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On conformal surfaces of annulus type

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ABSTRACT

Let $a > b > 0$ and $B_a \setminus B_b = \{x = (x_1, x_2) \in R^2\colon\ b < |x| < a\}$, and assume that f is a conformal map from $B_a \setminus B_b$ into $\mathbb{R}^n$, with $|\nabla f|^2 = 2e^{2u}$, then (e_1, e_2) with $e_1 = e^{-u}\frac{\partial f}{\partial r}$, and $e_2 = r^{-1}e^{-u}\frac{\partial f}{\partial \theta}$ is a moving frame on $f(B_a \setminus B_b)$ and it satisfies the following equation

$$d \star \langle de_1, e_2 \rangle = 0,$$

where $\star$ is the Hodge star operator on R^2 with respect to the standard metric.

We will study the Dirichlet energy of this frame and give some applications.

1. Introduction

Let Ω be a smooth bounded domain in R^2 and f be a $W^{2,2}$ map from Ω to R^n, and $(e_1, e_2) \in W^{1,2}(\Omega, \mathbb{R}^n \times \mathbb{R}^n)$ be a positively oriented basis of f. We define

$$\mathcal{K}(e_1, e_2) := \frac{\partial e_1}{\partial x^1}\frac{\partial e_2}{\partial x^2} - \frac{\partial e_1}{\partial x^2}\frac{\partial e_2}{\partial x^1} := \nabla e_1 \nabla^{\perp} e_2.$$

It is easy to check that $\mathcal{K}(e_1, e_2)$ is invariant under the group action $U(2)$, that is for any

$$e_1' = e_1 \cos\theta + e_2 \sin\theta, \qquad e_2' = -e_1 \sin\theta + e_2 \cos\theta,$$

where $\theta \in W^{1,2}(\Omega, R)$, we have

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$$\mathcal{K}(e_1', e_2') = \mathcal{K}(e_1, e_2), \tag{1.1}$$

see Appendix A for a proof. Hence we can write $\mathcal{K}(X_f) := \mathcal{K}(e_1, e_2)$, where X_f is the Gauss map of the surface $f(\Omega)$ defined from $f(\Omega)$ to the Grassmannian $G(2,n)$. Moreover, we have

$$K_f e^{2u} = \mathcal{K}(X_f) = \nabla e_1 \nabla^\perp e_2,$$

where K_f is the Gauss curvature of the immersed surface $f(\Omega)$ and $|\nabla f| = 2e^{2u}$ (see Appendix A for a proof).

Since $\mathit{div}\,\nabla^\perp e_1 = 0$, and $\mathit{rot}\,\nabla e_2 = 0$, $\mathcal{K}(X_f)$ has compensation compactness. Furthermore, Wente's type inequality can be applied here.

Recall Wente's type inequality, which states that if $a, b \in W^{1,2}(\Omega)$ and $u \in W_0^{1,2}(\Omega)$ solves the equation

$$-\Delta u = \nabla a \nabla^\perp b \quad \text{in } \Omega,$$

then u is continuous and we have

$$\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leqslant C(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}, \tag{1.2}$$

see [1,2,12].

It is easy to see that $C(\Omega)$ is invariant under translations and dilations. F. Bethuel and J.-M. Ghidaglia in [3,4] showed that there exists a constant C_1 which does not depend on Ω such that (1.2) holds true:

$$\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leqslant C_1\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}.$$

We denote by $C_\infty(\Omega)$ the best constant involving the L^∞ norm and by $C_2(\Omega)$ the best constant involving the L^2 norm. Then we have

$$\|u\|_{L^\infty(\Omega)} \leqslant C_\infty(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}, \tag{1.3}$$

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C_2(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}. \tag{1.4}$$

It is proved in [1] that $C_\infty(\Omega) \geqslant \frac{1}{2\pi}$, and when Ω is simply connected, $C_\infty(\Omega) = \frac{1}{2\pi}$. For the general Ω, it is proved by Topping [11] that $C_\infty(\Omega) = \frac{1}{2\pi}$. It is proved by Ge in [5] that $C_2(\Omega) = \sqrt{\frac{3}{64\pi}}$.

Let $B \subseteq R^2$ be the unit disk centered at the origin, then Li, Luo and Tang proved the following theorem by using the inequality (1.3):

Theorem A. *(See [9].) Let $\varphi \in W^{1,2}(B, G(2,n))$ with*

$$\int_B |\mathcal{K}(\varphi)|\, d\sigma \leqslant \gamma < 2\pi$$

(see Appendix A for information about $\mathcal{K}(\varphi)$), then there exists a map $(e_1, e_2) \in W^{1,2}(B, \mathbb{R}^n \times \mathbb{R}^n)$ such that for almost every $z \in B$, $(e_1(z), e_2(z))$ is a positively oriented basis of $\varphi(z)$. Furthermore, we have

$$\|d(e_1, e_2)\|_{L^2(B)} \leqslant C(\gamma)\|\nabla\varphi\|_{L^2(B)},$$

where $C(\gamma)$ is a constant depending on γ.

Note that $\mathcal{K}(\varphi) \leqslant \frac{1}{2}|\nabla\varphi|^2$, see (A.2). We have the following direct corollary:

Theorem B. *(See [9].) Let $\varphi \in W^{1,2}(B, G(2,n))$ with*

$$\int_B |\nabla\varphi|^2\, d\sigma \leqslant \gamma < 4\pi,$$

then there exists a map $(e_1, e_2) \in W^{1,2}(B, \mathbb{R}^n \times \mathbb{R}^n)$ such that for almost every $z \in B$, $(e_1(z), e_2(z))$ is a positively oriented basis of $\varphi(z)$. Furthermore, we have

$$\big\| d(e_1, e_2)\big\|_{L^2(B)} \leqslant C(\gamma)\|\nabla\varphi\|_{L^2(B)},$$

where $C(\gamma)$ is a constant depending on γ.

Remark 1.1. (1) The above theorem improves a theorem of Hélein [6, Chapter 5] by changing the constant from $\frac{8\pi}{3}$ to 4π (the same result also is proved in [8] by using a difficult result of [10]). The difference between these results is that in Hélein's original proof he used Wente's inequality (1.4) of L^2 norm, whereas we use Wente's inequality (1.3) of L^∞ norm.

(2) The constant 4π is sharp for $n > 3$ (see [8]).

Assume that $f : B \to R^n$ is a conformal map and $f \in C^\infty(\bar{B})$, and $X_f : f(B) \to G(2,n)$ is the Gauss map. Let $\varphi = X_f \circ f \in W^{1,2}(B, G(2,n))$, then we have (see (A.3), (A.4) in Appendix A):

$$\int_B |K_f|\, d\mu_f = \int_B \big|\mathcal{K}(\varphi)\big|\, d\sigma,$$

$$\int_B |\nabla\varphi|^2\, dx = \int_{f(B)} |\nabla_{g_f} X_f|^2\, d\mu_f = \int_B |A_f|^2\, d\mu_f,$$

where ∇_{g_f} is the gradient with respect to g_f, μ_f is the area measure on $f(B)$, and A_f is the second fundamental form of $f(B)$. Hence for such a conformal immersion f with the L^1 norm of the Gauss curvature bounded by 2π, there exists a moving frame on it whose Dirichlet energy is bounded by the L^2 norm of the second fundamental form. Hélein [6, Chapter 5] used this moving frame to derive the weak compactness of immersed conformal surfaces from B into R^n. Using his argument, we have

Theorem C. *(See [9].) Let $f_k \in C^\infty(\bar{B}, \mathbb{R}^n)$ be a sequence of conformal immersions with*

$$\sup_k \int_B |K_{f_k}|\, d\mu_{f_k} \leqslant \gamma < 2\pi, \qquad \sup_k \int_B |A_{f_k}|^2\, d\mu_{f_k} < \infty,$$

where $d\mu_{f_k}$ is the volume form deduced from the metric g_{f_k}. Assume that f_k converges to f_0 weakly in $W^{1,2}$, then f_0 is either a point or a conformal immersion.

In this paper, we are interested in generalizing these above results to immersed conformal surfaces from Ω into R^n when Ω is not simply connected. We will consider the easiest case, that is when Ω is an annuli. In the following we will let $a > b > 0$, and $B_a \setminus B_b = \{x \in R^2\colon\ b < |x| < a\}$. We have

Theorem 1.2. *For every conformal map $f : B_a \setminus B_b \to R^n$ satisfying*

$$\left\|\mathcal{K}(X_f)\right\|_{L^1(B_a\setminus B_b)} \leqslant \gamma < 2\pi,$$

there exists a map $b = (e_1, e_2)$ in $W^{1,2}(B_a \setminus B_b, \mathbb{R}^n \times R^n)$, such that for almost every $z \in B_a \setminus B_b$, $(e_1(z), e_2(z))$ is a positively oriented basis of $\varphi(z)$ and $\|d(e_1, e_2)\|_{L^2(B_a\setminus B_b)}$ is bounded.

Furthermore, if

$$\frac{\beta}{1-\sqrt{\frac{\gamma}{2\pi}}} < 1, \tag{1.5}$$

where

$$\int\limits_{B_a\setminus B_b} \left|\frac{\partial e_1}{\partial\theta}\, d\theta\right|^2 + \left|\frac{\partial e_2}{\partial\theta}\, d\theta\right|^2 dx = \beta^2\big(\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)}\big),$$

then we have that

$$\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)} \leqslant C\left(\beta, \gamma, \frac{a}{b}\right)\|A_f\|^2_{L^2(B_a\setminus B_b)},$$

where $C(\beta, \gamma, \frac{a}{b})$ is a constant depending on β, γ and $\frac{a}{b}$.

As a direct corollary, we have

Theorem 1.3. *For every conformal map $f : B_a \setminus B_b \to R^n$ satisfying*

$$\|A_f\|^2_{L^2(B_a\setminus B_b)} \leqslant \gamma < 4\pi,$$

where A_f is the second fundamental form of f, there exists a map $b = (e_1, e_2)$ in $W^{1,2}(B_a \setminus B_b, \mathbb{R}^n \times R^n)$, such that for almost every $z \in B_a \setminus B_b$, $(e_1(z), e_2(z))$ is a positively oriented basis of $f(z)$ and $\|d(e_1, e_2)\|_{L^2(B_a\setminus B_b)}$ is bounded.

Furthermore, if

$$\frac{\beta}{1-\sqrt{\frac{\gamma}{2\pi}}} < 1,$$

where

$$\int\limits_{B_a\setminus B_b} \left|\frac{\partial e_1}{\partial\theta}\, d\theta\right|^2 + \left|\frac{\partial e_2}{\partial\theta}\, d\theta\right|^2 dx = \beta^2\big(\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)}\big),$$

then we have that

$$\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)} \leqslant C\left(\beta, \gamma, \frac{a}{b}\right)\|A_f\|^2_{L^2(B_a\setminus B_b)},$$

where $C(\beta, \gamma, \frac{a}{b})$ is a constant depending on β, γ and $\frac{a}{b}$.

Remark 1.4. We can see that in the proof of the above two theorems these estimates hold true for any moving frame (e_1, e_2) satisfying

$$\begin{cases} d \star \langle de_1, e_2 \rangle = 0 & \text{in } B_a \setminus B_b, \\ \langle de_1, e_2 \rangle \left(\dfrac{\partial}{\partial \nu} \right) = 0 & \text{on } \partial(B_a \setminus B_b), \end{cases}$$

where $\star$ is the Hodge star operator on R^2 and $\frac{\partial}{\partial \nu}$ is the outward normal vector on the boundaries. In the following we will define such a moving frame to be a Coulomb frame.

It is nature to ask the following question:

Question. On which kind of conformal parametric surfaces from $B_a \setminus B_b$ to R^n, there exists a moving frame (e_1, e_2) on it and some $\beta \in [0, 1)$, such that

$$\int_{B_a \setminus B_b} \left| \frac{\partial e_1}{\partial \theta} d\theta \right|^2 + \left| \frac{\partial e_2}{\partial \theta} d\theta \right|^2 dx = \beta^2 \left(\|\nabla e_1\|^2_{L^2(B_a \setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a \setminus B_b)} \right)?$$

Definition 1.5. *Let Ω be a domain in R^2 and $f : \Omega \to R^n$ be a conformal immersion, and let (e_1, e_2) be a moving frame on $f(\Omega)$, then we call (e_1, e_2) to be a Coulomb frame of $f(\Omega)$ if*

$$\begin{cases} d \star \langle de_1, e_2 \rangle = 0 & \text{in } \Omega, \\ \langle de_1, e_2 \rangle \left(\dfrac{\partial}{\partial \nu} \right) = 0 & \text{on } \partial\Omega. \end{cases}$$

If we only have

$$d \star \langle de_1, e_2 \rangle = 0 \quad \text{in } \Omega,$$

then (e_1, e_2) is called a semi-Coulomb frame.

We have

Lemma 1.6. *Let $f : B_a \setminus B_b \to R^n$ be a conformal immersion, with $|\nabla f|^2 = 2e^{2u}$, and let $e_1 = e^{-u} \frac{\partial f}{\partial r}$ and $e_2 = r^{-1} e^{-u} \frac{\partial f}{\partial \theta}$, then (e_1, e_2) is a semi-Coulomb frame on $f(B_a \setminus B_b)$, and it is a Coulomb frame if and only if u are constants on the boundaries. We call (e_1, e_2) to be the canonical semi-Coulomb frame of $f(B_a \setminus B_b)$.*

The following theorem shows a relation between the canonical semi-Coulomb frame and the conformal factor of a conformal minimal immersion:

Theorem 1.7. *Let $f : B_a \setminus B_b \to R^n$ be a conformal minimal immersion, that is, $f(B_a \setminus B_b)$ is a conformal minimal surface in R^n, then the canonical semi-Coulomb frame of $f(B_a \setminus B_b)$, (e_1, e_2) satisfies that*

$$\int_{B_a \setminus B_b} \left| \frac{\partial e_1}{\partial \theta} d\theta \right|^2 + \left| \frac{\partial e_2}{\partial \theta} d\theta \right|^2 dx = \frac{1}{2} \int_{B_a \setminus B_b} |\nabla e_1|^2 + |\nabla e_2|^2 \, dx,$$

if and only if

$$\int\limits_{B_a\setminus B_b} \left(\frac{1}{r}+\frac{\partial u}{\partial r}\right)^2 dx = \int\limits_{B_a\setminus B_b} r^{-2}\left(\frac{\partial u}{\partial \theta}\right)^2 dx,$$

where $|\nabla f|^2 = 2e^{2u}$.

As a direct corollary we have

Corollary 1.8. *Let f be a conformal minimal immersion with $|\nabla f|^2 = 2e^{2u}$. Assume that u is radially symmetric and (e_1, e_2) is the canonical semi-Coulomb frame of f with*

$$\int\limits_{B_a\setminus B_b} \left|\frac{\partial e_1}{\partial \theta}\, d\theta\right|^2 + \left|\frac{\partial e_2}{\partial \theta}\, d\theta\right|^2 dx = \frac{1}{2}\int\limits_{B_a\setminus B_b} |\nabla e_1|^2 + |\nabla e_2|^2\, dx,$$

then we can get

$$u(x) = -\log r + c,$$

where $r = |x|$ and c is a constant. In addition we can deduce that $A = 0$ and e_1, e_2 are constant vectors.

Remark 1.9. (1) Let f be defined on $B_a \setminus B_b$ as $f(r,\theta) = \pm e^c(\log r, \theta, 0, \ldots, 0)$, where c is a constant, then f is a conformal immersion into R^n with $u(x) = -\log r + c$. It is easy to see that the canonical semi-Coulomb frame of $f(B_a \setminus B_b)$ is $((1,0)(0,1))$ and the second fundamental form of $f(B_a \setminus B_b)$ is zero.

(2) Assume that f is a conformal immersion from $B_a \setminus B_b$ to R^n, with $|\nabla f|^2 = 2e^{2u}$, and u is radially symmetric. Let (e_1, e_2) be the canonical semi-Coulomb frame of $f(B_a \setminus B_b)$, then if e_1 and e_2 are constant vectors, we have

$$\frac{\partial f}{\partial r} = e^u(a,b), \qquad \frac{\partial f}{\partial \theta} = re^u(c,d),$$

where a, b, c, d are constants with $a^2+b^2 = c^2+d^2 = 1$, $ac+bd = 0$. Thus $\frac{\partial^2 f}{\partial r \partial \theta} = \frac{\partial^2 f}{\partial \theta \partial r}$ implies that $u_r = -\frac{1}{r}$, and so $u(r) = -\log r + c$ for some constant c.

As a corollary of the above results, we have the following theorem, which partially answers the above question:

Theorem 1.10. *Let $f : B_a \setminus B_b \to R^n$ be a conformal minimal immersion, with $|\nabla f|^2 = 2e^{2u}$ and*

$$\int\limits_{B_a\setminus B_b} \left(\frac{1}{r}+\frac{\partial u}{\partial r}\right)^2 dx = \int\limits_{B_a\setminus B_b} r^{-2}\left(\frac{\partial u}{\partial \theta}\right)^2 dx, \tag{1.6}$$

$$u = c_a \quad on\ \partial B_a, \qquad u = c_b \quad on\ \partial B_b, \tag{1.7}$$

$$\int\limits_{B_a\setminus B_b} |K_f|\, d\mu_f \leqslant \gamma < (3-2\sqrt{2})\pi, \tag{1.8}$$

where c_a and c_b are constants.

Assume that (e_1, e_2) is the canonical semi-Coulomb frame of $f(B_a \setminus B_b)$, then we have

$$\int_{B_a \setminus B_b} |\nabla e_1|^2 + |\nabla e_2|^2 \, dx \leqslant C\left(\gamma, \frac{a}{b}\right) \int_{B_a \setminus B_b} |A_f|^2 \, d\mu_f, \tag{1.9}$$

where $C(\gamma, \frac{a}{b})$ is a constant depending on γ and $\frac{a}{b}$.

Remark 1.11. The properties of Coulomb frames on conformal surface $f(\Omega)$ have big difference between the case when Ω is simply connected and the case when Ω is not simply connected. Recall that if f is a conformal immersion from the unit disk $B \subseteq R^2$ into R^n, then the energy of a Coulomb frame on it can be controlled by the L^2 norm of the second fundamental form of $f(B)$, if the L^2 norm of the second fundamental form of $f(B)$ is below some constant [6,9]. But for $f : B_a \setminus B_b \to R^n$ with $f(x, y) = (x, y, 0, \ldots, 0)$, the second fundamental form of $f(B_a \setminus B_b)$ is zero, and $((\cos\theta, \sin\theta), (-\sin\theta, \cos\theta))$ is a Coulomb frame on $f(B_a \setminus B_b)$ with nonzero energy.

Now we will give an application of the above theorem.

Theorem 1.12. *Let $\{f_m \in C^\infty(\overline{B_a \setminus B_b})\}_{m\geqslant 1}$ be a sequence of minimal conformal immersions from $B_a \setminus B_b$ into R^n with $|\nabla f_m|^2 = 2e^{2u_m}$, and*

$$\int_{B_a \setminus B_b} \left(\frac{1}{r} + \frac{\partial u_m}{\partial r}\right)^2 dx = \int_{B_a \setminus B_b} r^{-2}\left(\frac{\partial u_m}{\partial \theta}\right)^2 dx, \tag{1.10}$$

$$u_m = c_{ma} \quad \text{on } \partial B_a, \qquad u_m = c_{mb} \quad \text{on } \partial B_b, \tag{1.11}$$

$$\sup_m \int_{B_a \setminus B_b} |K_{f_m}| \, d\mu_m < (3 - 2\sqrt{2})\pi, \qquad \sup_m \int_{B_a \setminus B_b} |A_{f_m}|^2 \, d\mu_m < \infty, \tag{1.12}$$

where c_{ma} and c_{mb} are constants, with $\sup_m\{|c_{ma}| + |c_{mb}|\} < \infty$.

Assume that f_m converges weakly to f_0 in $W^{1,2}$, then f_0 is a minimal conformal immersion with bounded conformal factor. Furthermore the metric induced by f_0 is continuous.

Remark 1.13. The minimal property will be kept under the weak convergence by the definition, if the limit immersion is conformal. Hence the difficult and non-obvious part is to prove that the limit immersion is conformal, and has bounded conformal factor.

Note that when the domain is a disk, the weak limit map may be a single point, that is there may be a collapsing in the convergence process. But when the domain is an annulus this cannot happen.

This paper is organized as follows: In Section 2 results are proved. In Appendix A we will give some basic computations which have been used in this paper and in Appendix B we will give an alternative proof about that f_0 is conformal in Theorem 1.12 by using a strong converge theorem of p-harmonic maps due to Hardt, Lin and Mou [7].

Notations. $\partial_r = \frac{\partial}{\partial r}$, $\partial_\theta = \frac{\partial}{\partial \theta}$, $f_r = \frac{\partial f}{\partial r}$, $f_\theta = \frac{\partial f}{\partial \theta}$, $\partial_{rr} = \ldots$.

2. Proof of the results

Proof of Theorem 1.2. Let $(e_1, e_2) \in W^{1,2}(B_a \setminus B_b, \mathbb{R}^n \times \mathbb{R}^n)$ be a positively oriented basis of f and X_f be the Gauss map of f. Let

$$F = \left\{(X_f, e_1, e_2) \in G(2, n) \times \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; (e_1, e_2) \text{ is a positively oriented orthonormal basis of } X_f\right\}.$$

Then $\mathcal{F}$ is a fibre bundle over $G(2,n)$ with fibre S^1. Since f is conformal, there exists a section $(\tilde{e_1}, \tilde{e_2})$ of $X_f^* \mathcal{F}$.

We consider for each $\theta \in W^{1,2}(B_a \setminus B_b, \mathbb{R})$ the frame (e_1, e_2) obtained by

$$(e_1, e_2) = (\tilde{e_1}, \tilde{e_2}) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Actually, we will minimize over $\theta \in W^{1,2}(B_a \setminus B_b, \mathbb{R})$ the functional

$$F(\theta) = \frac{1}{2} \int_{B_a \setminus B_b} \left(|\nabla e_1|^2 + |\nabla e_2|^2\right) d\sigma \tag{2.1}$$

$$= \int_{B_a \setminus B_b} \left|\omega_2^1\right|^2 d\sigma, \tag{2.2}$$

where $w_2^1 = \langle de_1, e_2 \rangle$. By the arguments in [6], the minimum of F is attained, and the minimizer (e_1, e_2) satisfies

$$\begin{cases} d\left(\star\omega_2^1\right) = 0 & \text{in } B_a \setminus B_b, \\ \omega_2^1\left(\dfrac{\partial}{\partial \nu}\right) = 0 & \text{on } \partial(B_a \setminus B_b), \end{cases}$$

where $\star$ is the Hodge star operator and $\frac{\partial}{\partial \nu}$ is the outward normal vector on the boundary. Then there exists some $v \in W^{1,2}(B_a \setminus B_b, \mathbb{R})$ such that

$$dv = \star\omega_2^1 - \alpha\, d\theta \quad \text{in } B_a \setminus B_b, \tag{2.3}$$

where $\alpha = \frac{1}{2\pi} \int_0^{2\pi} \star\omega_2^1$ is a constant. It is easy to check that $\frac{\partial v}{\partial \theta} = dv(\frac{\partial}{\partial \theta}) = -\alpha$ on $\partial(B_a \setminus B_b)$, and hence we have $v|_{\partial B_a} = c_a - \alpha\theta$, and $v|_{\partial B_b} = c_b - \alpha\theta$, where $c_a = v(a,0)$ and $c_b = v(b,0)$. A direct calculation yields

$$-\triangle v = \mathcal{K}(e_1, e_2). \tag{2.4}$$

Decompose v to be $v = v_1 + v_2$ where

$$\begin{cases} -\Delta v_1 = \mathcal{K}(e_1, e_2) & \text{in } B_a \setminus B_b, \\ v_1 = 0 & \text{on } \partial(B_a \setminus B_b), \end{cases}$$

and

$$\begin{cases} \Delta v_2 = 0 & \text{in } B_a \setminus B_b, \\ v_2 = c_a - \alpha\theta & \text{on } \partial B_a, \\ v_2 = c_b - \alpha\theta & \text{on } \partial B_b. \end{cases}$$

To estimate v_1, we decompose $v_1 = \Sigma_{k=1}^n v_1^k$ where v_1^k is the solution of

$$\begin{cases} -\Delta v_1^k = \mathcal{K}(e_1^k, e_2^k) & \text{in } B_a \setminus B_b, \\ v_1^k = 0 & \text{on } \partial(B_a \setminus B_b). \end{cases}$$

Applying Wente's inequality, we have

$$\|v_1^k\|_{L^\infty(B_a\setminus B_b)} \leqslant \frac{1}{2\pi}\|\nabla e_1^k\|_{L^2(B_a\setminus B_b)}\|\nabla e_2^k\|_{L^2(B_a\setminus B_b)},$$

which obviously implies that

$$\|v_1\|_{L^\infty(B_a\setminus B_b)} \leqslant \frac{1}{4\pi}\big(\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)}\big).$$

A simple calculation by integral by parts implies that

$$\begin{aligned}\int\limits_{B_a\setminus B_b} |\nabla v_1|^2\,d\sigma &= \int\limits_{B_a\setminus B_b} v_1\mathcal{K}(e_1,e_2)\,d\sigma \\ &\leqslant \|v_1\|_{L^\infty(B_a\setminus B_b)}\int\limits_{B_a\setminus B_b}\big|\mathcal{K}(e_1,e_2)\big|\,d\sigma \\ &\leqslant \frac{\gamma}{4\pi}\big(\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)}\big).\end{aligned}$$

For v_2, we have

$$v_2 = \frac{c_a - c_b}{\log\frac{a}{b}}\log|x| + \frac{c_b\log a - c_a\log b}{\log\frac{a}{b}} - \alpha\theta.$$

Noting that by calculations in Appendix A (see (A.1)) we have

$$\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)} = 2\|dv + \alpha\,d\theta\|^2_{L^2(B_a\setminus B_b)} + \|\nabla X_f\|^2_{L^2(B_a\setminus B_b)}.$$

We have

$$\begin{aligned}\|dv + \alpha\,d\theta\|_{L^2(B_a\setminus B_b)} &= \|dv_1 + dv_2 + \alpha\,d\theta\|_{L^2(B_a\setminus B_b)} \\ &\leqslant \|dv_1\|_{L^2(B_a\setminus B_b)} + \|dv_2 + \alpha\,d\theta\|_{L^2(B_a\setminus B_b)} \\ &\leqslant \left(\frac{\gamma}{4\pi}\right)^{\frac{1}{2}}\big(\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)}\big)^{\frac{1}{2}} + (2\pi)^{\frac{1}{2}}\frac{|c_a - c_b|}{(\log\frac{a}{b})^{\frac{1}{2}}}.\end{aligned}$$

Thus we obtain

$$\left(1 - \left(\frac{\gamma}{2\pi}\right)^{\frac{1}{2}}\right)\big(\|\nabla e_1\|^2_{L^2(B_a\setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a\setminus B_b)}\big)^{\frac{1}{2}} \leqslant (4\pi)^{\frac{1}{2}}\frac{|c_a - c_b|}{(\log\frac{a}{b})^{\frac{1}{2}}} + \|\nabla X_f\|_{L^2(B_a\setminus B_b)}. \tag{2.5}$$

Without loss of generality, we assume that

$$\int\limits_b^a \left|\frac{\partial e_2}{\partial\theta}\right|^2(r,0) + \left|\frac{\partial e_1}{\partial\theta}\right|^2(r,0)\,dr \leqslant \int\limits_b^a \left|\frac{\partial e_2}{\partial\theta}\right|^2(r,\theta) + \left|\frac{\partial e_1}{\partial\theta}\right|^2(r,\theta)\,dr,$$

for $0 \leqslant \theta < 2\pi$.

To estimate the number $|c_a - c_b|$, we note that

$$\begin{aligned}
|c_a - c_b| = \left| \int_b^a \frac{\partial v}{\partial r}(r,0)\,dr \right| &\leqslant \int_b^a \left| \left\langle e_1 \frac{\partial e_2}{\partial \theta} \right\rangle \right| (r,0) r^{-1}\,dr \\
&\leqslant \left(\frac{\log \frac{a}{b}}{2} \right)^{\frac{1}{2}} \left(\int_b^a \left(\left| \left\langle e_1 \frac{\partial e_2}{\partial \theta} \right\rangle \right|^2 + \left| \left\langle e_2 \frac{\partial e_1}{\partial \theta} \right\rangle \right|^2 \right) r^{-1}\,dr \right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{\log \frac{a}{b}}{2} \right)^{\frac{1}{2}} \left(\int_b^a \left| \frac{\partial e_2}{\partial \theta}\,d\theta \right|^2 + \left| \frac{\partial e_1}{\partial \theta}\,d\theta \right|^2 r\,dr \right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{\log \frac{a}{b}}{4\pi} \right)^{\frac{1}{2}} \left(\int_{B_a \setminus B_b} \left| \frac{\partial e_2}{\partial \theta}\,d\theta \right|^2 + \left| \frac{\partial e_1}{\partial \theta}\,d\theta \right|^2 dx \right)^{\frac{1}{2}} \\
&\leqslant \beta \left(\frac{\log \frac{a}{b}}{4\pi} \right)^{\frac{1}{2}} \left(\|\nabla e_1\|^2_{L^2(B_a \setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a \setminus B_b)} \right)^{\frac{1}{2}} \\
&\leqslant \frac{\beta}{1 - (\frac{\gamma}{2\pi})^{\frac{1}{2}}} |c_a - c_b| + C\left(\gamma, \frac{a}{b}\right) \|\nabla X_f\|_{L^2(B_a \setminus B_b)},
\end{aligned}$$

where we have used the fact that $\star d\theta = r^{-1}\,dr$ and that

$$\int_{B_a \setminus B_b} \left| \frac{\partial e_2}{\partial \theta}\,d\theta \right|^2 + \left| \frac{\partial e_1}{\partial \theta}\,d\theta \right|^2 dx = \beta^2 \left(\|\nabla e_1\|^2_{L^2(B_a \setminus B_b)} + \|\nabla e_2\|^2_{L^2(B_a \setminus B_b)} \right).$$

Hence if

$$\frac{\beta}{1 - \sqrt{\frac{\gamma}{2\pi}}} < 1,$$

$|c_a - c_b|$ is controlled by $C(\beta, \gamma, \frac{a}{b}) \|\nabla X_f\|_{L^2(B_a \setminus B_b)}$, and hence the energy of the Coulomb frame is controlled by $C(\beta, \gamma, \frac{a}{b}) \|\nabla X_f\|_{L^2(B_a \setminus B_b)}$. Noting that $\|\nabla X_f\|_{L^2(B_a \setminus B_b)} = \|A_f\|_{L^2(B_a \setminus B_b)}$, we complete the proof of Theorem 1.2. □

Proof of Lemma 1.6. Note that $e_1 = e^{-u} f_r$, and $e_2 = r^{-1} e^{-u} f_\theta$, then we have

$$\begin{aligned}
\langle de_1, e_2 \rangle &= \left\langle \frac{f_{rr} - f_r u_r}{e^u}, \frac{f_\theta}{re^u} \right\rangle dr + \left\langle \frac{f_{r\theta} - f_r u_\theta}{e^u}, \frac{f_\theta}{re^u} \right\rangle d\theta \\
&= \left\langle \frac{f_{rr}}{e^u}, \frac{f_\theta}{re^u} \right\rangle dr + \left\langle \frac{f_{r\theta}}{e^u}, \frac{f_\theta}{re^u} \right\rangle d\theta \\
&= \frac{-u_\theta}{r}\,dr + (1 + ru_r)\,d\theta,
\end{aligned}$$

hence

$$\begin{aligned}\star\langle de_1, e_2\rangle &= \star\left(\frac{-u_\theta}{r}\,dr + (1+ru_r)\,d\theta\right)\\ &= \frac{-u_\theta}{r}(-r\,d\theta) + (1+ru_r)r^{-1}\,dr\\ &= u_\theta\,d\theta + \left(r^{-1}+u_r\right)dr,\end{aligned}$$

finally we obtain

$$d\star\langle de_1, e_2\rangle = u_{\theta r}\,dr\wedge d\theta + u_{r\theta}\,d\theta\wedge dr = 0,$$

which implies that (e_1, e_2) is a semi-Coulomb frame on $f(B_a\setminus B_b)$. In addition, $\langle de_1, e_2\rangle(\frac{\partial}{\partial\nu}) = 0$ if and only if $\star\langle de_1, e_2\rangle(\frac{\partial}{\partial\theta}) = 0$. Note that $\star\langle de_1, e_2\rangle(\frac{\partial}{\partial\theta}) = u_\theta$, so $\langle de_1, e_2\rangle(\frac{\partial}{\partial\nu}) = 0$ if and only if u are constants on the boundaries. □

Proof of Theorem 1.7. Recall that $e_1 = e^{-u}\frac{\partial f}{\partial r}$ and $e_2 = r^{-1}e^{-u}\frac{\partial f}{\partial\theta}$, then we have

$$\begin{aligned}\frac{\partial e_1}{\partial\theta} &= \frac{f_{r\theta} - f_r u_\theta}{e^u},\\ \frac{\partial e_2}{\partial\theta} &= \frac{f_{\theta\theta} - f_\theta u_\theta}{re^u},\end{aligned}$$

hence we can obtain

$$\begin{aligned}\left|\frac{\partial e_1}{\partial\theta}\right|^2 &= e^{-2u}\left[f_{r\theta}^2 - 2\langle f_{r\theta}, f_r\rangle u_\theta + f_r^2 u_\theta^2\right]\\ &= e^{-2u}\left[f_{r\theta}^2 - 2e^{2u}u_\theta^2 + e^{2u}u_\theta^2\right]\\ &= e^{-2u}f_{r\theta}^2 - u_\theta^2,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial e_2}{\partial\theta}\right|^2 &= r^{-2}e^{-2u}\left[f_{\theta\theta}^2 - 2\langle f_{\theta\theta}, f_\theta\rangle u_\theta + f_\theta^2 u_\theta^2\right]\\ &= r^{-2}e^{-2u}\left[f_{\theta\theta}^2 - \frac{\partial r^2 e^{2u}}{\partial\theta}u_\theta + f_\theta^2 u_\theta^2\right]\\ &= r^{-2}e^{-2u}\left[f_{\theta\theta}^2 - 2r^2e^{2u}u_\theta^2 + r^2e^{2u}u_\theta^2\right]\\ &= r^{-2}e^{-2u}f_{\theta\theta}^2 - u_\theta^2.\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}\frac{\partial e_1}{\partial r} &= \frac{f_{rr} - f_r u_r}{e^u},\\ \frac{\partial e_2}{\partial r} &= \frac{f_{r\theta} - f_\theta(\frac{1}{r} + u_r)}{re^u},\end{aligned}$$

and

$$\left|\frac{\partial e_1}{\partial r}\right|^2 = e^{-2u} f_{rr}^2 - u_r^2,$$

$$\left|\frac{\partial e_2}{\partial r}\right|^2 = r^{-2} e^{-2u} f_{r\theta}^2 - \left(\frac{1}{r} + u_r\right)^2.$$

Summarizing the above computations and noting that $|d\theta|^2 = r^{-2}$, we have

$$\left|\frac{\partial e_2}{\partial \theta}\, d\theta\right|^2 + \left|\frac{\partial e_1}{\partial \theta}\, d\theta\right|^2 = r^{-2} e^{-2u} f_{r\theta}^2 + r^{-4} e^{-2u} f_{\theta\theta}^2 - 2r^{-2} u_\theta^2, \tag{2.6}$$

$$\left|\frac{\partial e_2}{\partial r}\, dr\right|^2 + \left|\frac{\partial e_1}{\partial r}\, dr\right|^2 = e^{-2u} f_{rr}^2 + r^{-2} e^{-2u} f_{r\theta}^2 - u_r^2 - \left(\frac{1}{r} + u_r\right)^2. \tag{2.7}$$

On the other hand, by the definition of the second fundamental form, we have

$$A_{rr} = f_{rr} - e^{-2u} \langle f_{rr}, f_r \rangle f_r - r^{-2} e^{-2u} \langle f_{rr}, f_\theta \rangle f_\theta,$$

$$A_{\theta\theta} = f_{\theta\theta} - e^{-2u} \langle f_{\theta\theta}, f_r \rangle f_r - r^{-2} e^{-2u} \langle f_{\theta\theta}, f_\theta \rangle f_\theta,$$

therefore we have

$$\begin{aligned} A_{rr}^2 &= f_{rr}^2 - 2\langle f_{rr}, f_r \rangle^2 e^{-2u} - 2\langle f_{rr}, f_\theta \rangle^2 r^{-2} e^{-2u} + \langle f_{rr}, f_r \rangle^2 e^{-2u} + \langle f_{rr}, f_\theta \rangle^2 r^{-2} e^{-2u} \\ &= f_{rr}^2 - \langle f_{rr}, f_r \rangle^2 e^{-2u} - \langle f_{rr}, f_\theta \rangle^2 r^{-2} e^{-2u} \\ &= f_{rr}^2 - u_r^2 e^{2u} - u_\theta^2 r^{-2} e^{2u}, \end{aligned}$$

and similar computations implies that

$$A_{\theta\theta}^2 = f_{\theta\theta}^2 - r^4 e^{2u} \left(\frac{1}{r} + u_r\right)^2 - r^2 e^{2u} u_\theta^2.$$

Note that f is minimal, so

$$Trace(A) = g^{rr} A_{rr} + 2g^{r\theta} A_{r\theta} + g^{\theta\theta} A_{\theta\theta} = g^{rr} A_{rr} + g^{\theta\theta} A_{\theta\theta} = 0,$$

where $g^{rr} = e^{-2u}$, $g^{\theta\theta} = r^{-2} e^{-2u}$ hence

$$A_{rr} = -r^{-2} A_{\theta\theta},$$

which implies that

$$A_{rr}^2 = r^{-4} A_{\theta\theta}^2.$$

Thus we obtain

$$f_{rr}^2 - u_r^2 e^{2u} = r^{-4} f_{\theta\theta}^2 - e^{2u} \left(\frac{1}{r} + u_r\right)^2. \tag{2.8}$$

Note that

$$\int_{B_a\setminus B_b} \left|\frac{\partial e_2}{\partial\theta}\,d\theta\right|^2 + \left|\frac{\partial e_1}{\partial\theta}\,d\theta\right|^2 dx = \frac{1}{2}\int_{B_a\setminus B_b} |\nabla e_1|^2 + |\nabla e_2|^2\,dx,$$

if and only if

$$\int_{B_a\setminus B_b} \left|\frac{\partial e_2}{\partial\theta}\,d\theta\right|^2 + \left|\frac{\partial e_1}{\partial\theta}\,d\theta\right|^2 dx = \int_{B_a\setminus B_b} \left|\frac{\partial e_2}{\partial r}\,dr\right|^2 + \left|\frac{\partial e_1}{\partial r}\,dr\right|^2 dx,$$

thus we can get by combining (2.6)–(2.8) that

$$\int_{B_a\setminus B_b} \left|\frac{\partial e_2}{\partial\theta}\,d\theta\right|^2 + \left|\frac{\partial e_1}{\partial\theta}\,d\theta\right|^2 dx = \frac{1}{2}\int_{B_a\setminus B_b} |\nabla e_1|^2 + |\nabla e_2|^2\,dx,$$

if and only if

$$\int_{B_a\setminus B_b} \left(\frac{1}{r} + \frac{\partial u}{\partial r}\right)^2 dx = \int_{B_a\setminus B_b} r^{-2}\left(\frac{\partial u}{\partial\theta}\right)^2 dx. \qquad \square$$

Proof of Corollary 1.8. From Theorem 1.7 we know that

$$\int_{B_a\setminus B_b} \left|\frac{\partial e_2}{\partial\theta}\,d\theta\right|^2 + \left|\frac{\partial e_1}{\partial\theta}\,d\theta\right|^2 dx = \frac{1}{2}\int_{B_a\setminus B_b} |\nabla e_1|^2 + |\nabla e_2|^2\,dx,$$

implies

$$\int_{B_a\setminus B_b} \left(\frac{1}{r} + \frac{\partial u}{\partial r}\right)^2 dx = \int_{B_a\setminus B_b} r^{-2}\left(\frac{\partial u}{\partial\theta}\right)^2 dx,$$

hence if u is radially symmetric we must have that

$$\int_{B_a\setminus B_b} \left(\frac{1}{r} + \frac{\partial u}{\partial r}\right)^2 dx = 0,$$

which implies that

$$\frac{1}{r} + \frac{\partial u}{\partial r} = 0,$$

and so there is a constant c such that

$$u(r) = -\log r + c.$$

Hence we have $\Delta u = 0$, which implies that the Gauss curvature $K = 0$, and so $A = 0$. By Theorem 1.10 (note that u are constants on boundaries and hence (e_1, e_2) is a Coulomb frame), we know that (e_1, e_2) has zero energy and so e_1 and e_2 are constant vectors. □

Proof of Theorem 1.10. We know that (e_1, e_2) is a Coulomb frame and so when (1.6) holds we have that

$$\int_{B_a \setminus B_b} \left| \frac{\partial e_2}{\partial \theta} \, d\theta \right|^2 + \left| \frac{\partial e_1}{\partial \theta} \, d\theta \right|^2 dx = \frac{1}{2} \int_{B_a \setminus B_b} |\nabla e_1|^2 + |\nabla e_2|^2 \, dx,$$

by Theorem 1.7. Then the constant β in Theorem 1.2 is $\frac{\sqrt{2}}{2}$ and so when

$$\int_{B_a \setminus B_b} |K_f| \, du_f \leqslant \gamma < (3 - 2\sqrt{2})\pi,$$

we have (1.5) holds, and then we get the desired inequality (1.9) from Theorem 1.2. □

Proof of Theorem 1.12. Let (e_{m1}, e_{m2}) be the canonical semi-Coulomb frame on $f_m(B_a \setminus B_b)$, then by Theorem 1.10 we have the following inequality

$$\int_{B_a \setminus B_b} |\nabla e_{m1}|^2 + |\nabla e_{m2}|^2 \, dx \leqslant C \int_{B_a \setminus B_b} |A_m|^2 \, d\mu_{f_m},$$

where C is independent of m.

Note that we have

$$-\Delta u_m = K_m e^{2u_m} = \mathcal{K}(e_{m1}, e_{m2}) \quad \text{in } B_a \setminus B_b,$$

where K_m is the Gauss curvature.

Let v_m solves the following equation

$$\begin{cases} -\Delta v_m = \mathcal{K}(e_{m1}, e_{m2}) & \text{in } B_a \setminus B_b, \\ v_m = 0 & \text{on } \partial(B_a \setminus B_b). \end{cases}$$

Let $e_{mi} = (e_{mi}^1, \dots, e_{mi}^n)$, $i = 1, 2$, and $v_m = v_m^1 + \dots + v_m^n$, such that for each $1 \leqslant k \leqslant n$,

$$\begin{cases} -\Delta v_m^k = \mathcal{K}\big(e_{m1}^k, e_{m2}^k\big) & \text{in } B_a \setminus B_b, \\ v_m^k = 0 & \text{on } \partial(B_a \setminus B_b), \end{cases}$$

then by Wente's inequality we obtain

$$\|v_m^k\|_{L^\infty(B_a \setminus B_b)} \leqslant \frac{1}{2\pi} \|\nabla e_{m1}^k\|_{L^2(B_a \setminus B_b)} \|\nabla e_{m2}^k\|_{L^2(B_a \setminus B_b)},$$

hence

$$\begin{aligned}
\|v_m\|_{L^\infty(B_a\setminus B_b)} &\leqslant \sum_k \|v_m^k\|_{L^\infty(B_a\setminus B_b)} \\
&\leqslant \sum_k \frac{1}{2\pi}\|\nabla e_{m1}^k\|_{L^2(B_a\setminus B_b)}\|\nabla e_{m2}^k\|_{L^2(B_a\setminus B_b)} \\
&\leqslant \frac{1}{2\pi}\|\nabla e_{m1}\|_{L^2(B_a\setminus B_b)}\|\nabla e_{m2}\|_{L^2(B_a\setminus B_b)},
\end{aligned}$$

where in the last inequality we have used Holder's inequality.

By using the equation satisfied by v_m and by integral by parts we have

$$\begin{aligned}
\int_{B_a\setminus B_b} -v_m\Delta v_m &= \int v_m \nabla e_{m1}\nabla^\perp e_{m2}\,dx \\
&\leqslant \|v_m\|_{L^\infty(B_a\setminus B_b)}\|\nabla e_{m1}\|_{L^2(B_a\setminus B_b)}\|\nabla e_{m2}\|_{L^2(B_a\setminus B_b)} \\
&\leqslant \frac{1}{2\pi}\|\nabla e_{m1}\|^2_{L^2(B_a\setminus B_b)}\|\nabla e_{m2}\|^2_{L^2(B_a\setminus B_b)}.
\end{aligned}$$

That is

$$\|\nabla v_m\|^2_{L^2(B_a\setminus B_b)} \leqslant \frac{1}{2\pi}\|\nabla e_{m1}\|^2_{L^2(B_a\setminus B_b)}\|\nabla e_{m2}\|^2_{L^2(B_a\setminus B_b)}.$$

On the other hand,

$$\begin{cases}
\Delta(u_m - v_m) = 0 & \text{in } B_a\setminus B_b, \\
u_m - v_m = c_{ma} & \text{on } \partial B_a, \\
u_m - v_m = c_{mb} & \text{on } \partial B_b,
\end{cases}$$

thus we have

$$u_m - v_m = \frac{c_{ma}-c_{mb}}{\log\frac{a}{b}}\log|x| + \frac{c_{mb}\log a - c_{ma}\log b}{\log\frac{a}{b}},$$

which implies that

$$\|\nabla u_m\|_{L^2(B_a\setminus B_b)} + \|u_m\|_{L^\infty(B_a\setminus B_b)} \leqslant C < \infty, \tag{2.9}$$

for some constant C independent of m.

Then by using an argument given by [6, Chapter 5], we can get that f_0 is a conformal immersion with bounded conformal factor as the following: Because f_m is conformal, there exists $0 \leqslant \theta_m \in C^\infty < 2\pi$ such that

$$df_m = e^{u_m}\big((\cos\theta_m e_{m1} + \sin\theta_m e_{m2})\,dx_1 + (-\sin\theta_m e_{m1} + \cos\theta_m e_{m2})\,dx_2\big). \tag{2.10}$$

In particular, projecting the equation $d^2 f_m = 0$ along e_{m1} and e_{m2} we obtain

$$\begin{cases}
\dfrac{\partial\theta_m}{\partial x_1} + \dfrac{\partial u_m}{\partial x_2} = \omega_{m2}^1\Big(\dfrac{\partial}{\partial x_1}\Big), \\
\dfrac{\partial\theta_m}{\partial x_2} - \dfrac{\partial u_m}{\partial x_1} = \omega_{m2}^1\Big(\dfrac{\partial}{\partial x_2}\Big),
\end{cases}$$

where $\omega_{m2}^1 = \langle de_{m2}, e_{m1}\rangle$.

Note that these equations imply θ_m is bounded in $W^{1,2}$, hence we have that (we do not distinguish a sequence and its subsequences)

$$(b_m, \theta_m, u_m) \rightharpoonup (b, \theta, u) \quad \text{weakly in } W^{1,2},$$

and so

$$(b_m, \theta_m, u_m) \to (b, \theta, u) \quad \text{in } L^2,$$

therefore we have

$$(b_m, \theta_m, u_m) \to (b, \theta, u) \quad \text{a.e. in } B_a \setminus B_b,$$

where $b_m = (e_{m1}, e_{m2})$, and $b = (e_1, e_2)$.

By passing to the limit in (2.10) we get

$$df_0 = e^u\big((\cos\theta e_1 + \sin\theta e_2)\,dx_1 + (-\sin\theta e_1 + \cos\theta e_2)\,dx_2\big), \tag{2.11}$$

which implies that f_0 is conformal, with bounded conformal factor e^u.

Because u satisfies the following Wente's type equation

$$-\Delta u = \nabla e_1 \nabla^{\perp} e_2 \quad \text{in } B_a \setminus B_b,$$

hence u is continuous.

Note that

$$\Delta f_m = 0 \quad \text{in } B_a \setminus B_b,$$

and

$$f_m \to f_0 \quad \text{weakly in } W^{1,2}(B_a \setminus B_b),$$

therefore we have that

$$\Delta f_0 = 0. \tag{2.12}$$

On the other hand, because f_0 is a conformal immersion with $|\nabla f_0|^2 = 2e^{2u}$, we have that

$$\Delta f_0 = e^{2u} H_{f_0}, \tag{2.13}$$

where H_{f_0} is the mean curvature vector of f_0.

By comparing (2.12) with (2.13) we get that $H_{f_0} = 0$, and so f_0 is a minimal immersion. □

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Appendix A

In this appendix, we review briefly some basic facts of Grassmannian manifolds. The concept in this appendix can be found in any textbook on the theory of Grassmannian manifolds.

Let

$$\Lambda^2 = \Lambda^2(\mathbb{R}^n) = \{a_{ij} v^i \wedge v^j \colon v^i, v^j \in \mathbb{R}^n\}.$$

Λ^2 is a linear space of dimension $\frac{n(n-1)}{2}$. If e_k is a normal basis of $\mathbb{R}^n$, then $\{e_i \wedge e_j \colon i < j\}$ is a basis of Λ^2. The standard inner product of Λ^2 is defined by:

$$\langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle := (v_1 \cdot w_1)(v_2 \cdot w_2) - (v_1 \cdot w_2)(v_2 \cdot w_1).$$

So, $\{e_i \wedge e_j\}$ is a normal basis of Λ^2.

Let $P(\Lambda^2)$ be the projective space getting from Λ^2. Recall that there is a nature map π from the unit sphere of Λ^2 to $P(\Lambda^2)$ which is a covering map.

Let ψ to be the Plücker embedding from $G(2,n)$ to $P(\Lambda^2)$, which endows $G(2,n)$ a Riemannian metric. Thus, given a $b = (e_1, e_2) \in W^{1,2}$, we think of $\varphi(x) = e_1 \wedge e_2$ as a map from Ω to the unit sphere of Λ^2 (also a map to Λ^2), then the normal of $\frac{\partial(e_1 \wedge e_2)}{\partial x}$ is just the normal of $\frac{\partial e_1}{\partial x} \wedge e_2 + e_1 \wedge \frac{\partial e_2}{\partial x}$ in Λ^2. By a direct calculation, we get

$$\begin{aligned}
\left|\frac{\partial(e_1 \wedge e_2)}{\partial x}\right|^2 &= \left|\frac{\partial e_1}{\partial x} \wedge e_2 + e_1 \wedge \frac{\partial e_2}{\partial x}\right|^2 \\
&= \left|\frac{\partial e_1}{\partial x} \wedge e_2\right|^2 + \left|e_1 \wedge \frac{\partial e_2}{\partial x}\right|^2 + 2\left\langle \frac{\partial e_1}{\partial x} \wedge e_2, e_1 \wedge \frac{\partial e_2}{\partial x}\right\rangle \\
&= \left|\frac{\partial e_1}{\partial x}\right|^2 + \left|\frac{\partial e_2}{\partial x}\right|^2 - 2\left|e_1 \frac{\partial e_2}{\partial x}\right|^2.
\end{aligned}$$

So we have

$$|\nabla\varphi|^2 = |\nabla b|^2 - 2\left|\langle de_1, e_2\rangle\right|^2. \tag{A.1}$$

Now, we prove (1.1). Let (e_1', e_2') be another positively oriented norm basis of X. Then we have

$$e_1' = \lambda e_1 + \mu e_2, \qquad e_2' = -\mu e_1 + \lambda e_2,$$

where $\lambda = (e_1', e_1)$ and $\mu = (e_1', e_2)$. We have

$$\begin{aligned}
\frac{\partial e_1'}{\partial x^i} &= \frac{\partial \lambda}{\partial x^i} e_1 + \lambda \frac{\partial e_1}{\partial x^i} + \frac{\partial \mu}{\partial x^i} e_2 + \mu \frac{\partial e_2}{\partial x^i}, \\
\frac{\partial e_2'}{\partial x^i} &= -\frac{\partial \mu}{\partial x^i} e_1 - \mu \frac{\partial e_1}{\partial x^i} + \frac{\partial \lambda}{\partial x^i} e_2 + \lambda \frac{\partial e_2}{\partial x^i}.
\end{aligned}$$

We have

$$\frac{\partial e_1'}{\partial x^1}\frac{\partial e_2'}{\partial x^2} = -\frac{\partial\lambda}{\partial x^1}\frac{\partial\mu}{\partial x^2} + \lambda\frac{\partial\lambda}{\partial x^1}e_1\frac{\partial e_2}{\partial x^2} - \lambda\mu\frac{\partial e_1}{\partial x^1}\frac{\partial e_1}{\partial x^2} + \lambda\frac{\partial\lambda}{\partial x^2}\frac{\partial e_1}{\partial x^1}e_2 + \lambda^2\frac{\partial e_1}{\partial x^1}\frac{\partial e_2}{\partial x^2}$$
$$-\mu\frac{\partial\mu}{\partial x^1}e_2\frac{\partial e_1}{\partial x^2} + \frac{\partial\mu}{\partial x^1}\frac{\partial\lambda}{\partial x^2} - \mu\frac{\partial\mu}{\partial x^2}\frac{\partial e_2}{\partial x^1}e_1 - \mu^2\frac{\partial e_2}{\partial x^1}\frac{\partial e_1}{\partial x^2} + \mu\lambda\frac{\partial e_2}{\partial x^1}\frac{\partial e_2}{\partial x^2},$$
$$\frac{\partial e_1'}{\partial x^2}\frac{\partial e_2'}{\partial x^1} = -\frac{\partial\lambda}{\partial x^2}\frac{\partial\mu}{\partial x^1} + \lambda\frac{\partial\lambda}{\partial x^2}e_1\frac{\partial e_2}{\partial x^1} - \lambda\mu\frac{\partial e_1}{\partial x^2}\frac{\partial e_1}{\partial x^1} + \lambda\frac{\partial\lambda}{\partial x^1}\frac{\partial e_1}{\partial x^2}e_2 + \lambda^2\frac{\partial e_1}{\partial x^2}\frac{\partial e_2}{\partial x^1}$$
$$-\mu\frac{\partial\mu}{\partial x^2}e_2\frac{\partial e_1}{\partial x^1} + \frac{\partial\mu}{\partial x^2}\frac{\partial\lambda}{\partial x^1} - \mu\frac{\partial\mu}{\partial x^1}\frac{\partial e_2}{\partial x^2}e_1 - \mu^2\frac{\partial e_2}{\partial x^2}\frac{\partial e_1}{\partial x^1} + \mu\lambda\frac{\partial e_2}{\partial x^2}\frac{\partial e_2}{\partial x^1}.$$

We have

$$\frac{\partial e_1'}{\partial x^1}\frac{\partial e_2'}{\partial x^2} - \frac{\partial e_1'}{\partial x^2}\frac{\partial e_2'}{\partial x^1} = -2\left(\frac{\partial\lambda}{\partial x^1}\frac{\partial\mu}{\partial x^2} - \frac{\partial\lambda}{\partial x^2}\frac{\partial\mu}{\partial x^1}\right) + 2\left(\lambda\frac{\partial\lambda}{\partial x^1} + \mu\frac{\partial\mu}{\partial x^1}\right)e_1\frac{\partial e_2}{\partial x^2}$$
$$+ 2\left(\lambda\frac{\partial\lambda}{\partial x^2} + \mu\frac{\partial\mu}{\partial x^2}\right)e_1\frac{\partial e_2}{\partial x^1} + \left(\lambda^2+\mu^2\right)\left(\frac{\partial e_1}{\partial x^1}\frac{\partial e_2}{\partial x^2} - \frac{\partial e_1}{\partial x^2}\frac{\partial e_2}{\partial x^1}\right).$$

Since $\lambda^2+\mu^2=1$, we have $\frac{\partial\lambda}{\partial x^1}\frac{\partial\mu}{\partial x^2} - \frac{\partial\lambda}{\partial x^2}\frac{\partial\mu}{\partial x^1} = 0$, and $\lambda\frac{\partial\lambda}{\partial x^i} + \mu\frac{\partial\mu}{\partial x^i} = 0$, then we get (1.1).

We extend e_1, e_2 to a normal basis $e_3,\dots,e_n \in W^{1,2}$. Such e_i ($i \geqslant 3$) exists because φ is also a $W^{1,2}$ map from B to $G(2,n)$.

We set

$$de_i = w_{ij}^k\,dx^j \otimes e_k + B_{ij}^\alpha\,dx^j \otimes e_\alpha,$$

where $i=1,2$ and $\alpha \in \{3,4,\dots,n\}$. Obviously, $w_{1i}^1 = w_{2i}^2 = 0$, $w_{2i}^1 = -w_{1i}^2 = \langle \frac{\partial e_1}{\partial x^i}, e_2\rangle$, hence (A.1) is equivalent to

$$|\nabla\varphi|^2 = \sum_{ij,\alpha} \left|B_{ij}^\alpha\right|^2.$$

We have

$$\mathcal{K}(\varphi) = \left(w_{11}^k e_k + B_{11}^\alpha n_\alpha\right)\left(w_{22}^k e_k + B_{22}^\alpha n_\alpha\right) - \left(w_{12}^k e_k + B_{12}^\alpha n_\alpha\right)\left(w_{21}^k e_k + B_{21}^\alpha n_\alpha\right)$$
$$= \sum_\alpha \left(B_{11}^\alpha \cdot B_{22}^\alpha - \left|B_{12}^\alpha\right|^2\right),$$

therefore we obtain

$$\mathcal{K}(\varphi) \leqslant \frac{1}{2}|\nabla\varphi|^2. \tag{A.2}$$

Now, we consider the Gauss map of a conformal map $f: \Omega \to \mathbb{R}^n$. Let $u = \frac{1}{2}\log(|\nabla f|^2/2)$ and denote by X_f the Gauss map induced by f.

X_f can be expressed as

$$X_f = \left(e^{-u}\frac{\partial f}{\partial x^1}\right) \wedge \left(e^{-u}\frac{\partial f}{\partial x^2}\right),$$

where $u=\frac{1}{2}\log|\frac{\partial f}{\partial x^1}|^2$. We will calculate $|\nabla X_f|^2$. Since

$$\frac{\partial^2 f}{\partial x^1\partial x^1}\cdot\frac{\partial f}{\partial x^1}=\frac{1}{2}\frac{\partial}{\partial x^1}\left|\frac{\partial f}{\partial x^1}\right|^2=e^{2u}\frac{\partial u}{\partial x^1},$$

$$\frac{\partial^2 f}{\partial x^1\partial x^1}\cdot\frac{\partial f}{\partial x^2}=-\frac{\partial f}{\partial x^1}\cdot\frac{\partial^2 f}{\partial x^1\partial x^2}=-\frac{1}{2}\frac{\partial}{\partial x^2}\left|\frac{\partial f}{\partial x^1}\right|^2=-e^{2u}\frac{\partial u}{\partial x^2},$$

and

$$\frac{\partial^2 f}{\partial x^1\partial x^1}=A_{11}+\frac{\partial^2 f}{\partial x^1\partial x_1}\cdot\frac{\partial f}{\partial x^1}e^{-2u}\frac{\partial f}{\partial x^1}+\frac{\partial^2 f}{\partial x^1}\cdot\frac{\partial f}{\partial x^2}e^{-2u}\frac{\partial f}{\partial x^2},$$

we get

$$\frac{\partial}{\partial x^1}\left(e^{-u}\frac{\partial f}{\partial x^1}\right)=e^{-u}\frac{\partial^2 f}{\partial x^1\partial x_1}-e^{-u}\frac{\partial u}{\partial x^1}\frac{\partial f}{\partial x^1}=e^{-u}\left(A_{11}-\frac{\partial f}{\partial x^2}\frac{\partial u}{\partial x^2}\right).$$

In the same way, we get

$$\frac{\partial}{\partial x^2}\left(e^{-u}\frac{\partial f}{\partial x^1}\right)=e^{-u}\left(A_{12}+\frac{\partial f}{\partial x^1}\frac{\partial u}{\partial x^2}\right),$$

$$\frac{\partial}{\partial x^1}\left(e^{-u}\frac{\partial f}{\partial x^2}\right)=e^{-u}\left(A_{21}+\frac{\partial f}{\partial x^2}\frac{\partial u}{\partial x^1}\right),$$

$$\frac{\partial}{\partial x^2}\left(e^{-u}\frac{\partial f}{\partial x^2}\right)=e^{-u}\left(A_{22}-\frac{\partial f}{\partial x^1}\frac{\partial u}{\partial x^1}\right).$$

Then, we get

$$\mathcal{K}(X_f)=e^{-2u}\left(A_{11}A_{22}-A_{12}^2\right)=Ke^{2u} \tag{A.3}$$

and

$$|\nabla X_f|^2=e^{-2u}\sum|A_{ij}|^2,\quad \text{i.e.}\quad |\nabla_{g_f}X_f|^2\,d\mu_{g_f}=|A|^2\,d\mu_{g_f}. \tag{A.4}$$

Appendix B

In this part, we will give an alternative proof about that f_0 is conformal in Theorem 1.12. We need a special case of the following theorem proved by Hardt, Lin and Mou [7].

Theorem B.1. *Let Ω be a smooth bounded domain in R^2, and suppose $1<p<\infty$ and for each $i=1,2,\ldots,$ $u_i\in W^{1,p}(\Omega)$ is a weak solution of*

$$div\left(|\nabla u|^{p-2}\nabla u\right)+f_i=0$$

with $\sup_i\|u_i\|_{W^{1,p}}+\sup_i\|f_i\|_{L^1}<\infty$. If $u_i\to u$ weakly in $W^{1,p}$, then $u_i\to u$ strongly in $W^{1,q}$, whenever $1<q<p$.

Note that f_m is conformal, that is

$$\left|\frac{\partial f_m}{\partial x_1}\right|^2 = \left|\frac{\partial f_m}{\partial x_2}\right|^2, \qquad \frac{\partial f_m}{\partial x_1} \cdot \frac{\partial f_m}{\partial x_2} = 0,$$

and f_m is minimal, that is

$$\Delta f_m = 0.$$

Thus by the above theorem we have that

$$f_m \to f_0 \quad \text{strongly in } W^{1,p},$$

whenever $1 < p < 2$, which implies that

$$\frac{\partial f_m}{\partial x_1} \to \frac{\partial f_0}{\partial x_1} \quad \text{a.e.,} \quad \text{and} \quad \frac{\partial f_m}{\partial x_2} \to \frac{\partial f_0}{\partial x_2} \quad \text{a.e.}$$

Therefore we obtain

$$\left|\frac{\partial f_0}{\partial x_1}\right|^2 = \left|\frac{\partial f_0}{\partial x_2}\right|^2, \qquad \frac{\partial f_0}{\partial x_1} \cdot \frac{\partial f_0}{\partial x_2} = 0,$$

implying that f_0 is conformal.

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