \cdots M_t \subset I$, from which it follows that the image of S in A/I has dimension at most $t<n$ over $R/\mathbf{m}=k$. Thus A/I has dimension less than n^2, which is what we wanted to show.

The problem remains of describing $\mathrm{Spec}_t A$ for $t<n$. While we cannot give a topological description in general, we can describe the irreducible representations of A whose kernels lie over branch points of $\mathrm{Spec}\, R$. The representations 2.3 are a special case of this more general construction.

Theorem 2.5. *Let* $\mathbf{m}$ *be maximal ideal of* R *and assume that there are* t *distinct maximal ideals of* S *lying over* R, *including* M_1. *Let* $H \subset G$ *be the stabilizer subgroup of* M_1. *Then inequivalent irreducible representations of* $(S/M_1) * H = k * H$ *of degree* d *yield inequivalent irreducible representations of* $A/\mathbf{m}A$ *of degree* dt.

Remark 2.6. In the above setting, even allowing R to be non-commutative, Lorenz and Passman have proved that there is a one-to-one correspondence between the prime ideals of $(S/M_1) * H$ and $A/\mathbf{m}A$ [6, Theorem 3.6]. Thus we can be sure that, up to equivalence, the representations produced in the proof of 2.5 are all the irreducible representations of A with kernel lying over $\mathbf{m}$.

In case $\mathbf{m}$ is tamely ramified, this can be seen directly. For then $|H| = n/t$ is relatively prime to the characteristic of k, and the usual proof of Maschke's theorem shows that $k * H$ is semisimple. Therefore, letting $d_1, \dots, d_r$ denote the degrees of the irreducible representation classes of $k * H$, we have

$$\sum_{i=1}^{r} d_i^2 = n/t.$$

Let $M_1, \dots, M_t$ be the distinct primes of S lying over $\mathbf{m}$. The proof of 2.4 shows that any prime in A containing $\mathbf{m}$ also contains $M = M_1 \cap \cdots \cap M_t$. The algebra A/MA is semisimple, and is spanned over S/M by the n independent elements of G, while S/M has dimension t over k, so A/MA has dimension tn. The irreducible representations of A/MA provided by 2.5 have degrees $td_1, \dots, td_r$, and

$$\sum_{i=1}^{r} (td_i)^2 = t^2(n/t) = tn,$$

so that these must be all the irreducible representations of $A/\mathbf{m}A$, as claimed.

Proof of 2.5. Let V' be a simple $k * H$ module of dimension d, with basis $v_1, \dots, v_d$, and write v^h for the vector obtained by applying $h \in H$ to a vector $v \in V'$. Let V be a space of dimension td, which we view as t copies of V', marked by the right cosets G/H. We will write $v(Hg)$ for the copy of $v \in V'$ associated to the coset Hg. Fixing a transversal $g_1 = e, g_2, \dots, g_t$ of H in G, we find that V has as basis

$$\{v_i(Hg_j) : 1 \le i \le d;\ 1 \le j \le t\}.$$

We define a diagonal action of S on V'. Denote by $\bar{s}$ the image in S/M_1 of $s \in S$, and let

$$v(Hg_j) \bullet s = v(Hg_j)\overline{(s^{g_j})}.$$

Let $M = \bigcap_j M_1^{g_j}$ as in 2.6. Then M annihilates V, so this defines an action of S/M and of $S/\mathbf{m}S$. The action of G on V is defined by

$$v(H)\bar{g}_j = v(Hg_j),$$

and for $h \in H$,

$$v(H)\bar{h} = v^h(H).$$

This defines the G-action completely. To see this, first observe that

$$v(Hg_i)\overline{hg_j} = v(H)\overline{g_i}\overline{hg_j} = v(H)\overline{g_ihg_j}f(g_i, hg_j).$$

Let $g_ihg_j = h'g_l$ for some $h' \in H$ and l. Then we obtain

$$v(Hg_i)\overline{hg_j} = v(H)\overline{h'g_l}f(g_i, hg_j) = v^{h'}(Hg_l)f(h', g_l)^{-1}f(g_i, hg_j).$$

To prove that V is simple, we will show that any vector $w \ne 0$ is cyclic. We may assume that w has a non-zero H-component. Choose $s \notin M_1$ with $s^{g_j} \in M_1$ for any $j > 1$. Then $w \bullet s$ is a non-zero scalar multiple of the H-component of w. The action of $(S/M_1) * H$ on $w \bullet s$ produces any other vector in the H-component, and G acts transitively on the components. Thus w is cyclic and V is simple.

Finally, let W' be another simple $k * H$ module, and construct W as above. If V and W are isomorphic as $A/\mathbf{m}A$-modules, they are also isomorphic as modules over the subalgebra $B = (S/\mathbf{m}S) * H$. Since H fixes the maximal ideal M_1 of S, we have $M_1\bar{h} = \bar{h}M_1$ for any $h \in H$, so that the annihilator of M_1 in any B-module is a B-submodule. In particular, the respective annihilators of M_1 in V and W must be isomorphic B-submodules. But the annihilators are precisely the H-components, so that V' and W' are isomorphic as B-modules, and as modules over $(S/M_1) * H$. This proves the theorem.

While 2.5 identifies all the irreducible representations of A, it does not shed light on the topological structure of the spaces $\mathrm{Spec}_t A$. Even when this is known, it would be of interest to describe how these spaces fit together inside $\mathrm{Spec}\, A$. This problem has been studied by Artin and Schelter in [3] for finitely-generated k-algebras.

3. Irreducible representations of some group rings

In this section we determine the finite-dimensional, irreducible representations of some group rings. We first obtain a general rule which will show that the representations of the examples factor through crossed products of the type in Section 2. Then, for each example, we analyze the resulting crossed products, determining which representations arise over the branch points of the center. The next theorem is stated in greater generality than is needed for our examples, but can be applied to many other examples as well.

Theorem 3.1. *Let T be a commutative k-algebra, G a group of k-automorphisms of T, and let $A=T*G$ be the associated skew group ring. Assume one of the conditions below holds:*

(i) *$G=G_1$ is abelian.*

(ii) *G is an extension of an abelian group G_1 by an element z of order two, with respect to the action $zgz=g^{-1}$.*

(iii) *G is an extension of an abelian group G_1 by a finite group F.*

Then for any prime ideal I of A and $x\in G_1$, if $I\cap T[x]$ is not generated by $I\cap T$, then x acts with finite order on $T/I\cap T$.

Remark. Of course, case (iii) includes (i) and (ii), but we use only (i) and (ii) in the examples to follow, and the proof in these cases is elementary, so we have stated them separately.

Proof. The ideal $I\cap T$ is G-invariant, so we may pass to the skew group ring $(T/I\cap T)*G$ and assume that $I\cap T=(0)$. Let

$$p(x)=\sum_{i=0}^{n} x^i t_i$$

be a non-zero element of $I\cap T[x]$ of minimal degree, with $t_n\neq 0$. We must have $t_0\neq 0$, by the minimality of degree, and $n\neq 0$, for otherwise I contains t_n, contrary to assumption.

Let $y\in G_1$. For any non-zero $t\in T$, consider the element

$$ty^{-1}py-y^{-1}pyt^{x^n}.$$

This lies in I, and since y commutes with x, it equals

$$\sum_{i=0}^{n} x^i t_i^y(t^{x^i}-t^{x^n}).$$

The degree is less than n, so the polynomial must be 0. Therefore, for all $t\in T$ and $y\in G_1$, we have

$$t_0^y(t-t^{x^n})=0. \tag{2}$$

In case (i), we deduce from (2) that

$$t_0A(t-t^{x^n})\subset I,$$

and since I is prime, we find that x^n acts as the identity on $T/I\cap T$. In case (ii), the same argument would work if we knew that

$$t_0^z(t-t^{x^n})=0.$$

A variation of the above argument shows this. Let $u=x^n$ and work with

$$u^{-1}pu=\sum x_i t_i^u$$

instead of p. Then I contains

$$tz^{-1}(u^{-1}pu)z-z^{-1}(u^{-1}pu)zt^{x^{-n}},$$

which equals

$$t\sum x^{-i}(t_i^u)^z-\sum x^{-i}(t_i^u)^z t^{x^{-n}}=\sum x^{-i}(t_i^u)^z(t^{x^{-i}}-t^{x^{-n}}).$$

Multiplying by x^n produces a lower degree polynomial in $I\cap T[x]$, which must be 0, so

$$(t_0^u)^z(t-t^{x^{-n}})=0.$$

Applying $u=x^n$ to this and using $x^nz=zx^{-n}$, we obtain $t_0^z(t^{x^n}-t)=0$ as desired.

Finally, in case (iii), observe that A can be viewed as a crossed product of $T*G_1$ by the finite group F. It follows by a result of Lorenz and Passman [6, 3.1] that $I\cap(T*G_1)$ is a finite intersection of prime ideals $\{I_j\}$ of $T*G_1$. Applying case (i) to each I_j, we find that x acts with finite order on $T/(I_j\cap T)$, and so it does on $T/I\cap T$ as well.

Corollary 3.2. *Let A be a skew group ring $T*G$, with T a commutative k-algebra, and assume* (i), (ii), *or* (iii) *of* 3.1 *is satisfied. In any finite-dimensional, irreducible representation of A, the elements of G_1 act with finite order on the image of T.*

Proof. Let $x\in G_1$, and let I be the kernel of a finite-dimensional, irreducible representation of A. Then the image of x in A/I must satisfy some equation over k, so that $I\cap k[x]\neq 0$. Theorem 3.1 now applies.

Remark. In case k is algebraic over a finite field, the conclusion of 3.2 holds for any element x of G, without assuming anything about G. For the equation for the image of x over k must divide t^q-1 for some q. Thus over such fields we may dispense with the arguments of 3.1.

Let us now see how the conclusion of 3.2 will be used. Suppose $T*G$ is a skew group ring for which we know that the elements of G all act with finite order on the image of T under any finite-dimensional, irreducible representation. In addition, assume that all the torsion images of G are finite. Given the kernel I of a finite-dimensional, irreducible representation, there is a normal subgroup H of finite index in G which acts as the identity on $T/I\cap T$, by 3.2. Let $I(H)$ be the ideal of T

generated by

$$\{t-h(t): h\in H, t\in T\}.$$

Then I is a maximal ideal of the crossed product

$$(T/I(H))[H]*G/H,$$

with respect to a 2-cocycle f with $f(g_1,g_2)=h$ in case $g_1g_2=h\in H$. Thus we can find all the finite-dimensional, irreducible representations by examining the crossed products above, provided that H is abelian and G/H acts faithfully on $T/I(H)$.

In the examples which we consider, $T/I(H)$ is the group ring of a finitely-generated, free abelian group and H is also free abelian of finite rank, so that the crossed products which arise are those of Section 2. Thus the space $\operatorname{Spec}_n A$ will include a disjoint union of spaces X_i, one for each subgroup H_i of index n in G, with X_i homeomorphic to an open subset of

$$\operatorname{Spec}(T/I(H_i))[H_i]^{G/H_i}.$$

In addition, we must determine the irreducible representations whose kernels contain branch points of the above space. This turns out to be possible in our examples, so that we are able to compute the dimension sequence $\{d_n\}$ discussed in the introduction. We now turn to the examples, noting that fuller information about the corresponding spaces of representations is contained in the proof of each theorem.

Example 1. Let $T=k[t,t^{-1},y_1,y_1^{-1},\dots,y_m,y_m^{-1}]$ and let G be the free abelian group of rank m generated by $x_1,\dots,x_m$. Define a G-action on T so that the skew group ring $A_m=T*G$ has the relations

t is central,

$$x_i^{-1}y_jx_i=y_j \qquad \text{if } i\neq j,$$

$$x_i^{-1}y_ix_i=ty_i.$$

We note that A_m satisfies (i) of Theorem 3.1, and may be viewed as the group ring of a polycyclic group, so is noetherian.

Theorem 3.3. *A_m has irreducible representations only in dimensions r^m, for $r=1,2,\dots$ in characteristic 0, and for r relatively prime to* char k *otherwise. The space* $\operatorname{Spec}_{r^m}A_m$ *has dimension $2m$, when it is non-empty.*

Proof. Let I be the kernel of a finite-dimensional, irreducible representation. By 3.2, some power of each x_i acts with finite order on $T/I\cap T$, so for some integer n, every x_i^n centralizes T in A_m/I. But

$$x_i^{-n}y_ix_i^n=t^ny_i,$$

so we must have $t^n=1$ in A_m/I. In particular, I contains $t-c$ for some root of unity c.

Consider, then, the algebra $A_m/(t-c)$ for c a primitive rth root of unity. It contains

$$S_r = k[y_i, y_i^{-1}, x_j^r, x_j^{-r}]$$

as a commutative subalgebra of dimension $2m$, and is obtained as a crossed product with respect to the finite group $\hat{G}$ of size r^m generated by elements $\hat{x}_j$ with $\hat{x}_j^r = 1$. Observe that the $\hat{G}$-orbit of any point in $\operatorname{Spec} S_r$ has size r^m, so that the cover

$$\operatorname{Spec} S_r \to \operatorname{Spec}(S_r)^{\hat{G}}$$

is unramified, and by 2.4,

$$\operatorname{Spec}_{r^m} A_m/(t-c) \approx \operatorname{Spec} S_r^{\hat{G}}.$$

Thus $\operatorname{Spec}_{r^m} A_m$ is a finite number of copies of spaces of the type above, one for each primitive rth root of unity, and has dimension $2m$.

This example was particularly simple because there was no ramification. In the next one, ramification does occur, but we can still determine the representations. We find that the sequence $\{d_n\}$ alternates between 1 and 3.

Example 2. We assume that the characteristic of k is not two. Let $T = k[t^{\pm 1}, y_1^{\pm 1}, y_2^{\pm 1}]$ and let G be the infinite dihedral group generated by x and z with $z^2 = 1$ and $zxz = x^{-1}$. Form the skew group ring $B = T * G$ with G acting on T so that t is central and

$$x^{-1}y_1x = ty_1, \qquad x^{-1}y_2x = t^{-1}y_2,$$
$$zy_1z = y_2, \qquad zy_2z = y_1.$$

Note that B is again finitely-generated noetherian, and is the group ring of a polycyclic group.

Theorem 3.4. *The space* $\operatorname{Spec} B_n$ *is non-empty only for n relatively prime to* char k, *in which case it contains one-dimensional components, and if n is even, a three-dimensional component.*

Proof. The group G satisfies (ii) of 3.1, so x acts with finite order on the image of T in any finite-dimensional, irreducible representation. As in Example 1, this means that t maps to a root of unity of some order r, and we are reduced to examining crossed products of $S_r = k[y_i^{\pm 1}, x^{\pm r}]$ by the dihedral group of size $2r$.

To be precise, let D be the dihedral group generated by $\hat{x}$ and z, with $z^2 = 1$, $\hat{x}^r = 1$, and $z\hat{x}z = \hat{x}^{-1}$. Let $S_r * D$ be the crossed product with the relations

$$\hat{x}^{-1}y_1\hat{x} = cy_1, \qquad \hat{x}^{-1}y_2\hat{x} = c^{-1}y_2$$

for c a primitive rth root of unity, and

$$zx^rz = x^{-r}, \qquad zy_1z = y_2, \qquad \hat{x}^r = x^r.$$

The last relation defines the appropriate 2-cocycle.

Any maximal ideal of $\operatorname{Spec} S_r$ is sent to r distinct maximal ideals by $\hat{x}$, so its D-orbit has size r or $2r$. Therefore, a ramification point has orbit of size r, with some conjugate fixed by z and having the form

$$M_1 = (y_1 - a, y_2 - a, x^r - \varepsilon)$$

with $\varepsilon = 1$ or -1. The distinct elements $M_1, \dots, M_r$ of the orbit have the form

$$M_i = (y_1 - c^{-i}a, y_2 - c^i a, x^r - \varepsilon),$$

and the ideal $\mathbf{m} = S_r^D \cap M_i$ contains the elements

$$u = (y_1^r - y_2^r)^2 \quad \text{and} \quad v_\varepsilon = x^r + x^{-r} - 2\varepsilon.$$

It is easily checked that, conversely, the branch locus in $\operatorname{Spec} S_r^D$ consists of the two closed sets

$$V(u, v_1) \cup V(u, v_{-1}).$$

By 2.4, $\operatorname{Spec}_{2r} S_r * D$ is homeomorphic to the complement of this branch locus in $\operatorname{Spec} S_r^D$, and it remains to find the maximal ideals of $S_r * D$ above the branch locus. This is answered, and the theorem proved, by the following result:

Proposition 3.5. *The irreducible representations of $S_r * D$ have degree r or $2r$, and* $\operatorname{Spec}_r S_r * D$ *is homeomorphic to two copies of the branch locus in* $\operatorname{Spec} S_r^D$.

Proof. Let I be the ideal of S_r^D vanishing on the branch locus. We know by 2.5 and 2.6 that for $\mathbf{m} \in V(I)$, there are two prime ideals of $S_r * D$ containing $\mathbf{m}$, each the kernel of an irreducible representation of degree r. Assume $\mathbf{m}$ contains $y_1^r - a^r$, $y_2^r - a^r$, and $x^r + x^{-r} - 2\varepsilon$. Then explicitly, the representations are the following:

$$x \mapsto \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ \varepsilon & & & & 0 \end{pmatrix}, \qquad y_1 \mapsto \begin{pmatrix} a & & & & & \\ & ca & & & & \\ & & c^2a & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & c^{r-1}a \end{pmatrix}$$

$$z \mapsto \delta \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & & & & & 1 \\ & & & & 1 & \\ \vdots & & & ⋰ & & \\ \vdots & & 1 & & & \\ \vdots & 1 & & & & \\ 0 & 1 & & & & \end{pmatrix},$$

for $\delta=1$ or -1. The subalgebra of $M_r(k)$ generated by the images of x and y_1 acts irreducibly, since it contains a diagonal matrix with distinct eigenvalues and a monomial matrix acting transitively on the basis. Thus both representations are irreducible, and they cannot be conjugate. For suppose a matrix w conjugates one to the other. Since w commutes with the images of x and y_1, it must be scalar, and cannot conjugate one image of z to the other.

Thus we have located all the maximal ideals of $S_r * D$, and we know set-theoretically that $\mathrm{Spec}_r S_r * D$ is two copies of the branch locus of $\mathrm{Spec}\, S_r^D$. Intuitively, the set of primes corresponding to $\delta=1$ and -1 respectively should be disjoint, and homeomorphic to the branch locus. We now show this.

The matrix y_2 maps to

$$\begin{bmatrix} a & & & & & \\ & c^{r-1}a & & & & \\ & & c^{r-2}a & & & \\ & & & \ddots & & \\ & & & & c^2a & \\ & & & & & ca \end{bmatrix}.$$

Consider the element

$$\begin{aligned} w_r &= \prod_{i=1}^{r-1} (y_2 - x^{-i}y_1x^i) \qquad r \text{ odd},\\ &= \prod_{i=1}^{r/2-1} (y_2 - x^{-2i}y_1x^{2i}) \quad r \text{ even}. \end{aligned}$$

For r odd, the image of w_r is a diagonal matrix with non-zero entries only in row 1. For r even, the image is diagonal with non-zero entries only in rows 1 and $r/2+1$. Specifically, the entries are

$$a^r \prod_{i=1}^{r-1} (1-c^i) \qquad \text{if } r \text{ is odd},$$

$$a^{r/2} \prod_{i=1}^{r/2-1} (1-c^{2i}) \quad \text{if } r \text{ is even}.$$

It follows that $(z-\delta)w_r$ maps to 0 and $(z+\delta)w_r$ does not. Thus we see that the two closed sets $V((z-1)w_r)$ and $V((z+1)w_r)$ are disjoint in $\mathrm{Spec}_r S_r * D$, and each maps bijectively to the branch locus of $\mathrm{Spec}\, S_r^D$.

We want further to know that the maps are homeomorphisms. Let I_δ be the ideal of $S_r * D$ vanishing on $V((z-\delta)w_r)$. Then every simple image of $S_r * D/I_\delta$ is isomorphic to $M_r(k)$. It follows by Artin's Theorem [2, p. 546] that this algebra is Azumaya over its center, which is $S_r^D/I_\delta \cap S_r^D$. Since the spectrum of an Azumaya algebra maps homeomorphically to the spectrum of the center, this proves the theorem.

We next discuss a related example, involving dihedral groups again, in which the ramification is more complicated and we do not obtain a complete topological description of the representation spaces.

Example 3. Assume that k does not have characteristic 2. Let $T=k[y_i^{\pm 1}, w_i^{\pm 1}]$ for $i \in \mathbb{Z}$ and let G be the infinite dihedral group generated by x and z as above. Form the skew group ring $C=T*G$ with respect to the action

$$x^{-1}y_i x = y_{i+1}, \qquad x^{-1}w_i x = w_{i-1}, \qquad zy_i z = w_i.$$

The algebra C is the group ring of a finitely-generated, solvable group, but is not noetherian.

Theorem 3.6. *For n even,* $\mathrm{Spec}_n C$ *contains components of dimension n and* $n+1$. *For n odd,* $\mathrm{Spec}_n C$ *has dimension n. Thus the sequence of dimensions has the form* $1, 3, 3, 5, 5, \ldots$.

Proof. Applying 3.2, we find that every finite-dimensional, irreducible representation factors through a crossed product of the form $S_r * D$, where

$$S_r = k[y_1^{\pm 1}, w_1^{\pm 1}, \ldots, y_r^{\pm 1}, w_r^{\pm 1}, x^{\pm r}]$$

and D is the dihedral group of size $2r$ generated by $\hat{x}$ and z with the obvious relations. Once again, we must investigate the representations of $S_r * D$ corresponding to branch points of S_r^D. Let M be a ramification point in Spec S_r whose $\hat{x}$-orbit has $t<r$ elements. Then M contains $y_i - y_{t+i}$ and $w_i - w_{t+i}$, where indices are taken modulo r. But then so does every D-conjugate of M, and the proof of 2.4 shows that every prime ideal I of $S_r * D$ containing $M \cap S_r^D$ must contain $\bigcap_{g \in D} M^g$. Thus x^t acts identically on the image of S_r in $S_r * D/I$, and we may pass to a crossed product of S_t by the dihedral group of size $2t$.

What this means is that the irreducible representations of C which are omitted when we look at the complements of branch loci in the various Spec S_r^D must arise from ramification points of Spec S_r with orbit of size r, where z fixes some ideal in the orbit. This is analogous to the situation of the preceding example. Let M_1 be such a ramification point, fixed by z, with

$$M_1 = (y_1 - a_1, w_1 - a_1, \ldots, y_r - a_r, w_r - a_r, x^r - \varepsilon).$$

By 2.5 and 2.6, we know that two primes of $S_r * D$ lie over $M_1 \cap S_r^D$, corresponding to irreducible representations of degree r. The representations are equivalent to the two below:

$$x \mapsto \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ \varepsilon & & & & 0 \end{pmatrix}, \qquad z \mapsto \delta \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ 0 & 1 & & & \end{pmatrix}$$

$$y_1 \mapsto \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & a_r \end{pmatrix}$$

As before, δ is 1 or -1. The images of x and $\{y_i, z_i\}$ generate an irreducible subalgebra of $M_r(k)$, so the two representations are inequivalent and yield distinct prime ideals.

In analogy to 3.5, we can find an element q with the property that $q(z-\delta)$ maps to 0 in all these representations. However it is only the case that $q(z+\delta)$ goes to a non-zero matrix for a dense open subset of the branch locus of $\operatorname{Spec} S_r^D$. This open set includes the points for which the a_i are distinct. In the same way as in 3.5, we can conclude that over this open set, $\operatorname{Spec}_r S_r * D$ breaks into two disjoint, homeomorphic copies of the open set, but we leave open the question of how $\operatorname{Spec}_r S_r * D$ fits together overall. In any case, we have that $\operatorname{Spec}_r S_r * D$ has the same dimension as the branch locus, which is r. This proves the theorem.

Our final family of examples E_m have the most poorly behaved dimension sequences, for $m > 1$, while E_1 has the sequence $0, 3, 4, \ldots$.

Example 4. Let T be the polynomial ring over k in variables $\{y_{i,1}^{\pm 1}, \ldots, y_{i,m}^{\pm 1} : i \in \mathbb{Z}\}$ and let G be the free abelian group of rank m generated by $x_1, \ldots, x_m$. Form the skew group ring $T * G$ with the relations

$$x_l^{-1} y_{i,j} x_l = y_{i,j} \quad \text{if } j \neq l,$$

$$x_j^{-1} y_{i,j} x_j = y_{i+1,j}.$$

Thus each x_j moves one sequence of y's up while leaving the rest alone. Define the algebra E_m to be $T * G$, with the inverses of $\{y_{i+1,j} - y_{i,j} : i \in \mathbb{Z}, j = 1, \ldots, m\}$ adjoined. This is a finitely-generated algebra over k, since $T * G$ is the group ring of a finitely-generated, solvable group, and the inverses are all x_j-conjugates of $\{(y_{1,j} - y_{0,j})^{-1}\}$.

Theorem 3.7. *E_m has irreducible representations of degree n only if n is the product of m integers >1, in which case* $\mathrm{Spec}_n E_m$ *has dimension equal to the maximum of* $m+\sum_{i=1}^{m} q_i$, *where* $q_i>1$ *and* $q_1 \cdots q_m = n$.

Proof. By 3.1(i), each x_j acts with finite order on the image of T in a finite-dimensional, irreducible representation. We thus are reduced to crossed products of the polynomial ring S over k in variables $\{y_{i,j}^{\pm 1} \mid i=1,\dots,q_j; j=1,\dots,m\} \cup \{x_j^{\pm q_j}\}$ by the abelian group $\hat{G}$ of size $q_1 \cdots q_m$ generated by elements $\hat{x}_j$ of order q_j. The effect of inverting the particular set of elements is to require that each q_j is >1. Let M be a ramification point in $\mathrm{Spec}\, S_r$. Then some x_j must send it to fewer than q_j elements, and we find that $\{y_{i,j}-y_{t+i,j}\} \subset M$ for some $t<q_j$. But then this set of elements lies in every $\hat{G}$-conjugate of M. Since $M \cap S_D^{\hat{G}}$ must contain

$$\bigcap_{g \in \hat{G}} M^g,$$

we have $x_j^t - 1 \in I$. So the prime ideals containing a branch point of $\mathrm{Spec}\, S^{\hat{G}}$ arise over the complement of the branch locus for a different S and $\hat{G}$. Thus, as we range over all sequences $\{(q_1,\dots,q_m) : q_j>1\}$ and look at the complement of the branch locus in the corresponding spaces $\mathrm{Spec}\, S^{\hat{G}}$, we obtain all the finite-dimensional, irreducible representations. The theorem now follows.

Remark. Let us consider the cases $m=1$ and 2. For $m=1$, the algebra $T*G$ is the group ring of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. The theorem implies that E_1 has irreducible representations of every degree $n>1$, with $\mathrm{Spec}_n E_1$ of dimension $n+1$. The effect of inverting $y_1 - y_0$ is to remove the 1-dimensional representations, so we see that $k[\mathbb{Z} \wr \mathbb{Z}]$ has the same irreducible representations as E_1, plus the obvious one-dimensional, irreducible representations obtained by sending x and y_0 to arbitrary non-zero scalars, forming a 2-dimensional space. The sequence $\{d_n\}$ in this case is $2, 3, 4, \dots$.

In case $m=2$, the sum q_1+q_2 is maximized when q_1 or q_2 are the least prime divisors of n. Thus we find that $d_n=0$ if n is prime, and otherwise $d_n = p + n/p + 2$, for p the least prime divisor of n. This sequence will bounce back and forth between $\frac{1}{2}n+4$ at even integers n and $2\sqrt{n}+2$ at squares of primes, when it is not zero. The sequences for $m>2$ can be analyzed similarly.

4. An observation for simple Lie algebras

The examples of Section 3 suggest that the dimension sequence $\{d_n\}$ can behave in varied ways for finitely-generated algebras. One might hope to say something about the sequence in terms of the associated generating function $\sum d_n t^n$. This function is not rational for the algebras A_m of Section 3, as the trivial lemma below shows.

Lemma 4.1. *A series $p(t) = \sum d_n t^n$ in which the set of indices n for which $d_n \neq 0$ has arbitrarily large gaps is not rational.*

Proof. Suppose $p(t)$ is rational. Then $p(t)g(t)$ is a polynomial for some polynomial $g(t)$. But for any pair of indices d_n, $d_m \neq 0$ with $d_i = 0$ for $n < i < m$ and $m - n > \deg g(t)$, we obtain a non-zero term of $p(t)g(t)$ of degree $m + \deg g(t)$. Thus $p(t)g(t)$ is not a polynomial.

Let k be an algebraically closed field of characteristic 0. In this section we show that for any simple Lie algebra besides sl_2, the polynomial ring $U(L)[t_1, \dots, t_m]$ with $m > 0$ has a dimension sequence with non-rational generating function. This follows from the following result, which may be well known, although we do not know a reference. We refer to [4] for standard facts from Lie algebra theory which we use.

Proposition 4.2. *Let L be a simple Lie algebra other than sl_2. Then the set of degrees of finite-dimensional, irreducible representations of L has density zero in the set of positive integers.*

Corollary 4.3. *Let A be the enveloping algebra of a simple Lie algebra other than sl_2, and let $m > 0$. Then the dimenion sequence $\{d_n\} = \{\dim \mathrm{Spec}_n A[t_1, \dots, t_m]\}$ does not have a rational generating function.*

Proof. By 4.2, $d_n = 0$ except for a set of numbers of density 0, so the sequence of indices for which $d_n \neq 0$ has arbitrarily large gaps. The Cartan–Weyl theory of the highest weight implies that L has infinitely many irreducible representations of finite degree, and the Weyl degree formula implies that there are finitely many in any degree. Thus, infinitely many of the d_n's are non-zero.

The approximations and estimates in the proof below were suggested by D. Harbater.

Proof of 4.2. Let $\alpha_1, \dots, \alpha_n$ be a set of simple roots for L, with dual roots $\bar{\alpha}_1, \dots, \bar{\alpha}_n$, and let $\lambda_1, \dots, \lambda_n$ be a dual basis to $\bar{\alpha}_i$. For any positive root α, decompose

$$\bar{\alpha} = \sum_{i=1}^{n} c_i^{\alpha} \bar{\alpha}_i.$$

The numbers c_i^{α} are non-negative integers. Define a polynomial

$$p(x_1, \dots, x_n) = \prod_{\alpha > 0} \left(\sum_{i=1}^{n} c_i^{\alpha} x_i \right). \tag{3}$$

Then Weyl's degree formula states that the irreducible representation of highest weight $\lambda = \sum m_i \lambda_i$ has degree

$$\frac{1}{N} p(m_1 + 1, \dots, m_n + 1),$$

where $N = p(1, \dots, 1)$. Thus the set of degrees is the set

$$\left\{\frac{1}{N}\, p(x_1, \dots, x_n) : x_i \text{ a positive integer}\right\}.$$

Let $D = (\mathbb{Z}^+)^n$. We claim that the inequality

$$\frac{1}{N}\, p(x_1, \dots, x_n) \geq n(x_1 \cdots x_n)^{1+1/n} \tag{4}$$

holds for $(x_i) \in D$.

Observe first that since the c_i^α are non-negative integers, at least one of which is non-zero for a fixed α, the αth term in the product (3) must take on values ≥ 1 on D. Thus

$$\frac{1}{N}\, p(x_1, \dots, x_n) \geq \prod_{\alpha = \alpha_i, \beta} \left(\sum_{i=1} c_i^\alpha x_i\right)$$

on D, where we take the product over the simple roots $\bar{\alpha}_i$ and the unique root $\bar{\beta}$ of greatest height in the dual root system. Of course, the terms corresponding to $\bar{\alpha}_i$ are simply x_i. The maximal root $\bar{\beta}$ involves every simple root non-trivially, so $c_i^\beta \geq 1$ for all i. Thus we find

$$\frac{1}{N}\, p(x_1, \dots, x_n) \geq (x_1 \cdots x_n)(x_1 + \cdots + x_n)$$

on D. But since the arithmetic mean of a set of positive numbers is greater than the geometric mean, we have

$$(x_1 + \cdots + x_n) \geq n(x_1 \cdots x_n)^{1/n},$$

which yields (4).

Therefore, for a fixed number $r > 0$, we have

$$\#\left\{(x_0) \in D : \frac{1}{N}\, p(x_i) \leq rn\right\} \leq \#\{(x_i) \in D : n(x_1 \cdots x_n)^{1+1/n} \leq rn\},$$

and it suffices to prove that

$$\lim_{r \to \infty} \frac{\#\{(x_i) \in D : (x_1 \cdots x_n)^{1+1/n} \leq r\}}{r} = 0.$$

The numerator is bounded by the volume under the hypersurface $(x_1 \cdots x_n)^{1+1/n} = r$ with $x_i \geq 1$, and this volume is no more than

$$\int_1^r \cdots \int_1^r \frac{r^{n/(n+1)}}{x_1 \cdots x_{n-1}}\, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} = r^{n/(n+1)} (\ln r)^{n-1}.$$

The resulting limit

$$\lim_{r\to\infty} \frac{(\ln r)^{n-1}}{r^{1/n+1}}$$

is 0, as $n-1$ applications of L'Hospital's rule show.

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