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# Representations of small braid groups in Temperley–Lieb algebra

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ABSTRACT

In this paper we consider the question of faithfulness of the Jones' representation of braid group $B_n$ into the Temperley–Lieb algebra $TL_n$. The obvious motivation to study this problem is that any non-trivial element in the kernel of this representation (for any $n$) would almost certainly yield a non-trivial knot with trivial Jones polynomial (see [S. Bigelow, Does the Jones polynomial detect the unknot? J. Knot Theory Ramifications 11 (4) (2002) 493–505], we will explain it in more detail in Section 1). As one of the two main results we prove Theorem 1 in which we present a method to obtain non-trivial elements in the kernel of the representation of $B_6$ into $TL_{9,2}$—to the authors' knowledge the first such examples in the second gradation of the Temperley–Lieb algebra. Theorem 2 which is a refinement of Theorem 1 may be used to produce smaller examples of the same kind. We also show briefly how some braids that are used in Section 4 to construct specific examples were generated with a computer program.



## 1. The Jones representation

We describe the definition of Temperley–Lieb algebra that is convenient for our considerations. Then we define the representations to be considered.

The Temperley–Lieb algebra $TL_n$ is a $\mathbb{Z}[A^{-1}, A]$ algebra. As a module it is generated by diagrams of non-intersecting strings connecting $2n$ points positioned on two sides of a rectangle (or rather by deformation classes of these diagrams). We will call these generators *Kauffman diagrams*, following [8]. We stress that the strings do not necessarily connect the points on one side to points on the other side. Multiplication is defined by putting one generator above the other (concatenation). If a closed curve is obtained, it is erased from the picture and the obtained generator is multiplied by $-A^{-2} - A^2$. This is illustrated in Fig. 1.

It may happen that a *number* of closed curves is obtained. In this case we erase all of these and multiply by a suitable power of $-A^{-2} - A^2$.

The number of generators of $TL_n$ for $n = 1, \dots, 12$ is 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 15796, 58786, 208012 (the $n$th Catalan number).

We will now define a filtration of $TL_n$ (see also [15]). In a given non-trivial generator of $TL_n$ there are certain strings that have both ends on the same side of the rectangle—the cups (top to top) and the caps (bottom to bottom). Of course the number of the cups and the number of the caps are equal. The trivial generator **1** (all strings just going straight down from the top to the bottom) is the only one with the number of cups (and caps) equal to 0. The 14 linear generators of $TL_4$ are shown in Fig. 2, split into 3 groups according to the number of cups/caps being equal to 0, 1 or 2.

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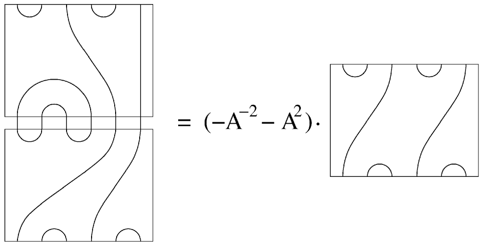


**Fig. 1.**

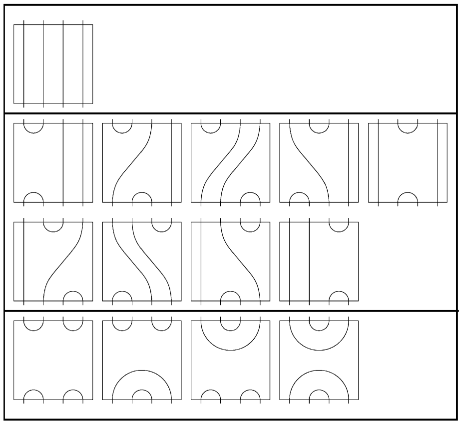

**Fig. 2.**

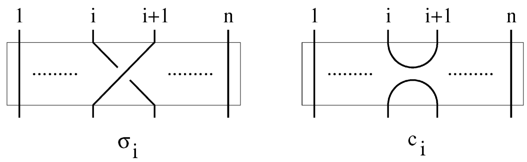


**Fig. 3.**

We will use the symbol $TL_{n,i}$ to denote the quotient of $TL_n$ by the ideal $I_{n,i}$ generated by all Kauffman diagrams with the number of cups exceeding $i$. The ideal $I_{n,i}$ is just the set of all possible combinations of generators with the number of cups exceeding $i$, because whenever a generator of such type is multiplied by any other generator (from the top or from the bottom), the result will obviously have the same or bigger number of cups. This observation makes it very easy to do calculations in $TL_{n,i}$. All elements are represented by linear combinations of generators with the number of cups smaller than or equal to $i$. When calculating the product we just ignore the generators with the number of cups greater than $i$.

We recall the definition of the Jones representation $\rho_n : B_n \to TL_n$. It is sufficient to define the image of a standard braid generator (as shown in Fig. 3) by the following formula:

$$J_n(\sigma_i) = A^{-2} \cdot \mathbf{1} + A^{-4} c_i. \tag{1.1}$$

The Jones polynomial $V$ of a link represented by a given braid $\alpha$ may be easily calculated from $J_n(\alpha)$. The way to do it is to take the standard closure of each generator, which produces a trivial link diagram and then use again the principle that every closed curve in the picture may be removed while at the same time the considered term is multiplied by $-A^{-2} - A^2$. When all closed curves are eliminated we obtain a polynomial in $A^{-1}, A$. This is the Kauffman version $\mathcal{L}$ of the Jones polynomial $V$. The original version of the Jones polynomial in variable $t$ is obtained as $V(t) = \mathcal{L}(t^{\frac{1}{4}})$.

There are many examples of different knots and links with identical Jones polynomial [1,6,9,12–14]. Examples of non-trivial $k$-component links, $k \geqslant 2$, with trivial Jones polynomial (that is with the Jones polynomial equal to that of the trivial $k$-component link, which is $(-A^{-2} - A^2)^{k-1}$) are also known [5]. Still, the question whether a non-trivial *knot* with Jones polynomial equal to 1 exists remains unsolved. Now, suppose that $\alpha \in B_n$ and $J_n(\alpha) = \mathbf{1}$. It is clear that the closure of $\alpha\sigma_1 \ldots \sigma_{n-1}$ would then have the Jones polynomial equal to 1. Of course adding $\sigma_1 \ldots \sigma_{n-1}$ to $\alpha$ is just one particular possibility. In fact, we can take any $n$-string braid $\beta$ representing the trivial knot instead and consider the closure of $\alpha\beta$. Like before, the obtained link will have the Jones polynomial equal to constant 1. This is the motivation to try to find non-trivial kernel elements for the Jones representation. It might possibly happen that the knot obtained in this way be trivial. In such case one could try to refine the approach in various ways (see [2]). First of all, we could replace $\alpha$ with some conjugate of $\alpha$—this surely would still be a non-trivial element in the kernel. We could also replace $\alpha$ with some power of $\alpha$ or combine both tricks. While we do not know how to prove that some of the knots obtained by combining the many different non-trivial elements in the kernel obtainable in this way with $\sigma_1 \dots \sigma_{n-1}$ would be non-trivial, it seems that priority should be given to the search for the kernel elements. One might also object that the link obtained as the closure of $\alpha\sigma_1 \dots \sigma_{n-1}$ could have more than one component—so we would obtain a link rather than a knot with the Jones polynomial equal to constant 1. In this case there is a simple argument disproving such a possibility. Obviously, the closure of $\alpha$ would have the Jones polynomial (in Kauffman variable $A$) equal to $(-A^{-2} - A^2)^{n-1}$, which is the polynomial of $n$-component trivial link. However, it is well known that the number of components $\mu$ is determined by the following formula for the Jones polynomial:

$$V(1) = (-2)^{\mu-1} \tag{1.2}$$

(see [7, Theorem 15]). It follows that the closure of $\alpha$ would be an $n$-component link. Therefore $\alpha$ would be a pure braid and the closure of $\alpha\sigma_1 \dots \sigma_{n-1}$ would be a knot.

We denote by $J_{n,i}$ the representation of $B_n$ into $TL_{n,i}$ defined as a natural composition $B_n \to TL_n \to TL_n/I_{n,i} = TL_{n,i}$. Apparently, for smaller $i$ the representation seems more forgetful. Thus it seems natural that the search for kernel elements of $J_n$ could begin from the search of kernel elements for $J_{n,1}$ or $J_{n,2}$. When we started this project we expected to find non-trivial elements in $\ker J_{n,1}$ rather easily and then to progress to higher gradations. In fact, it is known that $J_{n,1}$ has the same kernel as the Burau representation (we are very grateful to the referee for a clarification of this point). The Burau representation is now known to be non-faithful but this was a hard problem, solved finally by Moody [11] in 1991 for $B_9$ and higher, later for $B_6$ by Long and Paton [10], and for $B_5$ by Bigelow [3]. The only remaining unresolved case is that of $n = 4$ (see [2]).

## 2. Computer generated kernel elements

In this section we give a brief description of a computer program that has had some success in generating non-trivial braids in $TL_{6,1}$. In a similar manner (and perhaps more importantly for the further use in this paper) the program generates examples of braids whose algebraic images look as if the last string was a split string while geometrically this is not the case. These braids are used to construct examples in Section 4 by methods described in Theorems 1 and 2.

One very nice feature of the considered gradation of $TL_n$ is that when we calculate $J_n(\alpha)$ we can immediately see the image of $\alpha$ in various gradations of $TL_n$. Below we show as an example the image of

$$\alpha = 1\ 2\ -3\ -3\ 2\ -1\ -1\ 2\ 2\ -1.$$

We use a popular notation for braids here, which omits the symbols $\sigma_i$ and just lists the sequence of indices with the sign of its exponent. Another, even more simple notation is to write simply $A$ for $\sigma_1$ and $a$ for $\sigma_1^{-1}$, and so on. While we use this internally in our programs and in extensive lists of braids, it seems less convenient to read. Now, here is the announced image of $\alpha$ in the full algebra:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | | | | | | | |
| 2 | 1 | 2 | 1 | −1 | | | | | | |
| 3 | 1 | −8 | 1 | −3 | 5 | −4 | 1 | | | |
| 4 | 1 | 0 | 1 | −1 | | | | | | |
| 5 | 1 | −14 | −1 | 3 | −7 | 9 | −5 | 1 | | |
| 6 | 1 | −6 | 1 | −3 | 5 | −5 | 3 | −1 | | |
| 7 | 1 | −12 | −1 | 3 | −7 | 10 | −8 | 4 | −1 | |
| 8 | 2 | −4 | −1 | 3 | −3 | 1 | | | | |
| 9 | 2 | −2 | −1 | 4 | −5 | 3 | −1 | | | |
| 10 | 1 | −2 | 1 | −1 | | | | | | |
| 11 | 1 | −16 | −1 | 3 | −6 | 7 | −4 | 1 | | |
| 12 | 1 | −14 | −1 | 3 | −6 | 8 | −6 | 3 | −1 | |
| 13 | 2 | −10 | 1 | −3 | 5 | −4 | 1 | | | |
| 14 | 2 | −8 | 1 | −3 | 6 | −6 | 3 | −1 | | |

The image of $\alpha$—an element of $TL_4$—is given in the form of a 14-row array of integers. The first three columns contain some special information: the number of the generator, the number of cups of the considered generator and the degree of the first term with non-zero coefficient. Then in each row there is a sequence of coefficients starting from the lowest term, with step 4. Thus the coefficient of generator number 12 is: $-A^{-14} + 3A^{-10} - 6A^{-6} + 8A^{-2} - 6A^2 + 3A^6 - A^{10}$. The image of the considered braid in $TL_{4,1}$ is obtained immediately just by ignoring rows 8, 9, 13 and 14.

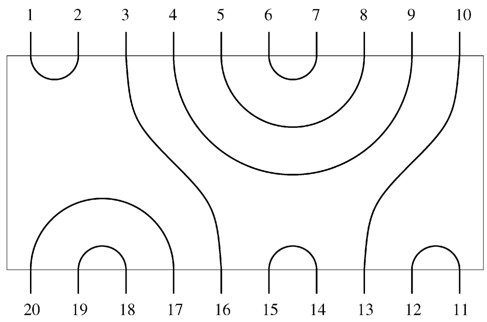


**Fig. 4.**

The linear generators of $TL_n$ are ordered according to the following rule. The inputs/outputs are numbered clockwise starting from top-left. Every generator is described as a sequence of $2n$ integers indicating where we go from a given input/output position. For example the code for a generator of $TL_{12}$ shown in Fig. 4 is

$$(2, 1, 16, 9, 8, 7, 6, 5, 4, 13, 12, 11, 10, 15, 14, 3, 20, 19, 18, 17).$$

Then we consider the inverse lexicographic order of the generators.

We developed computer programs suitable for the search of kernel elements for $J_{n,1} : B_n \to TL_{n,1}$ based on the following idea.

Suppose that we have a randomly generated braid $\alpha$ in $B_n$. First we calculate $J_{n,1}(\alpha)$ and record it as a sequence of coefficients (the coefficients being polynomials in $\mathbb{Z}[A^{-1}, A^1]$). This is easily recorded as a suitable array of integers. Now, one could forget the original braid $\alpha$ and then try to recover the inverse of $\alpha$ from $J_{n,1}(\alpha)$ alone. The obvious first step to do this would be to calculate and compare all possible values of $J_{n,1}(\alpha \cdot \sigma_i^{\pm 1})$ for $1 \leqslant i \leqslant n-1$. Then one could choose the result that seems to be the simplest and assume that the particular value of $\sigma_i^{\pm 1}$ used to obtain this result may be reasonably expected to be the first crossing in $\alpha^{-1}$. By iterating this operation one can hope to finally get the unit **1** of $TL_{n,1}$ as the image. If none of the values of $J_{n,1}(\alpha \cdot \sigma_i^{\pm 1})$ seems to get us closer to **1** we can use short braids rather than just the generators $\sigma_i^{\pm}$ in the same way. In each step we add the chosen crossing (or braid) to the braid word describing $\alpha$.

If the initial braid $\alpha$ is finally transformed into a longer braid with trivial image in $TL_{n,1}$ this may mean that $\alpha^{-1}$ was successfully recovered (and that is what usually happens) or that a non-trivial element in the kernel was obtained.

We tried many obvious parameters to measure the distance from the given element of $TL_{n,1}$ to the unit element **1**. Surprisingly, the best seems to be just the number of non-zero integer coefficients (in all polynomials).

To check whether the obtained kernel element is trivial we used the Dehornoy handle reduction algorithm, described in [4].

The `C++` program `kernel_find.cpp` for computer search for $J_{n,1}$ kernel elements is available at http://www.mimuw.edu.pl/~traczyk/brdTL. All details concerning compile conditions etc. are given there.

## 3. Some kernel elements in higher gradations

In this section we will introduce a method to construct some non-trivial elements in the kernel of the representation into higher gradations of the Temperley–Lieb algebras. The smallest example we know is one for $J_{9,2} : B_9 \to TL_{9,2}$.

We now describe the general construction. The idea is to start with two braids $\alpha \in \ker J_{k,r}$ and $\beta \in \ker J_{m,s}$ and combine the two into a braid in $\ker J_{k+m-1,r+s}$.

We will prove the following theorem.

**Theorem 1.** *Let $\alpha \in \ker J_{n,r}$ and $\beta \in \ker J_{n,s}$, where $2(r+s) \leqslant n$. Moreover, assume that the last $n-k+1$ strings of $\alpha$ go straight down and that the first $k$ strings of $\beta$ go straight down* (*without any interaction with the other strings*)*. Then $[\alpha, \beta] \in \ker J_{n,r+s}$.*

Before we prove Theorem 1 let us make some comments. The braids $\alpha$ and $\beta$ are described as $n$ string braids in the theorem. In fact, Theorem 1 is meant to be applied in the following way: we take two braids $\alpha_0$ and $\beta_0$, one of $k$ strings and another of $m$ strings so that their images under the Jones representation are trivial up to the $r$th (respectively $s$th) gradation. Then we make the required braids $\alpha$ and $\beta$ from the given braids by adding $m-1$ straight strings to the right of $\alpha_0$ and $k-1$ strings to the left of $\beta_0$. In this way we obtain two braids $\alpha$ and $\beta$ in $B_{k+m-1}$ that satisfy the assumptions of the theorem for $n = k+m-1$. This is best illustrated in Fig. 5. The braids in the big boxes are, reading from the top to the bottom, $\alpha, \beta, \alpha^{-1}, \beta^{-1}$. They are obtained from braids $\alpha_0, \beta_0, \alpha_0^{-1}, \beta_0^{-1}$ given in the smaller boxes.

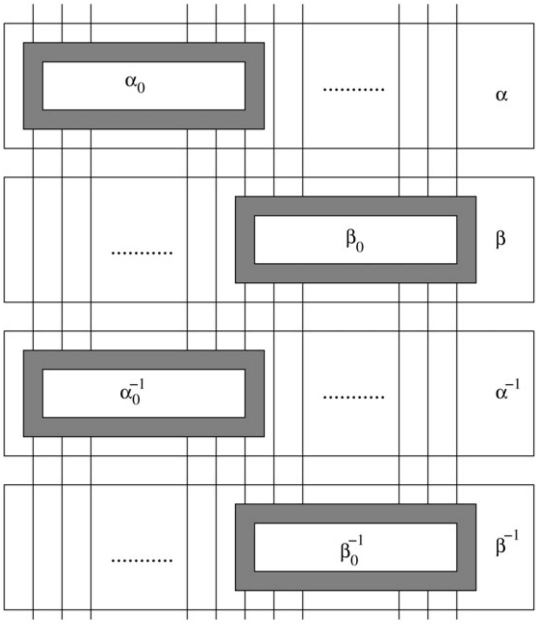


**Fig. 5.**

**Proof of Theorem 1.** $J_n(\alpha\beta\alpha^{-1}\beta^{-1})$ is a linear combination of the linear generators of $TL_n$. Obviously, the coefficient of the unit generator $\mathbf{1}$ is 1. What we need to show is that the coefficients of those non-unit generators that have at most $r+s$ cups are 0. We know that

$$J_n(\alpha\beta\alpha^{-1}\beta^{-1}) = J_n(\alpha)\cdot J_n(\beta)\cdot J_n(\alpha^{-1})\cdot J_n(\beta^{-1}) \tag{3.1}$$

and each of the four terms on the right is itself a linear combination of the generators of $TL_n$ with coefficients in $\mathbb{Z}[A, A^{-1}]$. Multiplying according to purely formal rules (but stopping one step short of using the product rules for $TL_n$ or collecting similar terms) will give a number of terms of the form $k(A, A^{-1})\cdot a\cdot v\cdot b\cdot w$ where $a, v, b, w$ are Kauffman diagrams. The notation $a, v, b, w$ is intended to ease keeping track of the source of the individual generators. Given the assumptions of the theorem many of the coefficients $k(A, A^{-1})$ will of course be 0. On the other hand, a non-unit generator with more than $r+s$ cups might appear somewhere in the sum with non-zero coefficient and we need to show that in the final sum such terms will cancel out. The idea of the proof of the theorem is to group some of the terms so that the cancellation will be obvious and then to analyze the remaining $k(A, A^{-1})\cdot a\cdot v\cdot b\cdot w$ terms individually to show that the product of $a\cdot v\cdot b\cdot w$ in $TL_n$ is a generator with more than $r+s$ cups.

We can rewrite $J_n(\alpha\beta\alpha^{-1}\beta^{-1})$ as

$$J_n(\alpha\beta\alpha^{-1}\beta^{-1}) = (\mathbf{1}+R(\alpha))\cdot(\mathbf{1}+R(\beta))\cdot(\mathbf{1}+R(\alpha^{-1}))\cdot(\mathbf{1}+R(\beta^{-1})). \tag{3.2}$$

The coefficient of $\mathbf{1}$ in all product terms is just 1 so in the $R$ summands the unit generator does not appear. Now, we can rewrite $J_n(\alpha\beta\alpha^{-1}\beta^{-1})$ again in the following form:

$$\begin{aligned} J_n(\alpha\beta\alpha^{-1}\beta^{-1}) &= \mathbf{1}\cdot(\mathbf{1}+R(\beta))\cdot\mathbf{1}\cdot(\mathbf{1}+R(\beta^{-1})) \\ &\quad + (\mathbf{1}+R(\alpha))\cdot\mathbf{1}\cdot(\mathbf{1}+R(\alpha^{-1}))\cdot\mathbf{1} - \mathbf{1} + S. \end{aligned} \tag{3.3}$$

Of course the term $S$ could be written more explicitly but we will not need this. The $-\mathbf{1}$ summand in the formula above is necessary because this (and only this) term is counted twice. Removing the Temperley–Lieb unit from the product where appropriate and making one step back in the notation of some terms we obtain the following:

$$\begin{aligned} J_n(\alpha\beta\alpha^{-1}\beta^{-1}) &= J_n(\beta)\cdot J_n(\beta^{-1}) + J_n(\alpha)\cdot J_n(\alpha^{-1}) - \mathbf{1} + S \\ &= \mathbf{1}+\mathbf{1}-\mathbf{1}+S \\ &= \mathbf{1}+S. \end{aligned} \tag{3.4}$$

It follows that the generators with more than $r+s$ cups could possibly appear only in the remainder sum $S$. The terms in this sum are of the form $k(A, A^{-1})\cdot a\cdot v\cdot b\cdot w$ where $a, b, v$ and $w$ are linear generators of $TL_n$. While some of $a, v, b, w$ may be equal to the unit generator $\mathbf{1}$, it is not possible that $a = b = \mathbf{1}$ or $v = w = \mathbf{1}$—in such case the whole

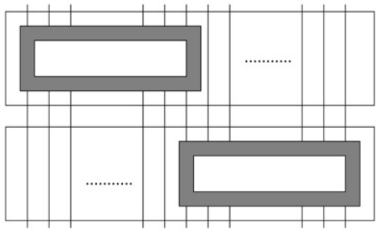

**Fig. 6.**

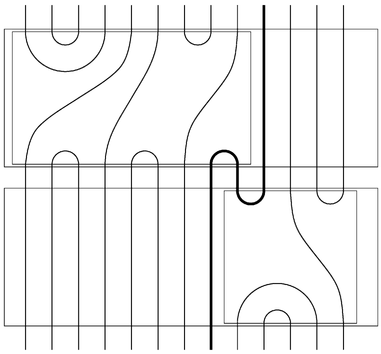

**Fig. 7.**

term $k(A, A^{-1}) \cdot a \cdot v \cdot b \cdot w$ would count to the sum considered separately earlier. Omitting the unit (possibly two units) whenever it does appear we arrive at the following possibilities with no unit appearing any more (among the listed terms): $(av)bw$, $v(bw)$, $a(bw)$, $(av)w$, $(av)b$, $(bw)$, $(aw)$, $(av)$, $(vb)$. Some terms are grouped in the above list. The reason is that we now intend to show that $av, bw, aw, vb \in I_{n,r+s}$. Since $I_{n,r+s}$ is an ideal in $TL_n$ it implies that all the listed products belong to $I_{n,r+s}$ and this is what we need to complete the proof of the theorem. Let us consider the product $av$ (the proof is similar in all cases). Fig. 6 illustrates the situation.

Of course we are interested only in those terms $k(A, A^{-1}) \cdot a \cdot v \cdot b \cdot w$ for which the coefficient $k(A, A^{-1})$ is non-zero. By the assumption made in the theorem this implies that there are at least $r+1$ cups in $a$ and at least $s+1$ cups in $v$. When we multiply $a$ and $v$ in $TL_n$ the cups of $a$ survive. The cups of $v$ also survive, except possibly one, which may cancel with a cap of $a$. This case is shown in Fig. 7. It follows that there are at least $r+s+1$ cups in $av$ which means that $av \in I_{n,r+s}$ as required. This completes the proof of the theorem. □

We will prove a minor refinement of Theorem 1, suitable for generating smaller examples. The idea is to slightly relax the assumptions about the two braids $\alpha_0$ and $\beta_0$. We will say that $\alpha_0$ is right $r$-split if the only Kauffman diagrams with $\leqslant r$ cups that do appear in $J_k(\alpha_0)$ are those in which the rightmost top input is connected with the rightmost bottom input. We define $\beta_0$ to be left $s$-split in a similar manner. We will also use the terms right split (left split) to mean the Kauffman diagrams with the rightmost (leftmost) string going straight down.

**Theorem 2.** *Let $\alpha_0 \in B_k$ and $\beta_0 \in B_m$ be two braids such that $\alpha_0$ is right $r$-split and $\beta_0$ is left $s$-split. Let $\alpha$ and $\beta$ be $n$-braids ($n = k+m-1$) obtained from the two given braids by adding $m-1$ straight strings to the right of $\alpha_0$ and $k-1$ strings to the left of $\beta_0$. Then $[\alpha, \beta] \in \ker J_{n,r+s}$.*

**Proof.** We want to prove that $\mathbf{1} - J_n(\alpha\beta\alpha^{-1}\beta^{-1}) \in I_{n,r+s}$. As in the proof of Theorem 1 we will write $J_n(\alpha\beta\alpha^{-1}\beta^{-1})$ as a linear combination of Kauffman diagrams,

$$J_n\left(\alpha\beta\alpha^{-1}\beta^{-1}\right) = \sum k\left(A, A^{-1}\right) \cdot a \cdot v \cdot b \cdot w. \tag{3.5}$$

Strictly speaking the coefficient polynomials $k(A^{-1}, A)$ do depend on the four terms and should be written as $k_{a,v,b,w}(A^{-1}, A)$ but we suppress the indices for simplicity. We can also write

$$\mathbf{1} = J_n\left(\alpha\alpha^{-1}\beta\beta^{-1}\right) = \sum k\left(A, A^{-1}\right) \cdot a \cdot b \cdot v \cdot w. \tag{3.6}$$

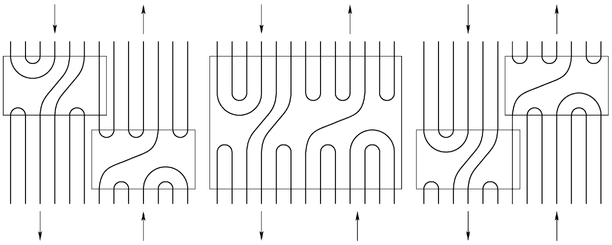

**Fig. 8.**

In the formulas (3.5) and (3.6) the corresponding terms $k(A, A^{-1}) \cdot a \cdot v \cdot b \cdot w$ and $k(A, A^{-1}) \cdot a \cdot b \cdot v \cdot w$ do have the same coefficient $k(A, A^{-1})$,

$$k_{a,v,b,w}\big(A^{-1}, A\big) = k_{a,b,v,w}\big(A^{-1}, A\big). \tag{3.7}$$

It follows that we need to prove that

$$\sum k\big(A, A^{-1}\big)(a \cdot b \cdot v \cdot w - a \cdot v \cdot b \cdot w) \in I_{n,r+s}. \tag{3.8}$$

We will prove that for every single term in the considered sum we have

$$a \cdot b \cdot v \cdot w - a \cdot v \cdot b \cdot w \in I_{n,r+s}. \tag{3.9}$$

If $b$ is right split or $v$ is left split, then $b$ and $v$ commute (this is illustrated in three stages in Fig. 8), so the considered difference is simply 0.

It remains to consider the case when $b$ is not right split and $v$ is not left split. This means that the number of cups in $v$ exceeds $s$ and the number of cups in $b$ exceeds $r$. Then $bv$ and $vb$ both belong to $I_{n,r+s}$. From this it follows that $a \cdot b \cdot v \cdot w$ and $a \cdot v \cdot b \cdot w$ both belong to $I_{n,r+s}$ and this implies that $a \cdot b \cdot v \cdot w - a \cdot v \cdot b \cdot w \in I_{n,r+s}$ as required. This completes the proof of the theorem. □

## 4. Examples

The 6 string braid −5 4 −3 −3 −2 1 1 −5 −5 −5 4 3 −2 −2 −2 1 2 −3 −4 5 2 3 5 5 3 −4 5 was found by one of our programs. It is right 1-split and it has 27 crossings. The left 1-split braid −6 7 −8 −8 −9 10 10 −6 −6 −6 7 8 −9 −9 −9 10 9 −8 −7 6 9 8 6 6 8 −7 6 (here given already in the form shifted to the right by 5 in $B_{11}$) was obtained from it in an obvious manner. The commutator of the two is the 108 crossing 11 string braid $\alpha =$ −5 4 −3 −3 −2 1 1 −5 −5 −5 4 3 −2 −2 −2 1 2 −3 −4 5 2 3 5 5 3 −4 5 −6 7 −8 −8 −9 10 10 −6 −6 −6 7 8 −9 −9 −9 10 9 −8 −7 6 9 8 6 6 8 −7 6 −5 4 −3 −5 −5 −3 −2 −5 4 3 −2 −1 2 2 2 −3 −4 5 5 5 −1 −1 2 3 3 −4 5 −6 7 −8 −6 −6 −8 −9 −6 7 8 −9 −10 9 9 9 −8 −7 6 6 6 −10 −10 9 8 8 −7 6.

This the shortest braid we know in any ker $J_{n,2}$. Its image $J_{11}(\alpha)$ is given in `http://www.mimuw.edu.pl/~traczyk/brdTL/TL11-108-2kernel.txt`. The (integer) coefficients of the polynomial coefficients of $J_{11}(\alpha)$ are all well within range of the `long long int` variable type of the `C++` compiler we used. We also found a 9 string braid with 380 crossings in ker $J_{9,2}$ using the same method.

## References

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