An Inequality for Permanents of (0, 1)-Matrices*

HENRYK MINC

Department of Mathematics,
University of California, Santa Barbara, California

Communicated by H. J. Ryser

ABSTRACT

Let $A$ be an $n$-square $(0, 1)$-matrix, let $r_i$ denote the $i$-th row sum of $A$, $i = 1, ..., n$, and let $\text{per}(A)$ denote the permanent of $A$. Then

$$\text{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + \sqrt{2}}{1 + \sqrt{2}}$$

where equality can occur if and only if there exist permutation matrices $P$ and $Q$ such that $PAQ$ is a direct sum of 1-square and 2-square matrices all of whose entries are 1.

If $A = (a_{ij})$ is an $n$-square matrix then the permanent of $A$ is defined by

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} . \quad (1)$$

An up-to-date survey of the theory of permanents was given in [2]. Many properties of the permanent function are similar to those of the determinant. In particular, there is the following analog of Laplace expansion by the $k$-th column

$$\text{per}(A) = \sum_{i=1}^{n} a_{ik} \text{per}(A(i|k)), \quad (2)$$

* This work was supported by the Air Force Office of Scientific Research under Grant AFOSR 432-63.
where \( A(i \mid k) \) denotes the \((n - 1)\)-square submatrix of \( A \) obtained by deleting the \( i \)-th row and the \( k \)-th column of \( A \).

Permanents and particularly permanents of \((0,1)\)-matrices are of combinatorial significance. Various bounds for permanents have been proved and conjectured [1, 2]. In [3] I have conjectured that

\[
\text{per}(A) \leq \prod_{i=1}^{n} \left( \frac{r_i!}{r_i} \right)^{1/r_i},
\]

where \( r_i = \sum_{j=1}^{n} a_{ij} \) is the \( i \)-th row sum of \( A \). I also proved that

\[
\text{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + 1}{2}
\]

with equality if and only if \( A \) is a permutation matrix. In [1] Jurkat and Ryser proved that

\[
\text{per}(A) \leq \prod_{i=1}^{n} \left( \frac{r_i!}{r_i} \right)^{1/n} \left( \frac{r_i + 1}{2} \right)^{(n-r_i)/n}
\]

which is better than the bound given by (4). In the present paper I prove the following improvement of (4).

**Theorem.** If \( A = (a_{ij}) \) is an \( n \)-square \((0,1)\)-matrix then

\[
\text{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + \sqrt{2}}{1 + \sqrt{2}}.
\]

Equality in (1) occurs if and only if there exist permutation matrices \( P \) and \( Q \) such that \( PAQ \) is a direct sum of 1-square and 2-square matrices all of whose entries are 1.

The bounds in (5) and (6) are not comparable. In fact, the inequality (5) becomes equality for matrices all of whose entries are 1, while (6) is equality in case \( A \) is a direct sum of 2-square matrices all of whose entries are 1.

We first prove two combinatorial lemmas.

**Lemma 1.** If \( A = (a_{ij}) \) is an \( n \)-square \((0,1)\)-matrix without a zero row then

\[
\min \sum_{i=1}^{n} a_{ij}/r_i \leq 1
\]
PROOF:

\[
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} / r_i \right) = \sum_{i=1}^{n} \frac{1}{r_i} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} \frac{1}{r_i} r_i = n.
\]

Thus \[
\min_{j} \sum_{i=1}^{n} a_{ij} / r_i \leq 1.
\]

Clearly, Lemma 1 implies that we can permute the rows and the columns of a \((0, 1)\)-matrix without a zero row so that

\[
\sum_{i=1}^{c} \frac{1}{r_i} \leq 1,
\]

where \(c\) is the first column sum and \(r_i\) the \(i\)-th row sum of the permuted matrix. Note that both sides of inequality (6) are invariant under permutations of rows and columns.

**Lemma 2.** Let \(A = (a_{ij})\) be an \(n\)-square \((0, 1)\)-matrix without zero rows for which (8) holds. Then

\[
(\sqrt{2}+1) \sum_{i=1}^{c} \frac{1}{r_i + \sqrt{2} - 1} \leq \prod_{i=1}^{c} \left( 1 + \frac{1}{r_i + \sqrt{2} - 1} \right).
\]

Equality holds in (9) if and only if either \(c = r_1 = 1\), or \(c = r_1 = r_2 = 2\).

**Proof:** Let \(d = \prod_{i=1}^{c} (r_i + 2^{1/2} - 1)\). Then

\[
(\sqrt{2}+1) \sum_{i=1}^{c} \frac{1}{r_i + \sqrt{2} - 1} = \frac{\sqrt{2}+1}{d} \sum_{i=1}^{c} \prod_{t=1}^{c} (r_t + \sqrt{2} - 1)
\]

\[
= \frac{1}{d} \left\{ (\sqrt{2}+1)E_{c-1} + 2E_{c-2} \right\}
\]

\[
+ \sum_{t=3}^{c} t(\sqrt{2}-1)^{t-2}E_{c-t},
\]

where \(E_k\) denotes the \(k\)-th elementary symmetric function of \(r_1, ..., r_c\); \(E_0 = 1\). Thus
\[(\sqrt{2} + 1) \sum_{i=1}^{\varepsilon} \frac{1}{r_i + \sqrt{2} - 1} \leq \frac{1}{d} \left\{ (\sqrt{2} + 1)E_{c-1} + 2E_{c-2} + \sum_{t=3}^{\varepsilon} \sqrt{2^t}E_{c-t} \right\} \]
\[= \frac{1}{d} (E_{c-1} - E_c) + \frac{1}{d} \sum_{i=1}^{\varepsilon} \sqrt{2^i}E_{c-t} \]
\[= \frac{1}{d} (E_{c-1} - E_c) + \frac{1}{d} \prod_{i=1}^{\varepsilon} (r_i + \sqrt{2}) \] (11)
\[\leq \prod_{i=1}^{\varepsilon} \left( 1 + \frac{1}{r_i + \sqrt{2} - 1} \right) \]

since

\[E_{c-1} - E_c = E_c \left( \sum_{i=1}^{\varepsilon} \frac{1}{r_i} - 1 \right),\]

which is nonpositive, by Lemma 1. If either \(c = r_1 = 1\) or \(c = r_1 = r_2 = 2\), then clearly (9) is equality. Conversely, if (9) is equality then (11) is equality which, together with (10), implies \(E_{c-3} = 0\), i.e., \(c \leq 2\). It easily follows, by direct computation, that either \(c_1 = r_1 = 1\) or \(c = r_1 = r_2 = 2\).

**Proof of the Theorem:** If \(A\) has a zero row then (6) is a strict inequality. Suppose now that \(A\) has no zero rows. Since both sides of (6) are invariant under permutations of rows and columns of \(A\), we can assume without loss of generality that

\[a_{11} = \cdots = a_{e1} = 1, \quad a_{e+1,1} = \cdots = a_{n,1} = 0\]

and that (8) holds. We shall prove (6) by induction on \(n\). Assume therefore that (6) holds for all \(k\)-square \((0, 1)\)-matrices, \(k < n\). Then

\[\text{per}(A) = \sum_{i=1}^{\varepsilon} \text{per}(A(i|1)) \]
\[\leq \sum_{i=1}^{\varepsilon} \left( \prod_{j=1}^{\varepsilon} \frac{r_j + \sqrt{2} - 1}{1 + \sqrt{2}} \right) \left( \prod_{j=\varepsilon+1}^{n} \frac{r_j + \sqrt{2}}{1 + \sqrt{2}} \right) \]
\[= \sum_{i=1}^{\varepsilon} \frac{1 + \sqrt{2}}{r_i + \sqrt{2} - 1} \left( \prod_{j=1}^{\varepsilon} \frac{r_j + \sqrt{2} - 1}{r_j + \sqrt{2}} \right) \left( \prod_{j=1}^{n} \frac{r_j + \sqrt{2}}{1 + \sqrt{2}} \right).\]
But
\[
\sum_{i=1}^{c} \frac{1 + \sqrt{2}}{r_i + \sqrt{2} - 1} \left( \prod_{j=1}^{c} \frac{r_j + \sqrt{2} - 1}{r_j + \sqrt{2}} \right) = (1 + \sqrt{2}) \sum_{i=1}^{c} \frac{1}{r_i + \sqrt{2} - 1} \prod_{j=1}^{n} \left( 1 + \frac{1}{r_j + \sqrt{2} - 1} \right) \leq 1
\]

by Lemma 2. Hence
\[
\text{per}(A) \leq \prod_{j=1}^{n} \frac{r_j + \sqrt{2}}{1 + \sqrt{2}}.
\]

If equality holds in (6), the inequality (12) must be equality and thus, by Lemma 2, either \(c = r_1 = 1\) or \(c = r_1 = r_2 = 2\). In the first case it follows immediately from the induction hypothesis that \(A\) is of the form described in the theorem. If \(c = r_1 = r_2 = 2\) then, by the induction hypothesis, both \(A(1|1)\) and \(A(2|1)\) must be of the required form. It is easily seen that in this case \(PAQ\) must be a direct sum of \(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\) and an \((n - 2)\)-square \((0, 1)\)-matrix and the result again follows from the induction hypothesis. Conversely, if \(PAQ\) is the direct sum of \(k\) 2-square and \(n - 2k\) 1-square matrices all of whose entries are 1, then
\[
\text{per}(A) = 2^k = \left( \frac{2 + \sqrt{2}}{1 + \sqrt{2}} \right)^{2k} \left( \frac{1 + \sqrt{2}}{1 + \sqrt{2}} \right)^{n-2k} = \prod_{i=1}^{n} \frac{r_i + \sqrt{2}}{1 + \sqrt{2}}.
\]

This completes the proof of the theorem.

Since \(\text{per}(A^n) = \text{per}(A)\) we have
\[
\text{per}(A) \leq \min \left\{ \prod_{i=1}^{n} \frac{r_i + \sqrt{2}}{1 + \sqrt{2}}, \prod_{i=1}^{n} \frac{c_i + \sqrt{2}}{1 + \sqrt{2}} \right\}
\]

where \(c_i\) denotes the \(i\)-th column sum of \(A\). However, the inequality
\[
\text{per}(A) \leq \prod_{i=1}^{n} \frac{s_i + \sqrt{2}}{1 + \sqrt{2}}
\]

where \(s_i = \min(r_i, c_i), i = 1, ..., n\), is false. For example, if
then \( \text{per}(A) = 4 \), while

\[
\prod_{i=1}^{n} \frac{s_i + \sqrt{2}}{1 + \sqrt{2}} = \sqrt{2}^{\frac{3 + \sqrt{2}}{1 + \sqrt{2}}} = 4\sqrt{2} - 2 < 4.
\]

\textbf{References}

