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An Inequality for Permanents of (0, 1)-Matrices*

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ABSTRACT

Let A be an n-square (0, 1)-matrix, let r_i denote the i-th row sum of A, $i = 1, ..., n$, and let per(A) denote the permanent of A. Then

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + \sqrt{2}}{1 + \sqrt{2}}$$

where equality can occur if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of 1-square and 2-square matrices all of whose entries are 1.

If $A = (a_{ij})$ is an n-square matrix then the *permanent* of A is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \,. \tag{1}$$

An up-to-date survey of the theory of permanents was given in [2]. Many properties of the permanent function are similar to those of the determinant. In particular, there is the following analog of Laplace expansion by the k-th column

$$\operatorname{per}(A) = \sum_{i=1}^{n} a_{ik} \operatorname{per}(A(i|k)), \tag{2}$$

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where $A(i \mid k)$ denotes the $(n-1)$-square submatrix of A obtained by deleting the i-th row and the k-th column of A.

Permanents and particularly permanents of (0,1)-matrices are of combinatorial significance. Various bounds for permanents have been proved and conjectured [1, 2]. In [3] I have conjectured that

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}, \tag{3}$$

where $r_i = \sum_{j=1}^{n} a_{ij}$ is the i-th row sum of A. I also proved that

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + 1}{2} \tag{4}$$

with equality if and only if A is a permutation matrix. In [1] Jurkat and Ryser proved that

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} (r_i!)^{1/n} \left(\frac{r_i + 1}{2}\right)^{(n-r_i)/n} \tag{5}$$

which is better than the bound given by (4). In the present paper I prove the following improvement of (4).

THEOREM. *If $A = (a_{ij})$ is an n-square* (0, 1)*-matrix then*

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + \sqrt{2}}{1 + \sqrt{2}}. \tag{6}$$

Equality in (1) *occurs if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of* 1*-square and* 2*-square matrices all of whose entries are* 1.

The bounds in (5) and (6) are not comparable. In fact, the inequality (5) becomes equality for matrices all of whose entries are 1, while (6) is equality in case A is a direct sum of 2-square matrices all of whose entries are 1.

We first prove two combinatorial lemmas.

LEMMA 1. *If $A = (a_{ij})$ is an n-square* (0, 1)*-matrix without a zero row then*

$$\min \sum_{i=1}^{n} a_{ij}/r_i \leq 1 \tag{7}$$

PROOF:

$$\sum_{=1}^{n}\left(\sum_{i=1}^{n} a_{ij}/r_i\right) = \sum_{i=1}^{n}\frac{1}{r_i}\sum_{j=1}^{n} a_{ij}$$
$$= \sum_{i=1}^{n}\frac{1}{r_i}\, r_i$$
$$= n.$$

Thus

$$\min_j \sum_{i=1}^{n} a_{ij}/r_i \leq 1.$$

Clearly, Lemma 1 implies that we can permute the rows and the columns of a (0, 1)-matrix without a zero row so that

$$\sum_{i=1}^{c}\frac{1}{r_i} \leq 1, \tag{8}$$

where c is the first column sum and r_i the i-th row sum of the permuted matrix. Note that both sides of inequality (6) are invariant under permutations of rows and columns.

LEMMA 2. *Let $A = (a_{ij})$ be an n-square* (0, 1)*-matrix without zero rows for which* (8) *holds. Then*

$$(\sqrt{2}+1)\sum_{i=1}^{c}\frac{1}{r_i+\sqrt{2}-1} \leq \prod_{i=1}^{c}\left(1+\frac{1}{r_i+\sqrt{2}-1}\right). \tag{9}$$

Equality holds in (9) *if and only if either $c = r_1 = 1$, or $c = r_1 = r_2 = 2$.*

PROOF: Let $d = \prod_{i=1}^{c}(r_i + 2^{1/2} - 1)$. Then

$$\begin{aligned}(\sqrt{2}+1)\sum_{i=1}^{c}\frac{1}{r_i+\sqrt{2}-1} &= \frac{\sqrt{2}+1}{d}\sum_{i=1}^{c}\prod_{\substack{t=1\\ t\neq i}}^{c}(r_t+\sqrt{2}-1)\\ &= \frac{1}{d}\{(\sqrt{2}+1)E_{c-1}+2E_{c-2}\\ &\quad+\sum_{t=3}^{c} t(\sqrt{2}-1)^{t-2}E_{c-t}\},\end{aligned} \tag{10}$$

where E_k denotes the k-th elementary symmetric function of $r_1, \ldots, r_c$; $E_0 = 1$. Thus

$$
\begin{aligned}
(\sqrt{2}+1)\sum_{i=1}^{c}\frac{1}{r_i+\sqrt{2}-1} &\leq \frac{1}{d}\{(\sqrt{2}+1)E_{c-1}+2E_{c-2}+\sum_{t=3}^{c}\sqrt{2}^tE_{c-t}\} \\
&= \frac{1}{d}(E_{c-1}-E_c)+\frac{1}{d}\sum_{t=0}^{c}\sqrt{2}^tE_{c-t} \\
&= \frac{1}{d}(E_{c-1}-E_c)+\frac{1}{d}\prod_{i=1}^{c}(r_i+\sqrt{2}) \qquad (11)\\
&\leq \prod_{i=1}^{c}\left(1+\frac{1}{r_i+\sqrt{2}-1}\right)
\end{aligned}
$$

since

$$E_{c-1}-E_c = E_c\left(\sum_{i=1}^{c}\frac{1}{r_i}-1\right),$$

which is nonpositive, by Lemma 1. If either $c = r_1 = 1$ or $c = r_1 = r_2 = 2$, then clearly (9) is equality. Conversely, if (9) is equality then (11) is equality which, together with (10), implies $E_{c-3} = 0$, i.e., $c \leq 2$. It easily follows, by direct computation, that either $c_1 = r_1 = 1$ or $c = r_1 = r_2 = 2$.

Proof of the Theorem: If A has a zero row then (6) is a strict inequality. Suppose now that A has no zero rows. Since both sides of (6) are invariant under permutations of rows and columns of A, we can assume without loss of generality that

$$a_{11} = \cdots = a_{c1} = 1, \quad a_{c+1,1} = \cdots = a_{n,1} = 0$$

and that (8) holds. We shall prove (6) by induction on n. Assume therefore that (6) holds for all k-square (0, 1)-matrices, $k < n$. Then

$$
\begin{aligned}
\operatorname{per}(A) &= \sum_{i=1}^{c}\operatorname{per}(A(i|1)) \\
&\leq \sum_{i=1}^{c}\left(\prod_{\substack{j=1\\ j\neq i}}^{c}\frac{r_j+\sqrt{2}-1}{1+\sqrt{2}}\right)\left(\prod_{j=c+1}^{n}\frac{r_j+\sqrt{2}}{1+\sqrt{2}}\right) \\
&= \sum_{i=1}^{c}\frac{1+\sqrt{2}}{r_i+\sqrt{2}-1}\left(\prod_{j=1}^{c}\frac{r_j+\sqrt{2}-1}{r_j+\sqrt{2}}\right)\left(\prod_{j=1}^{n}\frac{r_j+\sqrt{2}}{1+\sqrt{2}}\right).
\end{aligned}
$$

But

$$\sum_{i=1}^{c} \frac{1+\sqrt{2}}{r_i+\sqrt{2}-1}\left(\prod_{j=1}^{c} \frac{r_j+\sqrt{2}-1}{r_j+\sqrt{2}}\right)$$
$$= (1+\sqrt{2}) \sum_{i=1}^{c} \frac{1}{r_i+\sqrt{2}-1} \Big/ \prod_{j=1}^{n}\left(1+\frac{1}{r_j+\sqrt{2}-1}\right) \leq 1 \tag{12}$$

by Lemma 2. Hence

$$\operatorname{per}(A) \leq \prod_{j=1}^{n} \frac{r_j+\sqrt{2}}{1+\sqrt{2}}.$$

If equality holds in (6), the inequality (12) must be equality and thus, by Lemma 2, either $c = r_1 = 1$ or $c = r_1 = r_2 = 2$. In the first case it follows immediately from the induction hypothesis that A is of the form described in the theorem. If $c = r_1 = r_2 = 2$ then, by the induction hypothesis, both $A(1|1)$ and $A(2|1)$ must be of the required form. It is easily seen that in this case PAQ must be a direct sum of $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and an $(n-2)$-square (0, 1)-matrix and the result again follows from the induction hypothesis. Conversely, if PAQ is the direct sum of k 2-square and $n-2k$ 1-square matrices all of whose entries are 1, then

$$\operatorname{per}(A) = 2^k = \left(\frac{2+\sqrt{2}}{1+\sqrt{2}}\right)^{2k}\left(\frac{1+\sqrt{2}}{1+\sqrt{2}}\right)^{n-2k} = \prod_{i=1}^{n} \frac{r_i+\sqrt{2}}{1+\sqrt{2}}.$$

This completes the proof of the theorem.

Since $\operatorname{per}(A^T) = \operatorname{per}(A)$ we have

$$\operatorname{per}(A) \leq \min\left\{\prod_{i=1}^{n} \frac{r_i+\sqrt{2}}{1+\sqrt{2}},\ \prod_{i=1}^{n} \frac{c_i+\sqrt{2}}{1+\sqrt{2}}\right\} \tag{13}$$

where c_i denotes the i-th column sum of A. However, the inequality

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \frac{s_i+\sqrt{2}}{1+\sqrt{2}}$$

where $s_i = \min(r_i, c_i)$, $i = 1, ..., n$, is false. For example, if

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

then $\operatorname{per}(A) = 4$, while

$$\prod_{i=1}^{n} \frac{s_i + \sqrt{2}}{1 + \sqrt{2}} = \sqrt{2}\,\frac{3 + \sqrt{2}}{1 + \sqrt{2}}\,\sqrt{2} = 4\sqrt{2} - 2 < 4.$$

References

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