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Decomposability and structure of nonnegative bands in $\mathcal{M}_n(\mathbb{R})$

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Abstract

A standard subspace of $\mathbb{R}^n$ is a space spanned by a subset of the standard basis $\{e_1, e_2, \ldots, e_n\}$. A multiplicative semigroup $\mathcal{S}$ in $\mathcal{M}_n(\mathbb{R})$ is said to be decomposable if its members have a common nontrivial standard invariant subspace. Necessary and sufficient conditions for decomposability of nonnegative semigroups are given. In particular, decomposability of nonnegative bands (semigroups of idempotents) and their structure is discussed. It is proved that a nonnegative band with each member having rank greater than 1 is decomposable. Also, a geometric characterization of maximal, rank-one nonnegative bands is given. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

In what follows, $\mathcal{M}_n(\mathbb{R})$ will denote the space of all $n \times n$ matrices with entries from the field of real numbers. A matrix $A = (a_{ij})$ in $\mathcal{M}_n(\mathbb{R})$ is called nonnegative (resp. positive) if $a_{ij} \geqslant 0$ (resp. $a_{ij} > 0$) for $i, j = 1, 2, \ldots, n$. A vector $x = (x_i)$ in $\mathbb{R}^n$ is called nonnegative (resp. positive) if $x_i \geqslant 0$ (resp. $x_i > 0$) for all $i = 1, 2, \ldots, n$. A nonnegative semigroup in $\mathcal{M}_n(\mathbb{R})$ is a semigroup with nonnegative matrices. A matrix A in $\mathcal{M}_n(\mathbb{R})$ is said to be decomposable if there exists a proper subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ such that

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$$\bigvee\{Ae_{i_1}, Ae_{i_2}, \ldots, Ae_{i_k}\} \subseteq \bigvee\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}.$$

where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$. (For any set of vectors $\{v_1, v_2, \ldots, v_n\}$, $\bigvee\{v_1, v_2, \ldots, v_n\}$ denotes the linear span of the vectors $\{v_1, v_2, \ldots, v_n\}$).

The definition above of decomposability of a single matrix is extended in the obvious manner to a semigroup in $\mathcal{M}_n(\mathbb{R})$.

Definition 1.1. A band in $\mathcal{M}_n(\mathbb{R})$ is a multiplicative semigroup of idempotents i.e., matrices E such that $E = E^2$.

General bands of matrices have been the subject of study in recent years, and their structure seems to be quite complicated. See, for example, Refs. [2,3]. In this paper, we will be exclusively concerned with nonnegative semigroups, and in particular, nonnegative bands and conditions leading to their decomposability. Observe that decomposability of a semigroup implies reducibility i.e., the existence of a common nontrivial invariant subspace but the converse may not be true. A simple example to illustrate this point is the nonnegative semigroup (in fact, a band)

$$\left\{\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}$$

It is known [illegible]at if $\mathcal{S}$ is a semigroup in $\mathcal{M}_n(\mathbb{R})$ and f a nonzero functional on $\mathcal{M}_n(\mathbb{R})$ such t[illegible] the restriction of f to $\mathcal{S}$ is zero, then $\mathcal{S}$ is reducible. The proof of th[illegible]s an easy consequence of Burnside's theorem (cf. [6]). We will give an [illegible]e of this result which would imply decomposability of nonnegative semi[illegible]ps in $\mathcal{M}_n(\mathbb{R})$.

H. Radjavi pr[illegible] in Ref. [5] that a band in $\mathcal{M}_n(\mathbb{R})$ is reducible (in fact, simultaneously tri[illegible]arizable). In this paper, we will consider nonnegative bands in $\mathcal{M}_n(\mathbb{R})$ [illegible]ve that if in such a band, every member has rank greater than 1, it i[illegible]posable. Furthermore, the structure of such bands is described. Sectio[illegible]s a geometric characterization of maximal, nonnegative, indecomposab[illegible]ank-one bands.

2. Decomposability of nonnegative semigroups

We start with a simple proposition which will be used throughout the sequel as an equivalent form of defining decomposability.

Proposition 2.1. *An $n \times n$ matrix $A = (a_{ij})$ is decomposable if and only if there exists a permutation matrix P such that*

$$P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B and D are square matrices.

Proof. The proof is easy and therefore omitted. □

It follows naturally from the proposition above that a semigroup in $\mathcal{M}_n(\mathbb{R})$ is decomposable if and only if there exists a permutation matrix P in $\mathcal{M}_n(\mathbb{R})$ such that

$$P^{-1}SP = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \quad \text{for all } S \in \mathcal{S},$$

where S_{11} and S_{22} are square matrices of fixed sizes r and $n-r$, respectively.

The following lemma gives a necessary and sufficient condition for decomposability of nonnegative semigroups in $\mathcal{M}_n(\mathbb{R})$ which will be used repeatedly.

Lemma 2.2. *If a nonnegative semigroup $\mathcal{S}$ in $\mathcal{M}_n(\mathbb{R})$ has a common zero entry, that is, if for some fixed i and j, the (i, j) entry of every member of $\mathcal{S}$ is zero, then $\mathcal{S}$ is decomposable.*

Proof. Referring to Proposition III, 8.3 in Ref. [7], it is clear that the standard subspace generated by the set $\{Se_j : S \in \mathcal{S}\}$ is invariant under $\mathcal{S}$, which gives the decomposability of $\mathcal{S}$. □

Definition 2.3. A subset $\mathcal{J}$ of a semigroup $\mathcal{S}$ is called an ideal if JS and SJ belong to $\mathcal{J}$ for all J ∈ $\mathcal{J}$ and for all $S \in \mathcal{S}$.

It is a well known result that a nonzero ideal of an irreducible semigroup is irreducible (cf. [6]). We prove its counterpart for indecomposable semigroups of $n \times n$ matrices with nonnegative entries.

Lemma 2.4. *If $\mathcal{S}$ is an indecomposable semigroup of $n \times n$ nonnegative matrices, then so is every nonzero ideal of $\mathcal{S}$.*

Proof. Let $\mathcal{J}$ be a nonzero ideal of $\mathcal{S}$. If M is a nontrivial invariant subspace of $\mathcal{J}$, then the standard subspaces generated by the sets $\{JM : J \in \mathcal{J}\}$ and $\{x \in \mathbb{R}^n : J|x| = 0, J \in \mathcal{J}\}$ are both invariant under $\mathcal{S}$. Since $\mathcal{J}$ is nonzero ideal, at least one of them is nontrivial. □

Definition 2.5. By a nonnegative (resp. positive) linear functional f on $\mathbb{R}^n$, we mean a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}$ satisfying $f(x) \geqslant 0$ (resp. $f(x) > 0$) whenever $x \geqslant 0$ (resp. $0 \neq x \geqslant 0$) in $\mathbb{R}^n$.

The next lemma is a fundamental result which we state without proof.

Lemma 2.6. *Let f be a nonnegative linear functional on $\mathcal{M}_n(\mathbb{R})$. Then there exists a nonnegative matrix B in $\mathcal{M}_n(\mathbb{R})$ such that $f(A) = \operatorname{tr}(BA)$ for all $A \in \mathcal{M}_n(\mathbb{R})$.*

Proposition 2.7. *Let $\mathcal{S}$ be a semigroup in $\mathcal{M}_n(\mathbb{R})$ with nonnegative matrices and f a nonzero, nonnegative linear functional on $\mathcal{M}_n(\mathbb{R})$ whose restriction to $\mathcal{S}$ is zero. Then $\mathcal{S}$ has a common zero entry which makes it decomposable.*

Proof. By Lemma 2.6, there exists a nonnegative matrix B such that

$$f(A) = \operatorname{tr}(BA) \quad \text{for all } A \in \mathcal{M}_n(\mathbb{R}).$$

By our assumption, $\operatorname{tr}(BA) = 0$ for all $A \in \mathcal{S}$. Also f nonzero implies that B is nonzero. Suppose b_{ij} is a nonzero entry in B. Since the entries in BA are nonnegative and $\operatorname{tr}(BA) = 0$ for all $A \in \mathcal{S}$, all the diagonal entries of BA are zero for each $A \in \mathcal{S}$; in particular, the (i,i) entry is zero. Thus

$$b_{i1}a_{1i} + b_{i2}a_{2i} + \cdots + b_{ij}a_{ji} + \cdots + b_{in}a_{ni} = 0.$$

Each summand in the above sum being zero, we have

$$b_{ij}a_{ji} = 0 \Rightarrow a_{ji} = 0 \quad \text{as } b_{ij} \neq 0.$$

This shows that if the (i,j) entry of B is nonzero, then the (j,i) entry of each A in $\mathcal{S}$ is zero. Hence by Lemma 2.2, $\mathcal{S}$ is decomposable. □

We now list a few equivalent conditions for decomposability of nonnegative semigroups in $\mathcal{M}_n(\mathbb{R})$.

Theorem 2.8. *For a semigroup $\mathcal{S}$ in $\mathcal{M}_n(\mathbb{R})$ with nonnegative matrices, the following are equivalent:*

(i) *$\mathcal{S}$ is decomposable.*

(ii) *There exists a nonzero, nonnegative functional on $\mathcal{M}_n(\mathbb{R})$ whose restriction to $\mathcal{S}$ is zero.*

(iii) *$\mathcal{S}$ has a common zero entry.*

(iv) *$\mathcal{S}$ has a common nondiagonal zero entry.*

(v) *There exist A, B in $\mathcal{M}_n(\mathbb{R})$, both nonzero and nonnegative such that $A\mathcal{S}B = \{0\}$.*

Proof. (i) ⇒ (ii) If $\mathscr{S}$ is decomposable, then after a permutation of basis, every member S of $\mathscr{S}$ is of the form

$$\begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix},$$

where S_{11}, S_{22} are square matrices. Define a linear functional f on $\mathscr{M}_n(\mathbb{R})$ by $f(A) = a_{ij}$ where a_{ij} is the fixed (i,j) entry in the matrix representation of A with respect to the permuted basis from the block A_{21}. Clearly f is a nonzero, nonnegative functional on $\mathscr{M}_n(\mathbb{R})$ such that $f|_{\mathscr{S}} \equiv 0$.

(ii) ⇒ (iii) This has been proved in Proposition 2.7.

(iii) ⇒ (iv) If the common zero of $\mathscr{S}$ is a diagonal entry, then by permuting the basis, we can bring it to the (1, 1) slot. Now, if the first row is zero for every A in $\mathscr{S}$, we are done for then $\mathscr{S}$ is decomposable ($\bigvee\{e_2, e_3, \ldots, e_n\}$ being the nontrivial standard invariant subspace). Otherwise, $a_{1i_0} \neq 0$ for some $i_0 \neq 1$ and for some $A \in \mathscr{S}$. Now for any $B \in \mathscr{S}$,

$$0 = (AB)_{11} = \sum_{i=1}^{n} a_{1i} b_{i1}$$

$$\Rightarrow \quad a_{1i} b_{i1} = 0 \quad \text{for all } i \text{ and for all } B \in \mathscr{S}$$

$$\Rightarrow \quad b_{i_0 1} = 0 \quad \text{for all } B \in \mathscr{S} \text{ as } a_{1i_0} \neq 0$$

i.e., a nondiagonal entry is permanently zero in $\mathscr{S}$.

(iv) ⇒ (v) Let $s_{jk} = 0$ for all $S \in \mathscr{S}$ for some $j \neq k$. Construct an $n \times n$ matrix A such that $a_{i_0 j} > 0$ for some i_0 and the remaining entries are zero. Similarly, let $B \in \mathscr{M}_n(\mathbb{R})$ be such that $b_{kl_0} > 0$ for some l_0 and the remaining entries are zero. Then A, B are nonzero, nonnegative matrices and it can be easily verified that $A\mathscr{S}B = \{0\}$.

(v) ⇒ (i) We have $A\mathscr{S}B = \{0\}$ for some nonzero, nonnegative A, B in $\mathscr{M}_n(\mathbb{R})$. If a_{ij} and b_{kl} are nonzero entries in A and B respectively, then it is easy to see that the (j,k) entry in each $S \in \mathscr{S}$ is zero. This makes use of the fact that A, B and S are nonnegative matrices. By Lemma 2.2, $\mathscr{S}$ is decomposable. □

Remark 2.9. Clearly, if $\mathscr{S}$ is decomposable, it has a common nondiagonal zero entry but decomposability may not give a common diagonal zero entry.

For example,

$$\mathscr{S} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \right\}$$

is a singleton semigroup which is decomposable but no permutation of the basis will produce a zero on the diagonal.

3. Decomposability of nonnegative bands

We now confine our attention to nonnegative bands in $\mathcal{M}_n(\mathbb{R})$ with nonnegative matrices and prove their decomposability under certain conditions. We start with a singleton nonnegative band. In completeness, we include a simple proof of the following known lemma.

Lemma 3.1. *Let E be a nonnegative $n \times n$ idempotent with rank $r > 1$. Then E is decomposable.*

Proof. We first show that if $r > 1$, then the range of E contains a nonzero (column) vector z with nonnegative entries and at least one zero entry. Pick any two nonnegative linearly independent elements x and y in the range of E. Then $Ex = x$ and $Ey = y$. If either x or y has a zero entry, we are done. Otherwise, let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

and let $y_j/x_j = \max\{y_i/x_i : i = 1, 2, \ldots, n\}$.

Then the vector $z = y_j x - x_j y$ is nonzero, has nonnegative entries, and its jth entry is zero. Since $Ez = z$, it is the desired vector. With no loss of generality, we can assume that z is the vector with a minimal number of nonzero entries. After a permutation of the basis, we can assume that the entries (z_i) of z satisfy.

$$z_1 \geqslant \cdots \geqslant z_k > z_{k+1} = \cdots = z_n = 0.$$

Then the equation $Ez = z$, together with the nonnegativity of entries in E and z, implies that the (i, j) entry of E is zero whenever $i \geqslant k + 1$ and $j \leqslant k$. Thus the span of the first k basis vectors is invariant under E, i.e., E is decomposable. □

Remark 3.2. The above result can also be obtained using the Perron–Frobenius Theorem (Theorem 5.5.1(i) in Ref. [4], p. 124) part of which says that an $n \times n$ nonnegative indecomposable matrix has a real positive eigenvalue, say r, which is a simple root of its characteristic equation. Thus if E is indecomposable, then since an idempotent has only 0 and 1 as eigenvalues, the eigenvalue 1 will occur only once in its spectrum and so the trace of E is 1. But for an idempotent, its rank equals its trace and therefore, rank $(E) = 1$, which is a contradiction. Thus E must be decomposable.

We denote by $\mathscr{L}at'\mathscr{S}$ the lattice of all standard subspaces which are invariant under every member of $\mathscr{S}$, where $\mathscr{S}$ is a collection of matrices in $\mathscr{M}_n(\mathbb{R})$. It can be shown by simple induction that for any semigroup $\mathscr{S}$, $\mathscr{L}at'\mathscr{S}$ has a maximal chain. This chain may be nontrivial or trivial according as $\mathscr{S}$ has a nontrivial standard subspace or not. Each chain in $\mathscr{L}at'\mathscr{S}$ gives rise to a block triangularization for $\mathscr{S}$ and since the members in the chain are standard subspaces, we shall call it a standard block triangularization. Evidently, to say that $\mathscr{S}$ has a standard block triangularization is equivalent to saying that there exists a permutation matrix P such that for each S in $\mathscr{S}$, $P^{-1}SP$ has the upper block triangular form.

Suppose $\mathscr{C}$ is a chain in $\mathscr{L}at'\mathscr{S}$ and $\mathscr{M}, \mathscr{N}$ are two successive elements in $\mathscr{C}$ such that $\mathscr{M} \subset \mathscr{N}$, then $\mathscr{N} \ominus \mathscr{M}$ is called a gap in the chain. If P is the orthogonal projection onto $\mathscr{N} \ominus \mathscr{M}$, then the restriction of $P\mathscr{S}P$ to the range of P is called the compression of $\mathscr{S}$ to $\mathscr{N} \ominus \mathscr{M}$. Note that every compression corresponds to a diagonal block in the block triangularization of $\mathscr{S}$.

Theorem 3.3. *Let E be an $n \times n$ nonnegative idempotent of rank $r > 1$. Then*
1. *any maximal standard block triangularization of E has the two properties*
   (a) *each diagonal block is either zero or a positive idempotent of rank one.*
   (b) *there are exactly r nonzero diagonal blocks.*
2. *there exists a standard block triangularization of E with properties* (a) *and* (b) *such that no two consecutive diagonal blocks are zero* (*so that the total number of diagonal blocks is* $\leqslant 2r + 1$).

Proof. By Lemma 3.1, E is decomposable. Let $\mathscr{C}$ be a maximal chain in $\mathscr{L}at'E$ resulting in a maximal standard block triangularization of E. If $\mathscr{M}$ and $\mathscr{N}$ are in $\mathscr{C}$ such that $\mathscr{N} \ominus \mathscr{M}$ is a gap, and if the compression of E to $\mathscr{N} \ominus \mathscr{M}$ is nonzero, then it is an indecomposable idempotent. For otherwise, if it has an invariant subspace $\mathscr{K}$ of the desired kind, then $\mathscr{M} \oplus \mathscr{K}$ is a standard subspace, invariant under E which lies strictly between $\mathscr{M}$ and $\mathscr{N}$ and is comparable with every member of $\mathscr{C}$, thus contradicting the maximality of $\mathscr{C}$. Therefore, every nonzero compression (or diagonal block) is indecomposable and of rank one by Lemma 3.1. Since the rank of an idempotent equals its trace, it is apparent that the number of nonzero diagonal blocks is exactly r. (Observe that in any block triangularization of an idempotent, the diagonal blocks or the compressions are idempotents). It is easy to see that an indecomposable rank-one matrix cannot have any zeros in it. A zero entry would lead to a zero row (or a zero column) which after a permutation of basis can be brought to the position of the last row (or first column), thus rendering the matrix decomposable. Therefore, a nonzero diagonal block is a positive idempotent of rank one.

Lastly, the fact that a 2×2 block matrix whose (1, 1), (2, 1) and (2, 2) blocks are all zero is an idempotent if and only if it is zero proves Part 2 of the theorem. □

We now study the decomposability of a nonnegative band with more than a single member.

Theorem 3.4. *Suppose $\mathcal{S}$ is a band in $\mathcal{M}_n(\mathbb{R})$ with nonnegative members such that* rank $(S) > 1$ *for all $S \in \mathcal{S}$. Then $\mathcal{S}$ is decomposable.*

Proof. Let $m = \min\{\text{rank}(S); S \in \mathcal{S}\}$. Select a P in $\mathcal{S}$ of rank m. For an arbitrary $S \in \mathcal{S}$, consider PSP. This is an idempotent whose range is contained in the range of P and whose null space contains the null space of P. Since rank $(PSP) =$ rank (P) $= m$, we obtain $PSP = P$. Thus $P\mathcal{S}P = \{P\}$.

Further, since rank $(P) = m > 1$, by Theorem 3.3, we can see that P has the form

$$\begin{pmatrix} P_1 & A \\ 0 & P_2 \end{pmatrix}$$

with respect to some permutation of basis where both P_1 and P_2 are nonzero.

Let

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

be the representation of an arbitrary S in $\mathcal{S}$ with respect to this permuted basis. Then $PSP = P$ implies that $P_2S_{21}P_1 = 0$. As in the proof of Theorem 2.8 ((v) ⇒ (i)), we can show the existence of a zero entry in S_{21}. Since S is arbitrary and P fixed, this zero will occur commonly in each S_{21} and hence in $\mathcal{S}$. By Lemma 2.2, $\mathcal{S}$ is decomposable which proves the theorem. □

Remark 3.5. In the proof of the theorem above, if we consider $\mathcal{J}$ to be the collection of all rank m elements in $\mathcal{S}$, then $\mathcal{J}$ is a nonzero ideal of $\mathcal{S}$. By Proposition 2.4, $\mathcal{S}$ is decomposable if and only if $\mathcal{J}$ is decomposable. Thus, with no loss of generality, $\mathcal{S}$ can be assumed to be a nonnegative band of constant rank m.

Theorem 3.6. *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{M}_n(\mathbb{R})$ such that* $\operatorname{rank}(S) > 1$ *for all S in $\mathcal{S}$. Then any maximal standard block triangularization of $\mathcal{S}$ has the property that each nonzero diagonal block is a nonnegative band with at least one element of rank one in it.*

Proof. The proof is on the lines of the proof of Theorem 3.3. □

4. Structure of constant-rank nonnegative bands

In the previous section, we saw in Remark 3.5 that the question of decomposability for a nonnegative band reduces to the case of a constant-rank ideal in it. This fact shows the significance of constant- rank nonnegative bands and motivates us to study their structure. Some of the results on single nonnegative idempotents are similar to, and can be obtained from those in Ref. [1] (see for e.g. Theorem 3.1, p. 65) but the treatment given here is more appropriate to our purposes and is included for the sake of completeness.

Lemma 4.1. *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{M}_n(\mathcal{R})$ of constant rank one. Then there exists a permutation matrix P such that for each $S \in \mathcal{S}$, $P^{-1}SP$ has the block-triangular form*

$$\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix},$$

where the diagonal block $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ constitutes a rank-one indecomposable band and X and Y are nonnegative matrices of suitable size.

Proof. Let $\mathcal{B}_1$ consist of the elements of the standard basis $\mathcal{B}$ which are in ker $\mathcal{S}$ and let $\mathcal{B}_3$ consist of those elements of $\mathcal{B}$ which are in ker $\mathcal{S}^*$ but not in ker $\mathcal{S}$. Let $\mathcal{B}_2$ be the complement of $\mathcal{B}_1 \cup \mathcal{B}_3$ in $\mathcal{B}$. Then the arrangement $\mathcal{B}_1 \dot\cup \mathcal{B}_2 \dot\cup \mathcal{B}_3$ of the basis $\mathcal{B}$ gives rise to the permutation matrix P such that for each S in $\mathcal{S}$, $P^{-1}SP$ has the matrix form

$$\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix},$$

where X, Y, Z are matrices of suitable size.

The equations $E^2 = E, X = XE, Y = EY$ and $Z = XEY$ are obtained using the fact that each matrix in $\mathcal{S}$ is an idempotent. Lastly, the diagonal block $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ forms a rank-one band because $\mathcal{S}$ is a rank-one band. It is easily checked that $\mathscr{S}_0$ is decomposable, for otherwise, a zero entry in $\mathscr{S}_0$ will lead to a common zero row or a common zero column (using the fact that the rank of $\mathscr{S}$ is one), which is not possible as all the zero rows and zero columns have already been taken out. □

Lemma 4.2. *If $\mathscr{S}$ is a nonnegative band in $\mathscr{M}_n(\mathbb{R})$ with constant rank r, then $\mathscr{S}$ has a standard block triangular form with exactly r nonzero diagonal blocks, each constituting an indecomposable band of rank one. Furthermore, this can be done so that no two diagonal blocks are consecutively zero. Therefore, if k be the total number of diagonal blocks, then $k \leqslant 2r + 1$.*

Proof. We shall prove the lemma by induction on r. The case $r = 1$ is dealt with in Lemma 4.1. Suppose $r > 1$; then we know by Theorem 3.4 that $\mathscr{S}$ is decomposable. Therefore, after a permutation of basis, every $S \in \mathscr{S}$ is of the form

$$\begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix},$$

where S_1, S_2 are square matrices. Consider the two diagonal blocks, $\mathscr{S}_1 = \{S_1 : S \in \mathscr{S}\}$ and $\mathscr{S}_2 = \{S_2 : S \in \mathscr{S}\}$. Clearly, $\mathscr{S}_1$ and $\mathscr{S}_2$ form nonzero, nonnegative bands. We now prove that $\mathscr{S}_1$ and $\mathscr{S}_2$ are constant-rank bands.

Let

$$S = \begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_1 & Y \\ 0 & T_2 \end{pmatrix}$$

be two elements in $\mathscr{S}$ such that rank $(S_1) = m_1$ and rank $(T_1) = m_2$. Let us assume that $m_1 < m_2$. Then since the rank of S and T is r, rank $(S_2) = r - m_1$ and rank $(T_2) = r - m_2$. Consider

$$ST = \begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix}\begin{pmatrix} T_1 & Y \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} S_1T_1 & S_1Y + XT_2 \\ 0 & S_2T_2 \end{pmatrix}.$$

Now

$$\operatorname{rank}(S_1T_1) \leqslant \min\{\operatorname{rank}(S_1), \operatorname{rank}(T_1)\} = \min\{m_1, m_2\} = m_1$$

and

$$\operatorname{rank}(S_2T_2) \leqslant \min\{\operatorname{rank}(S_2), \operatorname{rank}(T_2)\} = \min\{r - m_1, r - m_2\} = r - m_2.$$

But then,

$$\operatorname{rank}(ST) = \operatorname{rank}(S_1T_1) + \operatorname{rank}(S_2T_2) \leqslant m_1 + r - m_2 < r,$$

which implies that $m_1 = m_2$. Therefore $\mathscr{S}_1$ has constant rank and by the same argument so does $\mathscr{S}_2$. Also since $\mathscr{S}_1$ and $\mathscr{S}_2$ are nonzero bands, their ranks are less than r. Thus induction applies and we obtain the desired result.

Lastly, the fact that a 2×2 block matrix all of whose blocks except (1, 2) are zero is an idempotent if and only if it is zero justifies the assertion that no two diagonal blocks are consecutively zero. □

Definition 4.3. A semigroup $\mathscr{S}$ in $\mathscr{M}_n(\mathbb{R})$ of nonnegative matrices will be called a full semigroup if $\mathscr{S}$ has no common zero row and no common zero column.

Lemma 4.4. *Let $\mathscr{S}$ be a full band of nonnegative matrices in $\mathscr{M}_n(\mathbb{R})$ with constant rank one. Then $\mathscr{S}$ is indecomposable.*

Proof. This follows immediately from the description of a rank-one nonnegative band in Lemma 4.1. □

Theorem 4.5. *Let $\mathscr{S}$ be a nonnegative band in $\mathscr{M}_n(\mathbb{R})$ with constant rank r.*

(i) *If $\mathscr{S}$ is full, then there exists a permutation matrix P such that for any $S \in \mathscr{S}$, $P^{-1}SP$ has the block diagonal form*

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where each $\mathscr{S}_i = \{S_i : S \in \mathscr{S}\}$ is an indecomposable band of rank-one matrices.

(ii) *In general, there is a permutation matrix Q such that for each $S \in \mathscr{S}$, $Q^{-1}SQ$ has the upper block triangular form*

$$\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix},$$

where matrices X, Y are of appropriate size and $\mathscr{S}_0 = \{E : S \in \mathscr{S}\}$ is as in case (i).

Proof. (i) If the rank r of $\mathscr{S}$ is one, then the result is true by Lemma 4.4. We shall prove the theorem by induction on r. Let $r > 1$, then by Lemma 4.2, each S in $\mathscr{S}$ can be assumed to have the form

$$\begin{pmatrix} S_1 & X_1 \\ 0 & S_2 \end{pmatrix},$$

where the diagonal blocks $\mathcal{S}_1 = \{S_1 : S \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{S_2 : S \in \mathcal{S}\}$ form nonzero bands of constant rank less than r. Also then, by the fullness of $\mathcal{S}$, $\mathcal{S}_1$ has no common zero column and $\mathcal{S}_2$ has no common zero row.

Let

$$E = \begin{pmatrix} E_1 & X \\ 0 & E_2 \end{pmatrix}$$

be arbitrary but fixed in $\mathcal{S}$.

Let

$$F = \begin{pmatrix} F_1 & Y \\ 0 & F_2 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} G_1 & Z \\ 0 & G_2 \end{pmatrix}$$

be arbitrary members in $\mathcal{S}$. Then

$$GEF = \begin{pmatrix} G_1 & Z \\ 0 & G_2 \end{pmatrix}\begin{pmatrix} E_1 & X \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} F_1 & Y \\ 0 & F_2 \end{pmatrix}$$
$$= \begin{pmatrix} G_1E_1F_1 & G_1E_1Y + G_1XF_2 + ZE_2F_2 \\ 0 & G_2E_2F_2 \end{pmatrix}.$$

The fact that GEF is an idempotent implies that

$$G_1E_1F_1(G_1E_1Y + G_1XF_2 + ZE_2F_2) + (G_1E_1Y + G_1XF_2 + ZE_2F_2)G_2E_2F_2$$
$$= G_1E_1Y + G_1XF_2 + ZE_2F_2.$$

Premultiplying the above equation by $G_1E_1F_1$ and postmultiplying by $G_2E_2F_2$, we obtain

$$G_1E_1F_1(G_1E_1Y + G_1XF_2 + ZE_2F_2)G_2E_2F_2 = 0$$
$$\Rightarrow G_1E_1F_1G_1E_1YG_2E_2F_2 + G_1E_1F_1G_1XF_2G_2E_2F_2$$
$$+ G_1E_1F_1ZE_2F_2G_2E_2F_2 = 0.$$

Since all the matrices are nonnegative, this gives

$$G_1E_1F_1G_1XF_2G_2E_2F_2 = 0. \qquad (4.1)$$

Now $G_1, E_1F_1 \in \mathcal{S}_1$ and $F_2, G_2E_2 \in \mathcal{S}_2$ both of which have constant rank. Therefore,

$$G_1E_1F_1G_1 = G_1 \quad \text{and} \quad F_2G_2E_2F_2 = F_2.$$

Thus Eq. (4.1) reduces to

$$G_1XF_2 = 0. \qquad (4.2)$$

Since $G_1 \in \mathscr{S}_1$ and $F_2 \in \mathscr{S}_2$ are arbitrary, Eq. (4.2) reduces to

$$\mathscr{S}_1 X \mathscr{S}_2 = 0.$$

But $\mathscr{S}_1$ has no common zero column, therefore $X\mathscr{S}_2 = 0$ and the fact that $\mathscr{S}_2$ has no common zero implies that $X = 0$. Thus

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}.$$

This shows that any general element S in $\mathscr{S}$ is of the form

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix},$$

where $\mathscr{S}_1 = \{S_1 : S \in \mathscr{S}\}$ and $\mathscr{S}_2 = \{S_2 : S \in \mathscr{S}\}$ are nonnegative full bands with constant rank less than r. Hence induction applies and $\mathscr{S}$ is of the desired form.

(ii) In the general case, we first consider the same arrangement of the basis $\mathscr{B}$ as in Lemma 4.1. Then with respect to this permutation of basis, every element S of $\mathscr{S}$ assumes the form

$$S = \begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $S^2 = S$, we have

$$E^2 = E, \; X = XE, \; Y = EY \text{ and } Z = XEY.$$

These equations imply that $\mathscr{S}_0 = \{E : S \in \mathscr{S}\}$ cannot have a common zero row or a common zero column. Thus $\mathscr{S}_0$ is a full nonnegative band of constant rank r and hence is of the form given in (i) above. □

Remark 4.6. 1. It is easily verified that the product of any two block matrices of the form exhibited in Part (ii) of Theorem 4.5 is again of the same form.

2. If in the statement of the theorem above, $\mathscr{S}$ is taken to be a maximal band, then it is readily observed that the bands $\mathscr{S}_i$ must be maximal. In part (ii), $\mathscr{S}_0$ and the collection of all X, Y are maximal too.

3. In Theorem 4.9, we show that the converse of Part (i) of Theorem 4.5 is also true in case the bands $\mathscr{S}_i$ are maximal. To prove this, we shall need a couple of lemmas, of which Lemma 4.8 may be of independent interest.

Lemma 4.7. *Let $\mathscr{S}$ be an indecomposable, nonnegative semigroup in $\mathscr{M}_n(\mathbb{R})$ and e_i be any basis vector. Then $\bigvee\{\mathscr{S}e_i\}$ contains a positive vector.*

Proof. Since $\mathscr{S}$ is indecomposable, no entry in the members of $\mathscr{S}$ is permanently zero. Therefore, for each $k = 1, 2, \ldots, n$, there exits $A^{(k)} \in \mathscr{S}$ such that its (k, i) entry is nonzero. It is evident that then $(A^{(1)} + A^{(2)} + \cdots + A^{(n)})e_i$ is the desired positive vector. □

Lemma 4.8. *Let $\mathscr{S}$ be a direct sum of r nonnegative, indecomposable semigroups $\mathscr{S}_1, \ldots, \mathscr{S}_r$, so that each member of $\mathscr{S}$ has block diagonal representation*

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where $S_i \in \mathscr{S}_i, i = 1, 2, \ldots, r$, with respect to a fixed decomposition $\mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_r$ of $\mathbb{R}^n$ into standard subspaces. Then every $\mathscr{M} \in \mathscr{L}at'\mathscr{S}$ is of the form $\mathscr{M} = \oplus_{i=1}^{r} \epsilon_i \mathscr{M}_i$, where each ϵ_i is either 0 or 1.

Proof. It is obvious that each $\mathscr{M}_i$ belongs to $\mathscr{L}at'\mathscr{S}$. Also, each $\mathscr{S}_i$ being indecomposable, $\mathscr{M}_i$ is a minimal standard subspace in $\mathscr{L}at'\mathscr{S}$ in the sense that $\mathscr{S}$ has no standard invariant subspace properly contained in it. Now let $\mathscr{M} \in \mathscr{L}at'\mathscr{S}$.

We define $\epsilon_i = 1$ if $\mathscr{M}_i \cap \mathscr{M}$ contains a basis vector and $\epsilon_i = 0$ otherwise. To prove the desired result, it is enough to show that if $e_j \in \mathscr{M}_i \cap \mathscr{M}$, then $\mathscr{M}_i \subseteq \mathscr{M}$. We write e_j with respect to the given decomposition of the space and suppose the resulting vector is

$$\begin{pmatrix} 0 \\ \vdots \\ x_i \\ \vdots \\ 0 \end{pmatrix},$$

where the column vector x_i has 1 at the appropriate place and zero elsewhere. Consider $\mathscr{S}e_j$. Then

$$\mathscr{S}e_j = \begin{pmatrix} 0 \\ \vdots \\ \mathscr{S}_i x_i \\ \vdots \\ 0 \end{pmatrix} \in \mathscr{M}.$$

Since $\mathscr{S}_i$ is a nonnegative, indecomposable semigroup, by Lemma 4.7, we obtain a positive vector y_i in $\mathscr{M}_i$ which is a nonnegative linear combination of $\{\mathscr{S}_i x_i\}$. Consider

$$y = \begin{pmatrix} 0 \\ \vdots \\ y_i \\ \vdots \\ 0 \end{pmatrix},$$

where y is a positive linear combination of all the basis vectors which span $\mathscr{M}_i$. Also $y \in \mathscr{M}$ and $\mathscr{M}$ being a standard subspace, it is spanned by a subset of basis vectors. Expressing y as a linear combination of the basis vectors that span $\mathscr{M}$, we observe by the linear independence of the basis vectors that there cannot be any basis vector which is in $\mathscr{M}_i$ but not in $\mathscr{M}$. Hence we must have $\mathscr{M}_i \subseteq \mathscr{M}$ which proves the lemma. □

Theorem 4.9. *A direct sum of r maximal, indecomposable, nonnegative rank-one bands is a maximal band of constant rank r.*

Proof. For $r = 1$, the result is obvious. Therefore, let $r > 1$. Suppose $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_r$ are r maximal indecomposable, nonnegative rank-one bands and consider their direct sum. Every member S of $\mathscr{S}$ is of the form

$$\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix},$$

where $S_i \in \mathscr{S}_i, i = 1, 2, \cdots, r$.

If $\mathscr{S}$ is not maximal, then let $\mathscr{S}' \supseteq \mathscr{S}$ be a band with constant rank r. Now observe that $\mathscr{S}$ is a full band. Therefore, $\mathscr{S}'$ is full too. By part (i) of Theorem 4.5, $\mathscr{S}'$ is a direct sum of r rank-one indecomposable, nonnegative bands, say, $\mathscr{S}'_1, \mathscr{S}'_2, \cdots, \mathscr{S}'_r$. Now $\mathscr{L}at'\mathscr{S}' \subseteq \mathscr{L}at'\mathscr{S}$. By the previous lemma, the cardinality of $\mathscr{L}at'\mathscr{S}$ is the same as that of $\mathscr{L}at'\mathscr{S}'$ which is 2^r. Therefore, we must have $\mathscr{L}at'\mathscr{S} = \mathscr{L}at'\mathscr{S}'$. Thus after permuting the basis if necessary, we obtain $\mathscr{S}_i \subseteq \mathscr{S}'_i$. But since the bands $\mathscr{S}_i$ are maximal, we have $\mathscr{S}_i = \mathscr{S}'_i$. Hence $\mathscr{S}$ is maximal. □

Theorem 4.9 and Remark 4.6 can be summed up to give the following characterization of maximal nonnegative bands of constant rank.

Theorem 4.10. *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{M}_n(\mathbb{R})$ of constant rank r.*

(i) *If $\mathcal{S}$ is full, then $\mathcal{S}$ is maximal if and only if*

$$\mathcal{S} = \left\{ \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix} : S_i \in \mathcal{S}_i, i = 1,2,\ldots,r \right\},$$

where $\mathcal{S}_i$ is a maximal rank-one indecomposable band for each i.

(ii) *In general, if $\mathcal{S}$ is maximal, then*

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix} : E \in \mathcal{S}_0, X \in \mathcal{X}, Y \in \mathcal{Y} \right\},$$

where $\mathcal{S}_0$ is a direct sum as in part (i) *and $\mathcal{X}$ and $\mathcal{Y}$ are the entire sets of nonnegative matrices of suitable size.*

In the next section, we shall give a geometric characterization of maximal bands of constant finite rank which in view of Theorem 4.10 gives a geometric characterization of constant-rank, nonnegative bands.

5. A geometric characterization of maximal, indecomposable, nonnegative rank-one bands

A nonzero, nonnegative, rank-one matrix in $\mathcal{M}_n(\mathbb{R})$ is of the form xy^*, where x, y are nonzero, nonnegative vectors in $\mathbb{R}^n$. Further, for xy^* to be an idempotent, x, y must satisfy the equation $\operatorname{tr}(xy^*) = y^*x = 1$.

Thus, if $\mathcal{S}$ is a nonnegative band of rank-one matrices in $\mathcal{M}_n(\mathbb{R})$, then we can find sets $\mathcal{X}, \mathcal{Y}$ in the nonnegative cone of $\mathbb{R}^n$, viz., $\mathbb{R}^n_+$, so that $\mathcal{S} \subseteq \mathcal{X}\mathcal{Y}^*$, where

$$\mathcal{X}\mathcal{Y}^* = \{xy^* : x \in \mathcal{X}, y \in \mathcal{Y}\}$$

and

$$y^*x = 1 \quad \text{for all } x \in \mathcal{X} \quad \text{and for all } y \in \mathcal{Y}.$$

(By the nonnegative cone of $\mathbb{R}^n$, we mean the set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geqslant 0\}$).

Further, if $\mathcal{S}$ is maximal, then we must have $\mathcal{S} = \mathcal{X}\mathcal{Y}^*$ for some $\mathcal{X}, \mathcal{Y}$ of the kind mentioned above. We wish to find the general form of $\mathcal{X}$ and $\mathcal{Y}$ for a maximal, nonnegative, indecomposable band $\mathcal{S}$ of rank-one matrices in $\mathcal{M}_n(\mathbb{R})$.

We observe that if $x_1, x_2 \in \mathcal{X}$, then $y^*(tx_1 + (1-t)x_2) = 1$ for $0 \leqslant t \leqslant 1$ and for all $y \in \mathcal{Y}$. Thus for a maximal $\mathcal{X}\mathcal{Y}^*$, $\mathcal{X}$ must contain all the convex combinations of its members too. Thus with no loss of generality we can assume that $\mathscr{X}$ has a positive vector, say $a = (a_i), a_i > 0$ for all i.

Any $x \in \mathscr{X}$ satisfies the system of equations $y^t x = 1, y \in \mathscr{Y}$. The vector $x = a$ is a particular solution to this system. Thus, for any $x \in \mathscr{X}$,

$$\begin{aligned} & y^t x = y^t a \quad \text{for all } y \in \mathscr{Y} \\ & \quad \Rightarrow y^t(x - a) = 0 \quad \text{for all } y \in \mathscr{Y} \\ & \quad \Rightarrow x - a \in \mathscr{Y}^\perp \\ & \quad \Rightarrow x \in a + \mathscr{Y}^\perp \quad \text{for all } x \in \mathscr{X} \\ & \quad \Rightarrow \mathscr{X} \subseteq a + \mathscr{Y}^\perp. \end{aligned}$$

Also, if $y' \in \mathscr{Y}^\perp$, then for any $y \in \mathscr{Y}, y^t(a + y') = y^t a = 1$. Thus, by the maximality of $\mathscr{S}$, we obtain

$$\mathscr{X} = \{a + \mathscr{Y}^\perp\} \cap \mathbb{R}^n_+. \tag{5.1}$$

By a similar reasoning applied to $\{S^t : S \in \mathscr{S}\}$, we can find a positive vector $b \in \mathscr{Y}$ and obtain

$$\mathscr{Y} = \{b + \mathscr{X}^\perp\} \cap \mathbb{R}^n_+. \tag{5.2}$$

Next, we show that if $\mathscr{X}$ and $\mathscr{Y}$ are given as in Eqs. (5.1) and (5.2) respectively, for some positive a, b and subspace $\mathscr{W}^\perp, \mathscr{Z}$, i.e.,

$$\mathscr{X} = \{a + \mathscr{W}^\perp\} \cap \mathbb{R}^n_+, \tag{5.3}$$

$$\mathscr{Y} = \{b + \mathscr{Z}\} \cap \mathbb{R}^n_+, \tag{5.4}$$

where $b^t a = 1, \mathscr{W}^\perp = \{b + \mathscr{Z}\}^\perp$ and $\mathscr{Z} = \{a + \mathscr{W}^\perp\}^\perp$, then $\mathscr{S} = \mathscr{X}\mathscr{Y}^t$ is a maximal band of nonnegative rank-one matrices in $\mathscr{M}_n(\mathbb{R})$. It is easy to see that $\mathscr{S}$ forms a nonnegative band of rank-one matrices. Suppose $\mathscr{S}$ is contained in a band $\mathscr{S}_0$ of rank-one matrices, where

$$\mathscr{S}_0 \subseteq \mathscr{X}'\mathscr{Y}'^t = \{x'y'^t : x' \in \mathscr{X}', y' \in \mathscr{Y}'\},$$

for some sets $\mathscr{X}', \mathscr{Y}' \subseteq \mathbb{R}^n_+$ and $y'^t x' = 1$. Let $s = zt^t \in \mathscr{S}_0$. Since $\mathscr{S}_0$ is a semigroup, $xy^t \cdot zt^t \in \mathscr{S}_0$ for all $xy^t \in \mathscr{S}$. Therefore,

$$1 = \operatorname{tr}(xy^t \cdot zt^t) = y^t z \cdot \operatorname{tr}(xt^t) = y^t z \cdot t^t x.$$

With no loss of generality, we can assume that $y^t z = 1$ and $t^t x = 1$ (for if, $y^t z = p(\neq 1)$, then $t^t x = \frac{1}{p}$; we can write $s = \frac{z}{p} \cdot pt^t = z't'^t$ where $z' = \frac{z}{p}, t' = pt$ and $y^t z' = 1, t'^t x' = 1$). Now

$$t^t x = 1 \quad \text{for all } x \in \mathscr{X}$$

$$\Rightarrow t^t a = 1 \quad \text{and} \quad t^t(a + w) = 1 \quad \text{for all } w \in \mathscr{W}^\perp$$

$$\Rightarrow t^t a = 1 \quad \text{and} \quad t^t w = 0 \quad \text{for all } w \in \mathscr{W}^\perp$$

$$\Rightarrow (t - b)^t(a + w) = 0 \quad \text{for all } w \in \mathcal{W}$$

$$\Rightarrow t - b \in \{a + \mathcal{W}\}^\perp = \mathcal{Z}$$

$$\Rightarrow t \in b + \mathcal{Z} = \mathcal{Y}.$$

Similarly, we can show that $z \in \mathcal{X}$. Thus $zt^t \in \mathcal{X}\mathcal{Y}^t = \mathcal{S}$ which implies that $\mathcal{S}_0 \subseteq \mathcal{S}$. Hence $\mathcal{S}$ is maximal.

Next, we would like to see which subspaces $\mathcal{W}$ and $\mathcal{Z}$ give rise to maximal indecomposable bands as in Eqs. (5.3) and (5.4). Suppose there is some $w = (w_i) \in \mathcal{W}$ such that $w_i \geqslant 0$ for all i, then since $w^t b = \sum w_i y_i = 0$ for all $y = (y_i) \in \mathcal{Y}$, a nonzero component of w, say w_{i_0} will render the i_0 component y_{i_0} of each $y \in \mathcal{Y}$ zero, thus yielding a decomposable band $\mathcal{X}\mathcal{Y}^t$. For the same reason, $\mathcal{W}$ cannot have a vector $w = (w_i)$ with $w_i \leqslant 0$ for all i. This shows that every vector of $\mathcal{W}$ must necessarily be a "mixed" vector, i.e., a vector having both negative and positive entries (possibly some zeros too). The same argument also applies to $\mathcal{Z}$. In other words, $\mathcal{W}$ and $\mathcal{Z}$ intersect $\mathbb{R}^n_+$ trivially.

We summarize the discussion above in the following theorem.

Theorem 5.1. *Let $\mathcal{S}$ be a maximal, nonnegative, indecomposable band of rank-one matrices in $\mathcal{M}_n(\mathbb{R})$. Denote the positive cone of $\mathcal{M}_n(\mathbb{R})$ by $\mathbb{R}^n_+$. Then there exist positive vectors a, b in $\mathbb{R}^n_+$ with $b^t a = 1$ and there exist mixed subspaces $\mathcal{W}$, $\mathcal{Z}$ of $\mathbb{R}^n$ with $\mathcal{W} = \{b + \mathcal{Z}\}^\perp$, $\mathcal{Z} = \{a + \mathcal{W}\}^\perp$ such that $\mathcal{S} = \mathcal{X}\mathcal{Y}^t$, where*

$$\mathcal{X} = \{a + \mathcal{W}\} \cap \mathbb{R}^n_+;$$

$$\mathcal{Y} = \{b + \mathcal{Z}\} \cap \mathbb{R}^n_+.$$

Conversely too, if $\mathcal{X}$ and $\mathcal{Y}$ are given in the form above, then $\mathcal{S} = \mathcal{X}\mathcal{Y}^t$ is a maximal, nonnegative indecomposable band of rank-one matrices.

Nontrivial examples of the above theorem are not easy to construct. However, an example where $\mathcal{W}$ is the zero space and $\mathcal{Z}$ can have any dimension is given below, preceded by an observation.

Suppose we have a band as given the theorem above. The positive vector

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

can be replaced with the vector

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

if the whole band $\mathcal{S}$ is transformed by an appropriate similarity to $L^{-1}\mathcal{S}L$, where L is given by

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}.$$

Then instead of working with $\mathcal{X}\mathcal{Y}^*$, we work with $L\mathcal{X}\mathcal{Y}^*L^{-1}$ which is again of the form $\mathcal{X}'\mathcal{Y}'^*$ where $\mathcal{X}' = L\mathcal{X}$ and $\mathcal{Y}' = ((L^{-1})^*\mathcal{Y})^*$.

A special case is obtained when $\mathcal{Y}^{\perp} = \{0\}$, i.e., when $\mathcal{X}$ is a singleton. In this case, $\mathcal{X}\mathcal{Y}^*$ is similar (upto a diagonal similarity with positive diagonal entries) to

$$\left\{ \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \dots & x_n \end{pmatrix} : x_1 + \cdots + x_n = 1, x_i \geqslant 0 \right\}.$$

References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in Mathematical Scieneces, Academic Press, New York, 1979.

[2] P. Fillmore, G. Macdonald, M. Radjabalipour, H. Radjavi, Towards a classification of maximal unicellular bands, Semigroup Forum, Springer, New York Inc. 49 (1994), 195–215.

[3] P. Fillmore, G. Macdonald, M. Radjabalipour, H. Radjavi, On principal-ideal bands, 1997, preprint.

[4] M. Marcus and H. Minc, A survey of matrix theory and matrix inequalities, Allyn and Bacon, 1964.

[5] H. Radjavi, A trace condition equivalent to simultaneous triangularizability, Canadian Journal of Mathematics 38 (2) (1986) 376–386.

[6] H. Radjavi, On reducibility of semigroups of compact operators, Indiana University Mathematics Journal 39 (2) (1990) 499–515.

[7] H.H. Schaefer, Banach Lattices and Positive Operators, Springer, New York, 1986.