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On the Number of Lattice Free Polytopes

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V. I. Arnold asked for estimates for the number of equivalence classes of lattice polytopes, under the group of unimodular affine transformations. What we investigate here is the analogous question for lattice free polytopes. Some of the results: the number of equivalence classes of lattice free simplices of volume at most v in dimension d is of order v^{d-1}, and the number of equivalence classes of lattice free polytopes of volume at most v in dimension d is $O(v^{2^d-1}(\log v)^{d-2})$.

1. INTRODUCTION AND RESULTS

There has recently been an increasing interest in integral polytopes. A particular class of them, the lattice free polytopes, play a special role in various areas: geometry of numbers (see [11]), integer programming [11, 16], singularities in algebraic geometry [9, 14], for example.

Given a polytope $P \subset R^d$ let vert P denote the set of its vertices. P is an *integral* or *lattice* polytope if vert $P \subset \mathbb{Z}^d$ and is *lattice free* if $\mathbb{Z}^d \cap P =$ vert P. Each lattice free polytope is, of course, a lattice polytope.

Two lattice polytopes, P and Q are said to be *equivalent* if there is a lattice-preserving affine transformation carrying P to Q. Equivalent polytopes have the same volume. Arnold [1] asked for estimates for the number $N(d, v)$ of equivalence classes of polytopes in R^d of volume (at most) v. (Of course, v is positive and is an integral multiple of $1/d!$.) After partial results by [1] and [12] the order of magnitude of $\log N(d, v)$ was determined in [3]: with suitable constants c_1, c_2 depending only on d

$$c_1 v^{\frac{d-1}{d+1}} \leq \log N(d, v) \leq c_2 v^{\frac{d-1}{d+1}}.$$

We will use Vinogradov's $\ll$ notation so that we can write this as

$$v^{\frac{d-1}{d+1}} \ll \log N(d, v) \ll v^{\frac{d-1}{d+1}}.$$

Here and in what follows the implied constants depend only on d, and all asymptotics are understood with d fixed and $v \to \infty$. One word of warning: the implied constants may be very large and we usually make no effort to compute or estimate them.

In this paper we study similar questions for lattice free simplices and, more generally, polytopes. Write $E(d, v)$ for the number of equivalence classes of lattice free simplices of dimension d and of volume at most v. Our main result is the following.

THEOREM 1. *When* $d \geq 3$

$$v^{d-1} \ll E(d, v) \ll v^{d-1}.$$

It is interesting to compare this with $S(d, v)$, the number of equivalence classes of integral simplices of dimension d and of volume at most v. The next result shows that a small, but not so minute, fraction of integral simplices are lattice free.

THEOREM 2. *When* $d \geq 2$

$$v^d \ll S(d, v) \ll v^d.$$

We also prove an upper bound on the number, $M(d, n, v)$, of (equivalence classes of) lattice free polytopes with n vertices and of volume at most v.

THEOREM 3. *When* $d \geq 3$

$$M(d, n, v) \ll v^{n-1}(\log v)^{d-2}.$$

For simplices, i.e., when $n = d+1$ this is weaker than Theorem 1. However as $|\operatorname{vert} P| \leq 2^d$ for a lattice free polytope (see [5, 16]), Theorem 3 implies the following.

COROLLARY. *When* $d \geq 3$, *the total number of equivalence classes of lattice free polytopes of volume at most* v *is* $\ll v^{2^d-1}(\log v)^{d-2}$.

This can probably be improved significantly. In three-dimensions, the truth is $O(v^2)$ and we see no reason why $O(v^{d-1})$ should not hold in general.

2. ORDERED SIMPLICES AND PROOF OF THEOREM 2

An *ordered* integral simplex $P \subset R^d$ is a simplex with a given ordering $z_0, z_1, \ldots, z_d$ of its vertices. Another ordered integral simplex Q with vertices $u_0, \ldots, u_d$ is *equivalent* to P if the unique affine map carrying z_i to u_i for all i is lattice preserving. Write $S^o(d, v)$ for the number of equivalence classes of ordered integral simplices of dimension d and of volume at most v. It is obvious that

$$\frac{1}{(d+1)!}S^o(d, v) \leq S(d, v) \leq S^o(d, v).$$

Given an ordered simplex P we first apply the unique translation carrying z_0 to the origin. The Hermite reduction theorem (see [4] for instance) states that there is a unique basis of $\mathbb{Z}^d$ such that in this basis, for each $i = 1, \ldots, d$ vertex z_i of P is the ith row of the lower triangular $d \times d$ matrix

$$M(P) = [a_{ij}]$$

where $a_{ij} \in \mathbb{Z}$, $a_{ij} = 0$ if $i < j$ and $0 \leq a_{ij} < a_{ii}$ if $i \geq j$. We call $M(P)$ the standard form of the ordered simplex P. It is evident that P and Q are equivalent as ordered simplices if and only if $M(P) = M(Q)$.

Clearly, $\operatorname{Vol} P = \frac{1}{d!}\det M(P)$. Write $M(d, V)$ for the number of distinct matrices in standard form with determinant at most V, here $V \in \mathbb{Z}$. Writing $d!v = V$ we have $S^o(d, v) = M(d, V)$. So in order to prove Theorem 2 it suffices to show the following.

THEOREM 4. *When* $d \geq 2$

$$\frac{1}{d}V^d \leq M(d, V) \leq V^d.$$

PROOF. Once $t_i = a_{ii}$ is fixed, the ith row of a matrix in standard form can be filled in t_i^{i-1} distinct ways. So there are $\prod_1^d t_i^{i-1}$ matrices with fixed diagonal $t_1, \ldots, t_d$. So

$$\begin{aligned} M(d, V) &= \sum_{t_1 \ldots t_d \leq V} \prod_1^d t_i^{i-1} \\ &= \sum_{t_d=1}^{V} t_d^{d-1} \sum_{t_{d-1}=1}^{\left\lfloor \frac{V}{t_d} \right\rfloor} t_{d-1}^{d-2} \cdots \sum_{t_2=1}^{\left\lfloor \frac{V}{t_3 \cdots t_d} \right\rfloor} t_2 \sum_{t_1=1}^{\left\lfloor \frac{V}{t_2 \cdots t_d} \right\rfloor} 1. \end{aligned}$$

The innermost sum here is

$$\left\lfloor \frac{V}{t_2 \ldots t_d} \right\rfloor \leq \frac{V}{t_2 \ldots t_d},$$

and so the second innermost sum is at most

$$\sum_{t_2=1}^{\left\lfloor \frac{V}{t_3 \ldots t_d} \right\rfloor} t_2 \frac{V}{t_2 \ldots t_d} \leq \left(\frac{V}{t_3 \ldots t_d} \right)^2.$$

Continuing the same way gives $M(d, V) \leq V^d$.

For the lower bound consider all matrices $M(P)$ where the ith row is e^i, the ith basis vector, for $i = 1, \ldots, d-1$ and the last row is $(x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$ with $0 \leq x_j < x_d \leq V$ for all $j = 1, \ldots, d-1$. These matrices are in standard form and have determinant at most V and their number is $\sum_1^V x_d^{d-1} \geq \frac{1}{d} V^d$. □

REMARK 1. This proof gives a more precise form of Theorem 2, namely

$$\frac{1}{d(d+1)!}(d!v)^d \leq S(d, v) \leq (d!v)^d.$$

REMARK 2. The number, $M^*(d, V)$, of matrices in standard form and with determinant exactly V can be computed precisely. It turns out that $M^*(d, V)$ is a multiplicative function of $V \in \mathbb{Z}$. An easy recursion shows that, when p is a prime and r a positive integer,

$$M^*(d, p^r) = \prod_{j=1}^{d-1} \frac{p^{r+j} - 1}{p^j - 1}.$$

3. FLATNESS AND THE PROOF OF THEOREM 1

We will use the lattice width of convex bodies and the flatness theorem throughout the paper. So we introduce the necessary concepts here (see [4] or [13] for explanations). Assume L is a d-dimensional lattice in R^d with a given basis $B = \{b^1, \ldots, b^d\}$. The *dual basis*, $C = \{c^1, \ldots, c^d\}$ is defined as to satisfy $b^i c^j = \delta_{ij}$ for all i and j. The dual basis spans a lattice L^* that is dual to L in the sense that, for all $x \in L$ and $y \in L^*$, $xy \in \mathbb{Z}$. Moreover, $\det L \det L^* = 1$ where $\det L$ and $\det L^*$ is the volume of any basis parallelotope of L and L^*. The *lattice width* of a convex body K with respect to $z^* \in L^*$ is defined as

$$w(K, z^*) = \max\{z^*(x - y) : x, y \in K\}.$$

The *lattice width*, $w(K, L)$ of K with respect to L is then

$$w(K, L) = \min\{w(K, z^*) : z^* \in L^*, z^* \neq 0\}.$$

For the upper bound on $E(d, v)$ we will need the so-called Flatness theorem.

FLATNESS THEOREM (CF. [10, 11]). *If K is convex and $K \cap \mathbb{Z}^d = \emptyset$, then $w(K, \mathbb{Z}^d) \leq w_0$ where w_0 is a constant depending only on d.*

The current best value of w_0 is $O(d^{3/2})$ in general, but for polytopes it is $O(d \log d)$ (see [2]). When P is a lattice free polytope, the Flatness theorem applies to the interior of P and we obtain $w(P, \mathbb{Z}^d) \leq w_0$.

To avoid some trivial complications we assume, from now on, that $d \geq 3$.

For the proof of the upper bound in Theorem 1 we use a modified version of the Hermite normal form. Let P be a lattice free simplex with $w(P, \mathbb{Z}^d) = w(P, e^1) = w$ (so e^1 is the lattice width direction of P). We order the vertices of P as $v_0, v_1, \dots, v_d$ in such a way that $\min\{e^1 x : x \in P\} = e^1 v_0$ and $\max\{e^1 x : x \in P\} = e^1 v_1$. We assume further that $v_0 = 0$.

We construct first a matrix $M(P)$. Set $b^1 = \frac{1}{w} v_1$ and let $L = \mathbb{Z}^d + \mathbb{Z} b^1$. This is the lattice spanned by $b^1, e^1, e^2, \dots, e^d$. Let L_0 denote the lattice spanned by $b^1 - e^1, e^2, \dots, e^d$. L_0 is a $(d-1)$-dimensional sublattice of L contained in the linear span of $e^2, \dots, e^d$. Define $\alpha_i = e^1 v_i$, the α_i are clearly integers satisfying $0 \leq \alpha_i \leq w$.

Note that $z_i = v_i - \alpha_i b^1 \in L_0$ (for every $i = 2, \dots, d$) and we can apply the Hermite reduction theorem in L_0: there is a unique basis $b^2, \dots, b^d$ of L_0 such that $z_i = \sum_2^i a_{ij} b^j$ with $0 \leq a_{ij} < a_{ii}$ with all a_{ij} integral.

Now we define the matrix $M(P)$:

$$M(P) = \begin{pmatrix} w & 0 & 0 & \dots & 0 \\ \alpha_2 & a_{22} & 0 & \dots & 0 \\ \alpha_3 & a_{32} & a_{33} & \dots 0 & \\ \dots & \dots & \dots & \ddots & \dots \\ \alpha_d & a_{d2} & a_{d3} & \dots & a_{dd} \end{pmatrix}$$

where the ith row gives the coefficients of v_i in the basis $b^1, b^2, \dots, b^d$ of L. Observe that this basis, and consequently $M(P)$, is uniquely determined. As $\det L = 1/w$, $\det M(P) = w \prod_2^d a_{ii} = wd!\,\mathrm{Vol}\,P$ implying $\prod_2^d a_{ii} \leq d!\,\mathrm{Vol}\,P$.

How many such matrices are there? The first column can be filled in $(w+1)^{d-1}$ different ways. As we saw in the proof of Theorem 4, the rest can be filled in at most V^{d-1} ways where V is an upper bound on $\prod a_{ii}$. So the number of such matrices is at most $(w+1)^{d-1}(d!v)^{d-1}$.

How to reconstruct P from the matrix $M(P)$? Or rather, given the matrix $M(P)$, how to find a sublattice L_1, of L, in which P is a lattice free simplex with $\det L_1 / \det L = w$, and how many such sublattices are there. The edge $[v_0, v_1]$ of P contains the points $\alpha b^1 \in L$ for $\alpha = 0, 1, \dots, w$, so $b^1, 2b^1, \dots, (w-1)b^1 \notin L_1$. This shows that for every $x \in L$ the L-lattice line $x + \mathbb{Z} b^1$ intersects L_1 in the L_1-lattice line $x + \beta(x) b^1 + w\mathbb{Z} b^1$ where $0 \leq \beta(x) < w$ is an integer. In particular, for every $i = 2, \dots, d$, $c^i = b^i + \beta(b^i) b^1 \in L_1$. It is straightforward to see that $wb^1, c^2, \dots, c^d$ span a lattice with determinant 1 so they form a basis of L_1. The number of such bases, and so the number of such sublattices L_1 is w^{d-2}. This shows that $M(P)$ determines at most w^{d-2} lattices in which P is lattice free. So

$$E(d, v) \leq \sum_{w=1}^{w_0} w^{d-2}(w+1)^{d-1}(d!v)^{d-1} \ll v^{d-1},$$

completing the proof of the upper bound.

REMARK 3. The above proof gives the following result: the number (of equivalence classes) of polytopes of volume at most v and lattice width at most w is $\ll (w^2 v)^{d-1}$. This implies Theorem 1 via the Flatness theorem.

Now let us consider the lower bound. We shall construct many lattice free simplices of volume at most v and of lattice width 1. Consider the ordered simplices $P = \mathrm{conv}\{z_0, z_1, \dots, z_d\}$ where $z_0 = 0$, $z_i = e^i$ for $i = 1, \dots, d-1$, and z_d is the vector $(1, x_2, x_3, \dots, x_d)$ such that

$$0 \leq x_2, \dots, x_{d-1} \leq x_d \text{ and } (x_2, \dots, x_d) \in \mathbb{Z}^{d-1} \text{ is primitive.}$$

(An integer vector is *primitive* if the greatest common divisor of its components is 1.) The flatness direction of P is e^1 with lattice width 1. This simplex is clearly lattice free. So a primitive vector $x = (x_2, \dots, x_d) \in \mathbb{Z}^{d-1}$ determines a lattice free ordered simplex in R^d. Further, $v \ge \operatorname{Vol} P = \frac{1}{d!} x_d$. It is well known that the number of such primitive vectors is $\gg v^{d-1}$, the proof given in [8, Theorems 330 and 332, p. 268] for the planar case works in any dimension. It gives that the density of primitive vectors in $\mathbb{Z}^d$ is $1/\zeta(d)$ (with Riemann's ζ-function).

We have to check that many of these vectors represent simplices from different equivalent classes. This is easy: if $x, y \in Z^{d-1}$ represent equivalent lattice free simplices P and Q then there is a lattice-preserving affine transformation T carrying the vertices of P to those of Q. This gives rise to a permutation π of $\{0, 1, \dots, d\}$ via $Tz_i = u_{\pi(i)}$ where z_i, resp. u_j are the vertices of P and Q. The permutation π determines T uniquely as one can readily check. This implies that at most $(d+1)!$ vectors may represent the same equivalence class. Consequently

$$E(d, v) \gg v^{d-1}.$$

4. Proof of Theorem 3

The key to this is the right choice of a representative from each equivalence class. This is given in Theorem 5 below; first we need some definitions.

Given a basis $B = \{b^1, \dots, b^d\}$ of $\mathbb{Z}^d$ and vectors $\alpha, \beta \in R^d$ with $\alpha_i \le \beta_i$ for all $i = 1, \dots, d$, we define the parallelotope

$$T(B, \alpha, \beta) = \{x = \sum x_i b^i : \alpha_i \le x_i \le \beta_i (\forall i)\}.$$

For a polytope $P \subset R^d$ let $T(B, P)$ be the smallest (with respect to inclusion) parallelotope $T(B, \alpha, \beta)$ containing P. Such a $T(B, P)$ obviously exists and is unique. Given a positive vector $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}^d$, we define the standard box

$$T(-\gamma, \gamma) = \{x \in R^d : -\gamma_i \le x_i \le \gamma_i \forall i\},$$

where x_i is the ith component of x in the standard basis.

THEOREM 5. *For every lattice polytope $P \subset R^d$ with positive volume, there exists an equivalent polytope Q contained in some standard box $T(-\gamma, \gamma)$ such that the origin is a vertex of Q, Q has another vertex whose max norm is at most γ_1 and*

$$\operatorname{Vol} T(-\gamma, \gamma) = 2^d \prod_1^d \gamma_i \ll \operatorname{Vol} P \qquad \textit{and} \qquad \gamma_1 \ll w(P).$$

The proof is given in the next section. We next show how to use this representation for the upper bound of Theorem 3.

PROOF OF THEOREM 3. Apply Theorem 5 to the lattice free polytope P with n vertices and volume at most v. The outcome is an equivalent Q sitting nicely in some standard box $T(-\gamma, \gamma)$ where $\gamma_1 \ll w(Q) \le w_0$ (w_0 coming from the Flatness theorem) and $\prod \gamma_i \ll \operatorname{Vol} P \le v$. So $\gamma_1 \le w_1$ for a suitable constant w_1 depending only on d. The number of such standard boxes is equal to the number of integer points $z \in \mathbb{Z}^{d-1}$ whose components are all positive and their product is $\ll v$ (the implied constant depending on d only). It is a simple computation to check that the number of such integers is $\ll v(\log v)^{d-2}$, for details see [6, Chapter 1, (3.5) and (8.1)]. Further, such a box contains

$$|T(-\gamma,\gamma)\cap\mathbb{Z}^d| = \prod_1^d (2\gamma_i + 1) \ll \operatorname{Vol} P \le v$$

integer points. The vertices of Q come from these integer points, but one vertex is at the origin and another one (whose max norm is at most γ_1) is in the finite box $[-w_1, w_1]^d$. Counting vertices without repetitions, this is altogether at most

$$\ll v(\log v)^{d-2}\binom{v}{n-2} < v^{n-1}(\log v)^{d-2}$$

lattice free polytopes. □

REMARK 4. The above proof works for polytopes having lattice width w: we used the fact that P is lattice free only through the Flatness theorem.

5. PROOF OF THEOREM 5

LEMMA. *For every lattice polytope $P \subset R^d$ of positive volume there exists a basis $B = \{b^1, \dots, b^d\}$ of $\mathbb{Z}^d$ such that*

$$\operatorname{Vol} T(B, P) \ll \operatorname{Vol} P \qquad \textit{and} \qquad \beta_1 - \alpha_1 \ll w(P).$$

Note that $\beta_1 - \alpha_1$ is the lattice width of P in direction c^1 (the first vector in the dual basis). Thus the lemma states that for every lattice polytope P there is an equivalent polytope contained in a nice box whose volume is $\ll \operatorname{Vol} P$ and whose lattice width in the 'first' direction is $\ll w(P)$. The lemma is a refinement of the representation theorem of lattice polytopes from [3]. The refinement comes from controlling the lattice width of P.

PROOF OF THE LEMMA. We repeat, almost word by word, the arguments of Theorem 3 from [3] with the necessary changes that take care of the lattice width. We assume first that P is 0-symmetric. Then there is an ellipsoid E, the Loewner–John ellipsoid (see [13] or [7]), centred at the origin, such that

$$d^{-1/2}E \subset P \subset E.$$

Apply a linear transformation τ that carries E to the Euclidean unit disk D of R^d. Obviously, $L = \tau\mathbb{Z}^d$ is a lattice again. It follows from the definition of the lattice width that $w(D, z^*) = 2||z^*||$. Consequently $w(D, L) = 2\lambda(L^*)$ where $\lambda(L^*)$ is the length of the shortest non-zero vector in L^*.

Let $\tilde{B} = \{\tilde{b}^1, \dots, \tilde{b}^d\}$ be a basis of L together with its dual basis $C = \{c^1, \dots, c^d\}$. Consider $T(\tilde{B}, D) = T(\tilde{B}, -\alpha, \alpha)$. Its facets touch D and the point $\alpha_i\tilde{b}^i$ is on such a facet. As the unit normal to this facet is $c^i/||c^i||$ we have $1 = \alpha_i\tilde{b}^i c^i/||c^i|| = \alpha_i/||c^i||$. Consequently

$$\operatorname{Vol} T(\tilde{B}, D) = \det L \prod_1^d 2\alpha_i = 2^d \det L \prod_1^d ||c^i||.$$

Choose the dual basis $C = \{c^1, \dots, c^d\}$ to be *reduced* in the sense of Definition 1.2.9 of Lovász ([13] p. 20) and compute the dual basis $\tilde{B}$. According to Theorem 1.2.10 of [13]

$||c^1|| \ll \lambda(L^*)$ and $\prod ||c^i|| \ll \det L^*$. Since $2\alpha_1 = 2||c^1|| = w(D, c^1)$, and $w(D, L) = 2\lambda(L^*)$, we obtain

$$\alpha_1 \ll w(D, L).$$

The other condition implies

$$\mathrm{Vol}\, T(\tilde{B}, D) \ll \det L \det L^* = 1.$$

Apply now τ^{-1} to $\tilde{B}$, D and L. We get a basis $B = \tau^{-1}\tilde{B}$ of $\mathbb{Z}^d = \tau^{-1}L$, and

$$\begin{aligned}\mathrm{Vol}\, T(B, P) \le \mathrm{Vol}\, T(B, E) &= \det \tau^{-1} \mathrm{Vol}\, T(\tilde{B}, D)\\ &\ll \det \tau^{-1} = \mathrm{Vol}\, E / \mathrm{Vol}\, D \ll \mathrm{Vol}\, P.\end{aligned}$$

On the other hand, the lattice width does not change under τ and so $2\alpha_1$ is the lattice width of E in direction $\tau^{-1}c^1$. So

$$2\alpha_1 \ll w(E) = d^{1/2} w(d^{-1/2}E) \ll w(P).$$

For a general, i.e., non-0-symmetric polytope P, one should consider $Q = P - P$ which is 0-symmetric. It is a well known result of Rogers and Shephard [15] that $\mathrm{Vol}\, Q \ll \mathrm{Vol}\, P$. Of course, $w(Q, z) = 2w(P, z)$. Moreover, a suitably translated copy of $T(B, P)$ is contained in $T(B, Q)$. Consequently the 'good' basis for Q is a good basis for P as well. □

For the proof of Theorem 5 we further refine this representation. Write $T(B, P) = T(B, \alpha, \beta)$ and set $\gamma = \beta - \alpha$. Then $T(B, \gamma)$ contains a translated copy, P' of P. The unimodular transformation carrying B to the standard basis $e^1, \dots, e^d$ carries $T(B, \gamma)$ to the standard box $T(\{e^1, \dots, e^d\}, \gamma) = T(\gamma)$ and P' to an equivalent polytope $P'' \subset T(\gamma)$. Note that $\gamma_1 \ll w(P)$. We may further assume that $\gamma_1 = \min_{1,\dots,d} \gamma_i$.

P'' has a vertex, z_0 say, whose first component is minimal. Translation by $-z_0$ carries P'' to P''' which is equivalent to P, has one vertex at the origin, and satisfies

$$P''' \subset T(-\gamma, \gamma).$$

Now let $z_1 = \zeta_1 e^1 + \dots + \zeta_d e^d$ be another vertex with maximal first component. Of course, $0 < \zeta_1 \le \gamma_1$. Divide ζ_i by ζ_1 (for each $i = 2, \dots, d$):

$$\zeta_i = \mu_i + m_i \zeta_1, \text{ where } 0 \le \mu_i < \zeta_1 \text{ and } \mu_i, m_i \in \mathbb{Z}.$$

Consider the unimodular linear transformation mapping $x = \sum_1^d x_i e^i$ to

$$\begin{aligned}&x_1 \to x_1\\ &x_i \to x_i - m_i x_1, \text{ for } i > 1.\end{aligned}$$

The image of z_1 has all of its components in $[0, \gamma_1]$. The image of every other vertex is in the box $T(-3\gamma, 3\gamma)$ as an easy and generous computation shows. So the image of P''' is a lattice polytope Q equivalent to P contained in the standard box $T(-3\gamma, 3\gamma)$, having a vertex at the origin, another vertex with max norm at most γ_1, and $w(Q, e^1) \le \gamma_1$. After rescaling the γ_i we obtain the theorem. □

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