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Extensions for Frobenius kernels

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1. Introduction

1.1. Let G be a simple algebraic group over an algebraically closed field k of characteristic $p > 0$. For $r \geqslant 1$, let G_r be the rth Frobenius kernel of G. It is well known that the representations for G_1 are equivalent to the restricted representations for $\operatorname{Lie} G$. Historically, the cohomology for Frobenius kernels has been best understood for large primes. Friedlander and Parshall [FP] first computed the cohomology ring $\mathrm{H}^\bullet(G_1, k)$ for $p \geqslant 3(h-1)$ where h is the Coxeter number of the underlying root system. They proved that the cohomology ring can be identified with the coordinate algebra of the nullcone. Andersen and Jantzen [AJ] later verified this fact for $p \geqslant h$. Furthermore, they generalized this calculation by looking at $\mathrm{H}^\bullet(G_1, H^0(\lambda))$ where $H^0(\lambda) = \operatorname{ind}_B^G \lambda$ for $p \geqslant h$. Their results had some restrictions on the type of root system involved. Kumar, Lauritzen, and Thomsen [KLT] removed the restrictions on the root systems through the use of Frobenius splittings.

The cohomology ring $\mathrm{H}^\bullet(G_1, k)$ modulo nilpotents can be identified in general with the coordinate algebra of the restricted nullcone $\mathcal{N}_1 = \{x \in \operatorname{Lie}(G)\colon x^{[p]} = 0\}$. For good primes, Nakano, Parshall, and Vella [NPV] proved that this variety is irreducible and can be identified with the closure of some Richardson orbit. Recently, Carlson, Lin, Nakano, and Parshall [CLNP] have given an explicit description of $\mathcal{N}_1$. These recent results provide

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some indication that one can systematically study extensions of Frobenius kernels for small primes by using general formulas which exhibit generic behavior for large primes.

1.2. This paper will first focus on the first extension groups (i.e., Ext^1) in the category of G_r-modules for arbitrary primes. The authors have shown that such computations are highly desirable because they can be used to provide vital information about extensions in the category of modules for the finite Chevalley group $G(\mathbb{F}_{p^r})$ (see [BNP1,BNP2,BNP3]). Of particular interest in this context are good upper bounds for the weights of $\mathrm{Ext}^1_{G_r}$ in arbitrary characteristic.

The paper is outlined as follows. In [Jan2], Jantzen provides extensive computations of the first cohomology groups of the first Frobenius kernel G_1 of G. In particular, he studies both $\mathrm{H}^1(G_1, L(\lambda))$ and $\mathrm{H}^1(G_1, H^0(\lambda))$ where $L(\lambda)$ is the simple G-module with highest weight λ. The first goal of the paper is to use Jantzen's computations to compute $\mathrm{H}^1(G_r, H^0(\lambda))$ for all Frobenius kernels G_r and dominant weights λ. These computations were inspired by statements of Andersen [And2]. To begin, in Section 2, we recall Jantzen's computations of $\mathrm{H}^1(B_1, \lambda)$ and then use those results to compute $\mathrm{H}^1(B_r, \lambda)$ for all r and all weights λ. In Section 3, the B_r-cohomology results are used to compute $\mathrm{H}^1(G_r, H^0(\lambda))$. As a special case, we determine all fundamental dominant weights ω for which $\mathrm{H}^1(G_r, H^0(\omega))$ is non-zero. Donkin conjectured that if V is a rational G-module with good filtration, then $\mathrm{H}^m(G_r, V)^{(-r)}$ has a good filtration for every $m \geqslant 0$ (see [Do, p. 79]). Van der Kallen [vdK] showed that this conjecture was not true in general by constructing a counterexample. Our results show that when $V = H^0(\lambda)$ for λ a dominant weight that indeed $\mathrm{H}^1(G_r, V)^{(-r)}$ has a good filtration for all primes. It would be an interesting question to determine to what extent Donkin's conjecture still remains valid.

In Section 4, an observation is made about the cohomology of simple modules. The second goal of the paper (discussed in Section 5) is to make use of the cohomology computations of induced modules to prove a general formula for extensions between two simple G_r-modules for arbitrary primes (see Theorem 5.4). More specifically, we can relate extensions (i.e., $\mathrm{Ext}^m_{G_r}$) between simple G_r-modules with the extensions between certain G-modules. In particular, for $m = 1$ and $\lambda, \mu \in X_r(T)$, we construct the following isomorphism (as vector spaces)

$$\mathrm{Ext}^1_{G_r}\big(L(\lambda), L(\mu)\big) \cong \bigoplus_{\nu \in \pi_h} \mathrm{Ext}^1_G\big(L(\lambda), I_h(\nu)^{(r)} \otimes L(\mu)\big) \otimes L(\nu)^{(r)}, \qquad (1.2.1)$$

where $I_h(\nu)$ is the injective hull of $L(\nu)$ in the bounded category $\mathcal{C}_h$, $\pi_h = \{\nu \in X(T)_+ \colon \langle \nu, \alpha_0 \rangle < h\}$, and $\mathcal{C}_h$ is the full subcategory of all G-modules whose composition factors $L(\lambda)$ have highest weights in π_h. For $p > h$, we can apply the explicit description of $\mathrm{H}^\bullet(G_1, H^0(\lambda))$ given in [KLT], to provide sharper results on the necessary bounds for our truncated categories. From this formula above, we can deduce that for $p \geqslant 2h - 1$,

$$\mathrm{Ext}^1_{G_r}\big(L(\lambda), L(\mu)\big) \cong \bigoplus_{\nu \in \pi_h} \mathrm{Ext}^1_G\big(L(\lambda), L(\nu)^{(r)} \otimes L(\mu)\big) \otimes L(\nu)^{(r)}. \qquad (1.2.2)$$

The preceding formulas significantly improve earlier results by the authors [BNP1] and Andersen [And1, Proposition 5.5].

1.3. Notation. Throughout this paper, let G be a simple simply connected algebraic group scheme defined and split over the finite field $\mathbb{F}_p$ with p elements. The field k is the algebraic closure of $\mathbb{F}_p$. For $r \geqslant 1$, let G_r be the rth Frobenius kernel of G. The basic definitions and notation can be found in [Jan1].

Let T be a maximal split torus and Φ be the root system associated to (G, T). Moreover, let Φ^+ (respectively Φ^-) be positive (respectively negative) roots and Δ be a base consisting of simple roots. For a given root system of rank n, the simple roots will be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ be the coroot corresponding to $\alpha \in \Phi$. In this case, the fundamental weights (basis dual to $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee$) will be denoted by ω_1, $\omega_2, \ldots, \omega_n$. We use the same ordering of roots as given in [Jan2] (following Bourbaki). In particular, for type B_n, α_n denotes the unique short simple root and for type C_n, α_n denotes the unique long simple root. For a generic simple root α, ω_α will denote the corresponding fundamental weight. Let B be a Borel subgroup containing T corresponding to the negative roots and U be the unipotent radical of B.

Let $\mathbb{E}$ be the Euclidean space associated with Φ and the inner product on $\mathbb{E}$ will be denoted by $\langle\,,\rangle$. Moreover, let $X(T)$ be the integral weight lattice obtained from Φ. The set $X(T)$ has a partial ordering defined as follows: if $\lambda, \mu \in X(T)$ then $\lambda \geqslant \mu$ if and only if $\lambda - \mu \in \sum_{\alpha\in\Delta} \mathbb{N}\alpha$. Set α_0 to be the highest short root. Moreover, let ρ be the half sum of positive roots and w_0 denote the long element of the Weyl group. The Coxeter number associated to Φ is $h = \langle\rho, \alpha_0^\vee\rangle + 1$. The set of dominant integral weights is defined by

$$X(T)_+ = \left\{\lambda \in X(T)\colon\ 0 \leqslant \left\langle\lambda, \alpha^\vee\right\rangle \text{ for all } \alpha \in \Delta\right\},$$

and the set of p^r-restricted weights is

$$X_r(T) = \left\{\lambda \in X(T)\colon\ 0 \leqslant \left\langle\lambda, \alpha^\vee\right\rangle < p^r \text{ for all } \alpha \in \Delta\right\}.$$

The simple modules for G are indexed by the set $X(T)_+$ and denoted by $L(\lambda)$, $\lambda \in X(T)_+$ with $L(\lambda) = \operatorname{soc}_G H^0(\lambda)$ where $H^0(\lambda) = \operatorname{ind}_B^G \lambda$. A complete set of non-isomorphic simple G_r-modules are easily obtained by taking $\{L(\lambda)\colon\ \lambda \in X_r(T)\}$. For $\lambda \in X(T)$, we will often use the notation $k_\lambda := \lambda$ to be the one-dimensional B-module obtained by taking the one-dimensional T-module λ and extending it to U-trivially.

2. B_r-cohomology

2.1. This section is concerned with computing B_r-cohomology. Specifically, we compute $\mathrm{H}^1(B_r, \lambda)$ for all $\lambda \in X(T)$. For an arbitrary $\lambda \in X(T)$, we may write $\lambda = \lambda_0 + p^r\lambda_1$ for a unique weight $\lambda_0 \in X_r(T)$. Furthermore,

$$\mathrm{H}^1(B_r, \lambda) = \mathrm{H}^1\big(B_r, \lambda_0 + p^r\lambda_1\big) \cong \mathrm{H}^1(B_r, \lambda_0) \otimes p^r\lambda_1.$$

Hence, it suffices to compute the cohomology for weights $\lambda \in X_r(T)$.

2.2. Special cohomology modules. We define certain cohomology modules which will be used throughout the rest of the paper. Jantzen's computations of the cohomology groups

$\mathrm{H}^1(B_1, \lambda)$ in [Jan2] begins with a computation of $\mathrm{H}^1(U_1, k)$ as a B-module. In most cases, $\mathrm{H}^1(U_1, k)$ decomposes as a direct sum of simple modules but not in all cases. In particular, certain indecomposable modules arise when the prime is small. This leads to the presence of certain indecomposable B-modules in the identification of $\mathrm{H}^1(B_1, \lambda)$. We list these modules here with our notation:

- Type B_n, $n \geqslant 3$, $p = 2$. Let M_{B_n} denote the 2-dimensional indecomposable B-module from [Jan2] having a filtration with factors k_{ω_n} on the top and $k_{\omega_{n-1}-\omega_n}$ on the bottom.
- Type C_n, $n \geqslant 2$, $p = 2$. Let M_{C_n} denote the n-dimensional indecomposable B-module from [Jan2] having a filtration with factors $k_{\omega_1}, k_{\omega_2-\omega_1}, k_{\omega_3-\omega_2}, \ldots,$ and $k_{\omega_n-\omega_{n-1}}$ from top to bottom.
- Type F_4, $p = 2$. Let M_{F_4} denote the 3-dimensional indecomposable B-module from [Jan2] having a filtration with factors k_{ω_4}, $k_{\omega_3-\omega_4}$, and $k_{\omega_2-\omega_3}$ from top to bottom.
- Type G_2, $p = 2, 3$. Let M_{G_2} denote the 2-dimensional indecomposable B-module from [Jan2] having a filtration with factors k_{ω_1} on the top and $k_{\omega_2-\omega_1}$ on the bottom. Note that there are properly two modules here, one for each prime. As the prime will be clear in context, we abusively use the same notation for both.

2.3. For a simple root α, whether or not the weight $p\omega_\alpha - \alpha$ is p-restricted affects the B_1-cohomology. For higher r, the question becomes whether $p^r\omega_\alpha - p^i\alpha$ is p^r-restricted for $0 \leqslant i \leqslant r-1$. More generally, if ω is a weight and $p^r\omega - p^i\alpha$ is p^r-restricted, one would like to know what conditions ω must satisfy. It is not hard to see that in fact ω is usually ω_α and in general is uniquely determined by p, r, α, and i.

Lemma. *Let $\lambda \in X_r(T)$. If $\lambda = p^r\omega - p^i\alpha \in X_r(T)$ for some $\omega \in X(T)$, $\alpha \in \Delta$, and $0 \leqslant i \leqslant r-1$, then $\omega = \omega_\alpha$* Except *in the following cases*:

(a) *$p = 2$, $i = r-1$, and the root system is of type B_n $(n \geqslant 3)$ with $\alpha = \alpha_{n-1}$. Then $\omega = \omega_{n-1} - \omega_n$.*
(b) *$p = 2$, $i = r-1$, and the root system is of type C_n $(n \geqslant 2)$ with $\alpha = \alpha_n$. Then $\omega = \omega_n - \omega_{n-1}$.*
(c) *$p = 2$, $i = r-1$, and the root system is of type F_4 with $\alpha = \alpha_2$. Then $\omega = \omega_2 - \omega_3$.*
(d) *$p = 2$ or 3, $i = r-1$, and the root system is of type G_2 with $\alpha = \alpha_2$. Then $\omega = \omega_2 - \omega_1$.*

Proof. By definition of $X_r(T)$, we must have $0 \leqslant \langle\lambda, \beta^\vee\rangle \leqslant p^r - 1$ for all simple roots β. Write $\omega = \sum_{\beta\in\Delta} n_\beta\omega_\beta$ for integers n_β. First, we have

$$\langle\lambda, \alpha^\vee\rangle = p^r\langle\omega, \alpha^\vee\rangle - p^i\langle\alpha, \alpha^\vee\rangle = p^r n_\alpha - 2p^i.$$

Hence $n_\alpha = 1$. If the underlying root system is of type A_1, we are done. Now, let $\beta \neq \alpha$ be another simple root. Then we have

$$\langle\lambda, \beta^\vee\rangle = p^r\langle\omega, \beta^\vee\rangle - p^i\langle\alpha, \beta^\vee\rangle = p^r n_\beta - p^i\langle\alpha, \beta^\vee\rangle.$$

Now $\langle\alpha,\beta^\vee\rangle = 0, -1, -2$, or -3. Hence n_β must be zero unless $p = 2$ or 3, $i = r-1$, and $\langle\alpha,\beta^\vee\rangle = -2$ or -3. Checking the various root systems, one obtains the above list of "exceptional" cases. □

2.4. Jantzen computed the cohomology groups $\mathrm{H}^1(B_1,\lambda)$ in [Jan2, Section 3] for all $\lambda \in X(T)$. For the reader's convenience, we recall these results. For small primes, the answer depends on the type of the root system and involves certain indecomposable B-modules which are identified in Section 2.2. Note that there is a "generic" answer for $p > 3$.

Theorem (A). *Let $p > 3$ and $\lambda \in X_1(T)$. Then*

$$\mathrm{H}^1(B_1,\lambda) \cong \begin{cases} k_{\omega_\alpha}^{(1)} & \textit{if } \lambda = p\omega_\alpha - \alpha \textit{ for } \alpha \in \Delta, \\ 0 & \textit{else.} \end{cases}$$

Theorem (B). *Let $p = 3$ and $\lambda \in X_1(T)$.*

(a) *Assume that the underlying root system of G is not of type A_2 or G_2. Then*

$$\mathrm{H}^1(B_1,\lambda) \cong \begin{cases} k_{\omega_\alpha}^{(1)} & \textit{if } \lambda = p\omega_\alpha - \alpha \textit{ for } \alpha \in \Delta, \\ 0 & \textit{else.} \end{cases}$$

(b) *Assume that the underlying root system of G is of type A_2. Then*

$$\mathrm{H}^1(B_1,\lambda) \cong \begin{cases} k_{\omega_1}^{(1)} \oplus k_{\omega_2}^{(1)} & \textit{if } \lambda = \omega_1 + \omega_2 = 3\omega_1 - \alpha_1 = 3\omega_2 - \alpha_2, \\ 0 & \textit{else.} \end{cases}$$

(c) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1(B_1,\lambda) \cong \begin{cases} k_{\omega_1}^{(1)} & \textit{if } \lambda = \omega_1 + \omega_2 = 3\omega_1 - \alpha_1, \\ M_{G_2}^{(1)} & \textit{if } \lambda = \omega_2 = 3(\omega_2 - \omega_1) - \alpha_2, \\ 0 & \textit{else.} \end{cases}$$

We remark that in this proposition, one sees two phenomena which lead to a non-generic answer. When $p\omega_i - \alpha_i = p\omega_j - \alpha_j$ for distinct i, j, there is a "doubling" of the cohomology (in the sense of a direct sum of modules). The second phenomenon involves the question of whether the weight $p\omega_j - \alpha_j$ is p-restricted. Notice that in type G_2 when $p = 3$, the weight $3\omega_2 - \alpha_2 = 3\omega_1$ is not p-restricted. And it gets "replaced" by the p-restricted weight $3(\omega_2 - \omega_1) - \alpha_2$. Furthermore, the cohomology involves a non-simple indecomposable B-module. We refer the reader to Lemma 2.3 which considers the question of whether $p\omega_j - \alpha_j$ is p-restricted and note that one sees the same phenomena for $p = 2$.

Theorem (C). *Let $p = 2$ and $\lambda \in X_1(T)$.*

(a) *Assume that the underlying root system of G is of type A_n with $n \neq 3$, E_6, E_7, or E_8. Then*

$$\mathrm{H}^1(B_1, \lambda) \cong \begin{cases} k_{\omega_\alpha}^{(1)} & \text{if } \lambda = p\omega_\alpha - \alpha \text{ for } \alpha \in \Delta, \\ 0 & \text{else.} \end{cases}$$

(b) *Assume that the underlying root system of G is of type A_3. Then*

$$\mathrm{H}^1(B_1, \lambda) \cong \begin{cases} k_{\omega_1}^{(1)} \oplus k_{\omega_3}^{(1)} & \text{if } \lambda = \omega_2 = 2\omega_1 - \alpha_1 = 2\omega_3 - \alpha_3, \\ k_{\omega_2}^{(1)} & \text{if } \lambda = \omega_1 + \omega_3 = 2\omega_2 - \alpha_2, \\ 0 & \text{else.} \end{cases}$$

(c) *Assume that the underlying root system of G is of type B_3. Then*

$$\mathrm{H}^1(B_1, \lambda) \cong \begin{cases} k_{\omega_1}^{(1)} \oplus k_{\omega_3}^{(1)} & \text{if } \lambda = \omega_2 = 2\omega_1 - \alpha_1 = 2\omega_3 - \alpha_3, \\ M_{B_3}^{(1)} & \text{if } \lambda = \omega_1 = 2(\omega_2 - \omega_3) - \alpha_2, \\ 0 & \text{else.} \end{cases}$$

(d) *Assume that the underlying root system of G is of type B_4. Then*

$$\mathrm{H}^1(B_1, \lambda) \cong \begin{cases} k_{\omega_1}^{(1)} \oplus M_{B_4}^{(1)} & \text{if } \lambda = \omega_2 = 2\omega_1 - \alpha_1 = 2(\omega_3 - \omega_4) - \alpha_3, \\ k_{\omega_j}^{(1)} & \text{if } \lambda = 2\omega_j - \alpha_j \text{ for } j \in \{2, 4\}, \\ 0 & \text{else.} \end{cases}$$

(e) *Assume that the underlying root system of G is of type B_n, $n \geqslant 5$. Then*

$$\mathrm{H}^1(B_1, \lambda) \cong \begin{cases} k_{\omega_j}^{(1)} & \text{if } \lambda = 2\omega_j - \alpha_j \text{ for } j \in \{1, 2, \ldots, n-2, n\}, \\ M_{B_n}^{(1)} & \text{if } \lambda = \omega_{n-2} = 2(\omega_{n-1} - \omega_n) - \alpha_{n-1}, \\ 0 & \text{else.} \end{cases}$$

(f) *Assume that the underlying root system of G is of type C_n, $n \geqslant 2$. Then*

$$\mathrm{H}^1(B_1, \lambda) \cong \begin{cases} k_{\omega_j}^{(1)} & \text{if } \lambda = 2\omega_j - \alpha_j \text{ for } j \in \{1, 2, \ldots, n-1\}, \\ M_{C_n}^{(1)} & \text{if } \lambda = 0 = 2(\omega_n - \omega_{n-1}) - \alpha_n, \\ 0 & \text{else.} \end{cases}$$

(g) *Assume that the underlying root system of G is of type D_4. Then*

$$\mathrm{H}^1(B_1, \lambda) \cong \begin{cases} k_{\omega_1}^{(1)} \oplus k_{\omega_3}^{(1)} \oplus k_{\omega_4}^{(1)} & \text{if } \lambda = \omega_2 = 2\omega_1 - \alpha_1 = 2\omega_3 - \alpha_3 = 2\omega_4 - \alpha_4, \\ k_{\omega_2}^{(1)} & \text{if } \lambda = \omega_1 + \omega_3 = 2\omega_2 - \alpha_2, \\ 0 & \text{else.} \end{cases}$$

(h) *Assume that the underlying root system of G is of type D_n, $n \geqslant 5$. Then*

$$\mathrm{H}^1(B_1,\lambda) \cong \begin{cases} k_{\omega_j}^{(1)} & \text{if } \lambda = 2\omega_j - \alpha_j \text{ for } j \in \{1,2,\ldots,n-2\},\\ k_{\omega_{n-1}}^{(1)} \oplus k_{\omega_n}^{(1)} & \text{if } \lambda = \omega_{n-2} = 2\omega_{n-1} - \alpha_{n-1} = 2\omega_n - \alpha_n,\\ 0 & \text{else.} \end{cases}$$

(i) *Assume that the underlying root system of G is of type F_4. Then*

$$\mathrm{H}^1(B_1,\lambda) \cong \begin{cases} k_{\omega_j}^{(1)} & \text{if } \lambda = 2\omega_j - \alpha_j \text{ for } j \in \{1,3,4\},\\ M_{F_4}^{(1)} & \text{if } \lambda = \omega_1 = 2(\omega_2 - \omega_3) - \alpha_2,\\ 0 & \text{else.} \end{cases}$$

(j) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1(B_1,\lambda) \cong \begin{cases} k_{\omega_1}^{(1)} & \text{if } \lambda = \omega_2 = 2\omega_1 - \alpha_1,\\ M_{G_2}^{(1)} & \text{if } \lambda = \omega_1 = 2(\omega_2 - \omega_1) - \alpha_2,\\ 0 & \text{else.} \end{cases}$$

Using these propositions, one can compute $\mathrm{H}^1(B_1,\lambda)$ for all weights λ by writing $\lambda = \lambda_0 + p\lambda_1$ and using $\mathrm{H}^1(B_1,\lambda) \cong \mathrm{H}^1(B_1,\lambda_0) \otimes p\lambda_1$.

2.5. With the aid of the Lyndon–Hochschild–Serre spectral sequence, we now extend the results of Section 2.4 for B_1 to B_r for all r. When $p > 3$, the answer fits a "generic" form that does not depend on the root system. We consider this case first.

Theorem. *Suppose $p > 3$ and $\lambda \in X_r(T)$. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-1,\\ 0 & \text{else.} \end{cases}$$

Proof. We proceed by induction on r with the $r = 1$ case being Theorem 2.4(A). To proceed inductively, consider the LHS spectral sequence

$$E_2^{i,j} = \mathrm{H}^i\big(B_r/B_{r-1}, \mathrm{H}^j(B_{r-1},\lambda)\big) \quad \Rightarrow \quad \mathrm{H}^{i+j}(B_r,\lambda)$$

and the corresponding five-term exact sequence

$$0 \to E^{1,0} \to E^1 \to E^{0,1} \to E^{2,0} \to E^2.$$

Write $\lambda = \lambda_0 + p^{r-1}\lambda_1$. By induction, we have:

$$\begin{aligned} E^{0,1} &= \mathrm{Hom}_{B_r/B_{r-1}}\big(k, \mathrm{H}^1(B_{r-1},\lambda)\big)\\ &\cong \mathrm{Hom}_{B_r/B_{r-1}}\big(k, \mathrm{H}^1(B_{r-1},\lambda_0) \otimes p^{r-1}\lambda_1\big) \end{aligned}$$

$$\cong \begin{cases} \operatorname{Hom}_{B_r/B_{r-1}}\big(k, k_{\omega_\alpha}^{(r-1)} \otimes k_{\lambda_1}^{(r-1)}\big) & \text{if } \lambda_0 = p^{r-1}\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else} \end{cases}$$

$$\cong \begin{cases} \operatorname{Hom}_{B_1}(k, k_{\omega_\alpha+\lambda_1})^{(r-1)} & \text{if } \lambda_0 = p^{r-1}\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else} \end{cases}$$

$$\cong \begin{cases} \big(k_\omega^{(1)}\big)^{(r-1)} & \text{if } \lambda_0 = p^{r-1}\omega_\alpha - p^i\alpha \text{ as above and } \omega_\alpha + \lambda_1 = p\omega \\ & \quad \text{for } \omega \in X(T), \\ 0 & \text{else} \end{cases}$$

$$\cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^{r-1}(\omega_\alpha + \lambda_1) - p^i\alpha = p^r\omega - p^i\alpha \text{ for } \omega \in X(T),\ \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

Since $\lambda \in X_r(T)$, applying Lemma 2.3, we get

$$E^{0,1} \cong \begin{cases} k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

On the other hand, we have

$$\begin{aligned} E^{1,0} &= \mathrm{H}^1\big(B_r/B_{r-1}, \operatorname{Hom}_{B_{r-1}}(k,\lambda)\big) \\ &\cong \begin{cases} \mathrm{H}^1\big(B_r/B_{r-1}, k_{\lambda'}^{(r-1)}\big) & \text{if } \lambda = p^{r-1}\lambda' \text{ for } \lambda' \in X(T), \\ 0 & \text{else} \end{cases} \\ &\cong \begin{cases} \mathrm{H}^1(B_1, k_{\lambda'})^{(r-1)} & \text{if } \lambda = p^{r-1}\lambda' \text{ for } \lambda' \in X(T), \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Note that since $\lambda \in X_r(T)$, the weight λ' must lie in $X_1(T)$. And so by induction (or simply Theorem 2.4(A)), we have

$$\begin{aligned} E^{1,0} &\cong \begin{cases} \big(k_{\omega_\alpha}^{(1)}\big)^{(r-1)} & \text{if } \lambda = p^{r-1}\lambda' \text{ as above and } \lambda' = p\omega_\alpha - \alpha \text{ for } \alpha \in \Delta, \\ 0 & \text{else} \end{cases} \\ &\cong \begin{cases} k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^{r-1}\alpha \text{ for } \alpha \in \Delta, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

If $E^{0,1} = 0$, then $E^1 \cong E^{1,0}$ and the above computations confirm the claim. On the other hand, if $E^{0,1} \neq 0$, we must have $\lambda = p^r\omega_\alpha - p^i\alpha$ for some $\alpha \in \Delta$ and $0 \leqslant i \leqslant r-2$. This implies that λ is not divisible by p^{r-1} and so $\operatorname{Hom}_{B_{r-1}}(k,\lambda) = 0$. Hence $E^{1,0} = 0 = E^{2,0}$ and then $E^1 \cong E^{0,1}$ and the result follows. □

2.6. For $p = 3$, one has to deal with the fact that $\mathrm{H}^1(B_1,\lambda)$ may not be of the form $k_\omega^{(1)}$ (when it is not zero). However, the same basic inductive argument still works.

Theorem. *Let $p=3$ and $\lambda\in X_r(T)$.*

(a) *Assume that the underlying root system of G is not of type A_2 or G_2. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases}k_{\omega_\alpha}^{(r)} & \text{if } \lambda=p^r\omega_\alpha-p^i\alpha \text{ for } \alpha\in\Delta,\ 0\leqslant i\leqslant r-1,\\ 0 & \text{else.}\end{cases}$$

(b) *Assume that the underlying root system of G is of type A_2. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases}k_{\omega_1}^{(r)}\oplus k_{\omega_2}^{(r)} & \text{if } \lambda=p^{r-1}(\omega_1+\omega_2)=p^r\omega_1-p^{r-1}\alpha_1=p^r\omega_2-p^{r-1}\alpha_2,\\ k_{\omega_j}^{(r)} & \text{if } \lambda=p^r\omega_j-p^i\alpha_j \text{ for } j\in\{1,2\},\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(c) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases}k_{\omega_1}^{(r)} & \text{if } \lambda=p^{r-1}(\omega_1+\omega_2)=p^r\omega_1-p^{r-1}\alpha_1,\\ M_{G_2}^{(r)} & \text{if } \lambda=p^{r-1}\omega_2=p^r(\omega_2-\omega_1)-p^{r-1}\alpha_2,\\ k_{\omega_j}^{(r)} & \text{if } \lambda=p^r\omega_j-p^i\alpha_j \text{ for } j\in\{1,2\},\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

Proof. The proof for part (a) is identical to the proof of Theorem 2.5. For part (b), we follow the same inductive argument. Let $\lambda=\lambda_0+p^{r-1}\lambda_1$. In this case, we get

$$E^{0,1}\cong\begin{cases}\mathrm{Hom}_{B_r/B_{r-1}}\big(k,\big(k_{\omega_1}^{(r-1)}\oplus k_{\omega_2}^{(r-1)}\big)\otimes k_{\lambda_1}^{(r-1)}\big) & \text{if } \lambda_0=p^{r-2}(\omega_1+\omega_2),\\ \mathrm{Hom}_{B_r/B_{r-1}}\big(k,k_{\omega_j}^{(r-1)}\otimes k_{\lambda_1}^{(r-1)}\big) & \text{if } \lambda_0=p^{r-2}\omega_j-p^i\alpha_j\\ & \quad\text{for } j\in\{1,2\},\ 0\leqslant i\leqslant r-3,\\ 0 & \text{else}\end{cases}$$

$$\cong\begin{cases}\mathrm{Hom}_{B_1}(k,k_{\omega_1+\lambda_1}\oplus k_{\omega_2+\lambda_1})^{(r-1)} & \text{if } \lambda_0=p^{r-2}(\omega_1+\omega_2),\\ \mathrm{Hom}_{B_1}(k,k_{\omega_j+\lambda_1})^{(r-1)} & \text{if } \lambda_0=p^{r-2}\omega_j-p^i\alpha_j \text{ for } j\in\{1,2\},\\ & \quad 0\leqslant i\leqslant r-3,\\ 0 & \text{else.}\end{cases}$$

Note that $\mathrm{Hom}_{B_1}(k,k_{\omega_1+\lambda_1}\oplus k_{\omega_2+\lambda_1})$ is at most one-dimensional since it is not possible to have both $\omega_1+\lambda_1=p\omega$ and $\omega_2+\lambda_1=p\omega'$ for weights $\omega,\omega'\in X(T)$. Hence, we have

$$E^{0,1}\cong\begin{cases}k_{\omega}^{(r)} & \text{if } \lambda_0=p^{r-2}(\omega_1+\omega_2) \text{ and } \omega_j+\lambda_1=p\omega \text{ for } j\in\{1,2\},\\ & \quad\omega\in X(T),\\ k_{\omega}^{(r)} & \text{if } \lambda_0=p^{r-2}\omega_j-p^i\alpha_j \text{ and } \omega_j+\lambda_1=p\omega \text{ for } j\in\{1,2\},\\ & \quad 0\leqslant i\leqslant r-3,\ \omega\in X(T),\\ 0 & \text{else.}\end{cases}$$

Since $p^{r-2}(\omega_1+\omega_2)=p^{r-1}\omega_1-p^{r-2}\alpha_1=p^{r-1}\omega_2-p^{r-2}\alpha_2$, the non-vanishing conditions above can be combined. The requirement is that

$$\begin{aligned}\lambda &= \lambda_0+p^{r-1}\lambda_1=p^{r-1}\omega_j-p^i\alpha_j+p^{r-1}\lambda_1=p^{r-1}(\omega_j+\lambda_1)-p^i\alpha_j\\ &= p^r\omega-p^i\alpha_j\end{aligned}$$

for $j=1$ or 2, $0\leqslant i\leqslant r-2$, and some $\omega\in X(T)$. Applying Lemma 2.3, ω must be ω_j. Hence, we get

$$E^{0,1}\cong\begin{cases}k_{\omega_j}^{(r)} & \text{if } \lambda=p^r\omega_j-p^i\alpha_j \text{ for } j\in\{1,2\},\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

Next we compute $E^{1,0}$. Again, we get

$$E^{1,0}\cong\begin{cases}\mathrm{H}^1(B_1,k_{\lambda'})^{(r-1)} & \text{if } \lambda=p^{r-1}\lambda' \text{ for } \lambda'\in X(T),\\ 0 & \text{else.}\end{cases}$$

And by induction,

$$E^{1,0}\cong\begin{cases}k_{\omega_1}^{(r)}\oplus k_{\omega_2}^{(r)} & \text{if } \lambda=p^{r-1}(\omega_1+\omega_2)=p^r\omega_1-p^{r-1}\alpha_1=p^r\omega_2-p^{r-1}\alpha_2,\\ 0 & \text{else.}\end{cases}$$

As in Theorem 2.5, $\mathrm{H}^1(B_r,\lambda)$ may be identified with either $E^{1,0}$ or $E^{0,1}$ and the claim follows.

For part (c), we proceed analogously. With $\lambda=\lambda_0+p^{r-1}\lambda_1$, we get

$$E^{0,1}\cong\begin{cases}\mathrm{Hom}_{B_r/B_{r-1}}\big(k,k_{\omega_1}^{(r-1)}\otimes k_{\lambda_1}^{(r-1)}\big) & \text{if } \lambda_0=p^{r-1}\omega_1-p^{r-2}\alpha_1,\\ \mathrm{Hom}_{B_r/B_{r-1}}\big(k,M_{G_2}^{(r-1)}\otimes k_{\lambda_1}^{(r-1)}\big) & \text{if } \lambda_0=p^{r-2}\omega_2=p^{r-1}(\omega_2-\omega_1)-p^{r-2}\alpha_2,\\ \mathrm{Hom}_{B_r/B_{r-1}}\big(k,k_{\omega_j}^{(r-2)}\otimes k_{\lambda_1}^{(r-2)}\big) & \text{if } \lambda_0=p^{r-1}\omega_j-p^i\alpha_j \text{ for } j\in\{1,2\},\\ & \quad 0\leqslant i\leqslant r-3,\\ 0 & \text{else,}\end{cases}$$

$$\cong\begin{cases}k_\omega^{(r)} & \text{if } \lambda_0=p^{r-1}\omega_1-p^{r-2}\alpha_1 \text{ and } \omega_1+\lambda_1=p\omega \text{ for } \omega\in X(T),\\ k_\omega^{(r)} & \text{if } \lambda_0=p^{r-2}\omega_2=p^{r-1}(\omega_2-\omega_1)-p^{r-2}\alpha_2 \text{ and } \omega_2-\omega_1+\lambda_1=p\omega\\ & \quad \text{for } \omega\in X(T),\\ k_\omega^{(r)} & \text{if } \lambda_0=p^{r-1}\omega_j-p^i\alpha_j \text{ for } j\in\{1,2\},\ 0\leqslant i\leqslant r-3 \text{ and } \omega_2+\lambda_1=p\omega\\ & \quad \text{for } \omega\in X(T),\\ 0 & \text{else.}\end{cases}$$

The condition in the second case above arises because the module $M_{G_2}\otimes k_{\lambda_1}$ is a two-dimensional indecomposable module with bottom factor being $k_{\omega_2-\omega_1}\otimes k_{\lambda_1}\cong k_{\omega_2-\omega_1+\lambda_1}$. Applying Lemma 2.3, we further get

$$E^{0,1} \cong \begin{cases} k_{\omega_1}^{(r)} & \text{if } \lambda = p^r\omega_1 - p^{r-2}\alpha_1, \\ k_{\omega_2}^{(r)} & \text{if } \lambda = p^r\omega_2 - p^{r-2}\alpha_2, \\ k_{\omega_j}^{(r)} & \text{if } \lambda = p^r\omega_j - p^i\alpha_j \text{ for } j \in \{1,2\},\ 0 \leqslant i \leqslant r-3, \\ 0 & \text{else,} \end{cases}$$

$$\cong \begin{cases} k_{\omega_j}^{(r)} & \text{if } \lambda = p^r\omega_j - p^i\alpha_j \text{ for } j \in \{1,2\},\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

Next we compute $E^{1,0}$. Again, we get

$$E^{1,0} \cong \begin{cases} \mathrm{H}^1(B_1, k_{\lambda'})^{(r-1)} & \text{if } \lambda = p^{r-1}\lambda' \text{ for } \lambda' \in X(T), \\ 0 & \text{else.} \end{cases}$$

And by induction,

$$E^{1,0} \cong \begin{cases} k_{\omega_1}^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1 + \omega_2) = p^r\omega_1 - p^{r-1}\alpha_1, \\ M_{G_2}^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r(\omega_2 - \omega_1) - p^{r-1}\alpha_2, \\ 0 & \text{else.} \end{cases}$$

As before, $\mathrm{H}^1(B_r, \lambda)$ may be identified with either $E^{1,0}$ or $E^{0,1}$ and the claim follows. □

2.7. For $p = 2$, the computation of $\mathrm{H}^1(B_r, \lambda)$ involves even more special cases. As for $p = 3$, one must deal with the presence of direct sums and non-simple indecomposable modules. The arguments are similar to those for $p = 3$ and left to the interested reader.

Theorem. *Let $p = 2$ and $\lambda \in X_r(T)$.*

(a) *Assume that the underlying root system of G is of type A_n with $n \neq 3$, E_6, E_7, or E_8. Then*

$$\mathrm{H}^1(B_r, \lambda) \cong \begin{cases} k_{\omega_\alpha}^{(r)} & \textit{if } \lambda = p^r\omega_\alpha - p^i\alpha \textit{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-1, \\ 0 & \textit{else.} \end{cases}$$

(b) *Assume that the underlying root system of G is of type A_3. Then*

$$\mathrm{H}^1(B_r, \lambda) \cong \begin{cases} k_{\omega_1}^{(r)} \oplus k_{\omega_3}^{(r)} & \textit{if } \lambda = p^{r-1}\omega_2 = p^r\omega_1 - p^{r-1}\alpha_1 = p^r\omega_3 - p^{r-1}\alpha_3, \\ k_{\omega_2}^{(r)} & \textit{if } \lambda = p^{r-1}(\omega_1 + \omega_3) = p^r\omega_2 - p^{r-1}\alpha_2, \\ k_{\omega_\alpha}^{(r)} & \textit{if } \lambda = p^r\omega_\alpha - p^i\alpha \textit{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \textit{else.} \end{cases}$$

(c) *Assume that the underlying root system of G is of type B_3. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases} k_{\omega_1}^{(r)}\oplus k_{\omega_3}^{(r)} & \text{if } \lambda=p^{r-1}\omega_2=p^r\omega_1-p^{r-1}\alpha_1=p^r\omega_3-p^{r-1}\alpha_3,\\ M_{B_3}^{(r)} & \text{if } \lambda=p^{r-1}\omega_1=p^r(\omega_2-\omega_3)-p^{r-1}\alpha_2,\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda=p^r\omega_\alpha-p^i\alpha \text{ for } \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(d) *Assume that the underlying root system of G is of type B_4. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases} k_{\omega_1}^{(r)}\oplus M_{B_4}^{(r)} & \text{if } \lambda=p^{r-1}\omega_2=p^r\omega_1-p^{r-1}\alpha_1\\ & \qquad =p^r(\omega_3-\omega_4)-p^{r-1}\alpha_3,\\ k_{\omega_j}^{(r)} & \text{if } \lambda=p^r\omega_j-p^{r-1}\alpha_j \text{ for } j\in\{2,4\},\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda=p^r\omega_\alpha-p^i\alpha \text{ for } \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(e) *Assume that the underlying root system of G is of type B_n, $n\geqslant 5$. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases} k_{\omega_j}^{(r)} & \text{if } \lambda=p^r\omega_j-p^{r-1}\alpha_j \text{ for } j\in\{1,2,\ldots,n-2,n\},\\ M_{B_n}^{(r)} & \text{if } \lambda=p^{r-1}\omega_{n-2}=p^r(\omega_{n-1}-\omega_n)-p^{r-1}\alpha_{n-1},\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda=p^r\omega_\alpha-p^i\alpha \text{ for } \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(f) *Assume that the underlying root system of G is of type C_n, $n\geqslant 2$. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases} k_{\omega_j}^{(r)} & \text{if } \lambda=p^r\omega_j-p^{r-1}\alpha_j \text{ for } j\in\{1,2,\ldots,n-1\},\\ M_{C_n}^{(r)} & \text{if } \lambda=0=p^r(\omega_n-\omega_{n-1})-p^{r-1}\alpha_n,\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda=p^r\omega_\alpha-p^i\alpha \text{ for } \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(g) *Assume that the underlying root system of G is of type D_4. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases} k_{\omega_1}^{(r)}\oplus k_{\omega_3}^{(r)}\oplus k_{\omega_4}^{(r)} & \text{if } \lambda=p^{r-1}\omega_2=p^r\omega_1-p^{r-1}\alpha_1\\ & \qquad =p^r\omega_3-p^{r-1}\alpha_3=p^r\omega_4-p^{r-1}\alpha_4,\\ k_{\omega_2}^{(r)} & \text{if } \lambda=p^{r-1}(\omega_1+\omega_3)=p^r\omega_2-p^{r-1}\alpha_2,\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda=p^r\omega_\alpha-p^i\alpha \text{ for } \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(h) *Assume that the underlying root system of G is of type D_n, $n \geqslant 5$. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_{\omega_j}^{(r)} & \text{if } \lambda = p^r\omega_j - p^{r-1}\alpha_j \text{ for } j \in \{1,2,\ldots,n-2\},\\ k_{\omega_{n-1}}^{(r)} \oplus k_{\omega_n}^{(r)} & \text{if } \lambda = p^{r-1}\omega_{n-2} = p^r\omega_{n-1} - p^{r-1}\alpha_{n-1}\\ & \qquad = p^r\omega_n - p^{r-1}\alpha_n,\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2,\\ 0 & \text{else.} \end{cases}$$

(i) *Assume that the underlying root system of G is of type F_4. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_{\omega_j}^{(r)} & \text{if } \lambda = p^r\omega_j - p^{r-1}\alpha_j \text{ for } j \in \{1,3,4\},\\ M_{F_4}^{(r)} & \text{if } \lambda = p^{r-1}\omega_1 = p^r(\omega_2-\omega_3) - p^{r-1}\alpha_2,\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2,\\ 0 & \text{else.} \end{cases}$$

(j) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_{\omega_1}^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r\omega_1 - p^{r-1}\alpha_1,\\ M_{G_2}^{(r)} & \text{if } \lambda = p^{r-1}\omega_1 = p^r(\omega_2-\omega_1) - p^{r-1}\alpha_2,\\ k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2,\\ 0 & \text{else.} \end{cases}$$

2.8. With the computations of $\mathrm{H}^1(B_r,\lambda)$ for all $\lambda \in X_r(T)$ above, one can readily compute $\mathrm{H}^1(B_r,\lambda)$ for arbitrary $\lambda \in X(T)$. First, we make the following observation.

Corollary. *Let $\lambda \in X(T)$. Then $\mathrm{H}^1(B_r,\lambda) \neq 0$ if and only if $\lambda = p^r\omega - p^i\alpha$ for some $\omega \in X(T)$, $\alpha \in \Delta$, and $0 \leqslant i \leqslant r-1$. Moreover, if $\lambda \in X_r(T)$, then the weight ω is the weight determined by p, r, α, and i in Lemma* 2.3.

Proof. Given $\lambda \in X(T)$, we first show that λ must have the desired form. Write $\lambda = \lambda_0 + p^r\lambda_1$ for (unique) $\lambda_0 \in X_r(T)$ and $\lambda_1 \in X(T)$. Then $\mathrm{H}^1(B_r,\lambda) \cong \mathrm{H}^1(B_r,\lambda_0) \otimes p^r\lambda_1$ and the answer depends on λ_0. From Theorems 2.5–2.7, $\mathrm{H}^1(B_r,\lambda_0) \neq 0$ if and only if $\lambda_0 = p^r\omega' - p^i\alpha$ for some $\alpha \in \Delta$ where ω' is the weight (corresponding to p, r, α, and i) determined in Lemma 2.3. Thus, $\lambda = \lambda_0 + p^r\lambda_1 = p^r\omega' - p^i\alpha + p^r\lambda_1 = p^r(\omega'+\lambda_1) - p^i\alpha$ has the form $p^r\omega - p^i\alpha$.

Conversely, given any weight $\lambda = p^r\omega - p^i\alpha$, one can always express ω as $\omega = \omega' + \lambda_1$ for the required weight ω' and some weight $\lambda_1 \in X(T)$. And since non-vanishing is independent of λ_1, $\mathrm{H}^1(B_r,\lambda)$ will be non-zero for all such λ. Finally, the “moreover” part follows immediately from Lemma 2.3. □

Given $\lambda = p^r\omega - p^i\alpha$, to compute $\mathrm{H}^1(B_r,\lambda)$, write $\lambda = (p^r\omega' - p^i\alpha) + p^r\lambda_1$ for the specific weight ω' (from Lemma 2.3) and some weight $\lambda_1 \in X(T)$. In terms of the given weight ω, $\lambda_1 = \omega - \omega'$. Thus we get

$$\begin{aligned}\mathrm{H}^1(B_r,\lambda) &\cong \mathrm{H}^1(B_r,\lambda_0)\otimes k_{\lambda_1}^{(r)} = \mathrm{H}^1\big(B_r, p^r\omega' - p^i\alpha\big)\otimes k_{\lambda_1}^{(r)}\\ &\cong \mathrm{H}^1\big(B_r, p^r\omega' - p^i\alpha\big)\otimes k_{\omega-\omega'}^{(r)}\end{aligned}$$

and one simply substitutes the answers from Theorem 2.5–2.7. For example, in the generic case, $\omega' = \omega_\alpha$ and $\mathrm{H}^1(B_r, p^r\omega_\alpha - p^i\alpha) \cong k_{\omega_\alpha}^{(r)}$ so that

$$\mathrm{H}^1\big(B_r, p^r\omega - p^i\alpha\big) \cong k_{\omega_\alpha}^{(r)} \otimes k_{\omega-\omega_\alpha}^{(r)} \cong k_\omega^{(r)}.$$

For completeness, we include the answers here omitting the straightforward details in the non-generic cases.

Theorem (A). *Let $p > 3$ and $\lambda \in X(T)$. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega\in X(T),\ \alpha\in\Delta,\ 0\leqslant i\leqslant r-1,\\ 0 & \text{else.}\end{cases}$$

Theorem (B). *Let $p = 3$ and $\lambda \in X(T)$.*

(a) *Assume that the underlying root system of G is not of type A_2 or G_2. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega\in X(T),\ \alpha\in\Delta,\ 0\leqslant i\leqslant r-1,\\ 0 & \text{else.}\end{cases}$$

(b) *Assume that the underlying root system of G is of type A_2. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} \oplus k_{\omega+\omega_\ell-\omega_j}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega\in X(T),\\ & \quad j,\ell\in\{1,2\},\ j\neq\ell,\\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega\in X(T),\\ & \quad \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(c) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_1 \text{ for } \omega\in X(T),\\ M_{G_2}^{(r)}\otimes k_{\omega+\omega_1-\omega_2}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_2 \text{ for } \omega\in X(T),\\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha_j \text{ for } \omega\in X(T),\\ & \quad j\in\{1,2\},\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

Theorem (C). *Let $p=2$ and $\lambda \in X(T)$.*

(a) *Assume that the underlying root system of G is of type A_n with $n \neq 3$, E_6, E_7, or E_8. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T),\ \alpha \in \Delta,\ 0 \leqslant i \leqslant r-1, \\ 0 & \text{else.} \end{cases}$$

(b) *Assume that the underlying root system of G is of type A_3. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} \oplus k_{\omega+\omega_\ell-\omega_j}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j,\ell \in \{1,3\},\ j \neq \ell, \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_2 \text{ for } \omega \in X(T), \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T), \\ & \quad \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(c) *Assume that the underlying root system of G is of type B_3. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} \oplus k_{\omega+\omega_\ell-\omega_j}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j,\ell \in \{1,3\},\ j \neq \ell, \\ M_{B_3}^{(r)} \otimes k_{\omega+\omega_3-\omega_2}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_2 \text{ for } \omega \in X(T), \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T), \\ & \quad \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(d) *Assume that the underlying root system of G is of type B_4. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} \oplus \big(M_{B_4}^{(r)} \otimes k_{\omega-\omega_1}^{(r)}\big) & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_1 \text{ for } \omega \in X(T), \\ k_{\omega+\omega_1+\omega_4-\omega_3}^{(r)} & \\ \quad \oplus\big(M_{B_4}^{(r)} \otimes k_{\omega+\omega_4-\omega_3}^{(r)}\big) & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_3 \text{ for } \omega \in X(T), \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j \in \{2,4\}, \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T),\ \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(e) *Assume that the underlying root system of G is of type B_n, $n \geqslant 5$. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j \in \{1,2,\ldots,n-2,n\}, \\ M_{B_n}^{(r)} \otimes k_{\omega+\omega_n-\omega_{n-1}}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_{n-1} \text{ for } \omega \in X(T), \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(f) *Assume that the underlying root system of G is of type C_n, $n \geqslant 2$. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j \in \{1,2,\ldots,n-1\}, \\ M_{C_n}^{(r)} \otimes k_{\omega+\omega_{n-1}-\omega_n}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_n \text{ for } \omega \in X(T), \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(g) *Assume that the underlying root system of G is of type D_4. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} \oplus k_{\omega+\omega_\ell-\omega_j}^{(r)} \oplus k_{\omega+\omega_s-\omega_j}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j,\ell,s \in \{1,3,4\},\ j \neq \ell,\ j \neq s,\ \ell \neq s, \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_2 \text{ for } \omega \in X(T), \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T), \\ & \quad \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(h) *Assume that the underlying root system of G is of type D_n, $n \geqslant 5$. Then*

$$\mathrm{H}^1(B_r,\lambda) \cong \begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j \in \{1,2,\ldots,n-2\}, \\ k_\omega^{(r)} \oplus k_{\omega+\omega_\ell-\omega_j}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega \in X(T), \\ & \quad j,\ell \in \{n-1,n\},\ j \neq \ell, \\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^i\alpha \text{ for } \omega \in X(T), \\ & \quad \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(i) *Assume that the underlying root system of G is of type F_4. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_j \text{ for } \omega\in X(T),\\ & \quad j\in\{1,3,4\},\\ M_{F_4}^{(r)}\otimes k_{\omega+\omega_3-\omega_2}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_2 \text{ for } \omega\in X(T),\\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{i}\alpha \text{ for } \omega\in X(T),\\ & \quad \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

(j) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1(B_r,\lambda)\cong\begin{cases} k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_1 \text{ for } \omega\in X(T),\\ M_{G_2}^{(r)}\otimes k_{\omega+\omega_1-\omega_2}^{(r)} & \text{if } \lambda = p^r\omega - p^{r-1}\alpha_2 \text{ for } \omega\in X(T),\\ k_\omega^{(r)} & \text{if } \lambda = p^r\omega - p^{i}\alpha \text{ for } \omega\in X(T),\\ & \quad \alpha\in\Delta,\ 0\leqslant i\leqslant r-2,\\ 0 & \text{else.}\end{cases}$$

We have chosen to present the results for $\lambda\in X_r(T)$ first and then those for general $\lambda\in X(T)$. If one prefers, this can be done in the opposite order: one can inductively obtain the results for arbitrary λ and then use Lemma 2.3 to deduce the results in Sections 2.5–2.7 for p^r-restricted weights.

3. G_r-cohomology of induced modules

3.1. According to Kempf's vanishing theorem, $H^0(\lambda)=\mathrm{ind}_B^G\lambda$ is zero unless $\lambda\in X(T)_+$. For dominant weights λ, the preceding computations of B_r-cohomology can now be used to compute $\mathrm{H}^1(G_r,H^0(\lambda))$ thanks to the isomorphism

$$\mathrm{H}^1\big(G_r,H^0(\lambda)\big)^{(-r)}\cong\mathrm{ind}_B^G\big(H^1(B_r,\lambda)^{(-r)}\big)$$

(cf. [Jan1, II.12.2]). Indeed, in the "generic" case, we simply have

$$\mathrm{ind}_B^G\big(\mathrm{H}^1(B_r,\lambda)^{(-r)}\big)\cong\mathrm{ind}_B^G(k_\omega)=H^0(\omega).$$

In general, for p^r-restricted weights, the computations follow readily from Theorems 2.5–2.7. However, some work is required when the B_r-cohomology involves a non-simple indecomposable module. For $p\geqslant 3(h-1)$, the following theorem (in conjunction with Lemma 2.3) is stated in [And2, p. 392].

Theorem (A). *Let $p>3$ and $\lambda\in X_r(T)$. Then*

$$\mathrm{H}^1\big(G_r,H^0(\lambda)\big)\cong\begin{cases} H^0(\omega_\alpha)^{(r)} & \text{if } \lambda=p^r\omega_\alpha-p^i\alpha \text{ for } \alpha\in\Delta,\ 0\leqslant i\leqslant r-1,\\ 0 & \text{else.}\end{cases}$$

Theorem (B). *Let $p = 3$ and $\lambda \in X_r(T)$.*

(a) *Assume that the underlying root system of G is not of type A_2 or G_2. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-1, \\ 0 & \text{else.} \end{cases}$$

(b) *Assume that the underlying root system of G is of type A_2. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_1)^{(r)} \oplus H^0(\omega_2)^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1+\omega_2) = p^r\omega_1 - p^{r-1}\alpha_1 \\ & \qquad = p^r\omega_2 - p^{r-1}\alpha_2, \\ H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r\omega_j - p^i\alpha_j \text{ for } j \in \{1,2\}, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(c) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_1)^{(r)} & \text{if } \lambda = p^r\omega_1 - p^{r-1}\alpha_1, \\ H^0(\omega_1)^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r(\omega_2-\omega_1) - p^{r-1}\alpha_2, \\ H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r\omega_j - p^i\alpha_j \text{ for } j \in \{1,2\}, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

Proof. There is one case where the computation involves inducing a non-simple indecomposable B-module. That is in part (c) when $\lambda = p^{r-1}\omega_2$ and $\mathrm{H}^1(B_r, \lambda) \cong M_{G_2}^{(r)}$. However, Jantzen shows in [Jan2, Proposition 5.2] that $\mathrm{ind}_B^G(M_{G_2}) \cong H^0(\omega_1)$ which gives the claim. □

Theorem (C). *Let $p = 2$ and $\lambda \in X_r(T)$.*

(a) *Assume that the underlying root system is of type A_n with $n \neq 3$, E_6, E_7, or E_8. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-1, \\ 0 & \text{else.} \end{cases}$$

(b) *Assume that the underlying root system of G is of type A_3. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_1)^{(r)} \oplus H^0(\omega_3)^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r\omega_1 - p^{r-1}\alpha_1 \\ & \qquad = p^r\omega_3 - p^{r-1}\alpha_3, \\ H^0(\omega_2)^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1+\omega_3) \\ & \qquad = p^r\omega_2 - p^{r-1}\alpha_2, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(c) *Assume that the underlying root system of G is of type B_3. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_1)^{(r)} \oplus H^0(\omega_3)^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r\omega_1 - p^{r-1}\alpha_1 \\ & \quad = p^r\omega_3 - p^{r-1}\alpha_3, \\ H^0(\omega_3)^{(r)} & \text{if } \lambda = p^{r-1}\omega_1 \\ & \quad = p^r(\omega_2 - \omega_3) - p^{r-1}\alpha_2, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(d) *Assume that the underlying root system of G is of type B_4. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_1)^{(r)} \oplus H^0(\omega_4)^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r\omega_1 - p^{r-1}\alpha_1 \\ & \quad = p^r(\omega_3 - \omega_4) - p^{r-1}\alpha_3, \\ H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r\omega_j - p^{r-1}\alpha_j \text{ for } j = 2, 4, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(e) *Assume that the underlying root system of G is of type B_n, $n \geqslant 5$. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r\omega_j - p^{r-1}\alpha_j \text{ for } j \in \{1, 2, \ldots, n-2, n\}, \\ H^0(\omega_n)^{(r)} & \text{if } \lambda = p^{r-1}\omega_{n-2} = p^r(\omega_{n-1} - \omega_n) - p^{r-1}\alpha_{n-1}, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(f) *Assume that the underlying root system of G is of type C_n, $n \geqslant 2$. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r\omega_j - p^{r-1}\alpha_j \text{ for } j \in \{1, 2, \ldots, n-1\}, \\ H^0(\omega_1)^{(r)} & \text{if } \lambda = 0 = p^r(\omega_n - \omega_{n-1}) - p^{r-1}\alpha_n, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(g) *Assume that the underlying root system is of type D_4. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_1)^{(r)} \oplus H^0(\omega_3)^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r\omega_1 - p^{r-1}\alpha_1 \\ \quad \oplus H^0(\omega_4)^{(r)} & \quad = p^r\omega_3 - p^{r-1}\alpha_3 \\ & \quad = p^r\omega_4 - p^{r-1}\alpha_4, \\ H^0(\omega_2)^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1 + \omega_3) \\ & \quad = p^r\omega_2 - p^{r-1}\alpha_2, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(h) *Assume that the underlying root system of G is of type D_n, $n \geqslant 5$. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r\omega_j - p^{r-1}\alpha_j \\ & \quad \text{for } j \in \{1, 2, \ldots, n-2\}, \\ H^0(\omega_{n-1})^{(r)} \oplus H^0(\omega_n)^{(r)} & \text{if } \lambda = p^{r-1}\omega_{n-2} \\ & \qquad = p^r\omega_{n-1} - p^{r-1}\alpha_{n-1} \\ & \qquad = p^r\omega_n - p^{r-1}\alpha_n, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(i) *Assume that the underlying root system of G is of type F_4. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r\omega_j - p^{r-1}\alpha_j \text{ for } j \in \{1, 3, 4\}, \\ H^0(\omega_4)^{(r)} & \text{if } \lambda = p^{r-1}\omega_1 = p^r(\omega_2 - \omega_3) - p^{r-1}\alpha_2, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta,\ 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

(j) *Assume that the underlying root system of G is of type G_2. Then*

$$\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \cong \begin{cases} H^0(\omega_1)^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r\omega_1 - p^{r-1}\alpha_1, \\ H^0(\omega_1)^{(r)} & \text{if } \lambda = p^{r-1}\omega_1 = p^r(\omega_2 - \omega_1) - p^{r-1}\alpha_2, \\ H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, \\ & \quad 0 \leqslant i \leqslant r-2, \\ 0 & \text{else.} \end{cases}$$

Proof. As in the previous proposition, the only difficulty arises in computing the induced module for the non-simple indecomposable modules. From [Jan2, 5.1, 5.2], we have

$$\mathrm{ind}_B^G\big(M_{B_n}\big) \cong H^0(\omega_n), \qquad \mathrm{ind}_B^G\big(M_{C_n}\big) \cong H^0(\omega_1),$$
$$\mathrm{ind}_B^G\big(M_{F_4}\big) \cong H^0(\omega_4), \qquad \mathrm{ind}_B^G\big(M_{G_2}\big) \cong H^0(\omega_1). \qquad \square$$

3.2. From Corollary 2.8 and Theorem 2.5, one immediately gets the following. Parts (a) and (b) lower the condition on p found in [And2, p. 392].

Corollary. *Suppose $\lambda \in X(T)_+$.*

(a) $\mathrm{H}^1\big(G_r, H^0(\lambda)\big) \neq 0$ *if and only if $\lambda = p^r\omega - p^i\alpha$ for some $w \in X(T)$, $\alpha \in \Delta$, and* $0 \leqslant i \leqslant r-1$.
(b) *If $p > 3$ and $\lambda = p^r\omega - p^i\alpha$, then* $\mathrm{H}^1(G_r, H^0(\lambda)) = H^0(\omega)^{(r)}$.
(c) *If $\lambda \in X_r(T)$, then the weight ω is the weight determined by p, r, α, and i in Lemma* 2.3.

As done in Section 3.1 for p^r-restricted weights, one can use Theorems 2.8(A–C) to compute $\mathrm{H}^1(G_r, H^0(\lambda))$ in terms of induced modules for all $\lambda \in X(T)_+$. As most cases simply involve inducing simple B-modules, for brevity, we do not include these results. However, some of the answers involve a module of the form $\mathrm{ind}_B^G(M_{X_n} \otimes k_\sigma)$. Specifically, the computation of $\mathrm{H}^1(G_r, H^0(\lambda))$ for the following dominant weights λ involves inducing the given module:

- $p = 3$, type G_2, $\lambda = p^r\omega - p^{r-1}\alpha_2$: $M_{G_2} \otimes k_{\omega+\omega_1-\omega_2}$.
- $p = 2$, type B_n, $n \geqslant 3$, $\lambda = p^r\omega - p^{r-1}\alpha_{n-1}$: $M_{B_n} \otimes k_{\omega+\omega_n-\omega_{n-1}}$.
- $p = 2$, type B_4, $\lambda = p^r\omega - p^{r-1}\alpha_1$: $M_{B_4} \otimes k_{\omega-\omega_1}$.
- $p = 2$, type C_n, $n \geqslant 2$, $\lambda = p^r\omega - p^{r-1}\alpha_n$: $M_{C_n} \otimes k_{\omega+\omega_{n-1}-\omega_n}$.
- $p = 2$, type F_4, $\lambda = p^r\omega - p^{r-1}\alpha_2$: $M_{F_4} \otimes k_{\omega+\omega_3-\omega_2}$.
- $p = 2$, type G_2, $\lambda = p^r\omega - p^{r-1}\alpha_2$: $M_{G_2} \otimes k_{\omega+\omega_1-\omega_2}$.

3.3. The following homological algebra fact will allow us to identify a filtration of the modules listed in Section 3.2 by $H^0(\gamma)$s. This strategy is based on the proof of [Jan2, Proposition 5.1].

Lemma. *Let M be a finite-dimensional B-module with a filtration* (*from top to bottom*) *by $k_{\sigma_1}, k_{\sigma_2}, \dots, k_{\sigma_n}$. Assume further that $R^i\,\mathrm{ind}_B^G(\sigma_j) = 0$ for all $i \geqslant 1$ and all j. Then $R^i\,\mathrm{ind}_B^G(M) = 0$ for all $i \geqslant 1$ and $\mathrm{ind}_B^G(M)$ has a filtration by factors* (*from top to bottom*)

$$H^0(\sigma_1), \quad H^0(\sigma_2), \quad \dots, \quad H^0(\sigma_n),$$

where any weights σ_j that are not dominant are omitted.

Proof. We argue by induction on n and are trivially done if $n = 1$. For $n > 1$, there is a short exact sequence

$$0 \to k_{\sigma_1} \to M \to N \to 0$$

for some module N. Associated to such a short exact sequence is a long exact sequence

$$0 \to \mathrm{ind}_B^G(\sigma_1) \to \mathrm{ind}_B^G(M) \to \mathrm{ind}_B^G(N) \to R^1\,\mathrm{ind}_B^G(\sigma_1) \to R^1\,\mathrm{ind}_B^G(M) \\ \to R^1\,\mathrm{ind}_B^G(N) \to \cdots.$$

By the hypothesis and induction, $R^i\,\mathrm{ind}_B^G(\sigma_1) = 0 = R^i\,\mathrm{ind}_B^G(N)$ for all $i \geqslant 1$. Hence $R^i\,\mathrm{ind}_B^G(M) = 0$ for all $i \geqslant 1$ and there is a short exact sequence

$$0 \to \mathrm{ind}_B^G(\sigma_1) \to \mathrm{ind}_B^G(M) \to \mathrm{ind}_B^G(N) \to 0.$$

Of course, $\mathrm{ind}_B^G(\sigma_1) = 0$ if σ_1 is not dominant. And by induction $\mathrm{ind}_B^G(N)$ has a filtration by $H^0(\sigma_2), H^0(\sigma_3), \dots, H^0(\sigma_n)$ for those σ_j which are dominant. Therefore, we get the desired filtration of M. □

3.4. Given a B-module M and a weight σ of M, we note that there are two easy conditions under which $R^i \operatorname{ind}_B^G(\sigma) = 0$ for all $i \geqslant 1$. First, this holds if σ is dominant (cf. [Jan1, II.4.5]). Secondly, if $\langle \sigma, \alpha^\vee \rangle = -1$ for some simple root α, then $R^i \operatorname{ind}_B^G(\sigma) = 0$ for all $i \geqslant 0$ by [Jan1, II.5.4(a)]. (Of course, for the latter σ, $H^0(\sigma) = 0$ and will not appear in the answer.) For the modules of interest to us, we will see that all weights satisfy one of these two conditions.

Proposition (A). *Suppose that* $p = 2$ *or* 3 *and the underlying root system of* G *is of type* G_2. *Let* $\omega \in X(T)$ *be such that* $p^r\omega - p^{r-1}\alpha_2$ *lies in* $X(T)_+$. *Then* $\langle \omega, \alpha_1^\vee \rangle \geqslant -1$ *and* $\langle \omega, \alpha_2^\vee \rangle \geqslant 1$. *Furthermore,*

(a) *if* $\langle \omega, \alpha_1^\vee \rangle \geqslant 0$, *then* $\operatorname{ind}_B^G(M_{G_2} \otimes k_{\omega+\omega_1-\omega_2})$ *has a filtration with factors* $H^0(\omega + 2\omega_1 - \omega_2)$ *on the top and* $H^0(\omega)$ *on the bottom.*
(b) *Whereas, if* $\langle \omega, \alpha_1^\vee \rangle = -1$, *then* $\operatorname{ind}_B^G(M_{G_2} \otimes k_{\omega+\omega_1-\omega_2}) \cong H^0(\omega + 2\omega_1 - \omega_2)$.

Proof. Let $\omega = m_1\omega_1 + m_2\omega_2$ for integers m_1, m_2. In order for the weight $p^r\omega - p^{r-1}\alpha_2 = (p^r m_1 + 3p^{r-1})\omega_1 + (p^r m_2 - 2p^{r-1})\omega_2$ to be dominant, it is necessary that $\langle \omega, \alpha_1^\vee \rangle = m_1 \geqslant -1$ and $\langle \omega, \alpha_2^\vee \rangle = m_2 \geqslant 1$ as claimed. Now, consider the B-module $M = M_{G_2} \otimes k_{\omega+\omega_1-\omega_2}$. This is a two-dimensional indecomposable B-module with a filtration having factors $k_{\omega_1+\omega+\omega_1-\omega_2} = k_{\omega+2\omega_1-\omega_2}$ on the top and $k_{\omega_2-\omega_1+\omega+\omega_1-\omega_2} = k_\omega$ on the bottom. From the conditions on ω, the weight $\omega + 2\omega_1 - \omega_2$ will be dominant but ω will be dominant only if $m_1 \geqslant 0$. On the other hand, if $m_1 = \langle \omega, \alpha_1^\vee \rangle = -1$, we are in the case of [Jan1, II.5.4(a)] mentioned above and have $R^i \operatorname{ind}_B^G(\omega) = 0$ for all i. And so the claims follow from Lemma 3.3. □

Proposition (B). *Suppose* $p = 2$.

(a) *Assume the underlying root system of* G *is of type* B_n *with* $n \geqslant 3$. *Let* $\omega \in X(T)$ *be such that* $p^r\omega - p^{r-1}\alpha_{n-1}$ *lies in* $X(T)_+$. *Then* $\langle \omega, \alpha_j^\vee \rangle \geqslant 0$ *for* $1 \leqslant j \leqslant n-2$, $\langle \omega, \alpha_{n-1}^\vee \rangle \geqslant 1$, *and* $\langle \omega, \alpha_n^\vee \rangle \geqslant -1$. *Further,*
   (i) *if* $\langle \omega, \alpha_n^\vee \rangle \geqslant 0$, *then* $\operatorname{ind}_B^G(M_{B_n} \otimes k_{\omega+\omega_n-\omega_{n-1}})$ *has a filtration with factors* $H^0(\omega + 2\omega_n - \omega_{n-1})$ *on the top and* $H^0(\omega)$ *on the bottom.*
   (ii) *Whereas, if* $\langle \omega, \alpha_n^\vee \rangle = -1$, *then* $\operatorname{ind}_B^G(M_{B_n} \otimes k_{\omega+\omega_n-\omega_{n-1}}) \cong H^0(\omega + 2\omega_n - \omega_{n-1})$.
(b) *Assume that the underlying root system of* G *is of type* B_4. *Let* $\omega \in X(T)$ *be such that* $p^r\omega - p^{r-1}\alpha_1$ *lies in* $X(T)_+$. *Then* $\langle \omega, \alpha_1^\vee \rangle \geqslant 1$ *and* $\langle \omega, \alpha_j^\vee \rangle \geqslant 0$ *for* $j = 2, 3, 4$. *Further,*
   (i) *if* $\langle \omega, \alpha_4^\vee \rangle \geqslant 1$, *then* $\operatorname{ind}_B^G(M_{B_4} \otimes k_{\omega-\omega_1})$ *has a filtration with factors* $H^0(\omega + \omega_4 - \omega_1)$ *on the top and* $H^0(\omega + \omega_3 - \omega_1 - \omega_4)$ *on the bottom.*
   (ii) *Whereas, if* $\langle \omega, \alpha_4^\vee \rangle = 0$, *then* $\operatorname{ind}_B^G(M_{B_4} \otimes k_{\omega-\omega_1}) \cong H^0(\omega + \omega_4 - \omega_1)$.
(c) *Assume that the underlying root system of* G *is of type* C_n, $n \geqslant 2$. *Let* $\omega \in X(T)$ *be such that* $p^r\omega - p^{r-1}\alpha_n$ *lies in* $X(T)_+$. *Then* $\langle \omega, \alpha_j^\vee \rangle \geqslant 0$ *for* $1 \leqslant j \leqslant n-2$, $\langle \omega, \alpha_{n-1}^\vee \rangle \geqslant -1$, *and* $\langle \omega, \alpha_n^\vee \rangle \geqslant 1$. *Furthermore, the module* $\operatorname{ind}_B^G(M_{C_n} \otimes k_{\omega+\omega_{n-1}-\omega_n})$ *has a filtration by factors* (*from top to bottom*):

$$H^0(\omega+\omega_1+\omega_{n-1}-\omega_n),\quad H^0(\omega+\omega_2-\omega_1+\omega_{n-1}-\omega_n),$$
$$H^0(\omega+\omega_3-\omega_2+\omega_{n-1}-\omega_n),\quad \ldots,\quad H^0(\omega+\omega_{n-2}-\omega_{n-3}+\omega_{n-1}-\omega_n),$$
$$H^0(\omega+\omega_{n-1}-\omega_{n-2}+\omega_{n-1}-\omega_n)=H^0(\omega+2\omega_{n-1}-\omega_{n-2}-\omega_n),$$
$$H^0(\omega+\omega_n-\omega_{n-1}+\omega_{n-1}-\omega_n)=H^0(\omega)$$

with not necessarily all factors present. Specifically,

- $H^0(\omega+\omega_1+\omega_{n-1}-\omega_n)$ *is always present.*
- *For* $n\geqslant 3$ *and* $1\leqslant j\leqslant n-2$, $H^0(\omega+\omega_{j+1}-\omega_j+\omega_{n-1}-\omega_n)$ *is present if* $\langle\omega,\alpha_j^\vee\rangle\geqslant 1$ *and not present if* $\langle\omega,\alpha_j^\vee\rangle=0$.
- $H^0(\omega)$ *is present if* $\langle\omega,\alpha_{n-1}^\vee\rangle\geqslant 0$ *and is not present if* $\langle\omega,\alpha_{n-2}^\vee\rangle=-1$.

(d) *Assume that the underlying root system of* G *is of type* F_4. *Let* $\omega\in X(T)$ *be such that* $p^r\omega-p^{r-1}\alpha_2$ *lies in* $X(T)_+$. *Then* $\langle\omega,\alpha_1^\vee\rangle\geqslant 0$, $\langle\omega,\alpha_2^\vee\rangle\geqslant 1$, $\langle\omega,\alpha_3^\vee\rangle\geqslant -1$, *and* $\langle\omega,\alpha_4^\vee\rangle\geqslant 0$. *Further* $\operatorname{ind}_B^G(M_{F_4}\otimes k_{\omega+\omega_3-\omega_2})$ *has a filtration by factors (from top to bottom):*

$$H^0(\omega+\omega_3+\omega_4-\omega_2),\quad H^0(\omega+2\omega_3-\omega_2-\omega_4),\quad H^0(\omega)$$

with not necessarily all factors present. Specifically,

- $H^0(\omega+\omega_3+\omega_4-\omega_2)$ *is always present.*
- $H^0(\omega+2\omega_3-\omega_2-\omega_4)$ *is present if* $\langle\omega,\alpha_4^\vee\rangle\geqslant 1$ *and not present if* $\langle\omega,\alpha_4^\vee\rangle=0$.
- $H^0(\omega)$ *is present if* $\langle\omega,\alpha_3^\vee\rangle\geqslant 0$ *and not present if* $\langle\omega,\alpha_3^\vee\rangle=-1$.

Proof. As in the preceding proof, we simply compute the conditions on ω and then apply Lemma 3.3. For part (a), write $\omega=\sum_{i=1}^n m_i\omega_i$. In order for

$$2^r\omega-2^{r-1}\alpha_{n-1}=2^rm_1\omega_1+2^rm_2\omega_2+\cdots+2^rm_{n-3}\omega_{n-3}+\big(2^rm_{n-2}+2^{r-1}\big)\omega_{n-2}$$
$$+\big(2^rm_{n-1}-2^r\big)\omega_{n-1}+\big(2^rm_n+2^r\big)\omega_n$$

to be dominant, we must have $m_i\geqslant 0$ for $1\leqslant i\leqslant n-2$, $m_{n-1}\geqslant 1$, and $m_n\geqslant -1$ as claimed. The module $M_{B_n}\otimes k_{\omega+\omega_n-\omega_{n-1}}$ has a filtration with $k_{\omega+2\omega_n-\omega_{n-1}}$ on the top and k_ω on the bottom. From above, the weight $\omega+2\omega_n-\omega_{n-1}$ is necessarily dominant. If $m_n\geqslant 0$, then ω is also dominant. Alternatively, we must have $\langle\omega,\alpha_n^\vee\rangle=m_n=-1$. In either case, the filtrations follow from Lemma 3.3.

For part (b), let $\omega=\sum_{i=1}^4 m_i\omega_i$. In order for the weight $2^r\omega-2^{r-1}\alpha_1=2^r\omega-2^r\omega_1+2^{r-1}\omega_2$ to be dominant, we must have $m_1\geqslant 1$ and $m_i\geqslant 0$ for $i=2,3,4$ as claimed. The module $M_{B_4}\otimes k_{\omega-\omega_1}$ has a filtration with $k_{\omega+\omega_4-\omega_1}$ on the top and $k_{\omega+\omega_3-\omega_1-\omega_4}$ on the bottom. The weight $\omega+\omega_4-\omega_1$ is dominant. However, the weight $\omega+\omega_3-\omega_1-\omega_4$ will be dominant only if $m_4=\langle\omega,\alpha_4^\vee\rangle\geqslant 1$. Alternatively, when $m_4=0$, we will have $\langle\omega+\omega_3-\omega_1-\omega_4,\alpha_4^\vee\rangle=-1$ which suffices.

For part (c), let $\omega=\sum_{i=1}^n m_i\omega_i$. In order for the weight $2^r\omega-2^{r-1}\alpha_n=2^r\omega+2^r\omega_{n-1}-2^r\omega_n$ to be dominant, we must have $m_i\geqslant 0$ for $1\leqslant i\leqslant n-2$, $m_{n-1}\geqslant -1$, and $m_n\geqslant 1$ as claimed. The module $M_{C_n}\otimes k_{\omega_{n-1}-\omega_n}$ has a filtration with factors (from top to bottom):

$$k_{\omega+\omega_{n-1}-\omega_n}, \quad k_{\omega+\omega_2-\omega_1+\omega_{n-1}-\omega_n}, \quad k_{\omega+\omega_3-\omega_2+\omega_{n-1}-\omega_n}, \quad \ldots,$$
$$k_{\omega+\omega_{n-2}-\omega_{n-3}+\omega_{n-1}-\omega_n}, \quad k_{\omega+\omega_{n-1}-\omega_{n-2}+\omega_{n-1}-\omega_n} = k_{\omega+2\omega_{n-1}-\omega_{n-2}-\omega_n},$$
$$k_{\omega+\omega_n-\omega_{n-1}+\omega_{n-1}-\omega_n} = k_\omega.$$

The weight $\omega + \omega_{n-1} - \omega_n$ is always dominant. On the other hand, the remaining weights need not be. Indeed, for $1 \leqslant j \leqslant n-2$, the weight $\sigma_j = \omega + \omega_{j+1} - \omega_j + \omega_{n-1} - \omega_n$ is dominant if and only if $m_j = \langle \omega, \alpha_j^\vee \rangle \geqslant 1$. Moreover, if $m_j = 0$, then $\langle \sigma_j, \alpha_j^\vee \rangle = -1$ as needed. Similarly, the weight ω is dominant if and only if $\langle \omega, \alpha_{n-1} \rangle \geqslant 0$. Alternatively, we have $\langle \omega, \alpha_{n-1} \rangle = -1$.

For part (d), let $\omega = \sum_{j=1}^4 m_j \omega_j$. In order for the weight $2^r\omega - 2^{r-1}\alpha_2 = 2^r\omega + 2^{r-1}\omega_1 - 2^r\omega_2 + 2^r\omega_3$ to be dominant, we must have $m_1 \geqslant 0$, $m_2 \geqslant 1$, $m_3 \geqslant -1$, and $m_4 \geqslant 0$ as claimed. The module $M_{F_4} \otimes k_{\omega+\omega_3-\omega_2}$ has a filtration by factors (from top to bottom): $k_{\omega+\omega_3+\omega_4-\omega_2}$, $k_{\omega+2\omega_3-\omega_2-\omega_4}$, k_ω. The weight $\omega + \omega_3 + \omega_4 - \omega_2$ is dominant. But the weight $\sigma = \omega + 2\omega_3 - \omega_2 - \omega_4$ is dominant only if $m_4 \geqslant 1$ whereas if $m_4 = 0$, then $\langle \sigma, \alpha_4^\vee \rangle = -1$. Lastly, if $m_3 \geqslant 0$, then ω is dominant. Alternatively, we have $\langle \omega, \alpha_3^\vee \rangle = -1$. □

3.5. In this section, we apply our results to provide a complete determination of when $\mathrm{H}^1(G_r, H^0(\omega_\beta))$ (or equivalently $\mathrm{H}^1(B_r, \omega_\beta)$) is non-zero for a fundamental dominant weight ω_β. For a given fundamental weight ω_β, by Corollary 3.2, $\mathrm{H}^1(G_r, H^0(\omega_\beta)) \neq 0$ if and only if $\omega_\beta = p^r\omega - p^i\alpha$ for some simple root α and $0 \leqslant i \leqslant r-1$ where ω is the corresponding weight from Lemma 2.3. Suppose $\omega_\beta = p^r\omega - p^i\alpha$. Then

$$1 = \langle \omega_\beta, \beta^\vee \rangle = p^r \langle \omega, \beta^\vee \rangle - p^i \langle \alpha, \beta^\vee \rangle.$$

Observe that the right-hand side is divisible by p unless $i = 0$. This reduces the assumption to $\omega_\beta = p^r\omega - \alpha$. By checking Lemma 2.3, notice that we always have $\langle \omega, \alpha^\vee \rangle = 1$. So

$$\langle \omega_\beta, \alpha^\vee \rangle = p^r \langle \omega, \alpha^\vee \rangle - \langle \alpha, \alpha^\vee \rangle = p^r - 2.$$

As the left-hand side equals 0 or 1, we must have $r = 1$ and $p = 2$ or 3. Therefore, we have the following result.

Proposition (A). *If $r > 1$ or $p > 3$, then*

$$\mathrm{H}^1(B_r, \omega_\beta) = 0 = \mathrm{H}^1\big(G_r, H^0(\omega_\beta)\big)$$

for all fundamental dominant weights ω_β.

The reader will have already observed that there are fundamental weights ω_β of the form $p\omega - \alpha$ when $p = 2$ or 3. We proceed to precisely identify these.

Case 1. Assume first that the weight ω equals ω_α. That is, we lie in the generic case. So, we are assuming $\omega_\beta = p\omega_\alpha - \alpha$. Then we have

$$\langle \omega_\beta, \alpha^\vee \rangle = p\langle \omega_\alpha, \alpha^\vee \rangle - \langle \alpha, \alpha^\vee \rangle = p - 2. \tag{3.5.1}$$

The left-hand side equals 1 if $\beta = \alpha$ and 0 if $\beta \neq \alpha$.

Case 1.1. Suppose $\beta = \alpha$. Then the left-hand side of (3.5.1) is 1 and we must have $p = 3$. In type A_1, one does indeed have $\omega_\alpha = 3\omega_\alpha - \alpha$. On the other hand, in any other type, we cannot have $\omega_\alpha = 3\omega_\alpha - \alpha$ for there exists a simple root $\sigma \neq \alpha$ with $\langle \alpha, \sigma^\vee \rangle \neq 0$ while $\langle \omega_\alpha, \sigma^\vee \rangle = 0$.

Case 1.2. Suppose $\beta \neq \alpha$. Then the left-hand side of (3.5.1) is zero and we must have $p = 2$. So $\omega_\beta = 2\omega_\alpha - \alpha$. In other words, $\alpha = 2\omega_\alpha - \omega_\beta$. One can readily identify all simple roots which have this form:

- Type A_n, $n \geqslant 2$: $\alpha_1 = 2\omega_1 - \omega_2$ so $\omega_2 = 2\omega_1 - \alpha_1$; $\alpha_n = 2\omega_n - \omega_{n-1}$ so $\omega_{n-1} = 2\omega_n - \alpha_n$.
- Type B_n, $n \geqslant 3$: $\alpha_1 = 2\omega_1 - \omega_2$ so $\omega_2 = 2\omega_1 - \alpha_1$; $\alpha_n = 2\omega_n - \omega_{n-1}$ so $\omega_{n-1} = 2\omega_n - \alpha_n$.
- Type C_n, $n \geqslant 2$: $\alpha_1 = 2\omega_1 - \omega_2$ so $\omega_2 = 2\omega_1 - \alpha_1$.
- Type D_n, $n \geqslant 4$: $\alpha_1 = 2\omega_1 - \omega_2$ so $\omega_2 = 2\omega_1 - \alpha_1$; $\alpha_{n-1} = 2\omega_{n-1} - \omega_{n-2}$ and $\alpha_n = 2\omega_n - \omega_{n-2}$ so $\omega_{n-2} = 2\omega_{n-1} - \alpha_{n-1} = 2\omega_n - \alpha_n$.
- Type E_n, $n = 6, 7, 8$: $\alpha_1 = 2\omega_1 - \omega_3$ so $\omega_3 = 2\omega_1 - \alpha_1$; $\alpha_2 = 2\omega_2 - \omega_4$ so $\omega_4 = 2\omega_2 - \alpha_2$; $\alpha_n = 2\omega_n - \omega_{n-1}$ so $\omega_{n-1} = 2\omega_n - \alpha_n$.
- Type F_4: $\alpha_1 = 2\omega_1 - \omega_2$ so $\omega_2 = 2\omega_1 - \alpha_1$; $\alpha_4 = 2\omega_4 - \omega_3$ so $\omega_3 = 2\omega_4 - \alpha_4$.
- Type G_2: $\alpha_1 = 2\omega_1 - \omega_2$ so $\omega_2 = 2\omega_1 - \alpha_1$.

Case 2. Suppose that $\omega_\beta = p\omega - \alpha$ and ω from Lemma 2.3 has non-generic form. For the list of "exceptional" weights ω, we simply check whether $p\omega - \alpha$ is a fundamental weight:

- Type B_n, $n \geqslant 3$, $p = 2$: $2(\omega_{n-1} - \omega_n) - \alpha_{n-1} = \omega_{n-2}$.
- Type C_n, $n \geqslant 2$, $p = 2$: $2(\omega_n - \omega_{n-1}) - \alpha_n = 0$ (not fundamental).
- Type F_4, $p = 2$: $2(\omega_2 - \omega_3) - \alpha_2 = \omega_1$.
- Type G_2, $p = 2$: $2(\omega_2 - \omega_1) - \alpha_2 = \omega_1$.
- Type G_2, $p = 3$: $3(\omega_2 - \omega_1) - \alpha_2 = \omega_2$.

Thus, we have all the fundamental dominant weights with non-vanishing cohomology.

Proposition (B). *Let $p = 2$ or 3 and ω_β be a fundamental dominant weight. Then*

$$\mathrm{H}^1(B_1, \omega_\beta) = 0 = \mathrm{H}^1\big(G_1, H^0(\omega_\beta)\big).$$

Except *for the following weights*:

(a) *Assume* $p = 2$.
- *Type* A_2: $\mathrm{H}^1(G_1, H^0(\omega_1)) \cong H^0(\omega_2)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$.
- *Type* A_3: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_3)^{(1)}$.
- *Type* A_n, $n \geqslant 4$: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_{n-1})) \cong H^0(\omega_n)^{(1)}$.
- *Type* B_3: $\mathrm{H}^1(G_1, H^0(\omega_1)) \cong H^0(\omega_3)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_3)^{(1)}$.
- *Type* B_4: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_4)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_3)) \cong H^0(\omega_4)^{(1)}$.
- *Type* B_n, $n \geqslant 5$: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_{n-2})) \cong H^0(\omega_n)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_{n-1})) \cong H^0(\omega_n)^{(1)}$.
- *Type* C_n, $n \geqslant 2$: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$.
- *Type* D_4: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_3)^{(1)} \oplus H^0(\omega_4)^{(1)}$.
- *Type* D_n, $n \geqslant 5$: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_{n-2})) \cong H^0(\omega_{n-1})^{(1)} \oplus H^0(\omega_n)^{(1)}$.
- *Type* E_n, $n = 6, 7, 8$: $\mathrm{H}^1(G_1, H^0(\omega_3)) \cong H^0(\omega_1)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_4)) \cong H^0(\omega_2)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_{n-1})) \cong H^0(\omega_n)^{(1)}$.
- *Type* F_4: $\mathrm{H}^1(G_1, H^0(\omega_1)) \cong H^0(\omega_4)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_3)) \cong H^0(\omega_4)^{(1)}$.
- *Type* G_2: $\mathrm{H}^1(G_1, H^0(\omega_1)) \cong H^0(\omega_1)^{(1)}$, $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$.

(b) *Assume* $p = 3$.
- *Type* A_1: $\mathrm{H}^1(G_1, H^0(\omega_1)) \cong H^0(\omega_1)^{(1)}$.
- *Type* G_2: $\mathrm{H}^1(G_1, H^0(\omega_2)) \cong H^0(\omega_1)^{(1)}$.

4. Simple G_r-modules

4.1. The computation of the cohomology groups $\mathrm{H}^1(G_r, L(\lambda))$ for $\lambda \in X(T)_+$ is not as straightforward. One strategy would be to extend results for G_1 to higher G_r as done for induced modules. Here, we simply present an observation based on [Jan2, 4.2, 4.3] that uses the computations for induced modules. Consider the short exact sequence

$$0 \to L(\lambda) \to H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0$$

and the long exact sequence in cohomology

$$\begin{aligned} 0 \to L(\lambda)^{G_r} \to H^0(\lambda)^{G_r} \to \big(H^0(\lambda)/L(\lambda)\big)^{G_r} \to \mathrm{H}^1\big(G_r, L(\lambda)\big) \to \mathrm{H}^1\big(G_r, H^0(\lambda)\big) \\ \to \mathrm{H}^1\big(G_r, H^0(\lambda)/L(\lambda)\big) \to \cdots. \end{aligned}$$

If $\lambda, \mu \in X_r(T)$ then $\mathrm{Hom}_{G_r}(L(\mu), H^0(\lambda))$ is zero if $\lambda \neq \mu$ and k otherwise. It follows that if $\lambda \in X_r(T)$ and $\lambda \neq 0$ then $H^0(\lambda)^{G_r} = 0$, and otherwise it is k. Consequently, for any $\lambda \in X_r(T)$, there is an exact sequence

$$0 \to \big(H^0(\lambda)/L(\lambda)\big)^{G_r} \to \mathrm{H}^1\big(G_r, L(\lambda)\big) \to \mathrm{H}^1\big(G_r, H^0(\lambda)\big) \to \mathrm{H}^1\big(G_r, H^0(\lambda)/L(\lambda)\big).$$

The following is now immediate.

Proposition. *If* $\lambda \in X_r(T)$ *and* $\mathrm{H}^1(G_r, H^0(\lambda)) = 0$, *then*

$$\mathrm{H}^1\big(G_r, L(\lambda)\big) \cong \big(H^0(\lambda)\big/L(\lambda)\big)^{G_r}.$$

4.2. By combining the previous results with Corollary 3.2, we obtain an identification of $\mathrm{H}^1(G_r, L(\lambda))$ for most weights (up to an understanding of the module $H^0(\lambda)$).

Corollary. *Suppose* $\lambda \in X_r(T)$. *If* $\lambda \neq p^r\omega - p^i\alpha$ *for* $\alpha \in \Delta$, *and* $0 \leqslant i \leqslant r-1$ *where* ω *is determined by* p, r, α, *and* i *from Lemma* 2.3, *then*

$$\mathrm{H}^1\big(G_r, L(\lambda)\big) \cong \big(H^0(\lambda)/L(\lambda)\big)^{G_r}.$$

We remark that it is still an open problem in terms of what happens to $\mathrm{H}^1(G_r, L(\lambda))$ when $\mathrm{H}^1(G_r, H^0(\lambda)) \neq 0$.

5. Ext^1-formula between simple modules

5.1. Let $k[G]$ be the coordinate algebra of G. For each $\nu \in X(T)_+$, let $I(\nu)$ be the injective hull of the simple G-module $L(\nu)$. As a G-module,

$$k[G] \cong \bigoplus_{\nu\in X(T)_+} I(\nu)^{\dim_k L(\nu)}. \tag{5.1.1}$$

Here $I(\nu)^{\dim_k L(\nu)} = \bigoplus_{i=1}^m I(\nu)$ where $m = \dim_k L(\nu)$. Therefore,

$$\mathrm{ind}_{G_r}^G k \cong k[G/G_r] \cong k[G]^{(r)} \cong \bigoplus_{\nu\in X(T)_+} \big(I(\nu)^{(r)}\big)^{\dim_k L(\nu)}.$$

Now let $\lambda, \mu \in X_r(T)$. By Frobenius reciprocity and the preceding isomorphism, we have for $m \geqslant 0$:

$$\begin{aligned}\mathrm{Ext}^m_{G_r}\big(L(\lambda), L(\mu)\big) &\cong \mathrm{Ext}^m_G\big(L(\lambda), L(\mu)\otimes \mathrm{ind}_{G_r}^G k\big)\\ &\cong \bigoplus_{\nu\in X(T)_+} \mathrm{Ext}^m_G\big(L(\lambda), L(\mu)\otimes I(\nu)^{(r)}\big)^{\dim_k L(\nu)}\\ &\cong \bigoplus_{\nu\in X(T)_+} \mathrm{Ext}^m_G\big(L(\lambda), L(\mu)\otimes I(\nu)^{(r)}\big)\otimes L(\nu)^{(r)}.\end{aligned}$$

Note that the last isomorphism is in general only an isomorphism of vector spaces.

5.2. Let $\pi_s = \{\nu \in X(T)_+\colon \langle \nu, \alpha_0^\vee\rangle < s\}$ and $\mathcal{C}_s$ be the full subcategory of all G-modules whose composition factors $L(\nu)$ have highest weights lying in π_s. For $L(\nu)$ in $\mathcal{C}_s$, let $I_s(\nu)$ be the injective hull of $L(\nu)$ in the category $\mathcal{C}_s$. We remark that $\mathcal{C}_s$ is a highest weight category as defined in [CPS]. The category $\mathcal{C}_s$ is equivalent to the module category for a finite-dimensional quasi-hereditary algebra. Moreover, the injective module $I_s(\nu)$ is a finite-dimensional G-module.

Proposition. *Let $\lambda, \mu \in X_r(T)$ and p be an arbitrary prime. Then $\mathrm{Ext}^m_{G_r}(L(\lambda), L(\mu))^{(-r)}$ is a G-module in $\mathcal{C}_{s(m)}$ where*

$$s(m) = \begin{cases} 1 & \text{if } m = 0, \\ h & \text{if } m = 1, \\ (m-1)(2h-3)+3 & \text{if } m > 1. \end{cases}$$

Proof. For $m = 0$ the statement is clear. The proof for the case $m = 1$ is inspired from the ideas in [And2, Lemma 2.3]. Set $\lambda^* = -w_0\lambda$. Consider the short exact sequence

$$0 \to L(\lambda^*) \otimes L(\mu) \to H^0(\lambda^*) \otimes H^0(\mu) \to N \to 0.$$

Since $\lambda, \mu \in X_r(T)$, we have $\mathrm{Hom}_{G_r}(L(\lambda), L(\mu)) \cong \mathrm{Hom}_{G_r}(V(\lambda), H^0(\mu))$. Therefore, from the short exact sequence above and the associated long exact sequence in cohomology, we obtain the following exact sequence

$$0 \to N^{G_r} \to \mathrm{Ext}^1_{G_r}\big(L(\lambda), L(\mu)\big) \to \mathrm{H}^1\big(G_r, H^0(\lambda^*) \otimes H^0(\mu)\big).$$

We first show that if ν is a weight of $\mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu))^{(-r)}$ then

$$p^r\langle \nu, \alpha_0^\vee\rangle \leqslant \langle \lambda^* + \mu, \alpha_0^\vee\rangle + 3p^{r-1}. \tag{5.2.1}$$

All the weights of N are less than $\lambda^* + \mu$ so (5.2.1) is true for the weights of $(N^{G_r})^{(-r)}$. Consequently it suffices to prove that (5.2.1) holds for all weights of $\mathrm{H}^1(G_r, H^0(\lambda^*) \otimes H^0(\mu))^{(-r)}$. Let $L(\sigma) = L(\sigma_0) \otimes L(\sigma_1)^{(r)}$ be a composition factor of $H^0(\lambda^*) \otimes H^0(\mu)$ where $\sigma_0 \in X_r(T)$. Then

$$\mathrm{H}^1\big(G_r, L(\sigma)\big) \cong \mathrm{H}^1\big(G_r, L(\sigma_0)\big) \otimes L(\sigma_1)^{(r)}.$$

Consider the short exact sequence

$$0 \to L(\sigma_0) \to H^0(\sigma_0) \to Q \to 0.$$

As above, this exact sequence induces an exact sequence of the form

$$0 \to Q^{G_r} \to H^1\big(G_r, L(\sigma_0)\big) \to \mathrm{H}^1\big(G_r, H^0(\sigma_0)\big).$$

Let μ be a weight of $\mathrm{H}^1(G_r, H^0(\sigma_0))^{(-r)}$. Then one can directly verify using Theorem 3.1(A–C) that

$$p^r\langle \mu, \alpha_0^\vee\rangle \leqslant \langle \sigma_0, \alpha_0^\vee\rangle + 3p^{r-1}. \tag{5.2.2}$$

Indeed, in the generic case, $\mathrm{H}^1(G_r, H^0(\sigma_0))$ is non-zero only when $\sigma_0 = p^r\omega_\alpha - p^i\alpha$ for a simple root α with $i < r$ in which case $\mathrm{H}^1(G_r, H^0(\sigma_0))^{(-r)} \cong H^0(\omega_\alpha)$. Since $p^r\omega_\alpha = \sigma_0 + p^i\alpha$, (5.2.2) readily holds. For $p > 3$ only the generic case occurs. For $p = 2, 3$, verification of the non-generic cases is left to the interested reader. Since every weight of Q^{G_r} is less than $p^r\sigma_0$, it follows that (5.2.1) holds for all weights of $\mathrm{H}^1(G_r, L(\sigma_0))^{(-r)}$.

If ν is a weight of $\mathrm{H}^1(G_r, H^0(\lambda^*) \otimes H^0(\mu))^{(-r)}$, then $p^r\nu \leqslant p^r\mu + p^r\sigma_1$ where $L(\sigma_0) \otimes L(\sigma_1)^{(r)}$ is a composition factor of $H^0(\lambda^*) \otimes H^0(\mu))$ and μ is a weight of $\mathrm{H}^1(G_r, L^0(\sigma_0))^{(-r)}$. Using (5.2.2), it follows that

$$\begin{aligned} p^r\langle\nu, \alpha_0^\vee\rangle &\leqslant p^r\langle\mu, \alpha_0^\vee\rangle + p^r\langle\sigma_1, \alpha_0^\vee\rangle \leqslant \langle\sigma_0, \alpha_0^\vee\rangle + 3p^{r-1} + p^r\langle\sigma_1, \alpha_0^\vee\rangle \\ &\leqslant \langle\lambda^* + \mu, \alpha_0^\vee\rangle + 3p^{r-1}. \end{aligned}$$

This verifies (5.2.1).

Consider the short exact sequence of G-modules

$$0 \to L(\mu) \to \mathrm{St}_r \otimes L(\hat{\mu}) \to R \to 0$$

where $\hat{\mu} = (p^r - 1)\rho - \mu^*$. By applying the long exact sequence in cohomology along with the projectivity of St_r over G_r, we see that for $m \geqslant 2$

$$\mathrm{Ext}^m_{G_r}\big(L(\lambda), L(\mu)\big) \cong \mathrm{Ext}^{m-1}_{G_r}\big(L(\lambda), R\big). \tag{5.2.3}$$

In fact, equation (5.2.3) also holds for $m = 1$. Since $\mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G_r}(L(\mu), L(\lambda))$, we may assume without loss of generality that $\lambda \not> \mu$. Then $\mathrm{Hom}_{G_r}(L(\lambda), \mathrm{St}_r \otimes L(\hat{\mu}))$ is trivial unless $\lambda = \mu$ in which case it is k. Hence the first map in the long exact sequence $\mathrm{Hom}_{G_r}(L(\lambda), L(\mu)) \to \mathrm{Hom}_{G_r}(L(\lambda), \mathrm{St}_r \otimes L(\hat{\mu}))$ is an isomorphism and (5.2.3) also holds for $m = 1$.

The highest weight of R is less than $2(p^r - 1)\rho - \mu^*$. Thus, any weight ν of $\mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu))^{(-r)} \cong \mathrm{Hom}_{G_r}(L(\lambda), R)^{(-r)}$ must satisfy

$$p^r\nu \leqslant 2\big(p^r - 1\big)\rho - \mu^* - \lambda.$$

Applying the inner product with $\alpha_0^\vee$, we get

$$p^r\langle\nu, \alpha_0^\vee\rangle \leqslant 2\big(p^r - 1\big)(h - 1) - \langle\mu^* + \lambda, \alpha_0^\vee\rangle. \tag{5.2.4}$$

Observe that for any weight ξ, $\langle\xi^*, \alpha_0^\vee\rangle = \langle\xi, \alpha_0^\vee\rangle$. Thus, combining (5.2.1) and (5.2.4) together yields

$$2p^r\langle\nu, \alpha_0^\vee\rangle \leqslant 2\big(p^r - 1\big)(h - 1) + 3p^{r-1}.$$

This implies that

$$\langle\nu, \alpha_0^\vee\rangle \leqslant (h - 1) + \frac{3}{2p} < h,$$

as required.

Next, we argue the case $m = 2$. By (5.2.3) we have

$$\operatorname{Ext}^2_{G_r}\big(L(\lambda), L(\mu)\big)^{(-r)} \cong \operatorname{Ext}^1_{G_r}\big(L(\lambda), R\big)^{(-r)}.$$

Let $L(\sigma) \cong L(\sigma_0) \otimes L(\sigma_1)^{(r)}$ be a composition factor of R. Then $p^r\sigma_1 \leqslant 2(p^r - 1)\rho - \mu^* - \sigma_0$ and one obtains

$$p^r\langle\sigma_1, \alpha_0^\vee\rangle \leqslant 2\big(p^r - 1\big)(h - 1) - \langle\mu^* + \sigma_0, \alpha_0^\vee\rangle.$$

It follows from (5.2.1) that any weight of ν of $\operatorname{Ext}^1_{G_r}(L(\lambda), L(\sigma_0))^{(-r)}$ satisfies

$$p^r\langle\nu, \alpha_0^\vee\rangle \leqslant \langle\lambda^* + \sigma_0, \alpha_0^\vee\rangle + 3p^{r-1}.$$

Thus, any weight ν of

$$\operatorname{Ext}^1_{G_r}\big(L(\lambda), L(\sigma)\big)^{(-r)} \cong \operatorname{Ext}^1_{G_r}\big(L(\lambda), L(\sigma_0)\big)^{(-r)} \otimes L(\sigma_1)$$

satisfies

$$p^r\langle\nu, \alpha_0^\vee\rangle \leqslant \langle\lambda^* + \sigma_0, \alpha_0^\vee\rangle + 3p^{r-1} + 2\big(p^r - 1\big)(h - 1) - \langle\mu^* + \sigma_0, \alpha_0^\vee\rangle. \quad (5.2.5)$$

As noted above, $\langle\xi^*, \alpha_0^\vee\rangle = \langle\xi, \alpha_0^\vee\rangle$ for any weight ξ. Further, we have assumed without loss of generality that $\langle\lambda, \alpha_0^\vee\rangle \leqslant \langle\mu, \alpha_0^\vee\rangle$. Thus, (5.2.5) yields that any weight ν of $\operatorname{Ext}^1_{G_r}(L(\lambda), R)^{(-r)}$ satisfies

$$p^r\langle\nu, \alpha_0^\vee\rangle \leqslant 3p^{r-1} + 2\big(p^r - 1\big)(h - 1),$$

which implies

$$\langle\nu, \alpha_0^\vee\rangle < 2h.$$

Finally, we apply (5.2.3) to the case $m > 2$. The highest weight of R is less than $2(p^r - 1)\rho$. Thus, any weight $\sigma = \sigma_0 + p^r\sigma_1$ of R must satisfy $\langle\sigma, \alpha_0^\vee\rangle \leqslant 2(p^r - 1)(h - 1)$, which implies that $\langle\sigma_1, \alpha_0^\vee\rangle \leqslant 2(h - 1) - 1 = 2h - 3$. Let $L(\sigma)$ be a composition factor of R and $m > 1$. Then, as G-modules,

$$\operatorname{Ext}^{m-1}_{G_r}\big(L(\lambda), L(\sigma)\big) \cong \operatorname{Ext}^{m-1}_{G_r}\big(L(\lambda), L(\sigma_0)\big) \otimes L(\sigma_1)^{(r)}.$$

Inductively we conclude that any weight ν of

$$\operatorname{Ext}^m_{G_r}\big(L(\lambda), L(\mu)\big)^{(-r)} \cong \operatorname{Ext}^{m-1}_{G_r}\big(L(\lambda), R\big)^{(-r)}$$

must satisfy

$$\langle\nu, \alpha_0^\vee\rangle < s(m - 1) + 2h - 3. \qquad \square$$

5.3. The previous proposition can be refined in the case when $p > h$ by using the work in [KLT].

Proposition. *Let $\lambda, \mu \in X_r(T)$ and $p > h$. Then $\operatorname{Ext}^m_{G_r}(L(\lambda), L(\mu))^{(-r)}$ is a G-module in $\mathcal{C}_{s(m)}$ where*

$$s(m) = \begin{cases} 1 & \text{if } m = 0, \\ h & \text{if } m = 1, \\ 2(h-1) + m\kappa & \text{if } m > 1, \end{cases}$$

where $\kappa = 3/2$ if G is of type G_2 and $\kappa = 1$ otherwise.

Proof. We will first prove the following.

Step 1. Let $\sigma \in X(T)_+$, $p > h$, and $m \geqslant 0$, then any weight $p\nu$ of $\mathrm{H}^m(G_1, H^0(\sigma))$ satisfies

$$p\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p(m+1)\kappa,$$

where $\kappa = 3/2$ if G is of type G_2 and $\kappa = 1$ otherwise.

We use [KLT, Theorem 8] which says that

$$\mathrm{H}^m\big(G_1, H^0(w \cdot 0 + p\lambda)\big) \cong \begin{cases} \operatorname{ind}_B^G\big(S^{(m-l(w))/2}\mathfrak{u}^* \otimes \lambda\big)^{(1)} & \text{if } m = l(w) \bmod 2, \\ 0 & \text{else.} \end{cases}$$

Here $\mathfrak{u} = \operatorname{Lie} U$. The weights of the ith symmetric powers $S^i\mathfrak{u}^*$ are just sums of i positive roots. Therefore, any weight γ of $S^i\mathfrak{u}^*$ satisfies

$$\langle \gamma, \alpha_0^\vee \rangle \leqslant i \cdot \max\{\langle \beta, \alpha_0^\vee \rangle \colon \beta \in \Phi^+\} \leqslant 2i\kappa.$$

Now let $\sigma = w \cdot 0 + p\lambda$ where $w \in W$ and $\lambda \in X(T)_+$.

If $l(w) = 0$ then $\sigma = p\lambda$ and it follows that any weight $p\nu$ of $\mathrm{H}^m(G_1, H^0(\sigma))$ satisfies $p\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + pm\kappa$.

If $l(w) > 0$ then $p\lambda \leqslant \sigma + 2\rho$ and it follows that any weight ν of $\mathrm{H}^m(G_1, H^0(\sigma))$ satisfies $p\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + 2(h-1) + p(m-1)\kappa < \langle \sigma, \alpha_0^\vee \rangle + p(m+1)\kappa$. This completes the proof of Step 1.

Consider the short exact sequence

$$0 \to L(\sigma) \to H^0(\sigma) \to Q \to 0,$$

which induces exact sequences of the form

$$\mathrm{H}^{m-1}(G_1, Q) \to \mathrm{H}^m\big(G_1, L(\sigma)\big) \to \mathrm{H}^m\big(G_1, H^0(\sigma)\big).$$

Any highest weight of Q is strictly less than σ and by using induction on m, we get the following.

Step 2. For $\sigma \in X(T)_+$, $p > h$ and $m \geqslant 0$, any weight $p\nu$ of $\mathrm{H}^m(G_1, L(\sigma))$ satisfies

$$p\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p(m+1)\kappa.$$

Next we will use induction on r to show the following.

Step 3. For $\sigma \in X(T)_+$, $p > h$ and $m \geqslant 1$, any weight $p^r\nu$ of $\mathrm{H}^m(G_r, L(\sigma))$ satisfies

$$p^r\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^r(m+1)\kappa.$$

For $r = 1$ the statement was proved in Step 2. Assume that $r > 1$. We will use the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{i,j} = \mathrm{H}^i\big(G_r/G_{r-1}, \mathrm{H}^j\big(G_{r-1}, L(\sigma)\big)\big) \quad \Rightarrow \quad \mathrm{H}^{i+j}\big(G_r, L(\sigma)\big).$$

The differentials in the spectral sequence are G-equivariant. Moreover, the cohomology is a subquotient of $\mathrm{E}_2^{i,j}$ where $i + j = m$. Therefore, any composition factor of $\mathrm{H}^m(G_r, L(\sigma))$ must be a composition factor of some $\mathrm{H}^i(G_r/G_{r-1}, \mathrm{H}^j(G_{r-1}, L(\sigma)))$ with $i + j = m$. It is sufficient to show that any weight $p^r\nu$ of $\mathrm{H}^i(G_r/G_{r-1}, \mathrm{H}^j(G_{r-1}, L(\sigma)))$ with $i + j = m$ satisfies $p^r\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^r(m+1)\kappa$.

We first discuss the case $j = 0$. $\mathrm{Hom}_{G_{r-1}}(k, L(\sigma))$ has a composition series with simple modules $L(\gamma)^{(r-1)}$. Clearly, the highest weights γ of each factor satisfy

$$p^{r-1}\langle \gamma, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle. \tag{5.3.1}$$

The composition factors of $\mathrm{H}^m(G_r/G_{r-1}, \mathrm{Hom}_{G_{r-1}}(k, L(\sigma)))$ are subquotients of some $\mathrm{H}^m(G_r/G_{r-1}, L(\gamma)^{(r-1)})$. By Step 2 any weight $p\nu$ of $\mathrm{H}^m(G_r/G_{r-1}, L(\gamma)^{(r-1)})^{(-r+1)} \cong \mathrm{H}^m(G_1, L(\gamma))$ satisfies

$$p\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \gamma, \alpha_0^\vee \rangle + p(m+1)\kappa.$$

Multiplying the inequality with p^{r-1} and using (5.3.1) yields

$$p^r\langle \nu, \alpha_0^\vee \rangle \leqslant p^{r-1}\langle \gamma, \alpha_0^\vee \rangle + p^r(m+1)\kappa \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^r(m+1)\kappa.$$

We conclude that any weight $p^r\nu$ of $\mathrm{H}^m(G_r/G_{r-1}, \mathrm{Hom}_{G_{r-1}}(k, L(\sigma)))$ satisfies

$$p^r\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^r(m+1)\kappa.$$

Next assume that $j > 0$. By the induction hypothesis we may assume that any weight $p^{r-1}\gamma$ of $\mathrm{H}^j(G_{r-1}, L(\sigma))$ satisfies

$$p^{r-1}\langle \gamma, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^{r-1}(j+1)\kappa. \tag{5.3.2}$$

$\mathrm{H}^j(G_{r-1}, L(\sigma))$ has a composition series with simple factors $L(\gamma)^{(r-1)}$. The composition factors of $\mathrm{H}^i(G_r/G_{r-1}, \mathrm{H}^j(G_{r-1}, L(\sigma)))$ are subquotients of some $\mathrm{H}^i(G_r/G_{r-1}, L(\gamma)^{(r-1)})$. By Step 2, any weight $p\nu$ of $\mathrm{H}^i(G_r/G_{r-1}, L(\gamma)^{(r-1)})^{(-r+1)} \cong \mathrm{H}^i(G_1, L(\gamma))$ satisfies

$$p\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \gamma, \alpha_0^\vee \rangle + p(i+1)\kappa.$$

Multiplying the inequality with p^{r-1} and using (5.3.2) yields

$$p^r\langle \nu, \alpha_0^\vee \rangle \leqslant p^{r-1}\langle \gamma, \alpha_0^\vee \rangle + p^r(i+1)\kappa \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^{r-1}(j+1)\kappa + p^r(i+1)\kappa.$$

Now $p^{r-1}(j+1) \leqslant p^{r-1}2j \leqslant p^r j$. Therefore,

$$p^r\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^r(j+i+1)\kappa = \langle \sigma, \alpha_0^\vee \rangle + p^r(m+1)\kappa.$$

We conclude that any weight $p^r\nu$ of $\mathrm{H}^i(G_r/G_{r-1}, \mathrm{H}^j(G_{r-1}, L(\sigma)))$ satisfies

$$p^r\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^r(m+1)\kappa.$$

Finally, we now prove our claim concerning the size of the weights in $\mathrm{Ext}^m_{G_r}(L(\lambda), L(\mu))$. Notice that the statement for $m \leqslant 1$ follows from Proposition 5.2. We assume that $m > 1$. One has the following sequence of isomorphisms:

$$\begin{aligned}\mathrm{H}^m\big(G_r, L(\lambda^*) \otimes L(\mu)\big) &\cong \mathrm{Ext}^m_{G_r}\big(L(\lambda), L(\mu)\big) \cong \mathrm{Ext}^m_{G_r}\big(L(\mu), L(\lambda)\big)\\ &\cong \mathrm{H}^m\big(G_r, L(\mu^*) \otimes L(\lambda)\big).\end{aligned}$$

Without a loss of generality we may assume that $\langle \lambda, \alpha_0^\vee \rangle \leqslant \langle \mu, \alpha_0^\vee \rangle$.

Consider the short exact sequence of G-modules

$$0 \to L(\lambda^*) \otimes L(\mu) \to L(\lambda^*) \otimes \mathrm{St}_r \otimes L(\hat{\mu}) \to R \to 0,$$

where $\hat{\mu} = (p^r - 1)\rho - \mu^*$. By applying the long exact sequence in cohomology along with the projectivity of St_r over G_r, we see that

$$\mathrm{Ext}^m_{G_r}\big(L(\lambda), L(\mu)\big) \cong \mathrm{H}^m\big(G_r, L(\lambda^*) \otimes L(\mu)\big) \cong \mathrm{H}^{m-1}(G_r, R).$$

Notice that for any composition factor $L(\sigma)$ of R, $\sigma < (p^r-1)\rho + \lambda^* + \hat{\mu} = (p^r-1)\rho + \lambda^* + (p^r-1)\rho - \mu^*$ and so $\sigma < 2(p^r-1)\rho + \lambda^* - \mu^* \leqslant 2(p^r-1)\rho$. Hence, $\langle \sigma, \alpha_0^\vee \rangle \leqslant 2(p^r-1)(h-1)$. It follows that any highest weight $p^r\nu$ of $\mathrm{H}^{m-1}(G_r, R)$ satisfies

$$p^r\langle \nu, \alpha_0^\vee \rangle \leqslant \langle \sigma, \alpha_0^\vee \rangle + p^r m\kappa \leqslant 2\big(p^r - 1\big)(h-1) + p^r m\kappa.$$

Dividing by p^r yields

$$\langle \nu, \alpha_0^\vee \rangle < 2(h-1) + m\kappa.$$

The assertion follows. □

5.4. We can now prove the following formula which relates extensions between simple modules in G_r with certain G-modules. Note that the isomorphism is in general only an isomorphism of vector spaces not necessarily of G-modules.

Theorem. *Let $\lambda, \mu \in X_r(T)$ and p be an arbitrary prime. Then for $m \geqslant 0$,*

$$\mathrm{Ext}^m_{G_r}\big(L(\lambda), L(\mu)\big) \cong \bigoplus_{\nu \in \pi_{s(m)}} \mathrm{Ext}^m_G\big(L(\lambda), I_{s(m)}(\nu)^{(r)} \otimes L(\mu)\big) \otimes L(\nu)^{(r)},$$

where

$$s(m) = \begin{cases} 1 & \text{if } m = 0,\\ h & \text{if } m = 1,\\ 2(h-1)+m & \text{if } m > 1,\ p > h \text{ and } G \text{ is not of type } G_2,\\ 2(h-1)+3/2m & \text{if } m > 1,\ p > h \text{ and } G \text{ is of type } G_2,\\ (m-1)(2h-3)+3 & \text{otherwise.} \end{cases}$$

Proof. Let N be a G-module. First consider the Lyndon–Hochschild–Serre spectral sequence:

$$E_2^{i,j} = \mathrm{Ext}^i_{G/G_r}\big(k, \mathrm{Ext}^j_{G_r}\big(L(\lambda), L(\mu)\big) \otimes N^{(r)}\big) \quad \Rightarrow \quad \mathrm{Ext}^{i+j}_G\big(L(\lambda), L(\mu) \otimes N^{(r)}\big).$$

For $i > 0$ and $0 \leqslant j \leqslant m$, let us look at

$$E_2^{i,j} = \mathrm{Ext}^i_{G/G_r}\big(k, \mathrm{Ext}^j_{G_r}\big(L(\lambda), L(\mu)\big) \otimes N^{(r)}\big).$$

When $N = I(\nu)$ we have $E_2^{i,j} = 0$ for $i > 0$ because $N^{(r)}$ is an injective G/G_r-module. On the other hand, if $N = I_{s(m)}(\nu)$ then $E_2^{i,j} = 0$ for $i > 0$ and $0 \leqslant j \leqslant m$ because $M^{(-r)} \equiv (\mathrm{Ext}^j_{G_r}(L(\lambda), L(\mu))^*)^{(-r)}$ is a G-module in $\mathcal{C}_{s(m)}$ (by Propositions 5.2 and 5.3) and N is injective in $\mathcal{C}_{s(m)}$. It follows that if $N = I(\nu)$ or $I_{s(m)}(\nu)$ then $E_2^{0,m} \cong E^m$. Consequently, if $\nu \notin \pi_{s(m)}$, then

$$\mathrm{Ext}^m_G\big(L(\lambda), L(\mu) \otimes I(\nu)^{(r)}\big) \cong \mathrm{Hom}_G\big(M^{(-r)}, I(\nu)\big) = 0$$

since the G-socle of $I(\nu)$ is $L(\nu)$ and $M^{(-r)}$ lies in $C_{s(m)}$. On the other hand, for $\nu \in \pi_{s(m)}$, we have

$$\begin{aligned}\mathrm{Ext}^m_G\big(L(\lambda), L(\mu) \otimes I(\nu)^{(r)}\big) &\cong \mathrm{Hom}_{G/G_r}\big(M, I(\nu)^{(r)}\big) \cong \mathrm{Hom}_{G/G_r}\big(M, I_{s(m)}(\nu)^{(r)}\big)\\ &\cong \mathrm{Ext}^m_G\big(L(\lambda), L(\mu) \otimes I_{s(m)}(\nu)^{(r)}\big).\end{aligned}$$

Note that the second isomorphism follows by Propositions 5.2 and 5.3. The statement of the theorem now follows by the isomorphisms given in Section 5.1. □

5.5. By specializing to the case when $m = 1$, we have the following corollary.

Corollary (A). *Let $\lambda, \mu \in X_r(T)$ and p be an arbitrary prime. Then*

$$\operatorname{Ext}^1_{G_r}\bigl(L(\lambda), L(\mu)\bigr) \cong \bigoplus_{\nu\in\pi_h} \operatorname{Ext}^1_G\bigl(L(\lambda), I_h(\nu)^{(r)} \otimes L(\mu)\bigr) \otimes L(\nu)^{(r)}.$$

Corollary (A) takes on even a nicer formulation when $p \geqslant 2(h-1)$.

Corollary (B). *Let $\lambda, \mu \in X_r(T)$ and $p \geqslant 2(h-1)$. Then $\operatorname{Ext}^1_{G_r}(L(\lambda), L(\mu))$ is a semi-simple G-module and*

$$\operatorname{Ext}^1_{G_r}\bigl(L(\lambda), L(\mu)\bigr) \cong \bigoplus_{\nu\in\pi_h} \operatorname{Ext}^1_G\bigl(L(\lambda), L(\nu)^{(r)} \otimes L(\mu)\bigr) \otimes L(\nu)^{(r)}.$$

For higher cohomologies we get the following.

Corollary (C). *Let $\lambda, \mu \in X_r(T)$, $m \geqslant 2$, and $p \geqslant 3(h-1) + m\kappa - 1$, where $\kappa = 3/2$ if G is of type G_2 and $\kappa = 1$ otherwise. Then $\operatorname{Ext}^m_{G_r}(L(\lambda), L(\mu))$ is a semisimple G-module and*

$$\operatorname{Ext}^m_{G_r}\bigl(L(\lambda), L(\mu)\bigr) \cong \bigoplus_{\nu\in\pi_{s(m)}} \operatorname{Ext}^m_G\bigl(L(\lambda), L(\nu)^{(r)} \otimes L(\mu)\bigr) \otimes L(\nu)^{(r)},$$

where $s(m) = 2(h-1) + m\kappa$.

The following proves both Corollaries (B) and (C).

Proof. For $m = 1$, set $s(m) = h$. Let $\overline{C}_{\mathbb{Z}} = \{\lambda \in X(T)\colon\ 0 \leqslant \langle\lambda + \rho, \alpha_0^\vee\rangle \leqslant p\}$ denote the closure of the "bottom alcove" under the action of the affine Weyl group. By the Strong Linkage Principle, if $\sigma_1, \sigma_2 \in \overline{C}_{\mathbb{Z}}$, then $\operatorname{Ext}^1_G(L(\sigma_1), L(\sigma_2)) = 0$. Let ν be such that $\langle\nu, \alpha_0^\vee\rangle < s(m)$. If $p \geqslant s(m) + (h-2)$, then

$$\bigl\langle\nu + \rho, \alpha_0^\vee\bigr\rangle < s(m) + h - 1 \leqslant p + 1,$$

which implies that $\nu \in \overline{C}_{\mathbb{Z}}$. Consequently, the category $\mathcal{C}_{s(m)}$ is semisimple and $I_{s(m)}(\nu) = L(\nu)$ for all $\nu \in \mathcal{C}_{s(m)}$. The result now holds by Theorem 5.4. □

As noted, the isomorphism in Theorem 5.4 is only an isomorphism of vector spaces. However, one obtains the composition factors of $\operatorname{Ext}^m_{G_r}(L(\lambda), L(\mu))$ via

$$\bigl[\operatorname{Ext}^m_{G_r}\bigl(L(\lambda), L(\mu)\bigr) : L(\nu)^{(r)}\bigr]_G = \dim \operatorname{Ext}^m_G\bigl(L(\lambda), I_{s(m)}(\nu)^{(r)} \otimes L(\mu)\bigr).$$

For $p \geqslant s(m) + (h-2)$ the category $\mathcal{C}_{s(m)}$ is semisimple. Therefore, the isomorphism in Corollaries (B) and (C) is actually an isomorphism of G-modules.

The preceding result improves results by the authors in [BNP1] and sharpens results by Andersen [And1] who proved this for $m = 1$ and $p \geqslant 3(h-1)$.

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