J. Math. Anal. Appl. 352 (2009) 614–619

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Aluthge iterations of weighted translation semigroups

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ARTICLE INFO

Article history:
Received 17 June 2008
Available online 6 November 2008
Submitted by J.H. Shapiro

Keywords:
Weighted translation semigroup
Aluthge transform
Aluthge iteration
Hyponormal operator

ABSTRACT

The problem whether Aluthge iteration of bounded operators on a Hilbert space $\mathcal{H}$ is convergent was introduced in [I. Jung, E. Ko, C. Pearcy, Aluthge transforms of operators, Integral Equations Operator Theory 37 (2000) 437–448]. And the problem whether the hyponormal operators on $\mathcal{H}$ with $\dim \mathcal{H} = \infty$ has a convergent Aluthge iteration under the strong operator topology remains an open problem [I. Jung, E. Ko, C. Pearcy, The iterated Aluthge transform of an operator, Integral Equations Operator Theory 45 (2003) 375–387]. In this note we consider symbols with a fractional monotone property which generalizes hyponormality and 2-expansivity on weighted translation semigroups, and prove that if $\{S_t\}$ is a weighted translation semigroup whose symbol has the fractional monotone property, then its Aluthge iteration converges to a quasinormal operator under the strong operator topology.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry. For $T = U|T|$ in $B(\mathcal{H})$, the *Aluthge transform* of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ (cf. [1,10]). Several operators related to such transforms are well developed and introduced in detail [8]. For every T in $B(\mathcal{H})$, the sequence of Aluthge iterates of T is defined by $\widetilde{T}^{(0)} = T$ and $\widetilde{T}^{(n+1)} = (\widetilde{T}^{(n)})^{\sim}$ for $n \in \mathbb{N}$. In [11] the authors continued to study this sequence $\{\widetilde{T}^{(n)}\}$ of iterates, and discussed the convergence of Aluthge iterations in some special cases. In particular, it was shown in [4] that the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ of iterated Aluthge transforms of T need not converge in the strong operator topology in general. However, it was proved that the sequence $\{\widetilde{T}^{(n)}\}$ (of $n \times n$ complex matrices) converges to a normal operator (cf. [3,2]). In this note we discuss Aluthge iteration of a weighted translation semigroup $\{S_t\}$ with symbol ϕ which will be defined below.

Let $\mathbb{R}_+ := (\mathbb{R}_+, \mu)$ be the Lebesgue measure space on the set of non-negative real numbers and let $L^2 := L^2(\mathbb{R}_+)$ be the Hilbert space of square integrable Lebesgue measurable complex valued functions on $\mathbb{R}_+$. Let $B(L^2)$ be the algebra of all bounded linear operators on L^2. A family $\{S_t\colon t \in \mathbb{R}_+\}$ in $B(L^2)$ is a *semigroup* if $S_0 = I$ and $S_t S_s = S_{t+s}$ for all t and s in $\mathbb{R}_+$. In particular, a *weighted translation semigroup* $\{S_t\}$ on L^2 is defined by

$$(S_t f)(x) = \begin{cases} \frac{\phi(x)}{\phi(x-t)} f(x-t) & \text{if } t \leqslant x, \\ 0 & \text{if } 0 \leqslant x < t, \end{cases}$$

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1 Supported by a grant (R01-2008-000-20088-0) from the Korea Research Foundation.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.10.055

where ϕ is a measurable, almost everywhere non-zero function from $\mathbb{R}_+$ into $\mathbb{C}$ that is called the *symbol* of $\{S_t\}$. A semigroup $\{S_t\}$ is *strongly continuous* if, for each f in L^2, the mapping $t \to S_t f$ is continuous from $\mathbb{R}_+$ into L^2. It follows from [5, p. 619] that $\{S_t\}$ is strongly continuous on $\mathbb{R}_+$ if and only if there exist $M, \omega > 0$ such that

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}_+}\left|\frac{\phi(x+t)}{\phi(x)}\right| \leqslant Me^{\omega t}, \quad t \in \mathbb{R}_+. \tag{1.1}$$

For brevity we will assume that ϕ is continuous on $\mathbb{R}_+$ throughout this article. Since the weighted translation semigroups with symbols ϕ and $|\phi|$ are unitarily equivalent, we will assume throughout this paper that all symbols of weighted translation semigroups are positive, and also assume that $\{S_t\}$ is a strongly continuous semigroup with symbol ϕ. (See [9] for more information about semigroups.)

This note is organized as follows: In Section 2 we introduce symbols with a fractional monotone property and discuss membership in various classes for semigroups with such symbols. In Section 3 the nth Aluthge iterations of a weighted translation semigroup $\{S_t\}$ are described in detail, and it is proved that if $\{S_t\}$ is a weighted translation semigroup whose symbol has the fractional monotone property, then its Aluthge iteration $\{\widetilde{S}_t^{(n)}\}_{n\geqslant 1}$ converges to a quasinormal operator in $B(L^2)$ under the strong operator topology.

2. Fractional monotone properties

Let ϕ be a symbol satisfying (1.1) and let

$$\Phi_t^{(k)}(x) := \begin{cases} \frac{\phi(x+(k+1)t)}{\phi(x+(k-1)t)} & \text{if } t \leqslant x, \\ 0 & \text{if } 0 \leqslant x < t, \end{cases} \tag{2.1}$$

for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}_+$. Then $\{\Phi_t^{(k)}(x)\}_{k=0}^{\infty}$ is a sequence of measurable functions on $\mathbb{R}_+$.

Definition 2.1. Let ϕ be a symbol satisfying (1.1). The symbol ϕ is said to have the *fractional monotone property* (we write *f.m.p.*) if the sequence $\{\Phi_t^{(k)}(x)\}_{k\geqslant 0}$ as in (2.1) is monotone pointwise on $[t,\infty)$, for each $t \in \mathbb{R}_+$. And, when $\{\Phi_t^{(k)}(x)\}_{k\geqslant 0}$ is monotone increasing (decreasing, resp.) pointwise on $[t,\infty)$ for each $t \in \mathbb{R}_+$, we say that the symbol ϕ has the *fractional monotone increasing* (*decreasing*, resp.) *property* (we write f.m.i.p. (f.m.d.p, resp.)).

Let ϕ be a symbol with the f.m.p. Then, since

$$\frac{\phi(x+(k+1)t)}{\phi(x+(k-1)t)} \leqslant \left\|\frac{\phi(x+2t)}{\phi(x)}\right\|_{\infty} \leqslant Me^{2\omega t}, \quad x \geqslant t,$$

$\{\Phi_t^{(k)}(x)\}_{k=0}^{\infty}$ is a bounded sequence pointwise on $\mathbb{R}_+$ for $t \in \mathbb{R}_+$. Therefore a measurable bounded function $\lim_{k\to\infty}\Phi_t^{(k)}(x)$ exists and we denote it by

$$\Phi_t^{(\infty)}(x) := \lim_{k\to\infty}\Phi_t^{(k)}(x). \tag{2.2}$$

In particular, if ϕ has the f.m.i.p. (or, f.m.d.p.), then $\Phi_t^{(\infty)}(x) = \sup_{k\geqslant 0}\Phi_t^{(k)}(x)$ (or, $\inf_{k\geqslant 0}\Phi_t^{(k)}(x)$).

Recall from [6, Lemma 3.3] that a weighted translation semigroup $\{S_t\}$ with symbol ϕ is hyponormal if and only if

$$\phi(x-t)\phi(x+t) \geqslant \phi^2(x), \quad x \geqslant t. \tag{2.3}$$

Proposition 2.2. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then the following assertions are equivalent*:

(i) *$\{S_t\}$ is hyponormal*;
(ii) *ϕ has the f.m.i.p.*;
(iii) *$\log\phi$ is convex.*

Proof. (i) $\Rightarrow$ (ii). Suppose $\{S_t\}$ is hyponormal. By (2.3)

$$\frac{\phi(x+2t)}{\phi(x)} \geqslant \frac{\phi(x+t)}{\phi(x-t)}, \quad x \geqslant t,$$

which implies that $\Phi_t^{(k+1)}(x) \geqslant \Phi_t^{(k)}(x)$, $k \in \mathbb{N}_0$. Hence ϕ has f.m.i.p.

(ii) $\Rightarrow$ (i). The method is similar to that used above and omitted.

(i) $\Leftrightarrow$ (iii). Condition (iii) is equivalent to (2.3). (Also see [6, Lemma 3.3].) □

Recall from [7] that T in $B(\mathcal{H})$ is *k-expansive* if

$$\sum_{0\leqslant p\leqslant k}(-1)^p\binom{k}{p}\left\|T^ph\right\|^2\leqslant 0,\quad h\in\mathcal{H}.$$

A simple computation shows that the k-expansivity of $\{S_t\}$ is equivalent to the inequality

$$\sum_{0\leqslant p\leqslant k}(-1)^p\binom{k}{p}\phi^2(x+pt)\leqslant 0,\quad x\in\mathbb{R}_+,\ t\in\mathbb{R}_+. \tag{2.4}$$

Proposition 2.3. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then $\{S_t\}$ is 2-expansive if and only if $\phi^2(x)$ is concave, and thus 2-expansivity implies that $\log\phi$ is concave.*

Proof. The first part is obvious by (2.4). For the second part, since $\phi^2(x)+\phi^2(x+2t)\leqslant 2\phi^2(x+t)$, $x\in\mathbb{R}_+$, $\phi(x)\phi(x+2t)\leqslant\phi^2(x+t)$, i.e., $\log\phi$ is concave. □

Corollary 2.4. *If $\log\phi$ is concave, then ϕ has f.m.d.p. Thus the symbol ϕ of any 2-expansive weighted translation semigroup $\{S_t\}$ has the f.m.d.p.*

Proof. Since the inequality $\phi(x)\phi(x+2t)\leqslant\phi^2(x+t)$ implies that

$$\frac{\phi(x+2t)}{\phi(x)}\leqslant\frac{\phi(x+t)}{\phi(x-t)}\quad\text{for all } x\geqslant t,$$

the sequence $\{\Phi_t^{(n)}(x)\}_{n\in\mathbb{N}_0}$ is decreasing pointwise on $[t,\infty)$. Hence ϕ has f.m.d.p. □

Example 2.5. This example will be continued in Example 3.5.

(i) Let $\phi(x)=e^{-x}$ for $x\in\mathbb{R}_+$ satisfying (1.1) with $M=1=\omega$. Then $\log\phi(x)=-x$ is convex and concave but ϕ^2 is not concave. However, the symbol ϕ has both f.m.i.p. and f.m.d.p.

(ii) Define $\phi(x)=\sqrt{\log(x+1)}$ for $0\leqslant x\leqslant 1$ and $\phi(x)=\sqrt{\log 2}$ for $x\geqslant 1$ satisfying (1.1). Then ϕ^2 is concave, and so the symbol ϕ has f.m.d.p.

(iii) Define $\phi(x)=e^{x^2}$ for $0\leqslant x\leqslant 1$ and $\phi(x)=e$ for $x\geqslant 1$ satisfying (1.1). Then ϕ^2 is not convex nor concave. And also, $\log\phi$ is not convex nor concave.

Remark 2.6. There are several classes of operators with weak hyponormality, for example, p-paranormal operators, absolutely p-paranormal operators, $A(p)$-class operators, etc. (The definitions of these classes will be given below.) The symbols of these weighted translation semigroups have f.m.p., too. Recall that T is *p-hyponormal* if $(T^*T)^p\geqslant(TT^*)^p$; *$p$-paranormal* if $\||T|^pU|T|^px\|\geqslant\||T|^px\|^2$ for all unit vectors $x\in\mathcal{H}$; *absolute p-paranormal* if $\||T|^pTx\|\geqslant\|Tx\|^{p+1}$ for all unit vectors $x\in\mathcal{H}$; and *$A(p)$-class* if $(T^*|T|^{2p}T)^{1/(p+1)}\geqslant|T|^2$ $(0<p<\infty)$ (cf. [8,12]). It is known that "p-hyponormal $\Rightarrow$ $A(p)$ class $\Rightarrow$ absolute p-paranormal"; "p-hyponormal $\Rightarrow$ p-paranormal". In fact, some direct computations show that if $\{S_t\}$ is a weighted translation semigroup, then $\{S_t\}$ is one of the above weak hyponormal semigroups if and only if $\log\phi$ is convex, which holds if and only if $\{S_t\}$ is hyponormal.

An operator $T\in B(\mathcal{H})$ is *normaloid* if $\|T^n\|=\|T\|^n$ for all $n\in\mathbb{N}$ (cf. [8]). It is easy to show that if $\{S_t\}$ is a weighted translation semigroup with symbol ϕ, then $\{S_t\}$ is normaloid if and only if $\|\phi(x)/\phi(x-t)\|_\infty^n=\|\phi(x)/\phi(x-nt)\|_\infty$ for all $n\in\mathbb{N}$. Also, it is known that "p-paranormal $\Rightarrow$ normaloid" (cf. [8]); however, in general, the normaloid of a weighted translation semigroup $\{S_t\}$ is not equivalent to its hyponormality (see Example 2.7).

Example 2.7. Let us consider a symbol $\phi(x)=2-x^2$ for $0\leqslant x\leqslant 1$ and $\phi(x)=1$ for $x\geqslant 1$ satisfying (1.1). Since $\log\phi$ is not convex, a weighted translation semigroup $\{S_t\}$ with symbol ϕ is not hyponormal. But, $\|S_t\|=\|\phi(x)/\phi(x-t)\|_\infty=1$ and $\|S_t^n\|=\|\phi(x)/\phi(x-nt)\|_\infty=1$. Thus $\{S_t\}$ is normaloid.

3. Convergence of Aluthge iterations

Let $\{U_t\}$ be the isometric semigroup in $B(L^2)$ defined by $(U_tf)(x)=f(x-t)$ for $x\geqslant t$ and 0 otherwise. Then the polar decomposition of a weighted translation semigroup S_t is represented by $U_t|S_t|$. Note that $S_t^*f(x)=\frac{\phi(x+t)}{\phi(x)}f(x+t)$ and $|S_t|f(x)=\frac{\phi(x+t)}{\phi(x)}f(x)$.

The following is the main theorem of this note.

Theorem 3.1. *Let $\{S_t\}$ be a weighted translation semigroup whose symbol ϕ has the f.m.p. Then the sequence $\{\widetilde{S}_t^{(n)}\}_{n\geqslant 1}$ of Aluthge iteration converges to a quasinormal operator Δ_t in $B(L^2)$ under the strong operator topology* (SOT), *where*

$$\Delta_t f(x) = \begin{cases} \Phi_t^{(\infty)}(x)^{1/2} U_t f(x) & \text{if } t \leqslant x, \\ 0 & \text{if } 0 \leqslant x < t. \end{cases} \tag{3.1}$$

We need several lemmas to prove this theorem.

Lemma 3.2. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Suppose that $n \in \mathbb{N}$. Then*

$$\widetilde{S}_t^{(n)} f(x) = \prod_{k=0}^{n-1} \left(\frac{\phi(x+(k+1)t)}{\phi(x+(k-1)t)} \right)^{\binom{n-1}{k}/2^n} f(x-t), \quad x \geqslant t, \tag{3.2}$$

and 0 *otherwise.*

Proof. Since U_t is isometric, by a simple computation, we have

$$\widetilde{S}_t f(x) = \left(\frac{\phi(x+t)}{\phi(x-t)} \right)^{1/2} f(x-t), \quad x \geqslant t,$$

and 0 otherwise. Hence (3.2) holds for $n = 1$. For mathematical induction, we assume that (3.2) holds for some $n > 1$. For brevity, write $\phi_k = \phi(x+kt)$, $k = -1, 0, 1, \ldots$. Since

$$\widetilde{S}_t^{(n)*} g(x) = \prod_{k=0}^{n-1} \left(\frac{\phi_{k+2}}{\phi_k} \right)^{\binom{n-1}{k}/2^n} \cdot g(x+t), \quad x \geqslant 0,$$

we have that

$$\left|\widetilde{S}_t^{(n)}\right|^{1/2} = \prod_{k=0}^{n-1} \left(\frac{\phi_{k+2}}{\phi_k} \right)^{\binom{n-1}{k}/2^{n+1}}, \quad x \geqslant 0.$$

Set $\widetilde{S}_t^{(n+1)} = |\widetilde{S}_t^{(n)}|^{1/2} U_n |\widetilde{S}_t^{(n)}|^{1/2}$ with the polar decomposition $\widetilde{S}_t^{(n)} = U_n |\widetilde{S}_t^{(n)}|$ of $\widetilde{S}_t^{(n)}$. Since

$$\widetilde{S}_t^{(n)} f(x) = U_n \left|\widetilde{S}_t^{(n)}\right| f(x) = \prod_{k=0}^{n-1} \left(\frac{\phi_{k+1}}{\phi_{k-1}} \right)^{\binom{n-1}{k}/2^n} \cdot f(x-t), \quad x \geqslant t,$$

and 0 otherwise, $U_n = U_t$. Hence, for $x \geqslant t$, we have

$$\begin{aligned}
\widetilde{S}_t^{(n+1)} f(x) &= \left|\widetilde{S}_t^{(n)}\right|^{1/2} U_t \left|\widetilde{S}_t^{(n)}\right|^{1/2} f(x) \\
&= \left[\prod_{k=0}^{n-1} \left(\frac{\phi_{k+2}}{\phi_k} \right)^{\binom{n-1}{k}/2^{n+1}} \right] \left[\prod_{k=0}^{n-1} \left(\frac{\phi_{k+1}}{\phi_{k-1}} \right)^{\binom{n-1}{k}/2^{n+1}} \right] f(x-t) \\
&= \left(\frac{\phi_2^{\binom{n-1}{0}} \phi_3^{\binom{n-1}{1}} \cdots \phi_{n+1}^{\binom{n-1}{n-1}}}{\phi_0^{\binom{n-1}{0}} \phi_1^{\binom{n-1}{1}} \cdots \phi_{n-1}^{\binom{n-1}{n-1}}} \cdot \frac{\phi_1^{\binom{n-1}{0}} \phi_2^{\binom{n-1}{1}} \cdots \phi_n^{\binom{n-1}{n-1}}}{\phi_{-1}^{\binom{n-1}{0}} \phi_0^{\binom{n-1}{1}} \cdots \phi_{n-2}^{\binom{n-1}{n-1}}} \right)^{1/2^{n+1}} f(x-t) \\
&= \left(\frac{\phi_1^{\binom{n-1}{0}} \phi_2^{\binom{n-1}{0}+\binom{n-1}{1}} \phi_3^{\binom{n-1}{1}+\binom{n-1}{2}} \cdots \phi_n^{\binom{n-1}{n-2}+\binom{n-1}{n-1}} \phi_{n+1}^{\binom{n-1}{n-1}}}{\phi_{-1}^{\binom{n-1}{0}} \phi_0^{\binom{n-1}{0}+\binom{n-1}{1}} \phi_1^{\binom{n-1}{1}+\binom{n-1}{2}} \cdots \phi_{n-2}^{\binom{n-1}{n-2}+\binom{n-1}{n-1}} \phi_{n-1}^{\binom{n-1}{n-1}}} \right)^{1/2^{n+1}} f(x-t) \\
&= \left(\frac{\phi_1^{\binom{n}{0}} \phi_2^{\binom{n}{1}} \phi_3^{\binom{n}{2}} \cdots \phi_n^{\binom{n}{n-1}} \phi_{n+1}^{\binom{n}{n}}}{\phi_{-1}^{\binom{n}{0}} \phi_0^{\binom{n}{1}} \phi_1^{\binom{n}{2}} \cdots \phi_{n-2}^{\binom{n}{n-1}} \phi_{n-1}^{\binom{n}{n}}} \right)^{1/2^{n+1}} f(x-t) \\
&= \prod_{k=0}^{n} \left(\frac{\phi_{k+1}}{\phi_{k-1}} \right)^{\binom{n}{k}/2^{n+1}} f(x-t) \quad (x \geqslant t),
\end{aligned}$$

which proves this lemma. □

The next lemma and proposition use the function Φ introduced in (2.1) and (2.2).

Lemma 3.3. *Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers that converges to a. Then*

(i) *we have*

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}a_k=a;$$

(ii) *if $\{\Phi_t^{(n)}(x)\}_{n\geqslant 0}$ is a sequence of positive real numbers that converges to $\Phi_t^{(\infty)}(x)$ pointwise on $\mathbb{R}_+$, then $\prod_{k=0}^{n-1}(\Phi_t^{(k)}(x))^{\binom{n-1}{k}/2^n}$ converges to $(\Phi_t^{(\infty)}(x))^{1/2}$ pointwise on $\mathbb{R}_+$ as $n\to\infty$.*

Proof. (i) Let $\epsilon>0$. Find $N\in\mathbb{N}$ such that $k\geqslant N$ implies $|a_k-a|<\epsilon$. Let $M=\max\{|a_0-a|,|a_1-a|,\dots,|a_{N-1}-a|\}$. Then when $n\geqslant N$ we have

$$\begin{aligned}\left|\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}a_k-a\right|&=\left|\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(a_k-a)\right|\leqslant\left|\frac{1}{2^n}\sum_{k=0}^{N-1}\binom{n}{k}(a_k-a)\right|+\left|\frac{1}{2^n}\sum_{k=N}^{n}\binom{n}{k}(a_k-a)\right|\\&\leqslant\frac{M}{2^n}\sum_{k=0}^{N-1}\binom{n}{k}+\frac{1}{2^n}\sum_{k=N}^{n}\binom{n}{k}\epsilon<\frac{M}{2^n}\sum_{k=0}^{N-1}\binom{n}{k}+\epsilon.\end{aligned}$$

Therefore $\limsup_{n\to\infty}|\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}a_k-a|\leqslant\epsilon$. Since $\epsilon>0$ was arbitrary, (i) holds.

(ii) Since

$$b_n:=\ln\left(\prod_{k=0}^{n-1}\left(\Phi_t^{(k)}(x)\right)^{\binom{n-1}{k}/2^n}\right)=\frac{1}{2}\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}\ln\left(\Phi_t^{(k)}(x)\right),$$

letting $n\to\infty$ and applying (i), we see that the sequence $\{b_n\}$ converges to $\frac{1}{2}\ln(\Phi_t^{(\infty)}(x))$. And, upon exponentiating, we obtain (ii). □

Remark 3.4. In fact if we consider an arbitrary monotone increasing (or, decreasing) sequence $\{a_k\}_{k=0}^{\infty}$ of positive real numbers (instead of $\{\Phi_t^{(k)}(x)\}_{k=0}^{\infty}$) and write $\sigma_n:=\prod_{k=0}^{n-1}(a_{k+1})^{\binom{n-1}{k}/2^n}$, then $\{\sigma_n\}_{n=1}^{\infty}$ is monotone increasing (or, decreasing). (Indeed, by mathematical induction, we may prove that $\{\sigma_n\}_{n=1}^{\infty}$ is monotone increasing. For example, if $\{a_k\}$ is increasing, then

$$\begin{aligned}\sigma_1=a_1^{1/2}&\leqslant(a_1a_2)^{1/4}=\sigma_2=a_1^{1/4}a_2^{1/4}\leqslant(a_1a_2)^{1/8}(a_2a_3)^{1/8}=\sigma_3=a_1^{1/8}a_2^{1/4}a_3^{1/8}\leqslant(a_1a_2)^{1/16}(a_2a_3)^{1/8}(a_3a_4)^{1/16}\\&=a_1^{1/16}a_2^{3/16}a_3^{3/16}a_4^{1/16}=\sigma_4\leqslant\cdots,\quad\text{etc.}\end{aligned}$$

The monotone decreasing case is similar and omitted.)

Proof of Theorem 3.1. Suppose that ϕ has the f.m.p. Let $t\in\mathbb{R}_+$ and let $E_t:=\{x\in\mathbb{R}_+:\ |\Phi_t^{(\infty)}(x)|\leqslant\|\Phi_t\|_\infty\}$. Then obviously the complement of E_t has measure zero. To show the SOT-convergence of $\{\widetilde{S}_t^{(n)}\}_{n\geqslant 1}$, for $f\in L^2$, we consider $\|\widetilde{S}_t^{(n)}f(x)-\Delta_t f(x)\|_{L^2}$, where Δ_t is as in (3.1). Note that $\Phi_t^{(\infty)}(x)=\Phi_t^{(\infty)}(x-t)$ since $\Phi_t^{(\infty)}(x)$ is periodic w.r.t. $t\in\mathbb{R}_+$. And we have that

$$\begin{aligned}\left\|\widetilde{S}_t^{(n)}f(x)-\Delta_t f(x)\right\|_{L^2}^2&=\int_{[t,\infty)}\left|\prod_{k=0}^{n-1}\left(\Phi_t^{(k)}(x)\right)^{\binom{n-1}{k}/2^n}-\Phi_t^{(\infty)}(x-t)^{1/2}\right|^2\left|f(x-t)\right|^2d\mu\\&=\int_{\mathbb{R}_+}\left|\prod_{k=0}^{n-1}\left(\Phi_t^{(k+1)}(x)\right)^{\binom{n-1}{k}/2^n}-\Phi_t^{(\infty)}(x)^{1/2}\right|^2\left|f(x)\right|^2d\mu.\end{aligned}$$

Applying Lemma 3.3, and by using the Lebesgue dominated convergence theorem, we see that $\|\widetilde{S}_t^{(n)}f(x)-\Delta_t f(x)\|_{L^2}^2\to 0$ as $n\to\infty$.

On the other hand, since $\Delta_t^*f(x)=\Phi_t^{(\infty)}(x)^{1/2}f(x+t)$, by direct computation, we find that

$$\left(\Delta_t^*\Delta_t\right)\Delta_t f(x)=\Delta_t\left(\Delta_t^*\Delta_t\right)f(x)=\begin{cases}\Phi_t^{(\infty)}(x)^{3/2}f(x-t) & \text{if } t\leqslant x,\\ 0 & \text{if } 0\leqslant x<t.\end{cases}$$

Hence Δ_t is quasinormal and the proof is complete. □

We close this note with the following examples and remark.

Example 3.5. (Continued from Examples 2.5 and 2.7.) Using the same symbols as in Example 2.5(i), (ii) and (iii), we obtain (i) $\Delta_t = e^{-2t}U_t$; (ii) $\Delta_t = U_t$; (iii) $\Delta_t = U_t$. However, even if the symbol ϕ in Example 2.7 does not have the f.m.p, we may obtain $\Delta_t = U_t$ because $\phi(x)$ is constant for $x \geqslant 1$.

Remark 3.6. We do not know whether the Aluthge iterations of a normaloid weighted translation semigroup $\{S_t\}$ with symbol ϕ converges under the SOT.

Example 3.7. Theorem 3.1 allows us to easily construct many examples of a weighted translation semigroup $\{S_t\}$ whose Aluthge iteration $\{\widetilde{S}_t^{(n)}\}_{n\geqslant 1}$ converges to a quasinormal operator in $B(L^2)$ under the SOT. For example, let $f_0(x) = x + 1$, $0 \leqslant x \leqslant 1$, and let $f_n(x) = \frac{1}{n+1}(x - n) + f_{n-1}(n)$, $n \leqslant x \leqslant n + 1$, for $n \in \mathbb{N}$. Define $\phi(x) = e^{f_n(x)}$ for $n \leqslant x \leqslant n + 1$ and $n \in \mathbb{N}_0$. Then obviously $\phi(x)$ satisfies (1.1) and $\log \phi(x)$ is concave. Thus, by Theorem 3.1, the Aluthge iteration converges to a quasinormal operator Δ_t in $B(L^2)$ under the SOT.

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