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Acyclic edge colouring of planar graphs without short cycles

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ABSTRACT

Let $G = (V, E)$ be any finite graph. A mapping $C : E \to [k]$ is called an *acyclic edge k-colouring* of G, if any two adjacent edges have different colours and there are no bichromatic cycles in G. In other words, for every pair of distinct colours i and j, the subgraph induced in G by all the edges which have colour i or j, is acyclic. The smallest number k of colours, such that G has an acyclic edge k-colouring is called the *acyclic chromatic index* of G, denoted by $\chi'_a(G)$.

In 2001, Alon et al. conjectured that for any graph G it holds that $\chi'_a(G) \le \Delta(G) + 2$; here $\Delta(G)$ stands for the maximum degree of G.

In this paper we prove this conjecture for planar graphs with girth at least 5 and for planar graphs not containing cycles of length 4, 6, 8 and 9. We also show that $\chi'_a(G) \le \Delta(G) + 1$ if G is planar with girth at least 6. Moreover, we find an upper bound for the acyclic chromatic index of planar graphs without cycles of length 4. Namely, we prove that if G is such a graph, then $\chi'_a(G) \le \Delta(G) + 15$.

1. Introduction

All graphs, which we consider, are finite, without loops or multiple edges. For a graph G, we denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For undefined concepts we refer the reader to [10].

As usual $[k]$ stands for the set $\{1, \dots, k\}$.

A mapping $C : E(G) \to [k]$ is called an *acyclic edge k-colouring* of a graph G, if any two adjacent edges have different colours and there are no bichromatic cycles in G, it means, for every pair of distinct colours i and j, the subgraph induced in G by all the edges which have colour i or j, is acyclic. The smallest number k of colours such that G has an acyclic edge k-colouring is called the *acyclic chromatic index* of G, denoted by $\chi'_a(G)$.

The acyclic chromatic index has been widely studied over past twenty years. The first general linear upper bound on $\chi'_a(G)$ was found by Alon et al. in [1]. Namely, they proved that $\chi'_a(G) \le 64\Delta(G)$. This bound was later improved to $16\Delta(G)$ by Molloy and Reed [5].

In 2001 Alon et al. [2] stated the Acyclic Edge Coloring Conjecture (AECC for short), which says that $\chi'_a(G) \le \Delta(G) + 2$ for all graphs G. In [2] this conjecture was proved to be true for random d-regular graphs and for graphs having sufficiently large girth. (The *girth* of a graph is defined as the length of its shortest cycle.)

The AECC was also verified for some special classes of graphs, including subcubic graphs [9], outerplanar graphs [8] and grid-like graphs [7]. Recently, Basavaraju and Chandran proved in [3] that any nonregular connected graph of maximum degree 3 is acyclically 4-edge colourable.

Since the problem of solving the AECC in general seems to be very difficult, it is interesting to prove it for some special classes of graphs or at least to give an upper bound for the acyclic chromatic index of a graph from a given class. In [4] such bounds for the class of planar graphs are given. Namely, the authors proved that the acyclic chromatic index of a planar graph G does not exceed $2\Delta(G) + 29$ and that for a planar graph G with girth at least 4 it holds that $\chi'_a(G) \le \Delta(G) + 6$.

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In this paper we prove that the AECC is true for planar graphs with girth at least 5 and for planar graphs not containing cycles of length 4, 6, 8 and 9. For the class of planar graphs with girth at least 6 we have a better bound. Namely, we show that if G is such a graph, then G is acyclically edge colourable with at most $\Delta(G)+1$ colours. Moreover, we prove that if G is planar and without cycles of length 4, then $\chi'_a(G) \le \Delta(G)+15$.

The paper is organized as follows: in Section 2 necessary notations are given. Section 3 contains the proofs concerning planar graphs with girth at least 5. The classes of planar graphs without specific short cycles are considered in Section 4.

2. Notations

In this section we introduce some necessary notations. Let G be a graph. For a vertex $v \in V(G)$, its *degree* is denoted by $d_G(v)$, or simply $d(v)$ when no confusion can arise. We call v a k-vertex if $d(v)=k$. Similarly, v is called a k^-- or a k^+-vertex if its degree is at most k or at least k, respectively. We denote by $l_k(v)$ (respectively, $l_{k^-}(v)$, $l_{k^+}(v)$) the cardinality of the set of these neighbours of v, which are k-vertices (respectively, k^--vertices, k^+-vertices).

Let G be a plane graph. We denote its face set by $F(G)$. Let $f \in F(G)$. We say that f is *incident* to a vertex x (or an edge e), if x (or e) belongs to the boundary of f. By the *degree* $d_G(f)$ (or simply $d(f)$) of a face f we mean the number of incident edges. A face of degree 3 will be called a *triangle*.

3. Planar graphs with girth at least 5

In this section we prove that the AECC is true for planar graphs with girth at least 5. Moreover, we show that if we have a planar graph G with girth at least 6, then its acyclic chromatic index does not exceed $\Delta(G)+1$.

Theorem 1. (a) *If the girth of a planar graph G is at least 6, then*

$$\chi'_a(G) \le \Delta(G)+1.$$

(b) *If the girth of a planar graph G is at least 5, then*

$$\chi'_a(G) \le \Delta(G)+2.$$

In order to prove Theorem 1, we recall the notion of the maximum average degree of a graph. Following [6], let us define for a given graph G the *maximum average degree* of G, denoted by $Mad(G)$, as follows:

$$Mad(G) = \max\left\{2\frac{|E(H)|}{|V(H)|} : H \subseteq G\right\}.$$

At the beginning we provide some structural properties of graphs with bounded maximum average degree in Lemmas 1 and 4. Next, we use these properties to obtain tight bounds for the acyclic chromatic index of such graphs. Finally, we prove Theorem 1, using the following basic observation, mentioned in [6], which shows the connection between the maximum average degree and the girth of a given planar graph. The proof of this observation is straightforward and follows easily from Euler's formula.

Proposition 1 ([6]). *If G is a planar graph with girth g, then $Mad(G) < \frac{2g}{g-2}$.*

In the proof of Lemma 1 a notion of a *weak* 3-vertex plays an important role. A vertex of degree 3 is called *weak*, if it has a neighbour of degree 2. For a vertex x, we denote by $l_{\overline{3}}(x)$ the cardinality of the set of weak 3-vertices in its neighbourhood.

Lemma 1. *Let G be a graph such that $Mad(G) < 3$ and $\delta(G) \ge 2$. Then G contains at least one of the configurations:*

$\mathcal{A}1$: *a 2-vertex adjacent to a 2-vertex,*
$\mathcal{A}2$: *a 3-vertex adjacent to a 2-vertex y_1 and a 3^--vertex y_2, $y_1 \ne y_2$,*
$\mathcal{A}3$: *a 4-vertex adjacent to a 2-vertex and three weak 3-vertices,*
$\mathcal{A}4$: *a 4-vertex adjacent to at least two 2-vertices and a weak 3-vertex,*
$\mathcal{A}5$: *a 4-vertex adjacent to at least three 2-vertices,*
$\mathcal{A}6$: *a 5-vertex adjacent to four 2-vertices and a weak 3-vertex,*
$\mathcal{A}7$: *a 5-vertex adjacent to five 2-vertices.*

Proof. We use the discharging method to prove the lemma. Let $G=(V,E)$ be a graph such that $Mad(G) < 3$ and $\delta(G) \ge 2$. Initially, we define a mapping f on the set of vertices of G as follows: for each $x \in V$ let $f(x)=d(x)$. From the definition of $Mad(G)$ it follows that

$$Mad(G) \ge \frac{2|E|}{|V|} = \frac{\sum_{x\in V} d(x)}{|V|} = \frac{\sum_{x\in V} f(x)}{|V|}. \tag{1}$$

In the discharging step, we redistribute the values of f between adjacent vertices, according to the rules described below, to obtain the function f'.

- If x is a 2-vertex, then x does not give anything to its neighbours;

- if x is a 3-vertex, then x gives $\frac{1}{2}$ to each 2-vertex in its neighbourhood;
- if x is a 4^+-vertex, then x gives $\frac{1}{2}$ to each 2-vertex in its neighbourhood and $\frac{1}{4}$ to each weak 3-vertex in its neighbourhood.

After this procedure, each vertex x has a new value $f'(x)$, but the sums of values of the functions f' and f, counting over all the vertices, are the same.

We show that if G does not contain any of the configurations $\mathcal{A}1$–$\mathcal{A}7$, then for each vertex x the value $f'(x)$ is greater than or equal to 3, contrary to the inequality (1). To calculate the value $f'(x)$ we consider a number of cases, depending on $d(x)$.

- If $d(x) = 2$, then $f'(x) = 2 + \frac{1}{2} \cdot l_{3^+}(x) = 3$, because $\mathcal{A}1$ does not hold.
- If $d(x) = 3$ and x is not weak, then $f'(x) = f(x) = 3$, therefore we can assume that x is a weak 3-vertex. Thus $f'(x) = 3 - \frac{1}{2} \cdot l_2(x) + \frac{1}{4} \cdot l_{4^+}(x)$. From the fact that $\mathcal{A}2$ does not hold we have that $l_2(x) < 2$ and that if $l_2(x) = 1$, then $l_3(x) = 0$. Hence, $l_{4^+}(x) = 2$ and $f'(x) = 3 - \frac{1}{2} + \frac{1}{4} \cdot 2 = 3$.
- If $d(x) = 4$, then $f'(x) = 4 - \frac{1}{2} \cdot l_2(x) - \frac{1}{4} \cdot l_{\overline{3}}(x)$. We can conclude, from the fact that $\mathcal{A}5$ does not hold, that $l_2(x) < 3$. Moreover, from the fact that $\mathcal{A}4$ cannot occur we have that if $l_2(x) = 2$, then $l_{\overline{3}}(x) = 0$ and then $f'(x) = 4 - 2 \cdot \frac{1}{2} = 3$. If $l_2(x) = 1$, then $l_{\overline{3}}(x) \leq 2$, because $\mathcal{A}3$ does not occur and therefore $f'(x) \geq 4 - \frac{1}{2} - 2 \cdot \frac{1}{4} = 3$. If $l_2(x) = 0$, then $f'(x) \geq 4 - 4 \cdot \frac{1}{4} = 3$.
- If $d(x) = 5$, then $f'(x) = 5 - \frac{1}{2} \cdot l_2(x) - \frac{1}{4} \cdot l_{\overline{3}}(x)$. From the fact that both $\mathcal{A}6$ and $\mathcal{A}7$ do not occur we have that $l_2(x) < 5$ and that if $l_2(x) = 4$, then $l_{\overline{3}}(x) = 0$. Hence, if $l_2(x) = 4$, then $f'(x) = 5 - \frac{1}{2} \cdot 4 = 3$. On the other hand, if $l_2(x) \leq 3$, then $f'(x) \geq 5 - \frac{1}{2} \cdot 3 - \frac{1}{4} \cdot 2 = 3$.
- If $d(x) = d \geq 6$, then $f'(x) \geq d - \frac{1}{2} \cdot l_2(x) - \frac{1}{4} \cdot l_{\overline{3}}(x) \geq d - \frac{1}{2} \cdot d = \frac{1}{2} \cdot d \geq 3$. ■

From now on, if we say a *colouring*, we always mean an *edge colouring*.

In what follows we frequently use the following notations. Let C be an acyclic k-colouring of a graph G. By $C(v)$, for any vertex $v \in V(G)$, we denote the set of colours assigned by C to the edges incident to v. If $W \subseteq V(G)$, then we define

$$C(W) = \bigcup_{w \in W} C(w).$$

If v and u are two distinct vertices of G, then let

$$W_G(v, u) = \{w \in N_G(v) : C(vw) \in C(u)\}.$$

Notice that the order of v and u is important here and that the set $W_G(v, u)$ could be empty.

The next useful lemma was stated in [4], as a modification of the Extension Lemma presented in [8].

Lemma 2 ([4]). *Let G be a graph, $vu \in E(G)$ and let C be an acyclic k-colouring of $G - vu$. If*

$$|C(v) \cup C(u) \cup C(W_{G-vu}(v, u))| < k,$$

then the colouring C can be extended to an acyclic k-colouring of G.

To shorten the proof of the next lemma we use a recent result on acyclic edge colourings of subcubic graphs.

Theorem 2 ([3]). *If G is a nonregular connected graph of maximum degree 3, then $\chi'_a(G) \leq 4$.*

Lemma 3. *If G is a graph such that $Mad(G) < 3$, then $\chi'_a(G) \leq \Delta(G) + 1$.*

Proof. Suppose H is a counterexample to the lemma with the minimum number of edges. Without loss of generality we can assume that H is connected and $\Delta(H) \geq 3$. Moreover, since each connected graph of maximum degree at most 3, but with maximum average degree less than 3, has a vertex of degree 2 or 1 and therefore, by Theorem 2, has the acyclic chromatic index at most 4, we can assume $\Delta(H) \geq 4$. Let $t = \Delta(H) + 1$.

We use Lemma 2 to prove that if for some edge $vu \in E(H)$ the graph $H - vu$ has an acyclic t-colouring, then the graph H also has an acyclic t-colouring, which contradicts its choice.

It is easy to see that H cannot contain vertices of degree one. Hence, we can assume $\delta(H) \geq 2$. According to Lemma 1 we have that H contains at least one of the configurations $\mathcal{A}1$–$\mathcal{A}7$.

$\mathcal{A}1$: If H contains a 2-vertex x, adjacent to a 2-vertex y_1, then let y_2 be the remaining neighbour of x. Let $H' = H - xy_1$. Since H' has less edges than H, by the minimality of H, we have that H' has an acyclic t-colouring C. Clearly, $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$, therefore, by Lemma 2, we conclude that the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

$\mathcal{A}2$: If H contains a 3-vertex x adjacent to a 2-vertex y_1 and to a 3^--vertex y_2, then let y_3 be the remaining neighbour of x. Let $H' = H - xy_1$. Similarly as above, from the minimality of H, we have that H' has an acyclic t-colouring C.

Let u be the neighbour of y_1 in H'. If $C(y_1u) \neq C(xy_3)$, then observe that $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$, therefore, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

In the opposite case, if $C(y_1u) = C(xy_3)$, then we can recolour, in H', the edge y_1u with a colour $\alpha \notin C(u) \cup \{C(xy_3)\}$, obtaining in this way an acyclic t-colouring C' of H', and we are back in the previous case.

$\mathcal{A}3$: If H contains a 4-vertex x which is adjacent to a 2-vertex y_1 and to three weak 3-vertices y_2, y_3 and y_4, then let z_i be the neighbour of degree 2 of y_i, for $i = 2, 3, 4$. Moreover, let $H' = H - xy_1$. From the minimality of H, it follows that H' has an acyclic t-colouring C. If $C(y_1) \cap C(x) = \emptyset$, then clearly the colouring C can be extended to an acyclic t-colouring of H. Therefore we can assume that $C(y_1) \cap C(x) = \{a\}$. There is no loss of generality in assuming that $C(xy_2) = a, C(xy_3) = b$ and $C(xy_4) = c$. Let z_1 be the neighbour, in H', of y_1. We can assume that $|C(x) \cup C(z_1)| = t$ and $|C(x) \cup C(y_2)| = t$, since otherwise the colouring C can be extended to an acyclic t-colouring of H.

One can observe that $|C(x) \cup C(y_2)|$ can equal t only if $t = 5$, because $d_H(y_2) = 3$ and $d_H(x) = 4$, hence we can now assume $\Delta(H) = 4$.

Clearly, one of the colours b, c, say b, cannot belong to $C(z_1)$. Moreover, it is easy to observe that if $|C(x) \cup C(y_3)| < 5$, then we can recolour, in H', the edge y_1z_1 with the colour b, obtaining in this way an acyclic 5-colouring of H', which can be extended to an acyclic 5-colouring of H.

Hence we can assume $|C(x) \cup C(y_3)| = 5$. Let u be the neighbour of z_2 different from y_2. One can observe that $C(z_2u) = a$, since otherwise the colouring C can be extended to an acyclic 5-colouring of H. Let $d = C(y_2z_2)$. Clearly $d \notin \{a, b, c\}$ and we can recolour, in H', the edge y_2z_2 with the colour b and then we can colour the edge xy_1 with the colour d, obtaining an acyclic 5-colouring of H, a contradiction.

$\mathcal{A}4$: If H contains a 4-vertex x which is adjacent to two 2-vertices y_1 and y_2, and to a weak 3-vertex y_3, then let y_4 be the remaining neighbour of x. Moreover, let $z_1 \neq x$ be the neighbour of y_1 and let z_2 be the neighbour of y_3 of degree 2. Let $H' = H - xy_1$. Since H' has less edges than H, by the minimality of H we have that H' has an acyclic t-colouring C. Assume that $C(xy_2) = a$, $C(xy_3) = b$ and $C(xy_4) = c$. If $C(x) \cap C(y_1) = \emptyset$ or $C(y_1z_1) = a$, then clearly the colouring C can be extended to an acyclic t-colouring of H, a contradiction. Hence there are two cases to consider: one if $C(y_1z_1) = b$ and another if $C(y_1z_1) = c$.

At the beginning we assume that $C(y_1z_1) = b$. One can observe that if $|C(x) \cup C(y_3)| < t$ or $|C(x) \cup C(z_1)| < t$ or $b \notin C(z_2)$, then the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

Therefore, similarly, as in the case $\mathcal{A}3$, we can assume that $t = 5$ and $\Delta(H) = 4$, since otherwise $|C(x) \cup C(y_3)| < t$.

Thus $|C(x) \cup C(y_3)| = 5$ and $|C(x) \cup C(z_1)| = 5$ and $b \in C(z_2)$. Moreover, $a \in C(z_1)$, since otherwise we can recolour, in H', the edge y_1z_1 with a, obtaining in this way an acyclic 5-colouring of H', which can be extended to an acyclic 5-colouring of H. Therefore, we can assume that $C(z_1) = \{a, b, d, e\}$ and $C(y_3) = \{b, d, e\}$, and there is no loss of generality in assuming $C(y_3z_2) = d$, with $\{a, b, c\} \cap \{d, e\} = \emptyset$.

Clearly, if $C(y_2) \neq \{a, b\}$, then we can recolour, in H', the edge y_3z_2 with a, obtaining in this way an acyclic 5-colouring of H', which can be extended to an acyclic 5-colouring of H. Hence $C(y_2) = \{a, b\}$.

It is also easy to see that if $d \notin C(y_4)$ (or, similarly, $e \notin C(y_4)$), then we can recolour, in H', the edge y_1z_1 with c and then we can colour, in H, the edge xy_1 with d (or e), obtaining an acyclic 5-colouring of H, a contradiction.

Hence we can assume $d, e \in C(y_4)$. Moreover, $a \in C(y_4)$, since otherwise we can recolour, in H', the edge y_1z_1 with c, xy_2 with c and xy_4 with a, obtaining in this way an acyclic 5-colouring of H', which can be extended to an acyclic 5-colouring of H, a contradiction.

Therefore, $C(y_4) = \{a, c, d, e\}$ and we can recolour, in H', the edge y_3z_2 with c to obtain an acyclic 5-colouring of H', which clearly can be extended, a contradiction.

In the opposite case, if $C(y_1z_1) = c$, then we can recolour, in H', the edge y_1z_1 with a colour $\alpha \notin C(z_1) \cup \{C(xy_4)\}$, and we are in the previous case.

$\mathcal{A}5$: If H contains a 4-vertex x adjacent to three 2-vertices: y_1, y_2 and y_3, then let y_4 be the remaining neighbour of x. Moreover, let $H' = H - xy_1$. Since H' has less edges than H, by the minimality of H we have that H' has an acyclic t-colouring C. Let z_1 be the neigbour, in H', of y_1. One can observe that if $C(y_1z_1) \neq C(xy_4)$, then the colouring C can be extended to an acyclic t-colouring of H, a contradiction. Hence we can assume that $C(y_1z_1) = C(xy_4)$. Clearly, we can recolour, in H', the edge y_1z_1 with a colour $\alpha \notin C(z_1) \cup \{C(xy_4)\}$ to obtain an acyclic t-colouring of H', which can be extended to an acyclic t-colouring of H, a contradiction.

$\mathcal{A}6$: If H contains a 5-vertex x adjacent to four 2-vertices: y_1, y_2, y_3 and y_4 and to a weak 3-vertex y_5, then let $H' = H - xy_1$. As above, from the minimality of H, we have that H' has an acyclic t-colouring C. Let z_1 be the neigbour, in H', of y_1. One can observe that if $C(y_1z_1) \neq C(xy_5)$, then the colouring C can be extended to an acyclic t-colouring of H, a contradiction. On the other hand, if $C(y_1z_1) = C(xy_5)$, then we can recolour, in H', the edge y_1z_1 with a colour $\alpha \notin C(z_1) \cup \{C(xy_5)\}$, and we are in the previous case.

$\mathcal{A}7$: If H contains a 5-vertex x adjacent to five 2-vertices, then let y be any of them. Moreover, let $H' = H - xy$. Since H' has less edges than H, by the minimality of H we have that H' has an acyclic t-colouring C. It is easy to observe that, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction. ■

Lemma 4. *Let G be a graph such that $Mad(G) < \frac{10}{3}$ and $\delta(G) \geq 2$. Then G contains at least one of the configurations:*

$\mathcal{B}1$: *a 2-vertex adjacent to a 3^--vertex,*
$\mathcal{B}2$: *a 3-vertex adjacent to at least two 3^--vertices,*
$\mathcal{B}3$: *a 4-vertex adjacent to a 2-vertex y_1 and a 3^--vertex y_2, $y_1 \neq y_2$,*
$\mathcal{B}4$: *a 5-vertex adjacent to at least three 2-vertices,*
$\mathcal{B}5$: *a 5-vertex adjacent to two 2-vertices y_1, y_2 and three 3^--vertices y_3, y_4, y_5, $y_i \neq y_j$, for $i \neq j$,*

$\mathcal{B}6$: *a* 6*-vertex adjacent to at least four* 2*-vertices* $y_1, \ldots, y_4$ *and a* 3^-*-vertex* y_5, $y_i \neq y_j$, *for* $i \neq j$,
$\mathcal{B}7$: *a* 7*-vertex adjacent to at least six* 2*-vertices,*
$\mathcal{B}8$: *an* 8*-vertex adjacent to seven* 2*-vertices* $y_1, \ldots, y_7$ *and a* 3^-*-vertex* y_8, $y_i \neq y_j$, *for* $i \neq j$,
$\mathcal{B}9$: *a* 9*-vertex adjacent to nine* 2*-vertices.*

Proof. Let $G = (V, E)$ be a graph such that $Mad(G) < \frac{10}{3}$ and $\delta(G) \geq 2$. We proceed similarly as in the proof of Lemma 1 using the discharging method. In the first step, we define a mapping f on the set of vertices of G as follows: for each $x \in V$ let $f(x) = d(x)$.

In the discharging step, we redistribute the values of f between adjacent vertices, according to the rules described below, to obtain the function f'.

- If x is either a 2-vertex or a 3-vertex, then x does not give anything to its neighbours;
- if $d(x) \geq 4$, then x gives $\frac{2}{3}$ to each 2-vertex in its neighbourhood and $\frac{1}{6}$ to each 3-vertex in its neighbourhood.

After this procedure, each vertex x has a new value $f'(x)$, but the sums of values of the functions f' and f, counting over all the vertices, are the same.

Now we check that if G does not contain any of the configurations $\mathcal{B}1$–$\mathcal{B}9$, then $f'(x) \geq \frac{10}{3}$ for each vertex x, which is an obvious contradiction with the inequality (1).

- If $d(x) = 2$, then $f'(x) = 2 + \frac{2}{3} \cdot l_{4^+}(x) = \frac{10}{3}$, because $\mathcal{B}1$ does not hold.
- If $d(x) = 3$, then $f'(x) = 3 + \frac{1}{6} \cdot l_{4^+}(x) \geq \frac{10}{3}$, since $\mathcal{B}2$ does not occur.
- If $d(x) = 4$, then $f'(x) = 4 - \frac{2}{3} \cdot l_2(x) - \frac{1}{6} \cdot l_3(x)$. We can conclude, from the fact that $\mathcal{B}3$ does not hold, that $l_2(x) \leq 1$ and moreover, that if $l_2(x) = 1$, then $l_3(x) = 0$. Therefore, $f'(x) = 4 - \frac{2}{3} = \frac{10}{3}$. On the other hand, if $l_2(x) = 0$, then $f'(x) \geq 4 - 3 \cdot \frac{1}{6} = \frac{7}{2}$.
- If $d(x) = 5$, then $f'(x) = 5 - \frac{2}{3} \cdot l_2(x) - \frac{1}{6} \cdot l_3(x)$. From the fact that $\mathcal{B}4$ does not occur we have that $l_2(x) \leq 2$. If $l_2(x) = 2$, then $l_3(x) \leq 2$, since $\mathcal{B}5$ cannot occur, and therefore, $f'(x) \geq 5 - \frac{2}{3} \cdot 2 - \frac{1}{6} \cdot 2 = \frac{10}{3}$. If we assume that $l_2(x) \leq 1$, then $f'(x) \geq 5 - \frac{2}{3} - \frac{1}{6} \cdot 4 = \frac{11}{3}$.
- If $d(x) = 6$, then $f'(x) = 6 - \frac{2}{3} \cdot l_2(x) - \frac{1}{6} \cdot l_3(x)$. From the fact that $\mathcal{B}6$ does not occur we have that $l_2(x) \leq 4$ and that if $l_2(x) = 4$, then $l_3(x) = 0$, and therefore, $f'(x) = 6 - \frac{2}{3} \cdot 4 = \frac{10}{3}$. Finally, if $l_2(x) \leq 3$, then $f'(x) \geq 6 - \frac{2}{3} \cdot 3 - \frac{1}{6} \cdot 3 = \frac{7}{2}$.
- If $d(x) = 7$, then $f'(x) = 7 - \frac{2}{3} \cdot l_2(x) - \frac{1}{6} \cdot l_3(x)$. From the fact that $\mathcal{B}7$ does not hold we have that $l_2(x) \leq 5$ and hence $f'(x) \geq 7 - \frac{2}{3} \cdot 5 - \frac{1}{6} \cdot 2 = \frac{10}{3}$.
- If $d(x) = 8$, then $f'(x) = 8 - \frac{2}{3} \cdot l_2(x) - \frac{1}{6} \cdot l_3(x)$. From the fact that $\mathcal{B}8$ does not occur we have that $l_2(x) \leq 7$ and that if $l_2(x) = 7$, then $l_3(x) = 0$ and $f'(x) = 8 - \frac{2}{3} \cdot 7 = \frac{10}{3}$. If $l_2(x) \leq 6$, then $f'(x) \geq 8 - \frac{2}{3} \cdot 6 - \frac{1}{6} \cdot 2 = \frac{11}{3}$.
- If $d(x) = 9$, then $f'(x) = 9 - \frac{2}{3} \cdot l_2(x) - \frac{1}{6} \cdot l_3(x)$. From the fact that $\mathcal{B}9$ does not occur we have that $l_2(x) \leq 8$. Hence $f'(x) \geq 9 - \frac{2}{3} \cdot 8 - \frac{1}{6} > \frac{10}{3}$.
- If $d(x) = d \geq 10$, then $f'(x) = d - \frac{2}{3} \cdot l_2(x) - \frac{1}{6} \cdot l_3(x) \geq d - \frac{2}{3} \cdot d = \frac{1}{3} \cdot d \geq \frac{10}{3}$. ■

Lemma 5. *If* $Mad(G) < \frac{10}{3}$, *then* $\chi'_a(G) \leq \Delta(G) + 2$.

Proof. First suppose H is a counterexample to the lemma with the minimum number of edges. Without loss of generality we can assume that H is connected. Let $t = \Delta(H) + 2$. We will use Lemma 2 to prove that if for some edge $vu \in E(H)$ the graph $H - vu$ has an acyclic t-colouring, then the graph H also has an acyclic t-colouring, which contradicts its choice.

It is easy to see that H cannot contain any vertex of degree one. Therefore, we can assume $\delta(H) \geq 2$.We can also assume $\Delta(H) \geq 4$, because the proof in the case $\Delta(H) \leq 3$ follows from the fact that the acyclic chromatic index of any subcubic graph is at most 5, see [1,9].

From Lemma 4 we have that H contains at least one of the configurations $\mathcal{B}1$–$\mathcal{B}9$.

$\mathcal{B}1$: If H contains a 2-vertex x, adjacent to a 3^--vertex y_1, then let y_2 be the remaining neighbour of x. Moreover, let $H' = H - xy_1$. H' has less edges than H, hence, by the minimality of H, we have that H' has an acyclic t-colouring C. Clearly, $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$. By Lemma 2, we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

$\mathcal{B}2$: If H contains a 3-vertex x, adjacent to two 3^--vertices y_1 and y_2, then let y_3 be the remaining neighbour of x. Moreover, let $H' = H - xy_1$. As above, by the minimality of H, we have that H' has an acyclic t-colouring C. If $|C(y_1) \cap C(x)| \leq 1$, then, by Lemma 2 and since $t \geq 6$, we can extend the colouring C to an acyclic t-colouring of H, a contradiction. Hence $|C(y_1) \cap C(x)| = 2$. Let $C(y_1) = \{a, b\}$ and $C(xy_2) = a$, $C(xy_3) = b$.

Let us initially assume that $|C(y_3) \cup C(y_2)| = t$. If $a \notin C(y_3)$, then there is a colour $\alpha \notin C(y_3)$, $\alpha \neq a$ such that we can recolour, in H', the edge xy_3 with α and obtain in this way an acyclic t-colouring C' of H' in which $|C'(y_1) \cap C'(x)| \leq 1$. This colouring can be extended to an acyclic t-colouring of H, a contradiction.

Therefore, we can assume that $a \in C(y_3)$. Hence $b \notin C(y_2)$ and there is a colour $\alpha \notin C(y_2)$, $\alpha \notin \{a, b\}$ such that we can recolour, in H', the edge xy_2 with α and obtain in this way an acyclic t-colouring C' of H' in which $|C'(y_1) \cap C'(x)| \leq 1$.

On the other hand, if $|C(y_3) \cup C(y_2)| < t$, then there is a colour $\alpha \notin C(y_1) \cup C(y_2) \cup C(y_3)$ and we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

$\mathcal{B}3$: If H contains a 4-vertex x adjacent to a 2-vertex y_1 and to a 3^--vertex y_2, then let y_3, y_4 be the remaining neighbours of x and let $H' = H - xy_1$. H' has less edges than H, hence, by the minimality of H, we have that H' has an acyclic t-colouring C. Assume that $C(y_1) = \{b\}$. One can observe that if $b \notin \{C(xy_3), C(xy_4)\}$, then $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$ and, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

Hence we can assume that $b = C(xy_3)$ (or, similarly, $b = C(xy_4)$). Let u be the neighbour of y_1 in H'. We can recolour, in H', the edge y_1u with a colour $\alpha \notin C(y_1) \cup C(u) \cup \{C(xy_3), C(xy_4)\}$ to obtain an acyclic t-colouring C' of H' which we can extend to an acyclic t-colouring of H, a contradiction.

$\mathcal{B}4$: If H contains a 5-vertex x adjacent to three 2-vertices y_1, y_2 and y_3, then let y_4, y_5 be the remaining neighbours of x and let $H' = H - xy_1$. H' has less edges than H, and, by the minimality of H, it follows that H' has an acyclic t-colouring C. Assume $C(y_1) = \{b\}$. If $b \notin \{C(xy_4), C(xy_5)\}$, then it is easy to observe that $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$, and, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

Hence we can assume that $b = C(xy_4)$ (or, similarly, $b = C(xy_5)$). Moreover, let u be the neighbour of y_1 in H'. Clearly, we can recolour, in H', the edge y_1u with a colour $\alpha \notin C(y_1) \cup C(u) \cup \{C(xy_4), C(xy_5)\}$ to obtain an acyclic t-colouring C' of H' and we are back in the previous case.

$\mathcal{B}5$: If H contains a 5-vertex x adjacent to two 2-vertices y_1 and y_2, and to three 3^--vertices y_3, y_4, y_5, then let $H' = H - xy_1$. H' has less edges than H, hence, by the minimality of H, we have that H' has an acyclic t-colouring C. It is easy to observe that $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$, therefore, by Lemma 2, we conclude that the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

$\mathcal{B}6$: If H contains a 6-vertex x adjacent to four 2-vertices y_1, y_2, y_3, y_4 and to a 3^--vertex y_5, then let y_6 be the remaining neighbour of x and let $H' = H - xy_1$. Similarly as above, H' has an acyclic t-colouring of C. We can assume that $C(y_1) = \{b\}$. If $b \notin \{C(xy_5), C(xy_6)\}$, then it is easy to observe that $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 2$, therefore, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

On the other hand, if $b = C(xy_5)$ (or, similarly, $b = C(xy_6)$), then let u be the neighbour of y_1 in H'. We can recolour, in H', the edge y_1u with a colour $\alpha \notin C(y_1) \cup C(u) \cup \{C(xy_5), C(xy_6)\}$ to obtain an acyclic t-colouring C' of H' and we are back in the previous case.

$\mathcal{B}7$: If H contains a 7-vertex x adjacent to six 2-vertices $y_1, y_2, \ldots, y_6$ and to a vertex y_7, then let $H' = H - xy_1$. H' has less edges than H, hence, by the minimality of H, we have that H' has an acyclic t-colouring C. We can assume that $C(y_1) = \{b\}$.

If $b \neq C(xy_7)$, then clearly $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 2$, and, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

In the opposite case, if $b = C(xy_7)$, then let u be the neighbour of y_1 in H'. We can recolour, in H', the edge y_1u with a colour $\alpha \notin C(y_1) \cup C(u) \cup \{C(xy_7)\}$ to obtain an acyclic t-colouring C' of H', which can be extended to an acyclic t-colouring of H, a contradiction.

$\mathcal{B}8$: If H contains an 8-vertex x adjacent to seven 2-vertices $y_1, y_2, \ldots, y_7$ and to a 3^--vertex y_8, then let $H' = H - xy_1$. Clearly, from the minimality of H, it follows that H' has an acyclic t-colouring C. It is easy to observe that $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$, therefore, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

$\mathcal{B}9$: If H contains a 9-vertex x adjacent to nine 2-vertices, then let y be any of them and let $H' = H - xy$. H' has less edges than H, and, by the minimality of H, it follows that H' has an acyclic t-colouring C. It is easy to observe that $|C(x) \cup C(y) \cup C(W_{H'}(x, y))| \leq t - 2$. Hence, by Lemma 2, the colouring C can be extended to an acyclic t-colouring of H, a contradiction. ■

Proof of Theorem 1. It follows from Lemmas 3, 5 and Proposition 1. ■

4. Planar graphs without short cycles

In this section we prove that the AECC is true for planar graphs not containing cycles of length 4, 6, 8, 9 and also present new upper bound for the acyclic chromatic index of planar graphs not containing cycles of length 4.

We start with the following lemma which presents structural properties of 2-connected planar graphs without cycles of length 4, 6, 8, 9.

Lemma 6. *Let G be a 2-connected planar graph not containing cycles of length 4, 6, 8, 9. Then G contains at least one of the configurations:*

$\mathcal{C}1$: *a triangle incident to at least two 3^--vertices,*
$\mathcal{C}2$: *a 2-vertex adjacent to a 3^--vertex,*
$\mathcal{C}3$: *a 3-vertex adjacent to at least two 3^--vertices,*
$\mathcal{C}4$: *a 4-vertex x adjacent to a 2-vertex y_1 and a 4^--vertex y_2, $y_1 \neq y_2$, and such that there is a triangle which is incident to both x and y_2, but not to y_1,*
$\mathcal{C}5$: *a 4-vertex x adjacent to a 2-vertex y and such that there is a triangle which is incident to both x and y,*
$\mathcal{C}6$: *a d-vertex, with $d \geq 4$, such that at least $d - 2$ of its neighbours are 3^--vertices and at least one of them is of degree 2.*

Proof. We use the discharging method to prove the lemma. Let $G = (V, E)$ be a 2-connected planar graph without cycles of length 4, 6, 8, 9. We fix a plane embedding of G. Let F be the face set of G. At the beginning, we define a mapping f on $V \cup F$ as follows: for each $x \in V$ let $f(x) = \frac{6}{5}d(x) - 4$ and for each $x \in F$ let $f(x) = \frac{4}{5}d(x) - 4$. Clearly,

$$\sum_{x\in V\cup F} f(x) = \sum_{x\in V}\left(\frac{6}{5}d(x) - 4\right) + \sum_{x\in F}\left(\frac{4}{5}d(x) - 4\right)$$
$$= \frac{6}{5}\sum_{x\in V} d(x) - 4|V| + \frac{4}{5}\sum_{x\in F} d(x) - 4|F| = 4(|E| - |V| - |F|) = -8,$$

which follows from Euler's formula and the fact that $\sum_{x\in V} d(x) = \sum_{x\in F} d(x) = 2|E|$. In the next step we will distribute the values of f between adjacent vertices, faces and between incident vertices and faces, according to the rules described below, to obtain the function f'.

- If x is either a cycle of length less than 10 or a 3^--vertex, then x does not give anything;
- if x is a cycle of length at least 10, then x gives $\frac{2}{5} \cdot l_a(x, y)$ to each adjacent triangle y, where $l_a(x, y)$ is the number of common edges (of the cycle x and the triangle y);
- if x is a 4-vertex, then x gives $\frac{1}{5}$ to each incident triangle, provided the triangle is not incident to two 5^+-vertices;
- if x is a 5^+-vertex, then x gives $\frac{1}{5}$ to each incident triangle;
- if x is a 4^+-vertex, then x gives $\frac{4}{5}$ to each adjacent 2-vertex and $\frac{1}{5}$ to each adjacent 3-vertex.

Now we calculate the values of the function f'. We show, that if the graph G does not contain any of the configurations $\mathcal{C}1$–$\mathcal{C}6$, then the value $f'(x)$ is nonnegative for each $x \in V \cup F$, contrary to the fact that the sums of values of the functions f' and f, counting over all the vertices and faces, remain the same.

At the beginning we calculate the value $f'(x)$ for each vertex x. Let us start with the following straightforward, but very useful, observation.

Proposition 2. *If G is a 2-connected planar graph without cycles of length 4 and v is a d-vertex, then v is incident to at most $\frac{d}{2}$ triangles.*

We consider a number of cases, depending on the degree of the vertex x. Let $l_t(x)$ stand for the number of triangles incident to x. Let $l_{\bar{t}}(x)$ denote the number of such triangles incident to x, which are not incident to two 5^+-vertices. Clearly, $l_{\bar{t}}(x) \le l_t(x)$.

- If $d(x) = 2$, then $f'(x) = \frac{6}{5} \cdot 2 - 4 + \frac{4}{5} \cdot l_{4^+}(x) = 0$, since $\mathcal{C}2$ does not hold.
- If $d(x) = 3$, then $f'(x) = \frac{6}{5} \cdot 3 - 4 + \frac{1}{5} \cdot l_{4^+}(x) \ge 0$, because $l_{4^+}(x) \ge 2$, from the absence of $\mathcal{C}3$.
- If $d(x) = 4$, then $f'(x) = \frac{6}{5} \cdot 4 - 4 - \frac{4}{5} \cdot l_2(x) - \frac{1}{5} \cdot l_3(x) - \frac{1}{5} \cdot l_{\bar{t}}(x)$. From Proposition 2 it follows that $l_{\bar{t}}(x) \le 2$. From the fact that $\mathcal{C}6$ does not hold it clearly follows that $l_2(x) < 2$ and that if $l_2(x) = 1$, then $l_3(x) = 0$.

  If we assume that $l_2(x) = 1$, $l_3(x) = 0 = l_{\bar{t}}(x)$, then clearly $f'(x) = 0$. Hence we can assume that if $l_2(x) = 1$, then $l_{\bar{t}}(x) = 1$ or $l_{\bar{t}}(x) = 2$.

  In the case when $l_2(x) = 1$ and $l_{\bar{t}}(x) = 1$, let y_1 be the 2-vertex adjacent to x and let T be the triangle incident to x. It is easy to observe that y_1 cannot be incident to T, because $\mathcal{C}5$ does not occur. Hence there must be another 4^--vertex, say y_2, which is adjacent to x and incident to T, but this situation is also impossible, since $\mathcal{C}4$ does not hold.

  It is easy to observe that the case $l_2(x) = 1$ and $l_{\bar{t}}(x) = 2$ is also impossible, since $\mathcal{C}5$ does not occur.

  Hence we can assume $l_2(x) = 0$. If moreover $l_{\bar{t}}(x) = 0$, then clearly $f'(x) \ge 0$. It is also easy to observe that if $l_3(x) = 4$, then $l_t(x) = 0$, because $\mathcal{C}1$ does not hold. Moreover, from the absence of $\mathcal{C}1$, we have that if $l_3 = 3$, then $l_t(x) \le 1$ and $f'(x) \ge 0$.

  Hence there is only one case left to consider, namely, when $0 < l_{\bar{t}}(x) \le 2$ and $l_3(x) \le 2$, but in this case we also have $f'(x) \ge 0$.
- If $d(x) = 5$, then $f'(x) = \frac{6}{5} \cdot 5 - 4 - \frac{4}{5} \cdot l_2(x) - \frac{1}{5} \cdot l_3(x) - \frac{1}{5} \cdot l_t(x)$.

  Observe at the beginning that from the absence of $\mathcal{C}6$ it follows that if $l_2(x) > 0$, then $l_2(x) + l_3(x) \le 2$. Moreover, from Proposition 2, we have $l_t(x) \le 2$.

  If we assume that $l_2(x) = 2$, then clearly $l_3(x) = 0$ and $f'(x) \ge 0$. It is also easy to calculate that if $l_2(x) = 1$, then $f'(x) > 0$, since $l_3(x) \le 1$ in this case.

  In the opposite case, when $l_2(x) = 0$, then it is easy to observe that $f'(x) = 2 - \frac{1}{5} \cdot l_3(x) - \frac{1}{5} \cdot l_t(x) > 0$, since $l_3(x) \le 5$ and $l_t(x) \le 2$.
- If $d(x) = d \ge 6$, then $f'(x) = \frac{6}{5} \cdot d - 4 - \frac{4}{5} \cdot l_2(x) - \frac{1}{5} \cdot l_3(x) - \frac{1}{5} \cdot l_t(x)$. As above, we have $l_t(x) \le \frac{d}{2}$. To calculate the value $f'(x)$ we need to consider two cases.

  If $l_2(x) > 0$, then from the fact that $\mathcal{C}6$ does not hold we have $l_2(x) + l_3(x) \le d - 3$. Hence $f'(x) \ge \frac{6}{5} \cdot d - 4 - \frac{4}{5} \cdot (d - 3) - \frac{1}{5} \cdot \frac{d}{2} = \frac{3}{10} \cdot d - \frac{8}{5} > 0$.

  In the opposite case, when $l_2(x) = 0$, we have $f'(x) \ge \frac{6}{5} \cdot d - 4 - \frac{1}{5} \cdot d - \frac{1}{5} \cdot \frac{d}{2} = \frac{9}{10} \cdot d - 4 > 0$.

To finish the proof it is enough to show that for each face x the value $f'(x)$ is nonnegative. It is easy to observe that from the absence of cycles of lengths 4, 6, 8 and 9, we have that a triangle cannot be adjacent to a cycle of length less than 10. Hence for each triangle x there are three adjacent cycles of length at least 10. Moreover, from the fact that $\mathcal{C}1$ does not hold, it follows that there are at least two 4^+-vertices incident to x. Hence, $f'(x) \geq \frac{4}{5} \cdot 3 - 4 + \frac{2}{5} \cdot 3 + \frac{1}{5} \cdot 2 = 0$.

Clearly, for each cycle x of length 5 or 7, the value $f'(x) = f(x)$ and is greater than or equal to 0.

Moreover, if x is a cycle of length $d \geq 10$, then it is obvious that $f'(x) \geq \frac{4}{5} \cdot d - 4 - \frac{2}{5} \cdot d = \frac{2}{5} \cdot d - 4 \geq 0$.

We have shown that for each $x \in V \cup F$ the value $f'(x)$ is nonnegative, which completes the proof. ■

Theorem 3. *If G is a planar graph not containing any cycle of length* 4, 6, 8 *and* 9, *then*

$$\chi'_a(G) \leq \Delta(G) + 2.$$

Proof. Let H be a counterexample to Theorem 3 with the minimum number of edges. There is no loss of generality in assuming that H is connected. We fix a plane embedding of G. As in the prove of Lemma 5, we can assume $\Delta(H) \geq 4$. Let $t = \Delta(H) + 2$. It is easy to observe that H is 2-connected, since otherwise we can obtain an acyclic t-colouring of each its 2-connected component and combine them (by permuting some colours, if needed) to get an acyclic colouring of the entire graph. Hence, by Lemma 6, we have that H contains at least one of the configurations $\mathcal{C}1$–$\mathcal{C}6$.

By Lemma 2, it is sufficient to show that there exists an edge vu and an acyclic t-colouring C of $H - vu$ such that $|C(v) \cup C(u) \cup C(W_{H-vu}(v, u))| < t$. We consider a number of cases, depending on which of the configurations $\mathcal{C}1$–$\mathcal{C}6$ occurs in H. In each case we point out such an edge which we can use with Lemma 2 to obtain a contradiction.

$\mathcal{C}1$: If H contains a triangle incident to the vertices x_1, x_2, x_3, such that $d(x_1), d(x_2) \leq 3$, then let $H' = H - x_1x_2$. From the minimality of H it follows that H' has an acyclic t-colouring C. Let $y_1 \neq x_3$ be the neighbour, in H', of x_1 and let $y_2 \neq x_3$ be the neighbour, in H', of x_2. We need to consider two cases.

Let us assume at the beginning that $C(x_1y_1) \neq C(x_2y_2)$.

If $C(x_2x_3) = C(x_1y_1)$ and $C(x_1x_3) = C(x_2y_2)$, then clearly $|C(x_1) \cup C(x_2) \cup C(x_3)| \leq t - 2$ and we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

If we assume that $C(x_2x_3) = C(x_1y_1)$ and $C(x_1x_3) \neq C(x_2y_2)$ (or, similarly, $C(x_2x_3) \neq C(x_1y_1)$ and $C(x_1x_3) = C(x_2y_2)$), then $|C(x_1) \cup C(x_2) \cup C(x_3)| \leq t - 1$ and, similarly as above, we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

In the case when $C(x_2x_3) \neq C(x_1y_1)$ and $C(x_1x_3) \neq C(x_2y_2)$ it is also easy to observe that the colouring C can be extended, a contradiction.

In the opposite case, namely, when $C(x_1y_1) = C(x_2y_2)$, then if $C(x_1x_3) \in C(y_2)$ (or, similarly, if $C(x_2x_3) \in C(y_2)$), then, from the fact that $|C(x_1) \cup C(x_2) \cup C(y_2)| \leq t - 1$, we clearly have that the colouring C can be extended to an acyclic t-colouring of H, a contradiction. On the other hand, if $C(x_1x_3) \notin C(y_2)$ and $C(x_2x_3) \notin C(y_2)$, then we can recolour, in H', the edge x_2y_2 with a colour $C(x_1x_3)$ and we are back in the previous case.

$\mathcal{C}2$: This case is equivalent to the case $\mathcal{B}1$ in the proof of Lemma 5.

$\mathcal{C}3$: This case is equivalent to the case $\mathcal{B}2$ in the proof of Lemma 5.

$\mathcal{C}4$: If H contains a 4-vertex x adjacent to a 2-vertex y_1 and a 4^--vertex y_2, $y_1 \neq y_2$, and such that there is a triangle xy_2y_3 which is not incident to y_1, then let y_4 be the remaining neighbour of x. Let $H' = H - xy_1$. From the minimality of H it follows that H' has an acyclic t-colouring C. Let z be the neighbour, in H', of y_1. We need to consider three cases.

At the beginning, we assume that $C(y_1z) = C(xy_2)$.

Moreover, if $C(xy_3), C(xy_4) \notin C(y_2)$ and $C(xy_2) \notin C(y_3)$, then we can extend C by colouring the edge xy_1 with the colour $C(y_2y_3)$. On the other hand, if $C(xy_3), C(xy_4) \notin C(y_2)$, but $C(xy_2) \in C(y_3)$, then we can recolour, in H', the edge y_1z with a colour $\alpha \notin C(z) \cup \{C(xy_2), C(xy_4)\}$. It can happen, that the only one possible $\alpha = C(xy_3)$, but in this case we can colour the edge xy_1 with the colour $C(y_2y_3)$ to extend the colouring C to an acyclic t-colouring of H, a contradiction. In the opposite case, if such an $\alpha \neq C(xy_3)$, then, by Lemma 2, we can also extend the colouring C, a contradiction.

If $C(xy_3) \in C(y_2)$ or $C(xy_4) \in C(y_2)$, then it is easy to see that $|C(x) \cup C(y_1) \cup C(y_2)| \leq t - 1$ and, by Lemma 2, we can extend the colouring C, a contradiction.

In the second case we assume that $C(y_1z) = C(xy_3)$ (or $C(y_1z) = C(xy_4)$). It is obvious that we can recolour, in H', the edge y_1z with a colour $\alpha \notin C(z) \cup \{C(xy_3), C(xy_4)\}$, and we are either in the first case or in the next case.

At the end, we assume that $C(y_1z) \notin C(x)$, but in this case it is obvious that the colouring C can be extended, because $W_{H'}(xy_1) = \emptyset$, a contradiction.

$\mathcal{C}5$: If H contains a 4-vertex x adjacent to a 2-vertex y_1 and such that there is a triangle xy_1y_2, then let y_3, y_4 be the remaining neighbours of x. Consider $H' = H - xy_1$. From the minimality of H it follows that H' has an acyclic t-colouring C. If $C(y_1y_2) \notin C(x)$, then clearly the colouring C can be extended. In the opposite case, when $C(y_1y_2) \in C(x)$, then $|C(x) \cup C(y_1) \cup C(y_2)| \leq t - 1$ and, by Lemma 2, the colouring C can be extended, again a contradiction.

$\mathcal{C}6$: Let us assume that H contains a 4^+-vertex x which has a neighbour y_1 of degree 2 and is adjacent to at most two 4^+-vertices. It is easy to observe that the worst case is when x is adjacent to exactly two 4^+-vertices, say y_2, y_3. Let $H' = H - xy_1$. Similarly as above, H' has an acyclic t-colouring C. Let z be the neighbour, in H', of y_1. We need to consider two cases.

Let us assume at the beginning that $C(y_1z) \notin \{C(xy_2), C(xy_3)\}$. Clearly, $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq t - 1$. Hence, by Lemma 2, we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

In the opposite case, if $C(y_1z) \in \{C(xy_2), C(xy_3)\}$, say $C(y_1z) = C(xy_2)$, then it is easy to observe that we can recolour, in H', the edge y_1z with the colour $\alpha \notin C(y_1) \cup C(z) \cup \{C(xy_2), C(xy_3)\}$, obtaining in this way an acyclic t-colouring C' of H', and we are back in the previous case. ■

For the class of planar graphs without cycles of length 4 we have the following upper bound for the acyclic chromatic index.

Theorem 4. *If G is a planar graph without cycles of length 4, then*

$$\chi'_a(G) \leq \Delta(G) + 15.$$

Before we prove Theorem 4, we show some structural properties of 2-connected planar graphs without cycles of length 4.

Lemma 7. *Let G be a 2-connected planar graph without cycles of length 4. Then G contains at least one of the configurations:*

$\mathcal{D}1$: *a triangle incident to at least two 3^--vertices,*
$\mathcal{D}2$: *a 2-vertex adjacent to at least one 9^--vertex,*
$\mathcal{D}3$: *a 3-vertex adjacent to at least two 9^--vertices,*
$\mathcal{D}4$: *a 4-vertex adjacent to at least three 7^--vertices,*
$\mathcal{D}5$: *a d-vertex, with $d \geq 10$, such that at least $d - 5$ of its neighbours are 4^--vertices and at least one of them is of degree 2.*

Proof. We use the discharging method to prove the lemma. Let $G = (V, E)$ be a 2-connected planar graph without cycles of length 4. We fix a plane embedding of G. Let F be the face set of G. At the beginning we define a function f on $V \cup F$ as follows: for each $x \in V \cup F$, let $f(x) = d(x) - 4$. Clearly,

$$\sum_{x\in V\cup F} f(x) = \sum_{x\in V}(d(x) - 4) + \sum_{x\in F}(d(x) - 4) = 4(|E| - |V| - |F|) = -8, \qquad (2)$$

which follows from Euler's formula and the fact that $\sum_{x\in V} d(x) = \sum_{x\in F} d(x) = 2|E|$. In the next step we will distribute the values of f between adjacent vertices, faces and between incident vertices and faces, according to the rules described below, to obtain the function f'.

- If x is a 3^--vertex, then x does not give anything;
- if x is a 4-,5-,6-,7-vertex, then x gives $\frac{1}{5}$ to each incident triangle;
- if x is a 8-,9-vertex, then x gives $\frac{1}{5}$ to each incident triangle and $\frac{1}{5}$ to each adjacent 4-vertex;
- if x is a 10^+-vertex, then x gives $\frac{1}{5}$ to each incident triangle, $\frac{1}{5}$ to each adjacent 4-vertex, $\frac{1}{2}$ to each adjacent 3-vertex and 1 to each adjacent 2-vertex;
- each non-triangular face x gives $\frac{1}{5} \cdot l_a(x, y)$ to each adjacent triangle y, where $l_a(x, y)$ is the number of common edges (of the face x and the triangle y).

Now each $x \in V \cup F$ has a new value $f'(x)$, but the sums of values of the functions f' and f, counting over all the vertices and faces, remain the same.

We show that if G does not contain any of the configurations $\mathcal{D}1$–$\mathcal{D}5$, then $f'(x)$ is nonnegative for each $x \in V \cup F$, contrary to the equality (2).

First we show that for each vertex x the value $f'(x)$ is nonnegative. We consider a number of cases depending on the degree of x. Let $l_t(x)$ stand for the number of triangles incident to a vertex x.

- If $d(x) = 2$, then $f'(x) = -2 + 1 \cdot l_{10^+}(x) = 0$, because $\mathcal{D}2$ does not hold.
- If $d(x) = 3$, then $f'(x) = -1 + \frac{1}{2} \cdot l_{10^+}(x) \geq -1 + \frac{1}{2} \cdot 2 = 0$ in all cases, because $\mathcal{D}3$ does not occur.
- If $d(x) = 4$, then $f'(x) = 0 + \frac{1}{5} \cdot l_{8^+}(x) - \frac{1}{5} \cdot l_t(x) \geq \frac{1}{5} \cdot 2 - \frac{1}{5} \cdot 2 = 0$, which follows from the facts that $l_{8^+}(x) \geq 2$, by the absence of $\mathcal{D}4$, and that $l_t(x) \leq 2$, by Proposition 2.
- If $5 \leq d(x) = d \leq 7$, then $f'(x) = d - 4 - \frac{1}{5} \cdot l_t(x) \geq d - 4 - \frac{1}{5} \cdot \frac{d}{2} > 0$, once again from Proposition 2.
- If $8 \leq d(x) = d \leq 9$, then $f'(x) = d - 4 - \frac{1}{5} \cdot l_4(x) - \frac{1}{5} \cdot l_t(x) \geq d - 4 - \frac{1}{5} \cdot d - \frac{1}{5} \cdot \frac{d}{2} \geq 0$, because obviously $l_4(x) \leq d$ and $l_t(x) \leq \frac{d}{2}$, by Proposition 2.
- If $d(x) = d \geq 10$, then $f'(x) = d - 4 - 1 \cdot l_2(x) - \frac{1}{2} \cdot l_3(x) - \frac{1}{5} \cdot l_4(x) - \frac{1}{5} \cdot l_t(x)$.

  If $l_2(x) = 0$ then $f'(x) = d-4-\frac{1}{2}\cdot l_3(x)-\frac{1}{5}\cdot l_4(x)-\frac{1}{5}\cdot l_t(x) \geq d-4-\frac{1}{2}\cdot(l_3(x)+l_4(x))-\frac{1}{5}\cdot l_t(x) \geq d-4-\frac{1}{2}\cdot d-\frac{1}{5}\cdot\frac{d}{2} = \frac{4}{10}\cdot d - 4 \geq 0$, because $l_3(x) + l_4(x) \leq d$ and, similarly as above, $l_t(x) \leq \frac{d}{2}$.

  Therefore, we can assume $l_2(x) > 0$. Moreover, one can observe that since both $\mathcal{D}1$ and $\mathcal{D}2$ do not hold it follows that in each triangle which is incident to x there can be at most one 3^--vertex. Hence $l_2(x) + l_3(x) \leq d - l_t(x)$. We consider two cases.

  First assume that $l_t(x) \geq \frac{1}{4} \cdot l_4(x) + 5$. Clearly, $f'(x) = d - 4 - l_2(x) - \frac{1}{2} \cdot l_3(x) - \frac{1}{5} \cdot l_4(x) - \frac{1}{5} \cdot l_t(x) \geq d - 4 - (l_2(x) + l_3(x)) - \frac{1}{5} \cdot l_4(x) - \frac{1}{5} \cdot l_t(x) \geq d - 4 - (d - l_t(x)) - \frac{1}{5} \cdot l_4(x) - \frac{1}{5} \cdot l_t(x) = \frac{4}{5} \cdot l_t(x) - \frac{1}{5} \cdot l_4(x) - 4 \geq 0.$

In the opposite case, if $l_t(x) < \frac{1}{4} \cdot l_4(x) + 5$, then $f'(x) = d - 4 - l_2(x) - \frac{1}{2} \cdot l_3(x) - \frac{1}{5} \cdot l_4(x) - \frac{1}{5} \cdot l_t(x) > d-4-l_2(x)-\frac{1}{2}\cdot l_3(x)-\frac{1}{5}\cdot l_4(x)-\frac{1}{5}\cdot(\frac{1}{4}\cdot l_4(x)+5) = d-5-l_2(x)-\frac{1}{2}\cdot l_3(x)-\frac{1}{4}\cdot l_4(x) \geq d-5-(l_2(x)+l_3(x)+l_4(x)) > 0$, the last inequality follows from the fact that $l_2(x) + l_3(x) + l_4(x) < d - 5$, by the absence of $\mathcal{D}5$.

To finish the proof it is enough to calculate the value $f'(x)$ for each $x \in F$.

If x is a triangle, then $f'(x) \geq -1 + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 3 = 0$, because the triangle has to be incident to at least two 4^+-vertices, since $\mathcal{D}1$ does not hold and, moreover, the triangle x cannot be adjacent to any other triangle, by the absence of 4-cycles in G.

On the other hand, if x is a cycle of degree $d \geq 5$, then clearly, $f'(x) \geq d - 4 - \frac{1}{5} \cdot d \geq 0$, since the cycle x can have with adjacent triangles at most d common edges.

Hence for each $x \in V \cup F$ we have $f'(x) \geq 0$ and the proof is complete. ■

Proof of Theorem 4. Let H be a counterexample to Theorem 4 with the minimum number of edges. There is no loss of generality in assuming that H is connected. Let us fix a plane embedding of G. As in the prove of Theorem 3, we can assume that H is 2-connected. Hence, by Lemma 7, H contains at least one of the configurations $\mathcal{D}1$–$\mathcal{D}5$. Let $t = \Delta(H) + 15$.

By Lemma 2, it is sufficient to show that there exists an edge vu and an acyclic t-colouring C of $H - vu$ such that $|C(v) \cup C(u) \cup C(W_{H-vu}(v, u))| < t$. We consider a number of cases, depending on which of the configurations $\mathcal{D}1$–$\mathcal{D}5$ occurs in H. In each case we point out such an edge which we can use with Lemma 2 to obtain a contradiction.

$\mathcal{D}1$: This case is equivalent to the case $\mathcal{C}1$ in the proof of Theorem 3.

$\mathcal{D}2$: If H contains a 2-vertex x which is adjacent to a 9^--vertex y, then let $H' = H - xy$. From the minimality of H it follows that H' has an acyclic t-colouring C.

One can easily observe that $|C(x) \cup C(y) \cup C(W_{H'}(x, y))| \leq \Delta(H) + 7$, and, by Lemma 2, we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

$\mathcal{D}3$: If H contains a 3-vertex x adjacent to two 9^--vertices y_1 and y_2, then let y_3 be the remaining neighbour of x. Consider $H' = H - xy_1$. As above, by the minimality of H, we have that H' has an acyclic t-colouring C. We need to consider two cases.

If $|C(x) \cap C(y_1)| \leq 1$, then clearly $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq \Delta(H) + 8$ and, by Lemma 2, we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

In the opposite case, if $|C(x) \cap C(y_1)| = 2$, then if we assume that $C(xy_3) \notin C(y_2)$, then it easy to observe that we can recolour, in H', the edge xy_2 with a colour $\alpha \notin C(x) \cup C(y_2) \cup C(y_1)$, and we are back in the previous case. On the other hand, if $C(xy_3) \in C(y_2)$, then $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq \Delta(H) + 14$ and, similarly as above, we can use Lemma 2 to show that the colouring C can be extended to an acyclic t-colouring of H, a contradiction.

$\mathcal{D}4$: If H contains a 4-vertex x adjacent to three 7^--vertices y_1, y_2, y_3, then let y_4 be the remaining neighbour of x. Let $H' = H - xy_1$. From the minimality of H it follows that H' has an acyclic t-colouring C. We consider two cases.

If we assume that $|C(x) \cap C(y_1)| \leq 2$, then it is quite easy to observe that $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq \Delta(H) + 12$ and, by Lemma 2, we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

If $|C(x) \cap C(y_1)| = 3$, then let us assume at the beginning that $C(xy_3) \notin C(y_2)$ (or $C(xy_4) \notin C(y_2)$). It easy to observe that in this case we can recolour, in H', the edge xy_2 with a colour $\alpha \notin C(x) \cup C(y_2) \cup C(y_1) \cup C(W_{H'}(x, y_2))$, and we are back in the previous case.

On the other hand, if $C(xy_3), C(xy_4) \in C(y_2)$, then $C(xy_2) \notin C(y_3)$, since otherwise $|C(x) \cup C(y_1) \cup C(W_{H'}(x, y_1))| \leq \Delta(H) + 14$ and, similarly as above, we can use Lemma 2 to show that the colouring C can be extended and obtain a contradiction. But, if $C(xy_2) \notin C(y_3)$, then we can recolour, in H', the edge xy_3 with a colour $\alpha \notin C(x) \cup C(y_3) \cup C(y_1) \cup C(W_{H'}(x, y_3))$ and we fall in the first case.

$\mathcal{D}5$: Let us assume that H contains a 10^+-vertex x which has a neighbour y of degree 2 and is adjacent to at most five 5^+-vertices. Clearly, the worst case is when x is adjacent to exactly five 5^+-vertices, say $z_1, z_2, \ldots, z_5$. Consider $H' = H - xy$. As above, by the minimality of H, we have that H' has an acyclic t-colouring C. Let z be the neighbour, in H', of y. We need to consider two cases.

Let us assume at the beginning that $C(yz) \notin \{C(xz_1), \ldots, C(xz_5)\}$. Clearly, $|C(x) \cup C(y) \cup C(W_{H'}(x, y))| \leq \Delta(H) + 2$. Hence, by Lemma 2, we can extend the colouring C to an acyclic t-colouring of H, a contradiction.

In the opposite case, if $C(yz) \in \{C(xz_1), \ldots, C(xz_5)\}$, say $C(yz) = C(xz_1)$, then it is easy to observe that we can recolour, in H', the edge yz with a colour $\alpha \notin C(y) \cup C(z) \cup \{C(xz_1), \ldots, C(xz_5)\}$, obtaining in this way an acyclic t-colouring C' of H', and we are back in the previous case. ■

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