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Krein parameters and antipodal tight graphs with diameter 3 and 4

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Abstract

We determine which Krein parameters of nonbipartite antipodal distance-regular graphs of diameter 3 and 4 can vanish, and give combinatorial interpretations of their vanishing. We also study tight distance-regular graphs of diameter 3 and 4. In the case of diameter 3, tight graphs are precisely the Taylor graphs. In the case of antipodal distance-regular graphs of diameter 4, tight graphs are precisely the graphs for which the Krein parameter q_{11}^4 vanishes. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Several classifications of distance-regular graphs with a prescribed first subconstituent were successful, see [5, p. 492]. Very often, the vanishing of certain Krein parameters in a distance-regular graph determines its local structure. When one considers a certain class $\mathscr{C}$ of distance-regular graphs, the Krein parameters of interest are the ones which are not zero for the whole class $\mathscr{C}$, but do vanish for certain members of $\mathscr{C}$. Let us mention two such results. In the case of strongly regular graphs only q_{11}^1 and q_{22}^2 are such Krein parameters. In one of the monuments in the study of strongly regular graphs Cameron et al. [6] showed that vanishing of either of the Krein parameters q_{11}^1 and q_{22}^2 implies that first *and* second subconstituent graphs are strongly regular. In the case of antipodal distance-regular graphs of diameter 3 (assume the eigenvalues satisfy $\theta_0 > \theta_1 > \theta_2 > \theta_3$), the Krein parameter q_{33}^3 is not zero for all members of this class, but does vanish for certain members, cf. Lemma 3.1, Godsil [9]. Taylor and Levingston [28], [5, Section 1.5] showed that vanishing of q_{33}^3 implies the first subconstituent of every vertex is strongly regular.

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In this paper we study nonbipartite antipodal distance-regular graphs of diameter 3 and 4. Let Γ be a distance-regular graph of diameter $d \in \{3,4\}$ and with eigenvalues $\theta_0 > \cdots > \theta_d$. We determine which Krein parameters of Γ can vanish. In the case of diameter 4 it turns out, see Sections 2 and 3, that only the vanishing of the Krein parameter q_{11}^4 has not yet been interpreted combinatorially. Dickie and Terwilliger [7, Lemma 3.3] showed that the local graphs of Q-polynomial antipodal distance-regular graphs are strongly regular. In the case when $d=4$ the graph Γ is Q-polynomial if and only if the size of antipodal classes r is equal to two and $q_{11}^4=0$, see Corollary 4.4. We generalized their result in the case of diameter 4 to arbitrary r [11, Theorem 4.5.7]. This led to the discovery of the following bound, whose equality was characterized by the local graphs being strongly regular with specific eigenvalues. Let X denote a distance-regular graph of diameter $d \geqslant 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. It was shown in [16,14] that the intersection numbers a_1, b_1 satisfy the following inequality:

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geqslant -\frac{ka_1b_1}{(a_1+1)^2} \tag{1}$$

called the *Fundamental Bound*, and X was defined to be *tight* whenever it is not bipartite, and equality holds in (1). Tight graphs were characterized by $a_1 \neq 0$, $a_d = 0$ and 1-homogeneous property in the sense of Nomura [20], and furthermore by their first subconstituent being a strongly regular graph with nontrivial eigenvalues

$$b^+ = -1 - \frac{b_1}{1+\theta_d} \quad \text{and} \quad b^- = -1 - \frac{b_1}{1+\theta_1}. \tag{2}$$

The Fundamental Bound (1) can also be written in the following form: $k(a_1+b^+b^-) \leqslant (a_1-b^+)(a_1-b^-)$. We prove in Section 2 that tight graphs of diameter 3 are precisely Taylor graphs, i.e., distance-regular antipodal double-covers of a complete graph.

Let Γ be a nonbipartite antipodal distance-regular graph of diameter 4. We show in Section 4 that vanishing of q_{11}^4 occurs precisely when Γ is tight (this is essentially the above mentioned generalization translated to the terminology of tight graphs). Let us now suppose additionally that Γ is tight, let r be the size of its antipodal classes, and let p and $-q$ be the nontrivial eigenvalues of a first subconstituent graph, where we assume $p > -q$. We call Γ an *antipodal tight graph* of diameter 4 and with parameters (p,q,r), $r \geqslant 2$ and denote it by AT4(p,q,r). We show that p and q are integral, $p \geqslant 1$, $q \geqslant 2$, and express all parameters and eigenvalues of Γ in terms of the parameters p, q and r. Furthermore, all the intersection parameters of the distance partition corresponding to two adjacent vertices, that were calculated in [14] in terms of cosine sequences, are also expressed in terms of parameters p, q and r, cf. [16]. We obtain among other feasibility conditions for p, q and r in Section 5 also that $r \leqslant p+q$, and $r(p+1) \leqslant q(p+q)$, with equality if and only if Γ is a Terwilliger graph. The above bounds motivate us to determine all antipodal tight graphs AT4$(p,2,r)$, AT4$(1,q,r)$, AT4$(p,q,p+q)$, and antipodal tight graphs AT4(p,q,r) that are Terwilliger graphs. In the last 3 cases we obtain only the Conway–Smith graph, i.e., the antipodal tight graph AT4$(1,2,3)$, i.e., the only connected graph of diameter 4 whose local graphs are the

Petersen graphs. This rules out an infinite family of feasible parameters of antipodal tight graphs AT4$(2s-2, 2s, 4s-2)$, for $s \geqslant 2$. The case $s=2$ was considered first by Brouwer [3].

In Section 6 we turn our attention to larger diameter. We show that a distance-regular graph of diameter at least 3 and strongly regular local graphs are a Taylor graph if there exists a nonintegral eigenvalue of a local graph. Then we give a new existence condition for antipodal distance-regular graphs, which shows that 1-homogeneous graphs can have only antipodal distance-regular covers of even diameter and that tight graphs have no distance-regular antipodal covers.

Finally, in Section 7, we search for interesting subfamilies of the AT4 family. From $q_{44}^4 \geqslant 0$ we derive $p \geqslant q-2$, with equality if and only if $q_{44}^4 = 0$. Then we show that local graphs of an antipodal tight graph AT4(p,q,r) are pseudogeometric with parameters $(q, p+1+p/q, p/q)$ if and only if $q|p$. It turns out that all examples of antipodal tight graphs AT4(p,q,r), except the Conway–Smith graph, together with all the feasible parameters from the table in [5, pp. 421–425], have either $q|p$ or $p=q-2$. Thus these two cases, i.e., the families AT4(qs,q,r), s integral, and AT4$(q-2,q,r)$, deserve special attention. In a follow up paper [17] we determine all antipodal tight graphs AT4(qs,q,q), $s \in \mathbb{N}$. We conclude this paper with some open problems and directions for further study of the AT4 family.

2. Definitions and notation

In this section we recall some basic definitions and notation. For a detailed treatment of distance-regular graphs, association schemes and all the terms which are not defined here see [2,5,10].

An *equitable partition* of a graph Γ is a partition of its vertices into cells $C_1, C_2, \ldots, C_s$ such that for all i and j the number c_{ij} of neighbours, which a vertex in C_i has in the cell C_j, is independent of the choice of the vertex in C_i. In other words each cell C_i induces a regular graph of valency c_{ii}, and between any two cells C_i and C_j there is a biregular graph, with vertices of the cells C_i and C_j having valencies c_{ij} and c_{ji}, respectively.

A graph Γ of diameter d is *distance-regular* when the distance partition corresponding to any vertex x of Γ is equitable and the parameters of the equitable partition do not depend on x. In a distance-regular graph for a pair of vertices (x,y) at distance h, the number p_{ij}^h of vertices at distance i from x and j from y depends only on integers i, j, h, and not on (x,y). We denote the intersection numbers p_{ii}^i, $p_{i,i+1}^i$, $p_{i,i-1}^i$ and p_{ii}^0, respectively by a_i, b_i, c_i and k_i, for $i=0,1,\ldots,d$. Note $b_0 = a_i + b_i + c_i$ $(i=0,1,\ldots,d)$ is the valency of the graph Γ and call $\{b_0,\ldots,b_{d-1}; c_1,\ldots,c_d\}$ the *intersection array* of Γ. A graph is *i-homogeneous* in the sense of Nomura [20] when the distance partition corresponding to any pair (x,y) of vertices at distance i is equitable and the parameters corresponding to such equitable partitions are independent of vertices x and y at distance i. For a graph Γ and a vertex x in Γ, we define the *local graph* $\Delta(x)$ as the subgraph of Γ induced by the neighbours of x. A graph Γ of diameter d is *antipodal* if the vertices at distance d from a given vertex are all at distance d from each other. Then 'being at distance d or zero' induces an equivalence relation on the vertices of Γ, and the equivalence classes are called *antipodal classes*. For an antipodal graph Γ we define the *antipodal quotient* Ω of Γ, to be the graph with the antipodal classes as vertices, where two classes are adjacent if they contain adjacent vertices, and say that Γ is a *distance-regular antipodal* cover of Ω.

A *d-class association scheme over the set X* is a set of symmetric binary matrices $I=A_0, A_1, \ldots, A_d$ of size $|X|\times|X|$, which sum to the all ones matrix and for which any product of these matrices lies in the span of these matrices, i.e., $A_iA_j=\sum_{h=0}^{d} p_{ij}^h A_h$ $(0\leqslant i,j\leqslant d)$ for some parameters p_{ij}^h, called the *intersection numbers*. These matrices form a basis of a semi-simple commutative subalgebra $\mathscr{M}$ of $\mathrm{Mat}_X(\mathbb{C})$, known as the *Bose–Mesner algebra*. Since the matrices A_i commute, they can be diagonalized simultaneously. Let $E_0, E_1, \ldots, E_d$ denote the minimal idempotents of the Bose–Mesner algebra M. Then $|X|E_0$ is the all ones matrix, matrices $E_0, E_1, \ldots, E_d$ sum to the identity matrix, and

$$E_i\circ E_j=\frac{1}{|X|}\sum_{h=0}^{d} q_{ij}^h E_h,\quad A_i=\sum_{h=0}^{d} P_{hi}E_h,$$

$$E_i=\frac{1}{|X|}\sum_{h=0}^{d} Q_{hi}A_h,\quad (0\leqslant i,j\leqslant d),$$

where "$\circ$" denotes the entry-wise multiplication, called *Schur product* of matrices. Constants q_{ij}^h are called the *Krein parameters* of the association scheme, $P_{0i},\ldots,P_{di}$ are the *eigenvalues* of A_i, and $Q_{0i},\ldots,Q_{di}$ are the *dual eigenvalues* of E_i. If $P_{i1}=\theta_j$, then $m_j=Q_{0i}$ is the multiplicity of θ_j. An association scheme is called *P-polynomial* when its intersection numbers p_{ij}^h satisfy the *triangle condition*, i.e., for all integers $h,i,j\in\{0,\ldots,d\}$, the intersection numbers $p_{ij}^h=0$ (resp. $p_{ij}^h\neq 0$) whenever one of h,i,j is greater than (resp. equal to) the sum of the other two. The distance matrices of a distance-regular graph form an association scheme that is P-polynomial. An association scheme is called *Q-polynomial* (with respect to the given permutation of indices of E_i's) when its Krein parameters q_{ij}^h satisfy the triangle condition.

Let Γ be a distance-regular graph of diameter d, with v vertices and eigenvalues $\theta_0,\ldots,\theta_d$. Then the entries of the matrix $P=P(\Gamma)$ of eigenvalues of the distance matrices of Γ are equal to $P_{ij}=v_j(\theta_i)$, $i,j\in\{0,\ldots,d\}$, where $v_i(x)$ is the polynomial for which the ith distance matrix $A_i=v_i(A)$. We can compute Krein parameters by using the entries of the matrix P and the following identity:

$$q_{ij}^h=\frac{m_im_j}{v}\sum_{a=0}^{d}\frac{v_a(\theta_i)v_a(\theta_j)v_a(\theta_h)}{k_a^2}. \tag{3}$$

3. Diameter 3 case

As we mentioned in the Introduction, Godsil, Taylor and Levingston showed that in the case of antipodal distance-regular graphs of diameter 3, vanishing of the Krein parameter q_{33}^3 implies that each local graph is strongly regular. The following result, that can be verified easily with a straightforward calculations using [5, Corollary 4.2.6] and (3), demonstrates that this is the only interesting Krein parameter for these graphs.

Lemma 3.1. *Let Γ be a nonbipartite antipodal distance-regular graph of diameter* 3, *with antipodal class size r and eigenvalues $k=\theta_0>\theta_1>\theta_2=-1>\theta_3$. Then the eigenvalues θ_1* and *θ_3 are the roots of $x^2-(a_1-c_2)x-k=0$, where $a_1=n-2-(r-1)c_2$ and $k=n-1$, and the following hold*:

(i) $q_{12}^2=0,\ q_{22}^3=0,\ q_{11}^2>0,\ q_{12}^3>0,\ q_{22}^2>0,\ q_{23}^3>0.$

(ii) $r\geqslant 2$, *with equality if and only if either of the Krein parameters q_{11}^1, q_{11}^3 and q_{13}^3 vanishes.*

(iii) *If $r=2$ then $q_{33}^3=0$. If $r>2$ then $q_{33}^3=0$ if and only if $k=\theta_1^3$.*

The above result implies that it remains to consider the vanishing of Krein parameters in the case when $r=2$, i.e., $q_{11}^1=q_{11}^3=q_{33}^1=q_{33}^3=0$. Let Γ be a *Taylor graph*, i.e., a distance-regular antipodal double-cover of a complete graph, i.e., a distance-regular graph with intersection array of the form $\{k,c,1;1,c,k\}$. Its local graphs are strongly regular graphs with intersection array $\{k-c-1,c/2;1,(k-c-1)/2\}$ (note c is even and k odd). For numerous examples and more information on Taylor graphs see [25] and [5, Section 7.6.C]. The following result characterizes Taylor graphs as tight graphs of diameter 3.

Theorem 3.2. *A nonbipartite distance-regular graph of diameter* 3 *is tight if and only if it is a Taylor graph.*

Proof. Let Γ be a tight graph of diameter 3. We will first prove $b_2=1$. In a tight graph of diameter 3 we have $a_3=0$ by [14, Theorem 10.4], so we obtain, by $p_{23}^3=k(b_2-1)/c_2$,

$$\frac{b_1b_2}{c_2}=k_3=p_{03}^3+p_{13}^3+p_{23}^3+p_{33}^3\geqslant 1+\frac{k(b_2-1)}{c_2}, \tag{4}$$

i.e., $a_2\geqslant a_1b_2$, with equality if and only if $p_{33}^3=0$. Let x and y be a pair of vertices of Γ at distance two. Let us count the edges between the sets $A_2=\Gamma_2(x)\cap\Gamma(y)$ and $B_2=\Gamma_3(x)\cap\Gamma(y)$, whose sizes are a_2 and b_2, respectively. The set B_2 is independent, since $a_3=0$, so each vertex in it has a_1 neighbours in A_2. The size s of the intersection $\Gamma(z)\cap B_2$, where $z\in A_2$, is independent of z by [14, Theorem 11.1] and the fact that the above intersection is equal to $\Gamma(y)\cap\Gamma(z)\cap\Gamma_3(x)$. Counting in two different ways give us $a_1b_2=a_2s$. This number cannot be zero, since in a tight graph $a_1\neq 0$ [14, Proposition 6.5], therefore $a_1b_2 \geqslant a_2$, with equality if and only if $s=1$. But this means $a_2=a_1b_2$ by (4), and so $p^3_{33}=0$ and $|\Gamma(z)\cap B_2|=1$. Hence, if $|B_2|\geqslant 2$, then two vertices in the independent set B_2 would have distance at least three in the local graph $\Delta(y)$, which is a connected strongly regular graph by [14, Theorem 12.4]. It follows $b_2=|B_2|=1$ and then also $a_1=a_2$. Thus Γ is a Taylor graph. The converse follows from a straightforward calculation. □

4. Diameter 4 case

Motivated by Lemma 3.1 we study the diameter 4 case. Let Γ be an antipodal distance-regular graph of diameter 4, with eigenvalues $k=\theta_0>\theta_1>\theta_2>\theta_3>\theta_4$ antipodal class size r. Then its intersection array is, by [5, Proposition 4.2.2], determined by parameters (k,a_1,c_2,r), and has the following form:

$$\{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\}=\{k,k-a_1-1,(r-1)c_2,1;1,c_2,k-a_1-1,k\}. \quad (5)$$

We summarize below the basic relations concerning the parameters of Γ.

Lemma 4.1. *Let Γ be an antipodal distance-regular graph of diameter* 4, *with v vertices, antipodal class size r. Let $k=\theta_0>\theta_1>\theta_2>\theta_3>\theta_4$ denote its eigenvalues and let m_i denote the multiplicity of θ_i. Then the following hold*:

(i) *The antipodal quotient is a connected strongly regular graph of diameter* 2 *and with parameters $(v/r,k,a_1,rc_2)$. Its eigenvalues are $\theta_0=k$ and θ_2,θ_4, which are the roots of $x^2-(a_1-rc_2)x-(k-rc_2)=0$. The remaining eigenvalues θ_1 and θ_3 of Γ are the roots of $x^2-a_1x-k=0$.*

(ii) *The following relations hold for the eigenvalues*:

$$\theta_0=-\theta_1\theta_3 \quad \text{and} \quad (\theta_2+1)(\theta_4+1)=(\theta_1+1)(\theta_3+1).$$

(iii) *Parameters of the antipodal quotient can be expressed in terms of eigenvalues and r*:

$$k=\theta_0,\quad a_1=\theta_1+\theta_3,\quad b_1=-(\theta_2+1)(\theta_4+1),\quad c_2=\frac{\theta_0+\theta_2\theta_4}{r}.$$

(iv) *The multiplicities are $m_0=1$,*

$$m_2=\frac{(\theta_4+1)k(k-\theta_4)}{rc_2(\theta_4-\theta_2)},\quad m_4=\frac{v}{r}-m_2-1,\ \text{ and }\ m_i=\frac{(r-1)v}{r(2+a_1\theta_i/k)}\quad \text{for } i=1,3.$$

(v) *The eigenvalues θ_2,θ_4 are integral, $\theta_4\leqslant -2$, $0\leqslant\theta_2$, with $\theta_2=0$ if and only if Γ is bipartite. Furthermore, $\theta_3<-1$, and the eigenvalues θ_1,θ_3 are integral when $a_1\neq 0$.*

Proof. (i), (iv) and the last part of (v) follow from [5, Theorem 1.3.1, Propositions 4.2.3, 4.2.4, Corollary 4.2.5]. (ii) and (iii) follow from (i).

(v): The claim that θ_2 and θ_4 are integral follows from two well known facts. The first one is that the conference graphs, i.e., the strongly regular graphs with parameters $(4c+1, 2c, c-1, c)$, are the only strongly regular graphs which can have nonintegral eigenvalues [5, Theorem 1.3.1(ii)]. The second fact is that a conference graph cannot have distance-regular antipodal covers, see [5, p. 180]. The latter can be derived directly from the fact that λ^2+4k is a square, see [5, Corollary 4.2.5]. An easy interlacing argument implies the least eigenvalue is at most $-\sqrt{2}$, therefore $\theta_4 \leqslant -2$ by its integrality.

Since $k=-\theta_1\theta_3$ and $k>\theta_1>\theta_3$ we conclude $-1>\theta_3$. On the other hand, $k-rc_2$ is equal to a_2 of the antipodal quotient and $-\theta_2\theta_4=k-rc_2$ by (i). Therefore $\theta_2 \geqslant 0$, with equality if and only if $a_2=0$, which means that the antipodal quotient is a complete multipartite graph. But such a graph can have distance-regular antipodal covers only when it is a complete bipartite graph, see [12, Proposition 2.7], in which case the original graph is bipartite. □

Remark 4.2. Let Γ be an antipodal distance-regular graph of diameter 4. The above result shows that we can express its intersection parameters by the size r of its antipodal classes and any three of its eigenvalues.

In the case of antipodal distance-regular graph of diameter 4 the matrix of eigenvalues $P(\Gamma)$ has the following form:

$$P(\Gamma)=\begin{pmatrix} 1 & \theta_0 & \theta_0 b_1/c_2 & \theta_0(r-1) & r-1 \\ 1 & \theta_1 & 0 & -\theta_1 & -1 \\ 1 & \theta_2 & -r(\theta_2+1) & \theta_2(r-1) & r-1 \\ 1 & \theta_3 & 0 & -\theta_3 & -1 \\ 1 & \theta_4 & -r(\theta_4+1) & \theta_4(r-1) & r-1 \end{pmatrix}. \tag{6}$$

Since the Krein parameter q_{ij}^h does not change sign when we permute superscripts and subscripts (see (3)), we need to check the 20 Krein bounds $q_{ij}^h \geqslant 0$ with $i,j,h \in \{1,2,3,4\}$ and $i \leqslant j \leqslant h$. They correspond to the following triples ijh: 111, 112, 113, 114, 122, 123, 124, 133, 134, 144, 222, 223, 224, 233, 234, 244, 333, 334, 344 and 444, all of which are described in the result below.

Theorem 4.3. *Let Γ be a nonbipartite antipodal distance-regular graph of diameter* 4, *with antipodal class size r and with eigenvalues $k=\theta_0>\theta_1>\theta_2>\theta_3>\theta_4$. Then the following hold*:

(i) $(\theta_2+1)^2(k^2+\theta_4^3) \geqslant (\theta_4+1)(k+\theta_2\theta_4)^2$, *with equality if and only if* $q_{44}^4=0$.
(ii) $\theta_3^2 \geqslant -\theta_4$, *with equality if and only if* $q_{11}^4=0$.
(iii) $q_{11}^2>0$, $q_{12}^3>0$, $q_{13}^4>0$, $q_{22}^2>0$, $q_{22}^4>0$, $q_{23}^3>0$, $q_{24}^4>0$, $q_{33}^4>0$.
(iv) $r \geqslant 2$, *with equality iff* $q_{11}^1=0$ *iff* $q_{11}^3=0$ *iff* $q_{13}^3=0$ *iff* $q_{33}^3=0$.
(v) $q_{12}^2=0$, $q_{12}^4=0$, $q_{14}^4=0$, $q_{22}^3=0$, $q_{23}^4=0$, $q_{34}^4=0$.

Proof. By (6), (3) and Lemma 4.1, we obtain the statements of items (i), (ii), (iv) for $r=2$, (v). The Krein bound $q_{22}^2 \geqslant 0$ translates, by Lemma 4.1, to

$$(\theta_4+1)^2(k^2+\theta_2^3) \geqslant (\theta_2+1)(k+\theta_2\theta_4)^2, \tag{7}$$

with equality if and only if $q_{22}^2=0$ (note that the above inequality can also be obtained from (i) by interchanging θ_2 and θ_4). We show that $q_{22}^2 \neq 0$. We use Lemma 4.1(i),(ii) to rewrite (7). In the case $a_1=0$ this is done in terms of k and θ_4 and transform (7) into the following inequality:

$$(k-\theta_4((\theta_4+1)^2+\theta_4))(k(1+2/\theta_4)+1) \geqslant 0,$$

which hold with strict inequality by $\theta_4 \leqslant -2$. In the case $a_1 \neq 0$ it is done in terms of $p{:=}\theta_2$, $q{:=}-\theta_3$ and θ_4:

$$\begin{aligned}(q(p+1)\theta_4+q^2+p(2q-1))[q(p+1)\theta_4^3+(q^2+p(2q+1)+2q)\theta_4^2\\ +(p^2(q-2)+(p+2q)q)\theta_4-p(p+q^2)] \geqslant 0.\end{aligned} \tag{8}$$

Note that $q \geqslant 2$ and $\theta_4 \leqslant \theta_3-1=-q-1 \leqslant -3$ by Lemma 4.1. For $q=2$ we have, by (ii), either $\theta_4=-3$ or $\theta_4=-4$, in which case it is easy to verify $q_{22}^2 \neq 0$ directly. Hence, we assume $q>2$. The sum of the first two terms of the second factor on the left-hand side of (8) is nonpositive when

$$q(p+1)\theta_4+(q^2+p(2q+1)+2q) \leqslant 0. \tag{9}$$

To verify this inequality substitute θ_4 with its upper bound $-q-1$ and get $pq(1-q)+p+q \leqslant 0$, which is obviously true for $q>2$. Therefore, the second factor on the left- hand side of (8) is always negative. The first factor on the left-hand side of (8) is smaller than the left-hand side of (9), thus negative as well and $q_{22}^2>0$.

We are going to prove $\theta_1^2>-\theta_4$. By Lemma 4.1, $a_1=\theta_1+\theta_3$ and $\theta_1>0>\theta_3$, so we conclude $\theta_1^2 \geqslant \theta_3^2$ with equality if and only if $a_1=0$. On the other hand we have, by (ii), $\theta_3^2 \geqslant -\theta_4$. Suppose both inequalities hold with equality, then $\theta_1=-\theta_3$ and thus also $k=\theta_3^2=-\theta_4$. But this contradicts the assumption that Γ is nonbipartite. Hence,

$$\theta_1^2>-\theta_4 \quad \text{and then obviously also } \theta_1^2>-\theta_3. \tag{10}$$

The cases q_{33}^3 in (iv) for $r \neq 2$ and q_{33}^4 in (iii) follow from (10). All other cases of (iii) and (iv) follow directly from Lemma 4.1. □

The above result shows that in the case of a nonbipartite antipodal distance-regular graph Γ of diameter 4 and with eigenvalues $\theta_0>\theta_1>\theta_2>\theta_3>\theta_4$ there are three interesting cases. These are $q_{11}^4=0$, $q_{44}^4=0$ and $q_{11}^4=0=q_{44}^4$. The Krein parameter $q_{2i,2j}^{2h}$ of Γ, where $i,j,h \in \{0,1,2\}$, is equal to the Krein parameter q_{ij}^h of the antipodal quotient of Γ. Furthermore, the local graphs of Γ that correspond to vertices in the same antipodal class are isomorphic to the local graph of the antipodal quotient of Γ

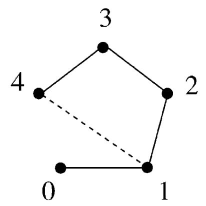

Fig. 1. The representation diagram $\Delta(E)$ for $E = E_h$ is the undirected graph with vertices $0, 1, 2, 3, 4$, where two are joined whenever $i \neq j$ and $q_{ij}^h = q_{ji}^h \neq 0$. Only the representation diagram $\Delta(E_1)$ can be a path, and this is the case if and only if $r = 2$ and $q_{11}^4 = 0$.

that correspond to this antipodal class. Therefore, $q_{44}^4 = 0$ in Γ implies, by Cameron et al. [6], that the local graphs of Γ are strongly regular. In the next section we will find a combinatorial interpretation of vanishing of q_{11}^4. In a follow up paper [13] we will study the simultaneous vanishing of both Krein parameters q_{11}^4 and q_{44}^4. We conclude this section with two corollaries. The first one is a special case of the result in [31, Theorem 3] and deals with the case $q_{11}^4 = 0$ and $r = 2$. The second one uses the fact that $q_{22}^2 \neq 0$ in Γ.

Corollary 4.4. *A nonbipartite antipodal distance-regular graph of diameter* 4 *is Q-polynomial if and only if $r = 2$ and $q_{11}^4 = 0$. If $r = 2$ and $q_{11}^4 = 0$, then θ_0, θ_1, θ_2, θ_3, θ_4 is a unique Q-polynomial ordering, and the Q-polynomial structure is dual bipartite, i.e., $q_{ij}^h = 0$ when $i + j + h$ is odd.*

Proof. Based on the information from Theorem 4.3 it is not difficult to draw the representation diagrams $\Delta(E)$ defined in [29], cf. [5, Theorem 2.11.6], corresponding to the primitive idempotents $E = E_1, E_2, E_3, E_4$. See Fig. 1 for the representation diagram $\Delta(E_1)$. In the representation diagram $\Delta(E_2)$ the edge 13 is a connected component, the representation diagram $\Delta(E_4)$ has a cycle on vertices $1, 2, 3$, and the representation diagram $\Delta(E_3)$ has a cycle on vertices $1, 2, 3, 4$. □

Corollary 4.5. *A strongly regular graph with eigenvalues $\theta_0 > \theta_1 > \theta_2$ and $q_{11}^1 = 0$ has no nonbipartite antipodal distance-regular covers of diameter* 4.

Example 4.6. The pentagon, the Clebsch graph, the Schläfli graph, the complement of the Higman–Sims graph, the complement of the point graph of generalized quadrangle $GQ(3, 9)$, the complement of the second subconstituent graph of the McLaughlin graph and the complement of the McLaughlin graph, i.e., the unique strongly regular graphs with parameters $(5, 2, 0, 1)$, $(16, 10, 6, 6)$, $(27, 16, 10, 8)$, $(100, 77, 60, 56)$, $(112, 81, 60, 54)$, $(162, 105, 72, 60)$, $(275, 162, 105, 81)$, are the first seven examples of strongly regular graphs, corresponding to the number of vertices, that satisfy Corollary 4.5.

The sequences $\{b_i\}$ and $\{-c_i\}$ of a nonbipartite antipodal distance-regular graph of diameter 4 are nonincreasing. This immediately implies that its antipodal quotient, which is a strongly regular graph with parameters (v,k,λ,μ), satisfies $k \leqslant \lfloor 2(v-1)/3 \rfloor$. See [4] and also [11, Proposition 4.1.2], where it is noted that for all feasible intersection arrays in [5, pp. 421–425] holds even sharper bound,

$$k \leqslant \left\lfloor \frac{v-1}{2} \right\rfloor \tag{11}$$

that can be written also in the form $k \geqslant \lambda + \mu + 1$. Only the Conway–Smith graph is known to satisfy the above bound with equality. In [11] it is conjectured that the inequality (11) holds for all antipodal quotients of nonbipartite antipodal distance-regular graphs Γ of diameter 4. This would give us an alternative proof of the fact that $q_{22}^2 \neq 0$ in Γ as shown in [6].

5. The Krein parameter q_{11}^4 and the 1-homogeneous property

We characterize the vanishing of the Krein parameter q_{11}^4 for antipodal distance- regular graphs of diameter 4 by the 1-homogeneous property in the sense of Nomura and calculate all the intersection parameters that correspond to a 1-homogeneous partition.

Lemma 5.1. *Let Γ be a nonbipartite antipodal distance-regular graph of diameter* 4 *and with eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Let b^+ and b^- be as defined in* (2). *Then $b^+ = \theta_2$ and $b^- = \theta_3$.*

Proof. By Lemma 4.1(iii), we have $b_1 = (\theta_2 + 1)(-\theta_4 - 1)$, and one immediately calculates b^+ and, by Lemma 4.1(ii), also b^-. □

As an immediate consequence of Lemmas 4.1 and 5.1 we obtain the following result.

Theorem 5.2. *Let Γ be a nonbipartite antipodal distance-regular graph of diameter* 4 *and with eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Then $q_{11}^4 = 0$ if and only if Γ is tight.*

Let Γ be an antipodal tight distance-regular graph of diameter four, with eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$ and antipodal class size r. For $p = \theta_2$ and $q = \theta_3$ we call Γ an *antipodal tight graph* of diameter four and with parameters (p,q,r), and denote it by AT4(p,q,r). Such a graph is not necessarily determined by its parameters.

Proposition 5.3. *Let Γ be an antipodal tight graph* AT4(p,q,r), *and eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Then $\theta_0 = q(pq+p+q)$, $\theta_1 = pq+p+q$, $\theta_2 = p$, $\theta_3 = -q$, $\theta_4 = -q^2$, the parameters p and q are integral, $p \geqslant 1$, $q \geqslant 2$, and the eigenvalue multiplicities are $m_0 = 1$,*

$$m_1 = (r-1)\frac{q(pq^2+q^2+pq-p)}{p+q},$$

$$m_2 = \frac{q(pq+p+q)(q^2-1)(2q+pq+p)}{(p+q)(q^2+p)},$$

$$m_3 = (r-1)\frac{(pq^2+q^2+pq-p)(pq+p+q)}{p+q},$$

$$m_4 = \frac{(p+1)(pq+p+q)(pq^2+q^2+pq-p)}{(p+q)(q^2+p)}.$$

Proof. In a tight graph $a_1 \neq 0$, see [14, Corollary 6.3], so all the eigenvalues of Γ are integral by Lemma 4.1. By Lemma 4.1(v), $-q = \theta_3 < -1$, i.e., $q \geqslant 2$, and since Γ is not bipartite also $p = \theta_2 \geqslant 1$. All the above formulas are obtained directly from Lemma 4.1. □

By (5) and Lemma 4.1 we derive an alternative characterization of antipodal tight graphs AT4(p,q,r).

Theorem 5.4. *Let Γ be an antipodal distance-regular graph of diameter* 4, *and let p, q and r be some real numbers. Then the following are equivalent*:

(i) *Γ is an antipodal tight graph* AT4(p,q,r).

(ii) *The intersection array of Γ equals*

$$\left\{q(pq+p+q),(q^2-1)(p+1),\frac{(r-1)q(p+q)}{r},1;\right.$$
$$\left.1,\frac{q(p+q)}{r},(q^2-1)(p+1),q(pq+p+q)\right\}.$$

(iii) *The antipodal quotient of Γ has the following parameters*

$$(k,\lambda,\mu) = (q(pq+p+q),\, p(q+1),\, q(p+q)). \qquad \square$$

Remark 5.5. For convenience we list here the remaining standard parameters of an antipodal tight graph AT4(p,q,r): $a_1 = a_3 = p(q+1)$, $a_2 = pq^2$, $a_4 = 0$, the valencies of the distance graphs are

$$k_1 = q(pq+p+q), \quad k_2 = \frac{(pq+p+q)(q^2-1)(p+1)r}{p+q},$$
$$k_3 = (r-1)q(pq+p+q), \quad k_4 = r-1$$

and they sum to $v = r(2q+pq+p)(pq^2+q^2+pq-p)/(p+q)$.

The above two results imply yet another characterization of tight graphs with parameters (p,q,r).

Corollary 5.6. *Let Γ be an antipodal distance-regular graph of diameter* 4, *and let p, q and r be some real numbers. Then the following are equivalent*:

(i) *Γ is an antipodal tight graph* AT4(p,q,r).

(ii) *The local graph of some vertex x of Γ is strongly regular with parameters* $(k',\lambda',\mu') = (p(q+1), 2p-q, p)$.

(iii) *The local graph of each vertex of Γ is strongly regular with parameters* $(k',\lambda',\mu') = (p(q+1), 2p-q, p)$.

Suppose (i) *and* (iii) *hold, then* p, q *and* r *are integral, and the nontrivial eigenvalues of local graphs in* Γ *are* $(\theta_1', \theta_2') = (p, -q)$, *with multiplicities*

$$m_1' = \frac{(q^2-1)(pq+p+q)}{p+q} \quad \text{and} \quad m_2' = k-1-m_1' = \frac{pq(q+1)(p+1)}{p+q}.$$

As we mentioned in the Introduction, tight graphs were characterized by $a_1 \neq 0$, $a_d = 0$ and 1-homogeneous property in the sense of Nomura, see [14, Theorem 11.7]. So we are now able to interpret, by Theorem 5.2, the vanishing of the Krein parameter q_{11}^4 combinatorially.

Corollary 5.7. *An antipodal distance-regular graph of diameter* 4 *and* $a_1 \neq 0$ *is* 1-*homogeneous in the sense of Nomura if and only if* $q_{11}^4 = 0$.

The *cosine sequence* $\{\psi_j\}$ *corresponding to* θ_i *is defined by* $\psi_j = v_j(\theta_i)/k_j$, $i, j \in \{0,1,2,3,4\}$, see [5, p. 128, 142]. The cosine sequences of an antipodal tight graph AT4(p,q,r) corresponding to eigenvalues θ_1 and θ_4, respectively are, by (6),

$$\left(1, \frac{1}{q}, 0, \frac{-1}{(r-1)q}, \frac{-1}{r-1}\right)$$

and

$$\left(1, \frac{-q}{(pq+p+q)}, \frac{p+q}{(p+1)(pq+p+q)}, \frac{-q}{(pq+p+q)}, 1\right). \tag{12}$$

Let Γ be an antipodal tight graph AT4(p,q,r). From the above information, formulas in [14, Theorem 11.2, Lemma 2.11], we can calculate the parameters corresponding to a 1-homogeneous partition of Γ in terms of the parameters p, q and r, see Fig. 2.

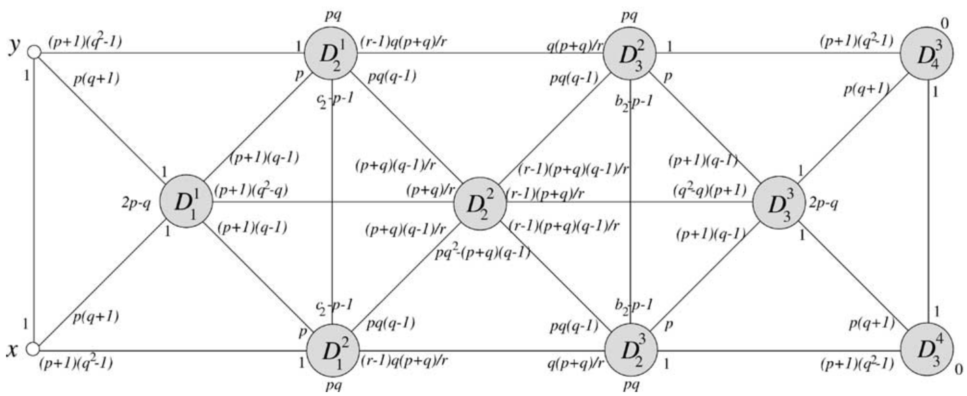

Fig. 2. The distance partition corresponding to an edge xy of Γ. $D_j^i = D_j^i(x,y) = \Gamma_i(x) \cap \Gamma_j(y)$. The number beside edges connecting cells D_i^j, indicates how many neighbours a vertex from the closer cell has in the other cell. We also put beside each cell the valency of the graph induced by the vertices of it. For convenience we mention here the intersection numbers needed for the above partition: $|D_1^1| = p_{11}^1 = a_1 = p(q+1)$, $|D_2^1| = p_{12}^1 = b_1 = (q^2-1)(p+1)$, $|D_3^2| = p_{23}^1 = (r-1)b_1 = (r-1)(q^2-1)(p+1)$, $|D_4^3| = p_{34}^1 = r-1$, $|D_2^2| = p_{22}^1 = rpq(q^2-1)(p+1)/(p+q)$.

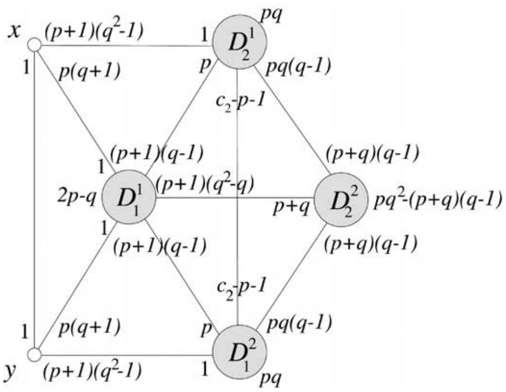

Fig. 3. The distance partition corresponding to a pair of adjacent vertices of the antipodal quotient of Γ.

Let Γ be an antipodal distance-regular graph of diameter d. Gardiner [8] proved that a vertex x of Γ, which is at distance $i \leqslant \lfloor d/2 \rfloor$ from one vertex in an antipodal class, is at distance $d-i$ from all other vertices in this antipodal class. This implies the following identity:

$$\Gamma_{d-i}(x) = \bigcup \{\Gamma_d(y) \mid y \in \Gamma_i(x)\} \quad \text{for } i = 0, 1, \ldots, \lfloor d/2 \rfloor. \tag{13}$$

Let us suppose Γ is 1-homogeneous. If we fold Γ, i.e., take the antipodal quotient of Γ, then by (13) the cells D^{d-j}_{d-i} and D^j_i fold together for $0 \geqslant i, j \geqslant \lfloor d/2 \rfloor$. This implies that in the case when we fold an antipodal tight graph AT4(p,q,r) we obtain an antipodal quotient that is 1-homogeneous as well.

Corollary 5.8. *The antipodal quotient of an antipodal tight graph* AT4(p,q,r) *is* 1-*homogeneous in the sense of Nomura*, *with parameters as in* Fig. 3.

6. Classification of the AT4 family

We collect feasibility conditions for the parameters of antipodal tight graphs AT4(p,q,r) in the following result. First we need two definitions. Let Γ be a graph of diameter at least two and let x, y be vertices of Γ at distance two. Then the $\mu(x,y)$-graph is the subgraph of Γ induced by the common neighbours of x and y. A distance-regular graph of diameter at least two is called *Terwilliger graph* when $\mu(x,y)$-graph is complete for every pair of vertices x and y at distance two, i.e., there are no induced quadrangles.

Theorem 6.1. *Let* $\Gamma = (X, R)$ *be an antipodal tight graph* AT4(p,q,r). *Then* p, q, r *are integers*, *such that* $p \geqslant 1$, $q \geqslant 2$, $r \geqslant 2$ *and*

(i) $pq(p+q)/r$ *is even*,

(ii) $r(p+1) \leqslant q(p+q)$, *with equality if and only if* Γ *is a Terwilliger graph*,

(iii) $r \mid p+q$,
(iv) $p \geqslant q-2$, *with equality if and only if* $q_{44}^4 = 0$,
(v) $p+q \mid q^2(q^2-1)$,
(vi) $p+q^2 \mid q^2(q^2-1)(q^2+q-1)(q+2)$.

Proof. (i), (iii) and the inequality in (ii) have already been proved in [16]. Let us give an alternative proof of (iii). Since Γ is a tight graph, it is 1-homogeneous in the sense of Nomura. Consider the distance partition corresponding to an edge $xy \in R$ and let $z \in D_2^2$. Then the number of neighbours of z in $D_1^1(x,y)$ equals $\sigma_2 = (p+q)/r$ and must therefore be an integer, so we obtained (iii). Since $\mu' = p$ and $c_2 = q(p+q)/r$, equality in (ii) translates to $\mu' + 1 = c_2$, i.e., Γ is a Terwilliger graph.

(iv): We express all the parameters in Theorem 4.3(i) in terms of p and q and obtain the desired inequality.

(v) and (vi) follow from the integrality of the multiplicities of a local graph and the antipodal quotient. The integrality of the nontrivial eigenvalue multiplicities of the local graph implies $p+q \mid q^2(q^2-1)$. We can express m_2 in the following way:

$$m_2 = -q(q^2-1)(q+1)^2 + \frac{(q^2-1)q^3}{p+q} - \frac{q^2(q^2-1)(q^2+q-1)(q+2)}{p+q^2}.$$

Therefore $p+q^2 \mid q^2(q^2-1)(q^2+q-1)(q+2)$. □

Remark 6.2. The remaining eigenvalue multiplicities of Γ give no new divisibility conditions. The integrality of the expression $(p+q)/r$, see Theorem 6.1(iii), implies integrality of all intersection parameters of the distance partition corresponding to an edge, see Fig. 2.

The information about the μ-graphs of the examples in Table 1 has been determined using GRAPE [27] and GAP [24], see also [22], cf. [5, p. 400]. The μ-graphs of Soicher2 graph are the 2-coclique extensions of the halved cube $\frac{1}{2}H(5,2)$, so they are *not* complete multipartite graph, see [3].

Let us now consider the cases $p=1$, i.e., $b^+ = 1$, and $q=2$, i.e., $b^- = -2$.

Theorem 6.3. *Let Γ be a tight graph of diameter d and with eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. If the second largest eigenvalue $\theta_1 = b_1 - 1$, i.e., $b^- = -2$, then Γ is either the Johnson graph, the Halved cube, the Gosset graph, or the Conway–Smith graph. If $b^+ = 1$ then Γ is the Conway–Smith graph.*

Proof. Each of the local graphs is connected, so we have $c_2 \geqslant 2$ and $a_1 \geqslant 2$. Now for the b^+ part see Koolen [18] and for the b^- part see Terwilliger [30], cf. [5, Theorem 4.4.11.]. □

The Conway–Smith graph, the Johnson graph and the Halved cube are the first three examples in Table 1.

Table 1
Known examples, where "!" indicates the uniqueness of the corresponding graph.

#	Graph	k	p	q	r	μ	Local graphs	μ-graphs
1	! Conway–Smith	10	1	2	3	2	Petersen graph	K_2
2	! $J(8,4)$	16	2	2	2	4	$GQ(3,1)$	$K_{2,2}$
3	! halved 8-cube	28	4	2	2	6	$T(8)$	$K_{3\times 2}$
4	$3.O_6^-(3)$	45	3	3	3	6	$GQ(4,2)$	$K_{3,3}$
5	! Soicher1 [26]	56	2	4	3	8	Gewirtz graph	$2\cdot K_{2,2}$
6	$3.O_7(3)$	117	9	3	3	12	SRG$(117,36,15,9)$	$K_{4\times 3}$
7	Meixner1 [19]	176	8	4	2	24	SRG$(176,40,12,8)$	$2\cdot K_{3\times 4}$
8	Meixner2 [19]	176	8	4	4	12	SRG$(176,40,12,8)$	$K_{3\times 4}$
9	Soicher2 [26]	416	20	4	3	32	SRG$(416,100,36,20)$	$\overline{K_2}$-ext. of $\frac{1}{2}Q_5$
10	$3.Fi_{24}^-$	31671	351	9	3	1080	SRG$(31671,3510,693,351)$	

Table 2
Remaining open case with $k \leqslant 416$ and $v \leqslant 4096$.

#	k	p	q	r	μ
1	81	6	3	3	9
2	96	4	4	2	16
3	96	4	4	4	8
4	115	3	5	2	20
5	115	3	5	4	10
6	117	9	3	2	18
7	175	5	5	5	10
8	176	8	4	3	16
9	189	15	3	3	18
10	204	4	6	2	30
11	204	4	6	5	12
12	261	21	3	2	36
13	288	6	6	2	36
14	288	6	6	3	24
15	329	5	7	2	42
16	336	16	4	2	40
17	416	20	4	2	48

Since we know all antipodal tight graphs AT4$(p,2,r)$ and AT4$(1,q,r)$, we can assume, when looking for new examples, that $p \geqslant 2$ and $q \geqslant 3$ by the above result. An antipodal tight graph AT4(p,q,r) is Q-polynomial if and only if $r = 2$.

Theorem 6.1(iii) implies $r \leqslant p+q$. Let us now consider the case when $r = p+q$ and the case of equality in Theorem 6.1(ii).

Theorem 6.4. *Let Γ be an antipodal tight graph* AT4(p,q,r). *Then the following are equivalent*:
(i) *Γ is the Conway–Smith graph, i.e.*, $(p,q,r) = (1,2,3)$,
(ii) $p = 1$,
(iii) $p+q = r$,
(iv) *Γ is a Terwilliger graph.*

Proof. (i) $\Rightarrow$ (ii), (iii), (iv): Let Γ be the Conway–Smith graph, i.e., a unique distance-regular graph with parameters $\{10, 6, 4, 1; 1, 2, 6, 10\}$, cf. [5, Section 13.2.B]. Then $(p, q, r) = (1, 2, 3)$, and hence (ii), (iii) and, since the valency of μ-graphs in p by [16, Theorem 3.1(i)], also (iv) hold.

(ii) $\Rightarrow$ (i): Follows directly from Theorem 6.3.

(iv) $\Rightarrow$ (i): Let Γ be an antipodal tight graph AT4(p, q, r). The μ-graphs have valency p by [11, Lemma 2.3] and $c_2 = q(p+q)/r$, where $(p+q)/r = \sigma_2$, is an integer. Let Γ be a Terwilliger graph, i.e., $c_2 = p+1$, i.e., $p+1 = q(p+q)/r$, i.e., $r = q(p+q)/(p+1)$. Hence q divides $p+1$ and $p+1$ divides $q(q-1)$.

The local graphs of Γ are strongly regular graphs with parameters $(k', \lambda', \mu') = (p(q+1), 2p-q, p)$ and are also Terwilliger graphs. Therefore, by [5, Theorem 1.16.3],

$$\lambda'(\lambda'-1) \geqslant (\mu'-1)(k'-\lambda'-1),$$
$$\text{i.e., } p^2(5-q) - 2p(2q+1) + q^2 + 2q - 1 \geqslant 0.$$

Since $\lambda' \geqslant 0$ implies $2p \geqslant q$, it follows $5-q > 0$. By Theorem 5.3, $q \geqslant 2$, so $q \in \{2, 3, 4\}$. The conditions $q | p+1$, $p+1 | q(q-1)$ and, by Theorem 6.1(v)(vi), the conditions $p+q | q^2(q^2-1)$ and $p+q^2 | q^2(q^2-1)(q^2+q-1)(q+2)$ are satisfied only by $(p, q, r) = (1, 2, 3)$.

(iii)$\Rightarrow$(i): Let now Γ be an antipodal tight graph AT4$(p, q, p+q)$. Then $c_2 = q$ and since the μ-graphs have valency p by [16, Theorem 3.1(i)], we conclude $q \geqslant p+1$. On the other hand, by Theorem 6.1(iv), $q \leqslant p+2$. Thus $q \in \{p+1, p+2\}$. If $q = p+1$, then Γ is a Terwilliger graph and we must have $(p, q, r) = (1, 2, 3)$.

Suppose now $q = p+2$. Then the complement of a μ-graph consists of $(p+2)/2$ copies of K_2, i.e., the μ-graph is $K_{(p+2)/2\times 2}$ and λ' (λ of the local graph) is $2p - q = p-2$, therefore $p \geqslant 2$ and p is even. Brouwer [3] proved that $(p, q) = (2, 4)$ implies $r = 3$, thus $p \geqslant 4$ and $r \geqslant 10$.

Let x and y be vertices of Γ at distance two. Let u and v be nonadjacent vertices in a local graph Δ of y. The μ-graph corresponding to u and v contains y, so it has only y and its antipodal vertex outside Δ. Hence, the set $D_1^1(u, v) \cap \Gamma(y)$ induces $K_{p/2\times 2}$ and the set $\{u, v\} \cup (D_1^1(u, v) \cap \Gamma(y))$ induces $K_{(p+2)/2\times 2}$ in Δ. The parameters λ and μ of the graph $K_{(p+2)/2\times 2}$ are the same as the parameters of the local graph Δ and $p \geqslant 4$, i.e., $(p+2)/2 \geqslant 3$, so each edge of Δ lies in at most one $K_{(p+2)/2\times 2}$ in Δ.

Let z be a vertex of $D_1^1(x, y)$. For each vertex $t \in D_1^3(x, y)$, the set $\{z, t\} \cup (D_1^1(z, t) \cap \Gamma(y))$ induces $K_{(p+2)/2\times 2}$ inside $\Gamma(y)$. Since the graphs $K_{(p+2)/2\times 2}$ inside $\Gamma(y)$ have no common edges and since z has $a_1 - \mu' = p(p+2)$ neighbours in $D_1^2(x, y)$, there are at most $p+2$ copies of $K_{(p+2)/2\times 2}$ containing z. But we have $|D_1^3(x, y)| = p_{13}^2 = b_2 = (r-1)(p+2)$ choices for t, so $r = 2$. Contradiction! $\square$

The above result rules out an infinite family of feasible parameters of antipodal tight graphs AT4$(2s-2, 2s, 4s-2)$, for $s \geqslant 2$. The case $s = 2$ was done first by Brouwer [3]. The implication (iv)$\Rightarrow$(i) in the above result can be derived also by [15, Corollary 4.10], where we classified 1-homogeneous Terwilliger graphs with $c_2 > 1$.

Corollary 6.5. *There is no distance-regular graph with intersection array*,

$$\{4s(2s^2-1),(2s+1)(2s-1)^2,(s-1)(2s-1),1;$$
$$1,2s-2,(2s+1)(2s-1)^2,4s(2s^2-1)\} \quad \text{for } s \geqslant 2.$$

In particular for $s=3$ we have the following intersection array: $\{204,175,40,1;1,4,175,204\}$.

7. Larger diameter

We started this paper with a study of Krein parameters of nonbipartite antipodal distance-regular graphs of diameter 3 or 4 that are not zero for each such graph, but do vanish in some cases. These two families have, respectively, 3 and 4 free parameters (in diameter 3 case see Lemma 3.1: (n,c_2,r), and in diameter 4 case see Lemma 4.1: (k,a_1,c_2,r)). It seems to be more difficult to consider nonbipartite antipodal distance-regular graphs of diameter 5, or the primitive distance-regular graphs of diameter 3 or 4, since the number of free parameters is, respectively, five, five or seven. A good hint which Krein parameters are especially interesting could be the nontrivial vanishing Krein parameters q^1_{14}, q^4_{14} and q^2_{14} of the Patterson graph, which is also tight, see [14, Section 13(xii), 5, Theorem 13.7.1]. This implies that $E_1 \circ E_4$ is a scalar multiple of E_3. Pascasio [21] obtained a generalization of this and also of our Theorem 5.2: a distance-regular graph Γ of diameter $d \geqslant 3$ is tight if and only if the entry-wise product $E_1 \circ E_d$ is a scalar multiple of E_{d-1}, i.e., $q^i_{1d}=0$ for $i \neq 0, d-1$. We use her result to simplify the Fundamental bound in the case of distance-regular graphs with vanishing Krein parameter q^d_{1d}.

Let Γ be an antipodal distance-regular graph of diameter d and let r be the size of the antipodal classes. If $r=2$ then $q^i_{jk}=0$ if and only if $i+j+k$ is odd (since 'bipartite' is dual of being 'antipodal', see Corollary 4.4 and [5, Section 8.2]). Therefore, when d is even and $r=2$ we have $q^d_{1d}=0$.

Theorem 7.1. *Let Γ be a distance-regular graph of diameter $d \geqslant 3$, eigenvalues $k=\theta_0 > \theta_1 > \cdots > \theta_d$ and $q^d_{d1}=0$. Then $\theta_1\theta_d \geqslant k\theta_{d-1}$ and equality holds if and only if Γ is tight.*

Proof. Let n be the number of vertices of Γ. Then

$$E_1 \circ E_d = \frac{1}{n}\sum_{i=0}^{d} q^i_{1d}E_i = \frac{1}{n}\sum_{i=0}^{d-1} q^i_{1d}E_i.$$

By looking at the (0,0)-entry and the (1,1)-entry of the above matrix equation, we conclude, respectively,

$$1=\frac{1}{n}\sum_{i=0}^{d-1} q^i_{1d} \quad \text{and} \quad \frac{\theta_1}{k}\,\frac{\theta_d}{k}=\frac{1}{n}\sum_{i=0}^{d-1} q^i_{1d}\frac{\theta_i}{k}.$$

Since θ_{d-1} is the smallest eigenvalue in the above sum, it follows that $\theta_1\theta_d \geqslant k\theta_{d-1}$, and equality holds if and only if $q^i_{1d} = 0$ for $i \leqslant d-2$. Finally, by the result of Pascasio [21], Γ is tight if and only if the entry-wise product $E_1 \circ E_d$ is a scalar multiple of E_{d-1}, i.e., $q^i_{1d} = 0$ for $i \neq 0, d-1$. □

Let Γ be a distance-regular graph of diameter at least 3, whose local graphs are strongly regular. We show that Γ is a Taylor graph if a local graph of Γ has a nonintegral eigenvalue. Strongly regular graphs with the same parameters as their complements are called *conference graphs*, see [10, Section 10.2]. They are the only strongly regular graphs which could have nonintegral eigenvalues. If we assume that the local graphs of a distance-regular graph Γ are conference graphs with parameters (v', k', λ', μ'), then Weetman [32, Corollary 3.7] shows that $2\mu' = k'$ implies $d \leqslant 3$. We will show that, in the case of diameter 3, Γ is a Taylor graph.

Proposition 7.2. *A distance-regular graph, whose local graphs are conference graphs, has diameter at most* 3, *and if the diameter is* 3, *then it is a Taylor graph.*

Proof. Let us assume Γ is a distance-regular graph of diameter at least 3 whose local graphs are conference graphs with parameters (v', k', λ', μ'), where $a_1 = k' = (v'-1)/2$ and $\lambda' + 1 = \mu' = (v'-1)/4$, cf. [9, Corollary 10.2.2]. Suppose $b_2 \geqslant 2$. Let x and y be vertices of Γ at distance two and

$$C_2(x,y) := \Gamma(x) \cap \Gamma(y), \quad A_2(x,y) := \Gamma_2(x) \cap \Gamma(y), \quad B_2(x,y) := \Gamma_3(x) \cap \Gamma(y).$$

Consider the partition $C_2(x,y) \cup A_2(x,y) \cup B_2(x,y)$. Let $u, v \in B_2(x,y)$ be two distinct vertices and $w \in C_2(x,y)$. Since μ' is the valency of the graph induced by $C_2(x,y)$, w has exactly μ' neighbours in $A_2(x,y)$. Therefore, these are the common neighbours of u and w and also the common neighbours of v and w. Since $\mu' > \lambda'$, vertices u and v cannot be adjacent, hence $B_2(x,y)$ is an independent set and $A_2(x,y)$ has at least $3\mu'$ vertices. But this is not possible, since $C_2(x,y)$ has at least $\mu'+1$ vertices and $|C_2(x,y) \cup A_2(x,y) \cup B_2(x,y)| = 4\mu'+1$. Thus $b_2 = 1$ and Γ is an antipodal distance-regular r-cover of a clique by Araya et al. [1]. Γ is 1-homogeneous by [15, Theorem 3.1], since the above partition is equitable and $a_3 = 0$. Therefore, Γ is tight by [14, Theorem 11.7], and finally Γ is a Taylor graph by Proposition 7.2. □

Corollary 7.3. *Let Γ be a distance-regular graph of diameter $d \geqslant 3$ whose local graphs are strongly regular. Then the eigenvalues of a local graph of Γ are integral unless Γ is a Taylor graph.*

Now we derive a new existence condition for a nonbipartite antipodal distance-regular graph.

Theorem 7.4. *Let Γ be a graph of diameter d and with eigenvalues $\theta_0 > \cdots > \theta_d$ that is an antipodal quotient of a nonbipartite antipodal distance-regular graph Γ.*

Let Δ be a local graph of $\tilde{\Gamma}$. Then

(i) $-1 - b_1/(\theta_1 + 1)$ *is not an eigenvalue of* Δ,
(ii) *if* $-1 - b_1/(\theta_d + 1)$ *is an eigenvalue of* Δ, *then the diameter* d *of* $\tilde{\Gamma}$ *is even.*

Proof. In the case of a distance-regular graph X of diameter e and with eigenvalues $\eta_0 > \eta_1 > \cdots > \eta_e$ we set

$$b^-(X) = -1 - \frac{b_1}{\eta_1 + 1} \quad \text{and} \quad b^+(X) = -1 - \frac{b_1}{\eta_e + 1}.$$

Let $\tilde{\Gamma}$ be an antipodal distance-regular graph whose antipodal quotient is Γ, let $\tilde{d}$ be its diameter and $\tilde{\theta}_0 > \cdots > \tilde{\theta}_{\tilde{d}}$ be its eigenvalues. Since $\theta_0 = \tilde{\theta}_0,\ \theta_1 = \tilde{\theta}_2, \ldots, \theta_d = \tilde{\theta}_{2d}$, i.e., the eigenvalues of Γ interlace the eigenvalues of $\tilde{\Gamma}$ which are not the eigenvalues of Γ, see [5, p. 142], we have

$$b^-(\Gamma) < b^-(\tilde{\Gamma}) \quad \text{and} \quad b^+(\Gamma) \geqslant b^+(\tilde{\Gamma})$$

The second inequality holds with equality in the case of even diameter, since $\theta_d = \theta_{\tilde{d}}$ and with strict inequality in the case of odd diameter, since $\theta_d > \tilde{\theta}_{\tilde{d}}$. □

The above result shows that for example the Johnson graphs, Grassmann graphs and halved cubes have no distance-regular antipodal covers, cf. Brouwer and Van Bon [4]. Furthermore, it shows that the folded Johnson graph $J(2n, n)$ and folded halved $2n$-cube can only have antipodal distance-regular covers of even diameter. Also it shows that 1-homogeneous graphs have only distance-regular antipodal covers with even diameter and that tight graphs, in particular the Patterson graph, cannot have a distance-regular antipodal cover.

The above result is in principle similar to the idea of C. Godsil (private communication, 1990) that a distance-regular graph of diameter $d \geqslant 2$ and with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$, which contains a clique C that meets the Delsarte's bound $|C| \leq 1 - k/\theta_d$, has no antipodal distance-regular covers of odd diameter, cf. Schade [23, Theorem 4.1].

Let us sketch an alternative proof of the fact that $q^2_{22} > 0$ in the case when Γ is a nonbipartite antipodal distance-regular graph of diameter 4 and with eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$, from Theorem 4.3. Let us suppose the opposite, i.e., $q^2_{22} = 0$ in Γ. Then we have $q^1_{11} = 0$ in the antipodal quotient Ω and the smallest eigenvalue of Ω is also the smallest eigenvalue of a local graph by Cameron et al. [6, Section 6]. Since $b_1 = -(\theta_2 + 1)(\theta_4 + 1)$ by Lemma 4.1, and therefore $\theta_4 = -1 - b_1/(\theta_2 + 1)$ is an eigenvalue of Ω (note that the eigenvalues of Ω are $\theta_0 > \theta_2 > \theta_4$). This means that Ω has no distance-regular antipodal covers by Theorem 7.4, which is not possible.

8. Conclusion

Now we search for interesting subfamilies of the AT4 family, see Tables 1 and 2. One example is the subfamily for which $p = q - 2$, i.e., AT4$(q - 2, q, r)$, i.e., the subfamily for which also the Krein parameter q^4_{44} vanishes. Another example is the subfamily AT4(qs, q, r), s integral, and it is related to finite geometries. We need to introduce a few basic terms and their properties before we can describe this relation.

An incidence structure I of points P and lines L is a *partial geometry* with parameters (R, K, T) when there are exactly R lines through each point, each line contains K points, any two points lie in at most one line (i.e., any two lines intersect in at most one point), and for a line m and a point p not on m there are exactly T lines through p that meet m. The *point graph* of a partial geometry is the graph with points as vertices and whose edges are the pairs of collinear points. The point graph of a partial geometry with parameters (R, K, T) is strongly regular with parameters $k = R(K-1)$, $\mu = RT$, $\lambda = (R-1)(T-1)+K-2$ and eigenvalues $t = K-1-T$ and $s = -R$. A strongly regular graph is called *pseudogeometric* with parameters (R, K, T) if its parameters k, t and s are given by the above formulas for some integers R, K and T.

Lemma 8.1. *Let Γ be an antipodal tight graph* AT4(p, q, r). *Then a local graph is pseudogeometric with parameters $(q, p+1+p/q, p/q)$ if and only if $q \mid p$.*

Proof. Let Γ be an antipodal tight graph AT4(p, q, r) and let us suppose that the parameters of a local strongly regular graph satisfy the following equations $k' = R(K-1)$, $\lambda' = (R-1)(T-1)+K-2$ and $\mu' = RT$, for some real numbers R, K and T. So the eigenvalues of the local graph are $r = K-1-T$ and $s = -R$. Then, by Corollary 5.6 which determines the parameters of local graphs, $p(q+1) = R(K-1)$, $p = RT$, $R = q$, so $T = p/q$ and $K = p+1+p/q$. Therefore, the parameters R, K and T are integral if and only if $q \mid p$. □

Remark 8.2. In the case of AT4$(4, 4, 2)$, whose existence is still open, a local graph could be the point graph of a generalized quadrangle GQ$(5, 3)$. In the case of AT4$(8, 4, 2)$ and AT4$(8, 4, 4)$, a local graph cannot be geometric, since it is known that there is no partial geometry with parameters $(4, 11, 2)$. However, there exists a nongeometric graph (constructed on nonisotropic points in $U_5(2^2)$, two adjacent when on a tangent), which is a local graph of the Meixner1 graph and the Meixner2 graph.

From the feasible parameters for AT4(p, q, r) given in Table 1 we notice that except for the Conway–Smith graph, all the known examples satisfy either

(1) $q \mid p$, i.e., a local graph is pseudogeometric by Lemma 8.1, or
(2) $p = q-2$, i.e., $q^4_{44} = 0$ by Theorem 6.1(iv).

Although the conditions (1) and (2) do not necessarily hold in general (the smallest such feasible example is AT4$(9, 6, 3)$), among the first 1000 of feasible cases of antipodal tight graphs AT4(p, q, r) there are 93% of parameter sets that satisfy either (1) or (2). Therefore, it seems reasonable to consider separately these two conditions. In a follow up paper [13] it will be shown that in antipodal tight graphs AT4$(q-2, q, r)$ each second subconstituent is again an antipodal distance-regular graph of diameter 4, and if we additionally assume $r = 2$, then the graph is 2-homogeneous. Antipodal tight graphs AT4(qs, q, q), s integral, are studied in another follow up paper [17], where we show that we already know all the examples. In particular, there are no graphs with feasible arrays 1, 3, 7, and 9 from Table 2. Furthermore, we determine all the antipodal tight graphs AT4(p, q, r), whose μ-graphs are complete multipartite graphs $K_{t\times n}$, for some $t, n \in \mathbb{N}$ (again, we already know all the examples), and show $r \leqslant q$ for all antipodal tight graphs AT4(p, q, r) except the Conway–Smith graph, i.e., AT4(1,2,3).

Finally, we propose two problems for further study of the AT4 family.

1. *Are there infinitely many antipodal tight graphs* AT4(p, q, r)?
2. *Find new existence conditions for antipodal tight graphs* AT4(p, q, r), *in particular, try to rule out any parameter set from Table* 1(b) (*beside* 1, 3, 7, 9), *or construct a new graph with such parameters.*

For example, an absolute bound rules out AT4(21, 3, 3), which satisfy all the conditions of Theorem 6.1.

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