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Point charge potential and weighting field of a pixel or pad in a plane condenser

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ABSTRACT

We derive expressions for the potential of a point charge as well as the weighting potential and weighting field of a rectangular pad for a plane condenser, which are well suited for numerical evaluation. We relate the expressions to solutions employing the method of image charges, which allows discussion of convergence properties and estimation of errors, providing also an illuminating example of a problem with an infinite number of image charges.

1. Introduction

In this report we derive the potential of a point charge and the so-called weighting potential and weighting field of a rectangular pad in a parallel plate geometry. These solutions are needed to calculate the signals in e.g. silicon pixel detectors as well as micropattern detectors with pixel or pad readout. The surface charge density σ induced on the metal planes by the presence of the point charge Q is related to the electric field E on the metal surface by $\sigma = \varepsilon_0 E$. Knowing the potential ϕ of a point charge Q at a (possibly time dependent) position x_0, y_0, z_0, the induced charge and current on a rectangular pad centred at zero is therefore given by

$$Q_{ind} = \int_{-w_x/2}^{w_x/2} \int_{-w_y/2}^{w_y/2} -\varepsilon_0 \vec{\nabla}\phi|_{z=0}\, dx\, dy, \quad I_{ind} = -\frac{dQ_{ind}}{dt} \tag{1}$$

Due to Green's reciprocity theorem [1] the charge and current are also given by

$$Q_{ind} = -\frac{Q}{V_w}\phi_w(\vec{x}_0), \quad I_{ind} = -\frac{Q}{V_w}\vec{E}_w(\vec{x}_0)_w \frac{d\vec{x}_0}{dt} \tag{2}$$

where ϕ_w and $E_w = -\vec{\nabla}\phi_w$ are the potential and electric field in the detector volume, respectively, in case all charges in the detector are removed, the pad is put to potential V_w and the rest stays grounded [2,3]. In the following we derive the expressions for ϕ, ϕ_w and E_w.

2. Potential

Fig. 1a shows a point charge between two grounded metal planes at a distance d. The potential is written as ϕ_1 in the region $0<z<z_0$ and ϕ_2 in the region $z_0<z<d$. We have

$$\phi_1(r,z) = \frac{Q}{2\pi\varepsilon_0}\int_0^\infty J_0(kr)\frac{\sin h(kz)\ \sin h(k(d-z_0))}{\sin h(kd)}\,dk \tag{3}$$

$$\phi_1(x,y,z) = \frac{Q}{\pi^2\varepsilon_0}\int_0^\infty\int_0^\infty \frac{\cos(k_x x)\ \cos(k_y y)}{k}\frac{\sin h(kz)\ \sin h(k(d-z_0))}{\sin h(kd)}\,dk_x\,dk_y \tag{4}$$

in cylindrical and cartesian coordinates, respectively [1,4]. $J_0(x)$ is the Bessel function of first kind and in Eq. (4) we have defined $k=\sqrt{k_x^2+k_y^2}$. For ϕ_2 we just have to swap z and z_0. The integrand has an infinite number of complex poles at $k_n = in\pi/d$ and by finding an appropriate contour in the complex plane the integral of Eq. (3) can be expressed as the sum of the residues which evaluates to [1]

$$\phi(r,z) = \frac{Q}{\pi\varepsilon_0 d}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{d}\right)\sin\left(\frac{n\pi z_0}{d}\right)K_0\left(\frac{n\pi r}{d}\right) \tag{5}$$

where $K_0(x)$ is the modified Bessel function of second kind. Since $K_0(x)$ has a logarithmic singularity at $x=0$ the expression diverges for $r=0$ and has slow convergence close to $r=0$. For numerical evaluation it is therefore easier to focus on the integral in Eqs. (3) and (4) for which very efficient methods are available. For large values of k the integrand of Eq. (3) behaves as $\sqrt{2/(kr\pi)}\cos(kr-\pi/4)e^{-k(z_0-z)}$. The exponential behaviour therefore allows to set the upper integration limit for k to a multiple of $1/(z_0-z)$ for precise numerical

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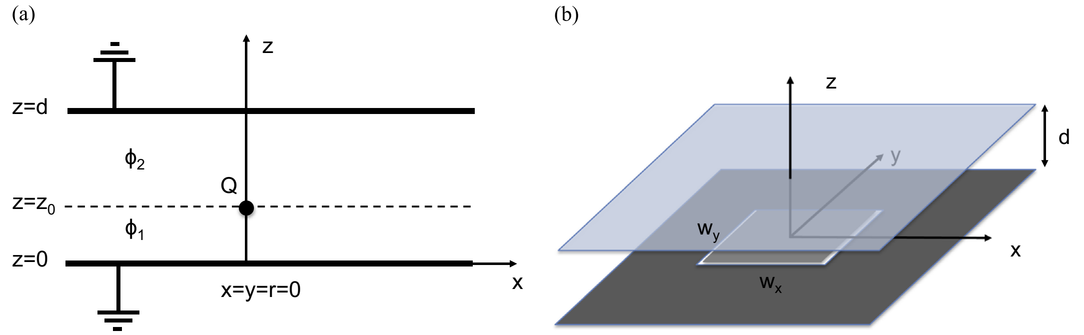

Fig. 1. (a) Point charge Q between two grounded metal planes. (b) Readout pad or pixel of dimension w_x and w_y centred at the origin.

evaluation. For values of $z=z_0$ i.e. in the plane of the point charge, the integral however shows very slow $1/\sqrt{kr}$ decay and numerical evaluation is difficult. We therefore apply the methods discussed in [4] where we subtract one or more exponential terms from the integrand which can be integrated explicitly. We can rewrite part of the integrand in the following form:

$$\frac{\sin h(kz)\ \sin h(k(d-z_0))}{\sin h(kd)} = \frac{1}{2}e^{-k(z_0-z)} - \frac{1}{2}e^{-k(z_0+z)} + \sum_{n=1}^{N}\left[\frac{1}{2}e^{-k(2nd-z_0+z)} + \frac{1}{2}e^{-k(2nd+z_0-z)} - \frac{1}{2}e^{-k(2nd-z_0-z)} - \frac{1}{2}e^{-k(2nd+z_0+z)}\right] - e^{-k(2N+1)d}\,\frac{\sin h(kz)\ \sin h(kz_0)}{\sin h(kd)} \tag{6}$$

where $N>0$ is an arbitrary positive integer. Inserting this expression into Eqs. (3) and (4) and using the relations [1]

$$\int_0^\infty J_0(kr)e^{-k|z|}\,dk = \frac{1}{\sqrt{r^2+z^2}}$$

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \cos(k_x x)\ \cos(k_y y)\frac{e^{-k|z|}}{k}\,dk_x\,dk_y = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

we find

$$\frac{4\pi\varepsilon_0}{Q}\phi(r,z) = \frac{1}{\sqrt{r^2+(z-z_0)^2}} - \frac{1}{\sqrt{r^2+(z+z_0)^2}} + \sum_{n=1}^{N}\left[\frac{1}{\sqrt{r^2+(z+2nd-z_0)^2}} + \frac{1}{\sqrt{r^2+(z-2nd-z_0)^2}} - \frac{1}{\sqrt{r^2+(z-2nd+z_0)^2}} - \frac{1}{\sqrt{r^2+(z+2nd+z_0)^2}}\right] - \int_0^\infty 2J_0(kr)e^{-k(2N+1)d}\,\frac{\sin h(kz)\ \sin h(kz_0)}{\sin h(kd)}\,dk \tag{7}$$

and

$$\frac{4\pi\varepsilon_0}{Q}\phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+(z-z_0)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+z_0)^2}} + \sum_{n=1}^{N}\left[\frac{1}{\sqrt{x^2+y^2+(z+2nd-z_0)^2}} + \frac{1}{\sqrt{x^2+y^2+(z-2nd-z_0)^2}} - \frac{1}{\sqrt{x^2+y^2+(z-2nd+z_0)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+2nd+z_0)^2}}\right] - \int_0^\infty\int_0^\infty \frac{4}{\pi}\,\frac{\cos(k_x x)\ \cos(k_y y)}{k}\,e^{-k(2N+1)d} \times \frac{\sin h(kz)\ \sin h(kz_0)}{\sin h(kd)}\,dk_x\,dk_y \tag{8}$$

Since the expressions are symmetric with respect to z and z_0 we do not have to distinguish between ϕ_1 and ϕ_2 and ϕ is therefore valid in the entire range of $0<z<d$. The above expressions represent the potential created by a point charge and $4N+1$ mirror charges together with a remaining integral part: charges of $-Q$ at positions $-z_0$ and $-z_0 \pm 2nd$ and charges of $+Q$ at positions z_0 and $z_0 \pm 2nd$.

For the maximum possible values of $z, z_0 = d$ the remaining integrand behaves as e^{-2kNd}, so for numerical evaluation of the integral an upper integration limit as a multiple of $1/(2Nd)$ will be sufficient for precise evaluation. Since $J_0(kr) \leq 1$ the integral part of Eqs. (7) and (8) is always smaller than

$$\int_0^\infty 2e^{-k(2N+1)d}\ \sin h(kd)\,dk = \frac{1}{2d}\,\frac{1}{N^2+N} \tag{9}$$

so we find the upper limit on the error $\Delta\phi$ of the calculated potential ϕ by terminating the series at N and neglecting the integral to be

$$|\Delta\phi| < Q/(8\pi\varepsilon_0 N^2 d) \tag{10}$$

By bringing N to ∞ the error becomes zero and the field is represented as an infinite number of mirror charges. This also provides the mathematical proof that the procedure of an infinite number of mirror charges converges to the correct potential. By moving the grounded plate at $z=d$ to infinity, only the first two terms in Eqs. (7) and (8) remain, which represents the correct result for a point charge in the presence of a single grounded plane i.e. a charge Q at $z=z_0$ and a single mirror charge of value $-Q$ at $z=-z_0$.

3. Induced charge and weighting field

Using Eqs. (1) and (4) we can now calculate the charge induced on the rectangular pad, as shown in Fig. 1b, according to

$$Q_{ind}(x_0,y_0,z_0) = \int_{-w_x/2}^{w_x/2}\int_{-w_y/2}^{w_y/2} -\varepsilon_0\frac{\partial\phi(x-x_0,y-y_0,z)}{\partial z}\Big|_{z=0}\,dx\,dy \tag{11}$$

With Eq. (2) we can express the result through the weighting potential ϕ_w as

$$\phi_w(x,y,z) = \frac{4V_w}{\pi^2}\int_0^\infty\int_0^\infty \cos(k_x x)\ \sin\left(k_x\frac{w_x}{2}\right)\ \cos(k_y y) \times \sin\left(k_y\frac{w_y}{2}\right)\frac{\sin h(k(d-z))}{k_x k_y\ \sin h\,(kd)}\,dk_x\,dk_y \tag{12}$$

We can now verify that this is indeed equal to the solution of the Laplace equation with boundary condition $\phi_w(x,y,z=0)=V_w$ in the range $-w_x/2 < x < w_x/2$ and $-w_y/2 < y < w_y/2$. The most general solution of the Laplace equation in cartesian coordinates that is symmetric in x and y and that satisfies the boundary condition of $\phi(x,y,z=d)=0$ is given by

$$\phi_w(x,y,z) = \int_0^\infty \int_0^\infty A(k_x,k_y)\ \cos(k_x x)\ \cos(k_y y)\ \sinh(k(d-z))\, dk_x\, dk_y \tag{13}$$

Now $A(k_x,k_y)$ has to be chosen such that the boundary condition $\phi(x,y,z=0)=V_w$ in the defined x,y-range is satisfied. If we write $B(k_x,k_y)=A(k_x,k_y)\sinh(kd)$ we see that Eq. (13) (with $z=0$) just expresses the 'two-dimensional box' as a cosine Fourier integral with the well known solution [1]

$$B(k_x,k_y) = \frac{4V_w}{\pi^2}\frac{\sin(k_x w_x/2)}{k_x}\frac{\sin(k_y w_y/2)}{k_y} \tag{14}$$

and the solution is equal to Eq. (12). We have therefore explicitly verified the reciprocity theorem for this geometry. Before discussing the numerical evaluation we check whether the solution yields the proper expression for a pad of infinite size. We change variables to $s_x = k_x w_x/2$ and $s_y = k_y w_y/2$ and let $w_x, w_y \to \infty$, which is equal to having $k_x, k_y \to 0$. With $\sin(k(d-z))/\sin(kd) \to (d-z)/d$ and using $\int_0^\infty (\sin s)/s\, ds = \pi/2$ we therefore recuperate the correct weighting potential and weighting field for an infinite plane

$$\phi_w(x,y,z) = V_w\left(1-\frac{z}{d}\right) \qquad E_w^z = -\frac{\partial\phi_w}{\partial z} = \frac{V_w}{d} \tag{15}$$

The weighting potential for a strip of infinite length in y-direction is derived by having $w_y \to \infty$. The integral can then be expressed in closed form and the resulting expression for the weighting potential can be found elsewhere [4,5].

For finite w_x, w_y the integral in Eq. (12) cannot be expressed in closed form and we have to use numerical evaluation. For small values of z the integral shows slow convergence and we therefore use Eq. (8) in Eq. (11) to find

$$\begin{aligned}\frac{\phi_w(x,y,z)}{V_w} &= \frac{1}{2\pi}f(x,y,z) - \frac{1}{2\pi}\sum_{n=1}^{N}[f(x,y,2nd-z)-f(x,y,2nd+z)] \\ &\quad -\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \cos(k_x x)\ \sin\left(k_x\frac{w_x}{2}\right)\ \cos(k_y y) \\ &\quad \times \sin\left(k_y\frac{w_y}{2}\right)\frac{e^{-k(2N+1)d}}{k_x k_y}\frac{\sinh(kz)}{\sinh(kd)}\, dk_x dk_y\end{aligned} \tag{16}$$

with

$$\begin{aligned}f(x,y,u) &= \int_{x-w_x/2}^{x+w_x/2}\int_{y-w_y/2}^{y+w_y/2}\frac{u}{(x'^2+y'^2+u^2)^{3/2}}\,dx'\,dy' \\ &= \arctan\left(\frac{x_1y_1}{u\sqrt{x_1^2+y_1^2+u^2}}\right)+\arctan\left(\frac{x_2y_2}{u\sqrt{x_2^2+y_2^2+u^2}}\right) \\ &\quad -\arctan\left(\frac{x_1y_2}{u\sqrt{x_1^2+y_2^2+u^2}}\right)-\arctan\left(\frac{x_2y_1}{u\sqrt{x_2^2+y_1^2+u^2}}\right)\end{aligned} \tag{17}$$

and

$$x_1 = x-\frac{w_x}{2}, \quad x_2 = x+\frac{w_x}{2}, \quad y_1 = y-\frac{w_y}{2}, \quad y_2 = y+\frac{w_y}{2} \tag{18}$$

Again we see that for large number of N the integral part tends to zero and the weighting potential can be expressed and an infinite sum of expressions involving $f(x,y,u)$. The integral term of Eq. (16) is smaller than

$$\leq \int_0^\infty\int_0^\infty \frac{4}{\pi^2}\frac{w_x}{2}\frac{w_y}{2}e^{-k(2N+1)d}\frac{\sinh(kz)}{\sinh(kd)}\,dk_x\,dk_y = \frac{w_xw_y}{8\pi d^2}\left[\psi\left(N+1-\frac{z}{2d}\right) -\psi\left(N+1+\frac{z}{2d}\right)\right] \tag{19}$$

where $\psi(x) = d/dx \ln\Gamma(x)$ is the digamma function [6]. Since the expression in brackets asymptotically approaches $z/(N^2d)$ from 'below' we find that the error on the weighting potential in case of terminating the series at N and neglecting the integral is smaller than

$$|\Delta\phi_w| < \frac{V_w}{8\pi}\frac{w_xw_y}{d^2}\frac{1}{N^2}\frac{z}{d} \tag{20}$$

It is interesting to observe that the error goes to zero for $z\to 0$, and indeed in Eq. (16) we see that for $z \to 0$ all expressions except for the first one i.e. $1/(2\pi)f(x,y,z\to 0)$ vanish, and this expression represents the correct 'box shaped' potential distribution on the metal surface.

The z-component of the weighting field $E_w^z = -\partial\phi_w/\partial z$ is then given by

$$\begin{aligned}\frac{E_w^z(x,y,z)}{V_w} &= \frac{1}{2\pi}g(x,y,z)+\frac{1}{2\pi}\sum_{n=1}^{N}[g(x,y,2nd+z)+g(x,y,2nd-z)] \\ &\quad +\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \cos(k_x x)\ \sin\left(k_x\frac{w_x}{2}\right)\ \cos(k_y y) \\ &\quad \times \sin\left(k_y\frac{w_y}{2}\right)\frac{e^{-k(2N+1)d}}{k_xk_y}k\frac{\cosh(kz)}{\sinh(kd)}\,dk_x\,dk_y\end{aligned} \tag{21}$$

with

$$\begin{aligned}g(x,y,u) &= -\frac{\partial f(x,y,u)}{\partial u} \\ &= \frac{x_1y_1(x_1^2+y_1^2+2u^2)}{(x_1^2+u^2)(y_1^2+u^2)\sqrt{x_1^2+y_1^2+u^2}} \\ &\quad +\frac{x_2y_2(x_2^2+y_2^2+2u^2)}{(x_2^2+u^2)(y_2^2+u^2)\sqrt{x_2^2+y_2^2+u^2}} \\ &\quad -\frac{x_1y_2(x_1^2+y_2^2+2u^2)}{(x_1^2+u^2)(y_2^2+u^2)\sqrt{x_1^2+y_2^2+u^2}}\end{aligned}$$

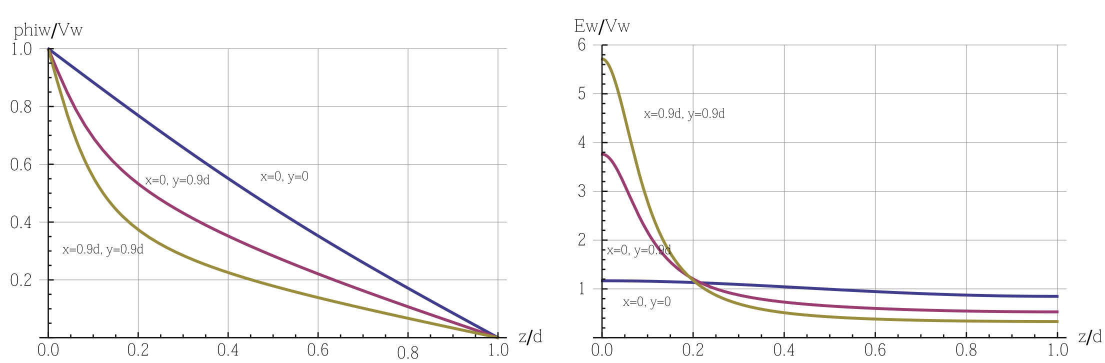

Fig. 2. Weighting potential (left) and z-component of the weighting field (right) for the geometry from Fig. 1 with $w_x = w_y = 2d$.

$$-\frac{x_2 y_1 (x_2^2+y_1^2+2u^2)}{(x_2^2+u^2)(y_1^2+u^2)\sqrt{x_2^2+y_1^2+u^2}} \tag{22}$$

Again we see that for large number of N the integral part tends to zero. The integral term is smaller than

$$\leq \int_0^\infty \int_0^\infty \frac{4}{\pi^2}\frac{w_x}{2}\frac{w_y}{2} e^{-k(2N+1)d} k \frac{\cos\mathrm{h}(kz)}{\sin\mathrm{h}(kd)}\, dk_x\, dk_y$$
$$= \frac{w_x w_y}{16\pi d^3}\left[-\psi'\left(N+1-\frac{z}{2d}\right)-\psi'\left(N+1+\frac{z}{2d}\right)\right] \tag{23}$$

The expression in brackets approaches the expression $2/N^2$ 'from below' for large values of N, so the error on the weighting field when terminating the series at N and neglecting the integral part is smaller than

$$|\Delta E_z^w| < \frac{V_w}{8\pi}\frac{w_x w_y}{d^3}\frac{1}{N^2} \tag{24}$$

Fig. 2 shows an example for the weighting potential and z-component of the weighting field for different x, y-positions. A value of $N=100$ was used for the evaluation.

4. 'Trouble' with the method of images

The expression for the weighting potential in Eq. (16) gives an interesting insight into the 'trouble with the method of images' discussed in [7]. Since the infinite series of mirror charges, i.e. Eq. (7) for $N\to\infty$, gives the correct potential one could assume that this expression can be used to calculate the total induced charge on the metal plane. This approach is equal to using Eq. (16) with $N\to\infty$, omitting the integral part and evaluating the expression for $w_x, w_y\to\infty$. Since

$$\lim_{w_x,w_y\to\infty} f(x,y,u) = 2\pi \tag{25}$$

we get $\phi_w(x,y,z)=V_w$, since only the first image charge term contributes and all other cancel. The result is clearly wrong since it refers to the situation where there is only one metal plane at $z=0$. It is correct that for a convergent infinite series of functions, also the sum of the integrals of these functions over a finite interval converges to the correct result, but for an integral over an infinite interval this is not the case. In this specific case this is very clear from expression (16), where for $w_x, w_y\to\infty$ the integral part approaches $-z/d$ independent from N, showing that the integral cannot be omitted even for an arbitrarily large value of N. Another indication of this fact comes from the expression for the error on the potential when terminating the series at N (Eq. (20)), which for $w_x, w_y\to\infty$ diverges for every value of N.

5. Conclusion

We have presented expressions for the potential of a point charge (Eqs. (7) and (8)) as well as the weighting potential (Eq. (16)) and weighting field (Eq. (21)) of a rectangular electrode for a parallel plate geometry. The expressions are well suited for numerical evaluation by either using numerical algorithms to evaluate the integrals with well defined upper limits, or by using the infinite sum representation. The errors on the calculated potentials or fields due to truncating the sum at N terms are derived, and it is shown that they are bounded by $\propto 1/N^2$.

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