A Mezei–Wright theorem for categorical algebras

S.L. Bloom\textsuperscript{a,}\textsuperscript{*}, Z. Ésik\textsuperscript{b}

\textsuperscript{a} Department of Computer Science, Stevens Institute of Technology, Hoboken, NJ, USA
\textsuperscript{b} Department of Computer Science, University of Szeged, Szeged, Hungary

\textbf{A R T I C L E  I N F O}

Article history:
Received 6 March 2008
Received in revised form 17 June 2009
Accepted 26 June 2009
Communicated by M.W. Mislove

\textbf{A B S T R A C T}

The main result of this paper is a generalization of the Mezei–Wright theorem, a result on solutions of a system of fixed point equations. In the typical setting, one solves a system of fixed point equations in an algebra equipped with a suitable partial order; there is a least element, suprema of \(\omega\)-chains exist, the operations preserve the ordering and least upper bounds of \(\omega\)-chains. In this setting, one solution of this kind of system is provided by least fixed points. The Mezei–Wright theorem asserts that such a solution is preserved by a continuous, order preserving algebra homomorphism.

In several settings such as (countable) words or synchronization trees there is no well-defined partial order but one can naturally introduce a category by considering morphisms between the elements. The generalization of this paper consists in replacing ordered algebras by “categorical algebras”; the least element is replaced by an initial element, and suprema of \(\omega\)-chains are replaced by colimits of \(\omega\)-diagrams. Then the Mezei–Wright theorem for categorical algebras is that initial solutions are preserved by continuous morphisms. We establish this result for initial solutions of parametric fixed point equations.

One use of the theorem is to characterize an “algebraic” element as one that can arise as a solution of some system of fixed point equations. In familiar examples, an algebraic element is one that is context-free, regular or rational. Then, if \(h : A \rightarrow B\) is a continuous morphism of categorical algebras, the algebraic objects in \(B\) are those isomorphic to \(h\)-images of algebraic objects in \(A\).

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

When studying properties of a \(\Sigma\)-algebra \(A\), it is natural to investigate those operations definable by means of a system of recursive fixed point equations such as

\[
F_1(x) = \sigma(x, F_1(\delta(x)))
\]

\[
F_2(x) = \sigma(F_1(\delta(x)), \delta(\delta(x))),
\]

a typographical variation on an example in [4]. The general case is a system of the form

\[
F_1(v_1, \ldots, v_k) = t_1
\]

\[
\vdots
\]

\[
F_n(v_1, \ldots, v_k) = t_n
\] (1)

* Corresponding author.
E-mail address: bloom@cs.stevens-tech.edu (S.L. Bloom).

0304-3975/ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2009.06.040
where \( t_1, \ldots, t_n \) are terms built from the given \( \Sigma \)-operations and the new function variables \( F_1, \ldots, F_n \) (see below for a precise account). For one familiar example, a “context-free grammar” is nothing but such a system of equations in the ordered algebra of languages, in which the “arity” of each \( F_i \) is 0. This algebra is equipped with the operations of binary union and concatenation, a constant for the empty set and constants for each letter in some alphabet. Typical questions about such systems are: when do solutions exist, and what are their properties.

In continuous ordered algebras, cf. [19,21,26], least solutions of such equations always exist. Further, it is a well-known fact that these operations are preserved by continuous homomorphisms, and are thus “implicit operations”. One of the first results in this area is by Mezei and Wright [24] who considered such systems \((1)\) only when the arity of each function variable \( F_i \) is zero, and when the algebras are subset algebras (see Section 4).

A partial order may be seen as a category with at most one morphism between two elements (objects). But in several applications, e.g., words or synchronization trees, there may be many morphisms between two elements (objects), and there is no canonical way of selecting one. For example, there are two possible embeddings of the word \( a \) into \( aba \) or of the word \( ab \) into \( aab \). For infinite words, we also face the problem that there are non-isomorphic words \( u \) and \( v \) with an embedding of \( u \) into \( v \) and an embedding of \( v \) into \( u \), even if we restrict ourselves to prefix or suffix embeddings. Consider the fixed point equation

\[
x = xaxbx
\]

in (countable) words over the alphabet \( \{a, b\} \). When words are equipped with the categorical structure provided by embeddings (so that concatenation becomes a bi-functor), this equation has a well-defined initial solution, unique up to isomorphism. This solution is the word obtained by labeling the rationals with the letters \( a, b \) such that between any two points there is both a point labeled \( a \) and a point labeled \( b \). There is no (natural) order on words that would provide this solution, or, in fact, any solution to this equation.

In the current paper, we generalize continuous ordered \( \Sigma \)-algebras to what we call continuous categorical \( \Sigma \)-algebras, and prove the corresponding preservation theorem, which is usually called a “Mezei–Wright type theorem”. The importance of such results to algebraic semantics is as usual that they show the equivalence of solving recursive systems of equations schematically and interpreting the solutions in the target categories on the one hand, and, on the other, interpreting schematic systems of recursive equations and then solving them in the target categories.

The paper is organized as follows. In Section 2, we fix the notation used in the paper. In Section 3 we define continuous categorical algebras as a generalization of continuous ordered algebras. At the end of this section, we list several properties of continuous categorical algebras that will be used in the sequel. Section 4 is devoted to examples: trees, words, subset algebras, synchronization trees and traces. In Section 5, we introduce recursion schemes and initial solutions of such schemes over continuous categorical algebras giving rise to algebraic objects and functors (operations). We state our main Mezei–Wright type result in Theorem 5.3. Section 6 is devoted to the proof of the main result. In Section 7, we discuss parts of the proof in the context of an extended example involving a version of Eq. (2). Some applications of the theorem are given in Section 8.

2. Notation

- When \( n \) is a nonnegative integer, we let

\[ [n] := \{1, 2, \ldots, n\}, \]

so that \([0] = \emptyset\).

- When \( f : A \to B \) is a functor, the identity natural transformation \( f \to f \) is denoted \( f \) also.

- If \( f : A \to B \) and \( g : B \to C \) are functors, we write their composite as \( g \circ f : A \to C \). Similarly, if \( f_i : A \to B_i \) and \( g_i : B_i \to C_i \) is a functor, for \( i \in [2] \), and if \( \varphi : f_1 \to f_2 \), and \( \psi : g_1 \to g_2 \) are natural transformations, then the horizontal composite is written

\[ \psi \circ \varphi : g_1 \circ f_1 \to g_2 \circ f_2. \]

- If \( f_i : A \to B_i \) is a functor, for \( i \in [3] \), and if \( \varphi : f_1 \to f_2 \) and \( \psi : f_2 \to f_3 \) are natural transformations, we write

\[ \psi \cdot \varphi : f_1 \to f_3 \]

for vertical composition.

- If \( f_i : A \to B_i \) is a functor, for \( i \in [n] \), then

\[ \langle f_1, \ldots, f_n \rangle : A \to B_1 \times B_2 \times \cdots \times B_n \]

denotes the target-tupling: \( x \mapsto (f_1(x), \ldots, f_n(x)) \), for an object or arrow \( x \) in \( A \). Similarly, if \( g_i : A \to B_i \) is also a functor and \( \tau_i : f_i \to g_i \) is a natural transformation, for all \( i \in [n] \), then \( \langle \tau_1, \ldots, \tau_n \rangle \) denotes the natural transformation \( \langle f_1, \ldots, f_n \rangle \to \langle g_1, \ldots, g_n \rangle \) which is the target-tupling of the \( \tau_i \).
3. Continuous categorical algebras

In this section, we recall the notion of continuous categorical $\Sigma$-algebra. These structures were used in [15,8,9] to give semantics to recursion schemes over synchronization trees and words.

Suppose that $\Sigma = \bigcup_n \Sigma_n$ is a ranked set (or “signature”). A categorical $\Sigma$-algebra, $c\Sigma a$ for short, is a small category $A$ equipped with a functor, sometimes called a “$\Sigma$-functor”, $\sigma^A : A^n \to A$ for each $\sigma \in \Sigma_n$, $n \geq 0$. A morphism between $c\Sigma a$’s $A$ and $B$ is a functor $h : A \to B$ such that for each $\sigma \in \Sigma_n$, the diagram

\[
\begin{array}{ccc}
A^n & \xrightarrow{\sigma^A} & A \\
| & | & | \\
| & h & | \\
B^n & \xrightarrow{\sigma^B} & B
\end{array}
\]

commutes up to a natural isomorphism. Here, $h^n : A^n \to B^n$ is the functor sending each object and morphism $(x_1, \ldots, x_n)$ of $A^n$ to $(h(x_1), \ldots, h(x_n))$ in $B^n$.

In more detail, there is a natural isomorphism $\pi_\sigma : h \circ \sigma^A \to \sigma^B \circ h^n$ such that if $f_i : a_i \to a'_i$ is a morphism in $A$, for $i \in [n]$, then

\[
\begin{array}{ccc}
\pi_\sigma(a_i) & \xrightarrow{\pi_\sigma} & \pi_\sigma(a'_i) \\
\pi_\sigma(a_\overline{a}) & \xrightarrow{\pi_\sigma} & \pi_\sigma(a_\overline{a'}) \\
\pi_\sigma(a_{\overline{f}}) & \xrightarrow{\pi_\sigma} & \pi_\sigma(a_{\overline{f'}})
\end{array}
\]

Here, we write $\overline{a}$ for $(a_1, \ldots, a_n)$, and $\overline{f}$ for $(f_1, \ldots, f_n)$, and $h^n(\overline{a})$ for $(h(a_1), \ldots, h(a_n))$, etc.

A morphism $h$ is **strict** if, for all $\sigma \in \Sigma$, the natural isomorphism $\pi_\sigma$ is the identity.

Let $A$ be a $c\Sigma a$. We call $A$ **continuous**, if $A$ has a distinguished initial object (denoted $\bot^n$) and colimits of all $\omega$-diagrams $(f_k : a_k \to a_{k+1})_{k \geq 0}$. Moreover, each functor $\sigma^A$ is continuous, i.e., preserves colimits of $\omega$-diagrams. We abbreviate “continuous $c\Sigma a$” by “$cc\Sigma a$”. A morphism of $cc\Sigma a$’s is a $c\Sigma a$ morphism which preserves the distinguished initial object and colimits of all $\omega$-diagrams. Thus, if $\sigma \in \Sigma_2$, say, and if

\[
x_0 \xrightarrow{f_0} x_1 \to x_2 \to \cdots
\]

\[
y_0 \xrightarrow{g_0} y_1 \to y_2 \to \cdots
\]

are $\omega$-diagrams in $A$ with colimits $(x_n \xrightarrow{\psi_n} x)$ and $(y_n \xrightarrow{\psi_n} y)$, then

\[
\sigma^A(x_0, y_0) \xrightarrow{\sigma^A(f_0, g_0)} \sigma^A(x_1, y_1) \to \cdots
\]

has colimit

\[
(\sigma^A(x_n, y_n))_{x \leq y} = \sup \sigma^A(x_n, y_n).
\]

Special cases of $c\Sigma a$’s are the ordered $c\Sigma a$’s. An **ordered** $c\Sigma a$ is a $c\Sigma a$ in which the categorical structure is determined by a partial order $\leq$; i.e., there is a morphism $x \to y$ in $A$ iff $x \leq y$. To say that an operation $\sigma_A$ is a functor is to say that it preserves the order. An ordered $c\Sigma a$ is continuous if $A$ has a least element and all $\omega$-chains have least upper bounds; further, each operation $\sigma_A$ preserves these least upper bounds: when $\sigma$ has rank 2, say,

\[
\sigma^A(x_0, y_0) = \sup \sigma^A(x, y),
\]

when $x_0 \leq x_1 \leq \cdots$ and $y_0 \leq y_1 \leq \cdots$.

The concept of a $\Sigma$-term over a set $V = \{v_1, v_2, \ldots\}$ is defined as usual, (see [20], for example). For each $n$, we let

\[
Tm_\Sigma(V_n)
\]

denote the set of all $\Sigma$-terms in the variables $V_n = \{v_1, \ldots, v_n\}$. When $A$ is a $c\Sigma a$ and $t \in Tm_\Sigma(V_n)$, $t$ induces a functor $t^A : A^n \to A$ in the usual way:
Definition 3.1. • If $t = v_i$, for some variable $v_i \in V_n$, then $t^A$ is the ith projection functor $\pi^n_i : A^n \to A$.

• If $t = \sigma(t_1, \ldots, t_m)$, where $\sigma \in \Sigma_m$ and $t_1, \ldots, t_m \in Tm_\Sigma(V_n)$, then

$$t^A = A^n \xrightarrow{(t_1^A, \ldots, t^A_m)} A^m \xrightarrow{\sigma^A} A.$$  

When $A$ is a cc $\Sigma$-a, $t^A$ is a continuous functor, for each term $t$.

3.1. Some facts

Below we list a few properties of categorical $\Sigma$-algebras that follow from well-known facts in category theory, cf. [1, 23, 27].

1. If $A$, $B$ are cc $\Sigma$-a’s, so is $A \times B$, where the $\Sigma$-functors are defined pointwise.

2. If $A$, $B$ are cc $\Sigma$-a’s, so is $[A \to B]$, the category of continuous functors $A \to B$, where the $\Sigma$-functors are defined as follows.

   Say for example that $\sigma \in \Sigma_2$, and $f, g : A \to B$ are continuous functors. Then the functor

   $$\sigma^{[A \to B]}(f, g) : A \to B,$$

   is defined on objects and morphisms $x$ in $A$ by:

   $$\sigma^{[A \to B]}(f, g)(x) := \sigma^B(f(x), g(x)).$$

   Thus,

   $$\sigma^{[A \to B]}(f, g) = \sigma^B \circ (f, g).$$

   The distinguished initial object in $[A \to B]$ is the constant functor whose value is $\perp^B$. Suppose now that $f'$ and $g'$ are also continuous functors $A \to B$ and $\tau$ and $\tau'$ are natural transformations $f \to f'$ and $g \to g'$, respectively. Then we define $\sigma^{[A \to B]}(\tau, \tau')$ as the natural transformation $\sigma^{[A \to B]}(f, g) \Rightarrow \sigma^{[A \to B]}(f', g')$ with $\sigma^{[A \to B]}(\tau, \tau')(a) = \sigma^B(\tau(a), \tau'(a))$ for all objects $a$ in $A$. In other words, denoting the identity natural transformation $\sigma^B \Rightarrow \sigma^B$ simply by $\sigma^B$, we have

   $$\sigma^{[A \to B]}(\tau, \tau') = \sigma^B \circ (\tau, \tau').$$

   the horizontal composite of $\sigma^B$ with the target-tupling of $\tau$ and $\tau'$.

3. It follows that, when $A$ is a cc $\Sigma$-a, so is

   $$[A^{k_1} \to A] \times [A^{k_2} \to A] \times \cdots \times [A^{k_n} \to A]$$

   for any $n \geq 1$ and nonnegative $k_1, \ldots, k_n$.

4. If $A$ is a cc $\Sigma$-a, and $F : A \to A$ is a continuous endofunctor, then $F$ has an initial fixed point $F^\uparrow$, i.e., there is an isomorphism

   $$\iota : F(F^\uparrow) \to F^\uparrow,$$

   and if $\alpha : F(a) \to a$ is any morphism in $A$, then there is a unique morphism $h : F^\uparrow \to a$ such that

   $$\begin{array}{ccc}
   F(F^\uparrow) & \xrightarrow{\iota} & F^\uparrow \\
   F(h) & \downarrow & h \\
   F(a) & \xrightarrow{\alpha} & a
   \end{array}$$

   commutes. $F^\uparrow$ is the colimit of the usual diagram

   $$\perp^A \xrightarrow{\alpha_0} F(\perp^A) \xrightarrow{\alpha_1} F^2(\perp^A) \to \cdots$$

   where $\perp^A$ is the distinguished initial object, $\alpha_0$ the unique map $\perp^A \to F(\perp^A)$, and $\alpha_{n+1} = F(\alpha_n), n \geq 0$.

5. When $A$, $B$ are cc $\Sigma$-a’s, an $\omega$-diagram in $[A \to B]$

   $$(F_n \xrightarrow{\tau_n} F_{n+1})_n$$

   has a colimit $(F_n \xrightarrow{\tau_n} F)_n$ iff for each object $a \in A$, the $\omega$-diagram

   $$(F_n(a) \xrightarrow{\tau_n(a)} F_{n+1}(a))_n$$

   in $B$ has a colimit $(F_n(a) \xrightarrow{\tau_n(a)} F(a))_n$. The colimit $F$ is continuous because of the interchangeability of colimits, see [23], Section IX.2(2).

4. Examples

   There are many examples of cc $\Sigma$-a’s.
4.1. Trees

For a ranked set $\Sigma$, the ordered algebra

$$T^\omega_\Sigma$$

consists of all (finite and infinite) $\Sigma$-trees. Let $\mathbb{N}$ denote the set of positive integers. The set of finite sequences of members of $\mathbb{N}$ is $\mathbb{N}^*$, including the empty sequence $\epsilon$. A tree $t$ in $T^\omega_\Sigma$ is a partial function $t : \mathbb{N}^* \rightarrow \Sigma$ whose domain is prefix closed and such that if $t(w) \in \Sigma_n$, $n > 0$, and if $t(wu)$ is defined, then $u \leq n$; if $t(w) \in \Sigma_0$, then $w$ is a leaf. A tree $t$ is finite if its domain is finite and complete if whenever $t(w) \in \Sigma_n$ then each of the words $w, \ldots, w_n$ is in dom($t$).

Trees are equipped with the following partial order $\sqsubseteq$: Given $t, t' \in T^\omega_\Sigma$ such that $t \neq t'$, we define $t \sqsubseteq t'$ iff for all words $u$, if $t(u)$ is defined, then $t(u) = t'(u)$. We consider $T^\omega_\Sigma$ as a category in the usual way: there is a morphism $t \rightarrow t'$ when $t \sqsubseteq t'$. Since $T^\omega_\Sigma$ has as least element the totally undefined tree $\bot$, the category $T^\omega_\Sigma$ has an initial object. It also has sups of all $\omega$-chains, i.e., colimits of all $\omega$-diagrams, cf. [19, 12].

$T^\omega_\Sigma$ is a cc $\Sigma$ $a$, where for each $\sigma \in \Sigma_n$, $t = \sigma^T(t_1, \ldots, t_n)$ is the tree defined on $u \in \mathbb{N}^*$ by:

$$t(u) = \begin{cases} 
\sigma & \text{if } u = \epsilon \\
t_1(u) & \text{if } u = iv, \quad i \in [n]. 
\end{cases}$$

(5)

In particular, each letter in $\Sigma_0$ is interpreted as the tree whose domain is the singleton set $\{\epsilon\}$ which maps the empty word $\epsilon$ to $\sigma$. If $\sigma \in \Sigma_n$, the operation $\sigma^T$ is a continuous functor $T^\omega_\Sigma \rightarrow T^\omega_\Sigma$.

**Remark 4.1.** $T^\omega_\Sigma$ is the initial cc $\Sigma$ $a$ in the following sense: For any cc $\Sigma$ $a$ $A$, there is a cc $\Sigma$ $a$ morphism

$$T^\omega_\Sigma \rightarrow A,$$

which is unique up to natural isomorphism.

To see this, first we assign an object $t^A$ in $A$ to each finite tree $t \in T^A_\Sigma$. When $t$ is the empty tree, we define $t^A := \bot^A$. Otherwise there are finite trees $t_1, \ldots, t_n$ with $t = t^A(t_1, \ldots, t_n)$ and we define $t^A := \sigma^T(t_1^A, \ldots, t_n^A)$. Next, if $t$ and $s$ are finite trees with $t \sqsubseteq s$, then we define a morphism $(t \sqsubseteq s)^A$ in $A$. When $t = \bot$, this is the unique morphism $\bot^A \rightarrow s^A$. If $t$ is of the form $\sigma^T(t_1, \ldots, t_n)$ then necessarily $s$ is of the form $\sigma^T(s_1, \ldots, s_n)$ with $t_i \sqsubseteq s_i$ for all $i$. We define

$$(t \sqsubseteq s)^A := \sigma^A((t_1 \sqsubseteq s_1)^A, \ldots, (t_n \sqsubseteq s_n)^A).$$

Now when $t$ is infinite, $t$ is the supremum of an $\omega$-chain of finite trees $(t_n)_n$, where for words $u$ of length $\leq n$ we have $t_n(u) = t(u)$ and $t_n(v)$ is undefined for all words $v$ of length greater than $n$. We define $t^A$ as the object part of a colimit of the $\omega$-diagram

$$t^A_n \xrightarrow{(t_n^A \sqsubseteq t_n^A)^A} t^A_1 \xrightarrow{(t_1^A \sqsubseteq t_2^A)^A} \cdots$$

It should be noted that when $t$ is finite, $t$ is still the supremum of the $\omega$-chain $(t_n)_n$ as defined above, and that $t^A$ is (isomorphic to) the object part of the same $\omega$-diagram. Finally, suppose that $t$ and $s$ are trees with $t \sqsubseteq s$ such that $s$ is infinite. Then consider the $\omega$-diagrams $(t^A_n \xrightarrow{(t_n^A \sqsubseteq t_n^A)^A} t^A_{n+1})_n$ and $(s^A_n \xrightarrow{(s_n^A \sqsubseteq s_n^A)^A} s^A_{n+1})_n$ with colimits $(t^A_n \xrightarrow{f_n} t^A)_n$ and $(s^A_n \xrightarrow{g_n} s^A)_n$. Since $t_n \sqsubseteq s_n$ for all $n$, there is a unique morphism $h : t^A \rightarrow s^A$ with

$$h \circ f_n = g_n \circ (t_n \sqsubseteq s_n)^A$$

for all $n$. We define $(t \sqsubseteq s)^A := h$. The assignment $t \mapsto t^A$, $(t \sqsubseteq s) \mapsto (t \sqsubseteq s)^A$ is the required cc $\Sigma$ $a$ morphism $T^\omega_\Sigma \rightarrow A$. For details of the above construction, see [5].

Below we will let $T^\Sigma_\omega$ denote the $\Sigma$-algebra of finite complete trees. (Note that a finite complete tree is just a term with no variables.)

4.2. Words

A word over $Z$ is a countable strict linear order $w = (W, <_w)$ equipped with a labeling function $\lambda_w : W \rightarrow Z$. (To force the collection of all words to be a small set, we require that the underlying set $W$ of a word $w$ is a subset of fixed set. Below, we will see that we may as well use the set $\{0, 1\}^*$. A morphism between words $v = (V, <_v, \lambda_v)$ and $w = (W, <_w, \lambda_w)$ is a function $h : V \rightarrow W$ which preserves the order (and is thus injective) and the labeling. A word $w$ is finite if the underlying set is finite. The category $W_Z$ of words over $Z$ has as initial object the empty word, denoted $\epsilon$. Moreover, $W_Z$ has colimits of all $\omega$-diagrams, cf. [8].

When $\Sigma_\omega = \{\}, \Sigma_0 = Z$ and $\Sigma_n = \emptyset$ for all $n \notin \{0, 2\}$, we turn $W_Z$ into a cc $\Sigma a$ by interpreting the binary symbol $\cdot$ as the concatenation functor $W^\omega_Z \rightarrow W_Z$, and each letter $a \in Z$ as a singleton word labeled $a$. The concatenation functor maps a pair of words $w = (W, <_w, \lambda_w)$ and $w' = (W', <_{w'}, \lambda_{w'})$ to the word

$$w \cdot w' := (W \cup W', <_{w', w'}, \lambda_{w', w'})$$

(6)
whose underlying set is the disjoint union of $W$ and $W'$ such that the restriction of $<_w$ to $W$ is the ordering $<_w$, the restriction of $<_w$ to $W'$ is the ordering $<_w'$, moreover, $x <_w x'$ holds for all $x \in W$ and $x' \in W'$. The restrictions of $\lambda_w$ to $W$ and $W'$ are respectively the functions $\lambda_w$ and $\lambda_w'$. The concatenation $h \cdot h'$ of morphisms $h : w \to v$ and $h' : w' \to v'$ is defined so that it agrees with $h$ on the underlying set of $w$ and it agrees with $h'$ on the underlying set of $h'$. It is known that the concatenation functor is continuous, cf. [8]. Below we will sometimes write just $ww'$ for $w \cdot w'$.

When $Z$ is an alphabet equipped with a linear order $<_Z$, there are three important orderings on the finite words on $Z$. The prefix order is denoted $<_p$; $u <_p v$ holds when there is a nonempty word $w$ with $v = uw$; the strict order, denoted $<_s$, holds when, for some $u_1, u_2, v \in Z^*$,

$$u = u_1au_2$$
$$v = u_1bu_2$$

and $a <_Z b$ in the alphabet $Z$. Last, the lexicographic order is defined as follows. If $u \neq v$,

$$u <_s v \iff u <_p v \text{ or } u <_s v.$$  

While both $<_p$ and $<_s$ are strict partial orderings on $Z^*$, $<_s$ is a strict linear order on all words.

The theorem below recalls a universal property of $<_s$ on $[0, 1]^*$.

**Theorem 4.2** ([12,7]). Each word over an alphabet $Z$ is isomorphic to a word whose underlying set is a subset of $[0, 1]^*$ ordered by the lexicographic order.

We call a word regular (deterministic context-free, context-free), cf. [12,8,9], if it is isomorphic to a word $w = (W, \lambda_w, \lambda_w)$ such that $W \subseteq [0, 1]^*$ and for each $z \in Z$, the language of those words in $W$ labeled $z$, $\lambda_w^{-1}(z)$, is regular (deterministic context-free, context-free, respectively).

The **yield of a tree** $t \in T^w_\Sigma$ is defined as the word over $Z$

$$\text{yield}(t) = (W, <_s, \lambda_z),$$

where $W$ is the set $\{u \in \mathbb{N}^* : t(u) \in Z\}$, linearly ordered by the strict partial order $<_s$. The labeling is defined by $\lambda_1(u) = t(u)$ for all $u \in W$. When $t \sqsubseteq t'$ in $T^w_\Sigma$, then we define $\text{yield}(t \sqsubseteq t')$ as the embedding of $W = \text{yield}(t)$ into $W' = \text{yield}(t')$. (Note that $W \subseteq W'$.)

**Proposition 4.3.** The function yield is the morphism $T^w_\Sigma \to W_Z$, unique up to a natural isomorphism; yield is strict, and dense (i.e., every word is isomorphic to yield(t), for some tree, by Theorem 4.2).

### 4.3. Subset algebras

If $A$ is a $\Sigma$-algebra, the powerset $\hat{A} = 2^A$ becomes an ordered cc $\Sigma$-a as follows. A morphism in $\hat{A}$ is given by the inclusion order. If $\sigma \in \Sigma_0$,

$$\sigma^\hat{A} := \{\sigma^A\}.$$

If $\sigma \in \Sigma_n$, $n \geq 1$, and $X_i \subseteq A$, for $i \in [n]$, then

$$\sigma^\hat{A}(X_1, \ldots, X_n) := \{\sigma^A(a_1, \ldots, a_n) : a_i \in X_i\}.$$ 

Then $\hat{A}$ is a cc $\Sigma$-a. Typically, subset algebras are enriched by the operation of binary union, denoted $\cup$.

Mezei and Wright [24] considered the solutions of recursion equations in subset algebras.

### 4.4. Synchronization trees

A **synchronization tree** $t = (V, v_0, E, I)$ over an alphabet $A$ of "action symbols" consists of

- a finite or countable set $V$ of "vertices" and an element $v_0 \in V$, the "root";
- a set $E \subseteq V \times V$ of "edges";
- a "labeling function" $I : E \to A \cup \{\text{ex}\}$.

These data obey the following restrictions.

- $(V, v_0, E)$ is a rooted tree: for each $u \in V$, there is a unique path $v_0 \rightsquigarrow u$.
- If $e = (u, v) \in E$ and $I(e) = \text{ex}$, then $v$ is a leaf, and $u$ is called an **exit vertex**.
- A path $p = (v_0, v_1, \ldots)$ from the root is a sequence, perhaps infinite, of vertices such that for each $i \geq 0$, $(v_i, v_{i+1}) \in E$. The **label** of $p$ is the finite or infinite word $a_0a_1 \cdots$ where $a_i = I(v_i, v_{i+1})$, for $i \geq 0$. 


A morphism $\varphi : t \to t'$ of synchronization trees is a function $V \to V'$ which preserves the root, the edges and the labels, so that if $(u, v)$ is an edge of $t$, then $(\varphi(u), \varphi(v))$ is an edge in $t'$, and $l(\varphi(u), \varphi(v)) = l(u, v)$.

The collection of synchronization trees over $A$ is equipped with two binary operations: $t + t'$, $t \cdot t'$ of “sum” and “sequential product”. The sum $t + t'$ of two trees with disjoint sets of vertices is obtained by taking the union of the vertices of $t$, $t'$ and identifying the roots. The edges and labeling are inherited. The sequential product $t \cdot t'$ of two trees is obtained by replacing each edge of $t$ labeled $ex$ by a copy of $t'$. For each letter $a \in A$, let $a$ denote the tree with 3 vertices $(v_0, v_1, v_2)$ and two edges: the edge $(v_0, v_1)$, labeled ‘a’, and the edge $(v_1, v_2)$, labeled $ex$. Let $0$ denote the tree with no edges.

It is known, cf. [6], that synchronization trees, equipped with the operations of sum and sequential product, and the constants $\{a : a \in A\}$, form a cc$\Sigma$A, which we denote here by ST(A). The tree $0$ is the initial object in ST(A).

Consider now the least equivalence relation $\sim$ on cc$\Sigma$A identifying any two trees connected by a surjective morphism. It can be seen that each equivalence class contains a tree $t_0$, unique up to isomorphism, such that for any tree $t$ in the equivalence class there is a surjective morphism $t \to t_0$. Then we can form a category BST(A) whose objects are the equivalence classes of the relation $\sim$. A morphism $[t] \to [s]$ in BST(A) is a morphism $t_0 \to s_0$, where there are surjective morphisms $t \to t_0$ and $s \to s_0$. Since the operations preserve equivalence, it follows that BST(A) is also a cc$\Sigma$A.

**Remark 4.4.** Suppose that $t = (V, v_0, E, I)$ and $t' = (V', v'_0, E', I')$ are synchronization trees over $A$. A simulation from $t$ to $t'$ is a relation $R \subseteq V \times V'$ such that

1. $v_0 R v'_0$.
2. If $(u, v) \in E$ and $u' \in V'$ and $u R u'$, then there is some $v' \in V'$ such that $(u', v') \in E' Ev'$ and $l(u, v) = l(u', v')$.

A bisimulation from $t$ to $t'$ is a simulation whose relation inverse is a simulation from $t'$ to $t$.

Two synchronization trees $t$, $t'$ are bisimilar if there is a bisimulation between $t$ and $t'$. It can be shown that two trees $t$ and $t'$ are bisimilar iff $t \sim t'$.

### 4.5. Traces

When $A$ is an alphabet, let $A^\omega$ denote the set of all $\omega$-words over $A$, and let $A^\infty = A^+ \cup A^\omega$. Given a synchronization tree $t$, we assign to $t$ the ordered pair

$$\text{trace}(t) := (X_t, Y_t),$$

where $X_t$ is the collection of all finite words formed from the labels of paths in $t$ from the root to an exit vertex, and $Y_t$ is the collection of all infinite words formed from the labels of infinite paths in $t$, whose source is the root, together with finite words which are labels of paths

$$(v_0, \ldots, v_k, v_{k+1})$$

where $k \geq 0$, whose target $v_{k+1}$ is a leaf but the label of the last edge $(v_k, v_{k+1})$ is not $ex$. (The trace of the tree $0$ is $(\emptyset, \emptyset)$ since there is no path of positive length from the root to a leaf.)

We thus define an object of the category Tr$A$ as an ordered pair $(X, Y)$ where $X \subseteq A^*$ and $Y \subseteq A^\infty$.

Suppose that $(X, Y)$ and $(X', Y')$ are objects. Then there is a morphism $(X, Y) \to (X', Y')$ only if $X \subseteq X'$. Assuming this, a morphism $(X, Y) \to (X', Y')$ is a relation $\phi : Y \to X' \cup Y'$ such that

- for each $y \in Y$, there is some $z \in X' \cup Y'$ such that $(y, z) \in f$;
- whenever $(y, z) \in f$ then $y$ is a prefix of $z$.

Alternatively, we may represent $f$ as a function from $Y$ to the set of nonempty subsets of $X' \cup Y'$ subject to the second condition. Composition is defined as follows. Suppose that $f : (X, Y) \to (X', Y')$ and $g : (X', Y') \to (X'', Y'')$. Then $X \subseteq X' \subseteq X''$. We define $g \circ f$ as the collection of all pairs $(y, z) \in f$ with $z \in X'$ together with all pairs $(y, z) \in g$ such that there exists some $y' \in Y'$ with $(y, y') \in f$ and $(y', z) \in g$.

The category Tr$A$ has as initial object the ordered pair $0 = (\emptyset, \emptyset)$. Given an $\omega$-dia gram $(f_i : (X_i, Y_i) \to (X_{i+1}, Y_{i+1}))_{i \geq 0}$, its colimit is constructed as follows. Let $X = \bigcup X_i$ and let $Y$ be the collection of all words $u$ in $A^\infty$ such that one of the following conditions holds for some integer $n_0 \geq 0$:

1. $(u, u) \in f_j$ for all $j \geq n_0$.
2. There is a sequence $(u_j)_{j \geq n_0}$ of finite words $u_j \in Y_j$ such that $(u_j, u_{j+1}) \in f_j$ for all $j \geq n_0$, whose limit is $u$.

Then for each $i \geq 0$, define the morphism $g_i : (X_i, Y_i) \to (X, Y)$ by $(u, v) \in g_i$ if one of the following holds for some $n_0 \geq i$:

1. $(u, v) \in f_i$ and $v \in X_{n_0}$, where $f_i = f_0 \circ \cdots \circ f_i$.
2. There exists some $v \in Y_{n_0}$ with $(u, v) \in f_i$ and $(u, v) \in f_j$ for all $j \geq n_0$.
3. There exists a sequence $(u_j)_{j \geq n_0}$ of finite words whose limit is $v$ such that $u$ is a prefix of $u_{n_0}$ and $(u_j, u_{j+1}) \in f_j$ for all $j \geq n_0$.

The morphisms $g_i$ form a colimit cone over the diagram $(f_i : (X_i, Y_i) \to (X_{i+1}, Y_{i+1}))_{i \geq 0}$. 
When $(X, Y)$, $(U, V)$ are objects in $\text{Tr}_A$, we define the sum $(X, Y) + (U, V)$ by

$$(X, Y) + (U, V) := (X \cup U, Y \cup V).$$

The sum of two morphisms $f : (X, Y) \to (X', Y')$ and $g : (U, V) \to (U', V')$ is the morphism which is the relation $f \cup g$.

The sequential product $(X, Y) \cdot (U, V)$ of two objects in $\text{Tr}_A$ is defined as follows.

$$(X, Y) \cdot (U, V) := (XU, Y \cup XV).$$

Moreover, for morphisms $f : (X, Y) \to (X', Y')$ and $g : (U, V) \to (U', V')$, let $f \cdot g$ be the morphism $(X, Y) \cdot (U, V) \to (X', Y') \cdot (U', V')$ defined as follows:

$$f \cdot g = \{(y, y') : (y, y') \in f, y' \in Y'\} \cup \{(y, x' z) : (y, x') \in f, x' \in X', z \in V'\} \cup \{(x u, x z) : (v, z) \in g, x \in X\}.$$

For each letter $a$, let the trace $a$ be $((a), \emptyset)$. We omit the routine proof that, equipped with the constants $\{a : a \in A\}$, and the sum and product operations, $\text{Tr}_A$ is a ccc $\Sigma A$.

## 5. Algebraic objects and functors

Suppose that $A$ is a fixed ccc $\Sigma A$. In this section we will consider the initial solutions in $A$ of finite systems of fixed point equations, called here the “algebraic objects and functors”. (Sometimes, the term “equational” has been used instead of “algebraic”.)

The main result of this paper is the following Mezei–Wright type theorem:

*Any morphic image of an algebraic element is algebraic.*

There is a more detailed statement below in Theorem 5.3.

Let $\Phi = \{F_1, \ldots, F_n\}$ be a ranked alphabet disjoint from $\Sigma$, and suppose that the rank of $F_i$ is $k_i$, for $i \in [n]$. Define

$$A^{\Phi(\Sigma)} = [A^{k_1} \to A] \times \cdots \times [A^{k_n} \to A].$$

Then $A^{\Phi(\Sigma)}$ is a ccc $\Sigma A$, as noted in Section 3.1.

Each term $(4) t \in \text{Tm}_{\Sigma A}(V_m)$ in the extended ranked set $\Sigma_{\Phi} = \Sigma \cup \Phi$ induces a continuous functor

$$t^A : A^{\Phi(\Sigma)} \to [A^m \to A].$$

When each $f_i : A^{k_i} \to A$, $i \in [n]$, is continuous, we interpret each letter $F_i$ that appears in $t$ as the functor $f_i$, and obtain a continuous functor $A^m \to A$. More precisely, we extend Definition 3.1 as follows. For $i \in [n]$,

$$(F_i(t_1, \ldots, t_k))^A := f_i \circ (t_1^A, \ldots, t_k^A).$$

To show that the functor $t^A$ depends on the choice of $f_1, \ldots, f_n$, we write $t^A(f_1, \ldots, f_n)$, not just $t^A$.

If $g_i : A^{k_i} \to A$ is also continuous, and

$$\alpha_i : f_i \to g_i$$

is a natural transformation, for each $i \in [n]$, then we define

$$t^A(\alpha_1, \ldots, \alpha_n)$$

to be the natural transformation

$$t^A(f_1, \ldots, f_n) \to t^A(g_1, \ldots, g_n)$$

defined inductively as follows.

- When $t$ is a variable $v_j$, where $j \in [m]$, then $t^A(\alpha_1, \ldots, \alpha_n)$ is the identity natural transformation from the $j$th projection functor $\pi^A_j : A^m \to A$ to itself.
- If $t = \sigma(t_1, \ldots, t_p)$, then

$$t^A(\alpha_1, \ldots, \alpha_n) = \sigma^A \circ (\gamma_1, \ldots, \gamma_p),$$

where $\gamma_j = t^A(\alpha_1, \ldots, \alpha_n)$ for each $j$.
- Finally, when $t = F_i(t_1, \ldots, t_k), i \in [n]$, then

$$t^A(\alpha_1, \ldots, \alpha_n) = \alpha_i \circ (\gamma_1, \ldots, \gamma_m),$$

where each $\gamma_j$ is $t_j^A(\alpha_1, \ldots, \alpha_n)$, and where $\circ$ denotes the horizontal composition of natural transformations.
Note that if each $\alpha_i : f_i \to f_i$ is an identity natural transformation (so that $f_i = g_i$, for all $i \in [n]$), then $t^\ell (\alpha_1, \ldots, \alpha_n)$ is the identity natural transformation $t^\ell (f_1, \ldots, f_n) \to t^\ell (f_1, \ldots, f_n)$.

**Definition 5.1.** A *recursion scheme* over $\Sigma$ is a sequence $E$ of equations

$$
F_1(v_1, \ldots, v_{k_1}) = t_1 \\
\vdots \\
F_n(v_1, \ldots, v_{k_n}) = t_n
$$

(7)

where $t_i$ is a term in $T \Sigma_{\omega}^c (V_{k_i})$, for $i \in [n]$. A recursion scheme is *scalar* if $k_i = 0$, for each $i \in [n]$.

We may think of a recursion scheme as defining the functor $F_1$ by means of the parameters $F_2, \ldots, F_n$. When $k_1$, the arity of $F_1$, is zero, then the functor defined by the scheme may be identified with an object of $A$.

In any cc $\Sigma a A$, by target-tupling the right sides of the recursion scheme in (7), we obtain a continuous functor

$$E^A : A^{\rho(\Phi)} \to A^{\rho(\Phi)}.$$ 

Indeed, for $i \in [n]$, $t_i^A : A^{\rho(\Phi)} \to [A^{k_i}] \to A$, so that

$$E^A = (t_1^A, \ldots, t_n^A) : A^{\rho(\Phi)} \to A^{\rho(\Phi)}.$$ 

Thus, by Section 3.1, Fact 7, $E^A$ has initial fixed point which we denote by

$$|E^A| = (|E|^1_1, \ldots, |E|^n_n) \in A^{\rho(\Phi)},$$

so that, in particular,

$$|E|^A_i = t_i^A(|E|^1_1, \ldots, |E|^n_n),$$

at least up to isomorphism, for each $i \in [n]$.

For example, in the case that $\Sigma_2 = \{\cdot\}$ and $\Sigma_0 = Z$, a scalar recursion scheme is a system of equations of the form

$$F_i = u_i, \quad i \in [n],$$

where $u_i$ is a finite (parenthesized) word on the alphabet $Z \cup \{F_1, \ldots, F_n\}$.

**Definition 5.2.** Suppose that $A$ is a cc $\Sigma a A$. If $m > 0$, we call a functor $f : A^m \to A$, *algebraic* if there is a recursion scheme $E$ such that $f$ is isomorphic to $|E|^1_1$, the first component of the above initial solution. An object $a$ in a cc $\Sigma a A$ is *(0-)algebraic* if there is a (scalar) recursion scheme $E$ such that $a$ is isomorphic to $|E|^1_1$, the first component of the above initial solution.

In certain examples, 0-algebraic objects are the “regular” objects, and in others the “context-free” objects, see Section 7.

Note that any algebraic functor is continuous since it is constructed in a category $[A^m \to A]$ of continuous functors.

In Section 6 we will prove the following theorem.

**Theorem 5.3.** Suppose that $A, B$ are cc $\Sigma a$'s, and that $h : A \to B$ is a cc $\Sigma a$ morphism. Then an object $b$ of $B$ is (0-)algebraic iff there is an (0-)algebraic object $a$ in $A$ such that $b$ is isomorphic to $h(a)$. More generally, if $m \geq 0$ and if $f : A^m \to A$ is an algebraic functor, then there is an algebraic object $a$ in $A$ such that $f \circ h \circ a$ is naturally isomorphic to $h(a)$; conversely, if $g : B^m \to B$ is algebraic, there is an algebraic $f : A^m \to A$ such that $h \circ f \circ g$ is naturally isomorphic to $h(a)$.

Given a recursion scheme $E$, we may first solve it $A$, obtaining $f$ and then interpret the solution in $B$ by means of $h$, obtaining $h \circ f$. Conversely, we may solve it in $B$ obtaining $g$, and interpret in $A$, obtaining $g \circ h$. The theorem says that, up to a natural isomorphism, the results are the same.

**Remark 5.4.** Given a category $E$ of ordinary algebras, an *$m$-ary implicit operation* is an assignment of a function $\sigma^A : A^m \to A$ to each algebra $A$ in $E$ such that for any morphism $h : A \to B$ in $E$, $h \circ \sigma^A = \sigma^B \circ h^m$. Thus, Theorem 5.3 asserts that each recursion scheme determines an implicit operation over the category of cc $\Sigma a$’s, at least up to isomorphism.

**Example.** Suppose that $h : A \to B$ is a cc $\Sigma a$ morphism, and $f : A^m \to A$ and $g : B^m \to B$ are functors. It is not true that if $f$ is algebraic and $h \circ f$ is naturally isomorphic to $g \circ h^m$, then $g$ is algebraic. Indeed, let $\Sigma$ be the signature with one unary symbol $\sigma$ and one constant symbol $\bot$. Let $A$ be the initial cc $\Sigma a$ of all $\Sigma$-trees, namely the finite trees $\sigma^n \bot$, $n \geq 0$, and the infinite tree $\sigma^\omega$. Let $B$ be the ordered cc $\Sigma a$ on the set $0, 1, \ldots$, ordered as usual, and let $1^B = 0$, $\sigma^B(n) = 2n$, where $2^\omega = \infty$. The unique cc $\Sigma a$ morphism $h : A \to B$ maps every tree to $0$, since $h(\bot) = 0$ and, assuming $h(\sigma^n \bot) = 0$, we have $h(\sigma(\sigma^n \bot)) = h(\sigma^0) = 0$. Since $h$ is continuous, $h(\sigma^\omega) = \sup_n h(\sigma^n \bot) = 0$.

Now a continuous functor $g : B \to B$ is an order preserving function such that when $n_1 \leq n_2 \leq \cdots$, $g(\sup_n n_k) = \sup_n g(n_k)$. There are uncountably many such functions such that $g(0) = 0$. Indeed, if $X$ is a subset of the positive integers, define the function $g_X : N_\infty \to N_\infty$ as follows:

$$g_X(n) := \begin{cases} 
  n & \text{if } n = 0 \text{ or } n = \infty \\
  g_X(n-1) + 1 & \text{if } n > 0 \text{ and } n \in X \\
  g_X(n-1) + 2 & \text{if } n > 0 \text{ and } n \notin X.
\end{cases}$$

Then, if $X, Y$ are distinct subsets, $g_X(n) \neq g_Y(n)$, where $n$ is the least positive integer in just one of the two sets.
Thus, for any set $X$, and any algebraic functor $f : A \rightarrow A$, for all $a \in A$,
\[ 0 = h(f(a)) = g_X(0) = g_X(h(a)). \]

There is only a countable number of algebraic functors $B \rightarrow B$, so there are uncountably many nonalgebraic functors $g$ such that $h \circ f = g \circ h$.

This same idea can be used to show that it is not true that if $g : B \rightarrow B$ is algebraic and $f : A \rightarrow A$ is a functor such that $h \circ f = g \circ h$.

**Remark 5.5.** If $h : A \rightarrow B$ is dense (i.e., for each object $b$ of $B$, there is some object $a$ in $A$ such that $h(a)$ is isomorphic to $b$ and if $h$ is full (surjective on hom-sets) we may make a stronger statement). A functor $g : B^m \rightarrow B$ is algebraic if and only if there is an algebraic functor $f : A^m \rightarrow A$ such that $h \circ f$ is naturally isomorphic to $g \circ h^m$. Indeed, when an algebraic $f$ exists, since there is at least one algebraic functor $g' : B^m \rightarrow B$ with $g' \circ h^m$ naturally isomorphic to $h \circ f$, the functors $g$ and $g'$ are naturally isomorphic.

### 6. Proof of the Mezei-Wright theorem

**Theorem 6.1.** Suppose that $E$ is a recursion scheme as in (7). Let $A$ and $B$ be $\Sigma \Sigma a$'s, and let $h$ be a continuous morphism $A \rightarrow B$. For each $i \in [n]$, let $|E|^i_A$ denote the $i$th component of the initial solution of $E^A$ and let $|E|^i_B$ be the corresponding component of the initial solution of $E^B$. Then $h \circ |E|^i_A$ is naturally isomorphic to $|E|^i_B \circ h^i$.

We will only prove the claim when there is just one equation, since the argument in the general case is only notationally more complicated. So let $E$ consist of the single equation
\[ F(v_1, \ldots, v_m) = t, \tag{8} \]
where $t$ is a term in $T_{\Sigma \cup \{f\}}(V_m)$. Thus, $E^A$ and $E^B$ are the continuous functors $t^A : A^m \rightarrow A$ and $t^B : B^m \rightarrow B$, respectively.

Let $\bot$ denote a symbol of rank $0$ not in $\Sigma$ and let $\Sigma_\bot$ denote the ranked set obtained by adding the symbol $\bot$ to $\Sigma$. By interpreting $\bot$ as initial objects in $A$ and $B$, we turn $A$ and $B$ into $\Sigma \Sigma \bot a$'s. Since $h$ maps initial objects to initial objects, it is a morphism of $\Sigma \Sigma \bot a$'s. Thus, we have natural isomorphisms
\[ \pi_\sigma : h \circ \sigma^A \rightarrow \sigma^B \circ h^\bot, \quad \sigma \in \Sigma_n, \ n \geq 0 \]
\[ \pi_\bot : h \circ \bot^A \rightarrow \bot^B. \]

The isomorphism $\pi_\bot$ gives rise to natural isomorphisms $h \circ \bot^A \rightarrow \bot^B \circ h^\bot$, where $\bot^A \rightarrow A$ is the constant functor $A^m \rightarrow A$ with value $\bot^A$, etc. For ease of notation, we will just write $\pi_\bot$ to denote these natural isomorphisms, moreover, we will just write $\bot^A$ for $\bot^A \rightarrow A$ as well; similarly for $\bot^B$.

**Lemma 6.2.** Suppose that $m \geq 0$, and $f : A^m \rightarrow A$ and $g : B^m \rightarrow B$ are continuous such that there is a natural isomorphism $\tau : h \circ f \rightarrow g \circ h^\bot$. Then for each term $t$ in $T_{\Sigma_\bot \cup \{f\}}(V_m)$ there is a natural isomorphism
\[ \pi_\tau(\tau) : h \circ t^A \rightarrow t^B(\tau) \circ h^\bot. \tag{9} \]

**Proof.** We define $\pi_\tau(\tau)$ by induction on the structure of $t$.

- If $t$ is the variable $v_i$, where $i \in [m]$, then $h \circ t^A = t^B \circ h^\bot$, and we define $\pi_\tau(\tau)$ as the appropriate identity natural transformation.
- If $t$ is the symbol $\bot$, then $\pi_\tau(\tau)$ be the natural isomorphism $\pi_\bot$ given above.
- Suppose now that $t$ is of the form $\sigma(t_1, \ldots, t_n)$. By induction, the natural isomorphisms $\pi_j(\tau), j \in [m]$, have been defined. In this case we define $\pi_\tau(\tau)$ as the horizontal composition of the natural isomorphisms: $\pi_\sigma \circ (\pi_{t_1}(\tau), \ldots, \pi_{t_n}(\tau))$.
- Finally, when $t$ is of the form $F(t_1, \ldots, t_n)$, we let $\pi_\tau(\tau) = \tau \circ (\pi_{t_1}(\tau), \ldots, \pi_{t_n}(\tau))$, where the $\pi_{t_i}(\tau)$ are again well-defined by the induction hypothesis.

Since the horizontal composition of natural isomorphisms is again a natural isomorphism, the proof is complete. \[ \square \]

In the case of one recursion equation (8), $|E|^A$ is a colimit of the diagram
\[ \bot^A \xrightarrow{t^A} t^A(\bot^A) \xrightarrow{t^A(t^A)} t^A(t^A(\bot^A)) \rightarrow \cdots \]
and, similarly, $|E|^B$ is a colimit of
\[ \bot^B \xrightarrow{t^B} t^B(\bot^B) \xrightarrow{t^B(t^B)} t^B(t^B(\bot^B)) \rightarrow \cdots \]
Each functor in diagram (10) is a functor $A^m \rightarrow A$. If we compose each with $h$, we obtain functors $A^m \rightarrow A \rightarrow B$. Similarly, if we precompose each functor in diagram (11) with $h^m$, we get functors $A^m \rightarrow B^m \rightarrow B$.

We would like to prove that the diagram

\[
\begin{array}{c}
\begin{array}{cccc}
& h \circ \bot^A & \rightarrow & h \circ t^A(\bot^A) \\
\downarrow & & \downarrow & \\
& \pi_{\bot} & \rightarrow & \pi_{t}(\pi_{\bot}) \\
\end{array} \\
\begin{array}{cccc}
\downarrow & & \downarrow & \\
\downarrow & & \downarrow & \\
\downarrow & & \downarrow & \\
\bot^B \circ h^m & \rightarrow & t^B(\bot^B) \circ h^m & \rightarrow & t^B(t^B(\bot^B)) \circ h^m \\
\end{array}
\end{array}
\]

commutes, and, since each vertical natural transformation is a natural isomorphism by Lemma 6.2, it follows that the top and bottom horizontal diagrams have isomorphic colimits.

We prove a more general result. (The fact that diagram (12) commutes is obtained by instantiating Lemma 6.3 by letting $f_1 = \bot^A$, $f_2 = \bot^B$, $\varphi = \pi_{\bot}$ and $\psi = \pi_{t}(\pi_{\bot})$.)

**Lemma 6.3.** Suppose that $t$ is a term as above, and suppose that $f_1, f_2 : A^m \rightarrow A$, $g_1, g_2 : B^m \rightarrow B$ are continuous functors with natural isomorphisms $\tau_1 : h \circ f_1 \rightarrow g_1 \circ h^m$ and $\tau_2 : h \circ f_2 \rightarrow g_2 \circ h^m$. Moreover, suppose that $\varphi : f_1 \rightarrow f_2$ and $\psi : g_1 \rightarrow g_2$ are natural transformations. Then, if the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
& h \circ f_1 & \rightarrow & h \circ f_2 \\
\downarrow & & \downarrow & \\
& \tau_1 & \rightarrow & \tau_2 \\
\end{array} \\
\begin{array}{ccc}
& g_1 \circ h^m & \rightarrow & g_2 \circ h^m \\
\end{array}
\end{array}
\]

commutes, then so does the diagram

\[
\begin{array}{c}
\begin{array}{cccc}
& h \circ t^A(f_1) & \rightarrow & h \circ t^A(f_2) \\
\downarrow & & \downarrow & \\
& \pi_{t}(\tau_1) & \rightarrow & \pi_{t}(\tau_2) \\
\end{array} \\
\begin{array}{cccc}
& t^B(g_1) \circ h^m & \rightarrow & t^B(g_2) \circ h^m \\
\end{array}
\end{array}
\]

**Proof.** We argue by induction on the structure of $t$. Assume first that $t$ is a variable $v_i$. Then $t^A(\varphi)$, $t^B(\psi)$, $\pi_{t}(\tau_1)$ and $\pi_{t}(\tau_2)$ are all identity natural transformations and our claim is thus obvious.

Assume next that $t = \bot$. Then $t^A(\varphi) = !^A$, $t^B(\psi) = !^B$, $\pi_{t}(\tau_1) = \pi_{t}(\tau_2) = \pi_{\bot}$. Since both $h \circ \bot^A$ and $!^B \circ h^m$ are the corresponding identity natural transformations, our claim is again obvious. See diagram (15).
Suppose now that $t$ is of the form $\sigma(t_1, \ldots, t_n)$. By the induction assumption, the following diagram commutes, for $i = 1, \ldots, n$.

![Diagram](image)

(16)

Consider the diagram (17).

![Diagram](image)

(17)

Note that $\pi_1(\tau_1)$ is the composite of the left sides, and $\pi_1(\tau_2)$ is the composite of the right sides. It is clear that the second and fourth squares commute. The first square commutes by the interchange rule, cf. [23]. Finally, the third square commutes by the commutativity of the squares (16) and the interchange rule.

The last case, when $t$ is of the form $F(t_1, \ldots, t_m)$, is similar to the previous one. □

**Proof of Theorem 6.1.** As before, let $E$ consist of the single equation $F(v_1, \ldots, v_m) = t$. Let $t^A$ denote the unique natural transformation $\bot^A \to t^A(\bot^A)$ (here $\bot^A$ denotes the constant functor $A^m \to A$ whose value is the distinguished initial object of $A$, also denoted $\bot^A$). We know that the $\omega$-diagrams

$$((t^A)^n(\bot^A) \xrightarrow{(t^A)^n(\bot^A)} (t^A)^{n+1}(\bot^A))_n$$

(18)
and
\[(t^A)^n(\bot A) \xrightarrow{(t^A)^n(\bot)} (t^B)^{n+1}(\bot B)_n\] (19)

have colimit diagrams
\[(t^A)^n(\bot A) \xrightarrow{\varphi_n} |E|^A_n\] (20)

\[(t^B)^n(\bot B) \xrightarrow{\psi_n} |E|^B_n\] (21)

respectively. Since \(h\) is continuous, the cone \((h \circ \varphi_n : (t^A)^n(\bot A) \to h \circ |E|^A_n)\) is a colimit cone of the diagram obtained by composing \(h\) with the diagram (18):
\[(h \circ (t^A)^n(\bot)) : h \circ (t^A)^n(\bot A) \to h \circ (t^A)^{n+1}(\bot A)_n.\] (22)

Also, the cone \((\psi_n \circ h^m : (t^B)^n(\bot B) \to |E|^B \circ h^m)_n\) is a colimit cone of the diagram
\[((t^B)^n(\bot) \circ h^m : (t^B)^n(\bot B) \circ h^m \to (t^B)^{n+1}(\bot B) \circ h^m)_n.\] (23)

Now we show each square in diagram (12) commutes. By assumption, there is a natural isomorphism \(\pi_\bot : h \circ \bot A \to \bot B \circ h^m\).

Let \(\tau_0\) denote this isomorphism. Then, for each \(n > 0\), define
\[\tau_n := \pi_1(\tau_{n-1}),\]
so that \(\tau_n\) is a natural isomorphism \(h \circ (t^A)^n(\bot A) \to (t^B)^n(\bot B) \circ h^m\).

If we can show that for each \(n \geq 0\),
\[\tau_{n+1} : (h \circ (t^A)^n(\bot)) = ((t^B)^n(\bot) \circ h^m) \cdot \tau_n,\] (23)

then it follows that \((\psi_n \cdot \tau_n)_n\) is a also a cone over the diagram (22). Thus, there results a unique natural transformation \(\tau : h \circ |E|^A \to |E|^B \circ h^m\) with
\[\tau \cdot (h \circ \varphi_n) = (\psi_n \circ h^m) \cdot \tau_n\] (24)

for all \(n\). In the same way, there is a unique natural transformation \(\tau' : |E|^B \circ h^m \to h \circ |E|^A\) with
\[\tau' \cdot (\psi_n \circ h^m) = (h \circ \varphi_n) \cdot \tau_n^{-1}\] (25)

for all \(n\). It follows now that \(\tau\) is a natural isomorphism with inverse \(\tau'\).

It remains only to prove (23). When \(n = 0\), this equation holds by initiality. Moreover, assuming that (23) holds for \(n\), it also holds for \(n + 1\) by Lemma 6.3. This completes the proof. □

Recall from Section 3.1 that there is a unique natural transformation
\[t^A : t^A(|E|^A) \to |E|^A\]
such that
\[\varphi_n = t^A(\varphi_n),\]
for all \(n\). Similarly, for \(t^B : t^B(|E|^B) \to |E|^B\). In fact, both \(t^A\), \(t^B\) are natural isomorphisms.

**Corollary 6.4.** The square
\[
\begin{array}{ccc}
  h \circ t^A(|E|^A) & \xrightarrow{h \circ \alpha} & h \circ |E|^A \\
  \pi_t(\tau) & & \tau \\
  t^B(|E|^B) \circ h^m & \xrightarrow{t^B \circ h^m} & |E|^B \circ h^m
\end{array}
\]
commutes.

**Proof.** There is a unique mediating morphism
\[\alpha : h \circ t^A(|E|^A) \to |E|^B \circ h^m\]
such that, for each \( n \geq 0 \),
\[
\alpha \cdot (h \circ t^A(\varphi_n)) = (\psi_{n+1} \circ h^m) \cdot \tau_{n+1}.
\]

But, both \( \tau \cdot (h \circ \iota^A) \) and \( (t^B \circ h^m) \cdot \pi_1(\tau) \) work. Indeed,
\[
\tau \cdot (h \circ \varphi_{n+1}) = (\psi_{n+1} \circ h^m) \cdot \tau_{n+1},
\]
by definition of \( \tau \). But
\[
h \circ \varphi_{n+1} = (h \circ \iota^A) \cdot (h \circ t^A(\varphi_n)),
\]
by definition of \( t^A \). Thus,
\[
\tau \cdot (h \circ \iota^A) \cdot (h \circ t^A(\varphi_n)) = (\psi_{n+1} \circ h^m) \cdot \tau_{n+1},
\]
proving one part of the claim. Also,
\[
\pi_1(\tau) \cdot (h \circ \iota^A) = (\psi_{n+1} \circ h^m) \cdot \tau_{n+1},
\]
by Lemma 6.3, and
\[
(t^B \circ h^m) \cdot (t^B(\psi_n) \circ h^m) = \psi_{n+1} \circ h^m.
\]
by the definition of \( t^B \). Thus,
\[
(t^B \circ h^m) \cdot \pi_1(\tau) \cdot (h \circ t^A(\varphi_n)) = (\psi_{n+1} \circ h^m) \cdot \tau_{n+1},
\]
proving the second part. Thus,
\[
(t^B \circ h^m) \cdot \tau(\tau) = \tau \cdot (h \circ \iota^A),
\]
as claimed. \( \square \)

**Corollary 6.5.** In the case that \( m = 0 \), the objects \( h(|E|^A) \) and \( |E|^B \) are isomorphic.

7. An example

Both trees, \( A = T_{\Sigma}^\Sigma \) and words, \( B = W_Z \) are \( cc \Sigma \) a’s, where \( \Sigma \) is the signature with one binary relation symbol \( \cdot \), and two constant symbols \( \perp, a \) and where \( Z \) is the single letter alphabet \( \{a\} \). See Sections 4.1 and 4.2. We have discussed the yield morphism \( h : A \to B \). See Proposition 4.3.

Consider the scalar fixed point equation
\[
x = x \cdot (a \cdot x).
\]
In this case, the Mezei–Wright theorem asserts the existence of an isomorphism between the \( h \)-image of the colimit (10) and the colimit (11). Let \( \theta(x) \) be the term \( x \cdot (a \cdot x) \). The functor \( \theta^A \) in this case maps \( t \subseteq t' \) in \( A \) to
\[
t \cdot (a \cdot t) \subseteq t' \cdot (a \cdot t').
\]
In \( B \), \( \theta^B \) maps \( w \xrightarrow{\beta} w' \) to
\[
w \cdot a \cdot w \xrightarrow{\beta a \cdot \beta} w' \cdot a \cdot w'.
\]
If \( t \) is a tree, an indication of the tree \( \theta^A(t) = t \cdot (a \cdot t) \) is

```
    / \     / \     / \     / \\
 t   / \   / \   / \   / \\
 a o o a o
```
Thus, \( |E|^A \) is the colimit in \( A \) of the diagram

```
o -> o -> o -> o -> ...
    / \   / \   / \   / \\
    o   o   o   o   o
    / \   / \   / \\
   a o o a o
```

```
    / \   / \   / \\
   a o o a o
```

Here the “o” is the initial tree, ⊥. Each tree in the chain is obtained by replacing any leaf labeled o in the previous tree by

```
/ \  
o  
```

```
/ \  
a o
```

It is not difficult to see that the vertices of the tree $|E|^A$ are all prefixes of the set of words denoted by the regular expression $(0 + 11)^* 10$. In fact, the leaves labeled $a$ in the tree $((θ^A)^k)(⊥)$ are the words in $(0 + 11)^* 10$ of length at most $2k$.

In $B$, we write just $uv$ instead of $u \cdot v$, and since $\cdot B$ is associative, we see that $|E|^B$ is the colimit in $B$ of the diagram

```
o \rightarrow a \rightarrow a a \rightarrow a a a a a a \rightarrow \cdots .
```

In this diagram $o$ is the initial word $ε$. The word $((θ^B)^k)(⊥) = ([2^k - 1], <, λ)$ is the word on $\{a\}$ whose underlying linear order is $1 < 2 < \cdots < 2^k - 1$, but it is useful to regard this word as having $2^k$ occurrences of $o$ in appropriate positions:

```
((θ^B)^k)(⊥) = (oa)^{2^k - 1} o,
```

and $((θ^B)^{k+1})(⊥)$ is obtained from this word by replacing every occurrence of $o$ by $oao$; the morphism $α_k : ((θ^B)^k)(⊥) \rightarrow ((θ^B)^{k+1})(⊥)$ is the order preserving function $i \mapsto 2i$. From this description, one may find an argument independent of the Mezei–Wright theorem Theorem 5.3 that the colimit $|E|^B$ is the diagram $((θ^B)^k \rightarrow A)_k$, where $A = \{i/2^k : 1 \leq i < 2^k, k \geq 1\}$, ordered as usual, and

```
((θ^B)^k)\rightarrow A
i \mapsto i/2^k.
```

Note that $A$ is countable, has no first or last element, and between any two elements is a third, i.e., it is isomorphic to the rationals. But, by Theorem 5.3, another description of $|E|^B$ is $((L_w, <, λ_w))$, where $L_w = (0 + 11)^* 10$ and $<_w$ is the strict order. This is just the yield of $|E|^A$.

Because the yield functor is strict, Lemma 6.3 for the term $θ(x)$ becomes the following:

**Lemma 7.1.** Suppose that $f_1, f_2 : A \rightarrow A, g_1, g_2 : B \rightarrow B$ are continuous functors with natural isomorphisms $τ_1 : h \circ f_1 \rightarrow g_1 \circ h$ and $τ_2 : h \circ f_2 \rightarrow g_2 \circ h$. Moreover, suppose that $ψ : f_1 \rightarrow f_2$ and $ψ' : g_1 \rightarrow g_2$ are natural transformations. Then, if the diagram

```
\begin{array}{c}
h \circ f_1 \overset{h \circ ψ}{\longrightarrow} h \circ f_2 \\
\tau_1 \downarrow \hspace{1cm} \tau_2 \\
g_1 \circ h \overset{ψ \circ h}{\longrightarrow} g_2 \circ h
\end{array}
```

(26)

commutes, then so does the diagram

```
\begin{array}{c}
h \circ (f_1 \cdot (a \cdot f_1)) \overset{h \circ θ^A(ψ)}{\longrightarrow} h \circ (f_2 \cdot (a \cdot f_2)) \\
π_θ(τ_1) \downarrow \hspace{1cm} π_θ(τ_2) \\
(g_1 \cdot (a \cdot g_1)) \circ h \overset{θ^B(ψ) \circ h}{\longrightarrow} (g_2 \cdot (a \cdot g_2)) \circ h
\end{array}
```

(27)
Now, for any tree \( t \), \( \theta^A(\varphi_t) \) is the morphism
\[
\varphi_t \cdot (a \cdot \varphi_t) : f_1(t) \cdot (a \cdot f_1(t)) \to f_2(t) \cdot (a \cdot f_2(t))
\]
so that
\[
h(\theta^A(\varphi_t)) : h(f_1(t) \cdot (a \cdot f_1(t))) \to h(f_2(t) \cdot (a \cdot f_2(t)))
\]
is
\[
h(\theta^A(\varphi_t)) = h(\varphi_t \cdot (a \cdot \varphi_t)) = h(\varphi_t) \cdot (a \cdot h(\varphi_t)).
\]  
(28)

Also, \( \theta^B(\psi_{h(t)}) : g_1(h(t)) \cdot (a \cdot g_1(h(t))) \to g_2(h(t)) \cdot (a \cdot g_2(h(t))) \) is
\[
\theta^B(\psi_{h(t)}) = \psi_{h(t)} \cdot (a \cdot \psi_{h(t)}).
\]  
(29)

According to the lemma, if for a tree \( t \), the diagram
\[
\begin{array}{ccc}
h(f_1(t)) & \xrightarrow{h(\varphi_t)} & h(f_2(t)) \\
\tau_1(t) \downarrow & & \downarrow \tau_2(t) \\
g_1(h(t)) & \xrightarrow{\psi_{h(t)}} & g_2(h(t))
\end{array}
\]
commutes, so does the diagram
\[
\begin{array}{ccc}
h(f_1(t) \cdot (a \cdot f_1(t))) & \xrightarrow{h(\varphi_t) \cdot (a \cdot h(\varphi_t))} & h(f_2(t) \cdot (a \cdot f_2(t))) \\
\pi_\theta(\tau_1(t)) \downarrow & & \downarrow \pi_\theta(\tau_2(t)) \\
g_1(h(t)) \cdot (a \cdot g_1(h(t))) & \xrightarrow{\psi_{h(t)} \cdot (a \cdot \psi_{h(t)})} & g_2(h(t)) \cdot (a \cdot g_2(h(t)))
\end{array}
\]  
(30)

Also, since \( h \) is strict, for \( i = 1, 2 \),
\[
\pi_\theta(\tau_i(t)) = \tau_i(t) \cdot (a \cdot \tau_i(t)).
\]

Hence, we can write diagram (30) as
\[
\begin{array}{ccc}
h(f_1(t) \cdot (a \cdot f_1(t))) & \xrightarrow{h(\varphi_t) \cdot (a \cdot h(\varphi_t))} & h(f_2(t) \cdot (a \cdot f_2(t))) \\
\tau_1(t) \cdot (a \cdot \tau_1(t)) \downarrow & & \downarrow \tau_2(t) \cdot (a \cdot \tau_2(t)) \\
g_1(h(t)) \cdot (a \cdot g_1(h(t))) & \xrightarrow{\psi_{h(t)} \cdot (a \cdot \psi_{h(t)})} & g_2(h(t)) \cdot (a \cdot g_2(h(t)))
\end{array}
\]  
(31)

and the conclusion follows by the interchange law. \( \Box \)
8. Some applications

By “application” here, we mean a triple, $A, B, h$, where $A, B$ are $\Sigma\Sigma$ a’s, and $h : A \rightarrow B$ is a $\Sigma\Sigma$ a morphism. So, by Theorem 5.3, objects in $B$ are $(0)$-algebraic iff they are isomorphic to $h$-images of $(0)$-algebraic objects in $A$. The first application, considered by Mezei and Wright [24] is the case when $A$ and $B$ are subset algebras of some $\Sigma\Sigma$-algebras $A_0$ and $B_0$, respectively, so that using the notation introduced in Section 4, $A = \hat{A}_0$, $B = \hat{B}_0$. In that case a 0-algebraic object in $A$ is an equational subset of $A_0$. Any morphism $A_0 \rightarrow B_0$ may be lifted to a $\Sigma\Sigma$ a morphism $A \rightarrow B$, and the theorem says that a subset of $B_0$ is equational iff it is the image of an equational subset of $A_0$ under a morphism $h : A \rightarrow B$. It is shown in [24] that the equational subsets of $T_\Sigma$, the finite $\Sigma$-trees, are the recognizable sets. Both the theorem and its proof are special cases of our Theorem 5.3. It is always useful and interesting to find independent descriptions of the 0-algebraic and algebraic objects in both $A$ and $B$, if possible.

A subset of a $\Sigma$-algebra $A_0$ is “algebraic” if it is an algebraic object of the corresponding subset algebra $\hat{A}_0$ (see Section 4.3). The algebraic subsets of $T_\Sigma$ are known as the $\Sigma$-context-free tree languages, and their homomorphic images are the algebraic subsets of any $\Sigma$-algebra. See [14].

Now consider an alphabet $Z$ and let $\Sigma_2$ consist of the binary symbol $\cdot$, and let $\Sigma_0$ consist of the letters in $Z$ and the symbol $\epsilon$. Then the free monoid $Z^*$ may naturally be seen as a $\Sigma$-algebra where $\epsilon$ is the empty word. It is well-known that the 0-algebraic subsets of $Z^*$ are the context-free languages over $Z$, and the algebraic subsets are the $\Sigma$-macrolanguages of $[18, 14]$. An instance of the Mezei–Wright theorem may be obtained by considering the morphism that maps a tree $t \in T_\Sigma$ to its frontier, i.e. to the left-to-right sequence of leaf labels. By lifting this morphism to subset algebras, we obtain the result that a language is context-free iff it is the frontier of a recognizable subset of $T_\Sigma$, and it is an $\Sigma$-macrolanguage iff it is the frontier of an $\Sigma$-context-free tree language, cf. [18, 14].

For theorems of the Mezei–Wright type which involve recursive program schemes and their semantics we refer to [19, 26, 21, 10]. In Guessarian [21], $A$ is $T_\Sigma^0$, ordered as in Section 4.1, and $B$ is any continuous ordered $\Sigma$-algebra. Since $A$ is initial, $h$ is the unique morphism. Again, the Mezei–Wright theorem given here is a special case of our Theorem 5.3. More generally, Theorem 4.26 of [21] also allows $A$ to be any continuous ordered $\Sigma$-algebra of trees, freely generated by a set $V$. The $\Sigma\Sigma$ a of $\Sigma$-trees plays the role of abstract syntax in both [19, 10], where this structure is usually denoted $CT_\Sigma$. In [19], Section 3.1, $A$ is $T_\Sigma^0$, where $\Sigma$ is derived from a context-free grammar over some alphabet, and where $B$ is the algebra of all languages over that alphabet, equipped with suitable operations, so that both $A$ and $B$ are continuous ordered algebras. In Section 3.2, $A$ is again an algebra of trees, containing a symbol corresponding to each construct of a prototype of a higher order programming language with recursion, called SAL. Moreover, $B$ is a continuous ordered algebra of “environments” over a complete lattice defined as the initial solution of a recursive domain equation. The unique morphism $A \rightarrow B$ provides the semantics of each program in SAL.

The setting in [10] is continuous theories, not continuous categorical algebras. A system of recursion equations interpreted in a continuous theory $T$ is a morphism

$$\varphi : CT_\Sigma \rightarrow CT_\Sigma + T,$$

where the right side is the coproduct in the category of continuous theories. A solution of $\varphi$ is a morphism $\alpha : CT_\Sigma \rightarrow T$ such that

$$\alpha = CT_\Sigma \varphi \rightarrow CT_\Sigma + T \overset{[\alpha, 1]}{\rightarrow} T.$$

Least solutions exist, and the least solution of $\varphi$ is denoted $\varphi^\dagger$. If $h : T \rightarrow T'$ is a continuous theory morphism, $h$ induces the morphism $(1 + h) : CT_\Sigma + T \rightarrow CT_\Sigma + T'$. The Mezei–Wright type theorem in [10] is that

$$((1 + h) \circ \varphi)^\dagger = h \circ \varphi^\dagger.$$

Equational or 0-algebraic elements of additive algebras were studied in Bozapalidis [11] and a Mezei–Wright type theorem was proved. The application in this paper concerns the category of complete $K\Sigma$-algebras $A$, where $K$ is a complete semiring (“additive”, in the author’s terminology) and $A$ is a complete $K$-module enriched with the structure of a $\Sigma$-algebra. Thus $A$ has all sums, with the usual properties. Each $\Sigma$-functor preserves the $K$-action and all sums in each argument. $A$ is ordered by the sum order: $x \leq y$ iff $y = x + z$, for some $z$. A morphism $h : A \rightarrow B$ of complete $K\Sigma$-algebras is a $\Sigma$-algebra morphism that preserves the $K$-action and all sums, and hence preserves the order. $A_0 = K(\langle T_\Sigma \rangle)$, is the ordered category of all formal power series over finite $\Sigma$-trees, considered as a $K\Sigma$-algebra. It is shown that $A_0$ is initial in the category of complete $K\Sigma$-algebras. Thus, if $B$ is a complete $K\Sigma$-algebra, there is a unique morphism $h : A_0 \rightarrow B$. The author proves that the 0-algebraic elements of $A_0$ are the recognizable power series, and the Mezei–Wright theorem given here states that any 0-algebraic element in $B$ is the image under $h$ of a recognizable series in $A_0$. Thus, this theorem is a special case of Theorem 5.3, and the standard proof for ordered algebras applies to this case as well. The author does not consider algebraic elements in $A_0$.

When $S$ is a continuous semiring, in the sense of [17], then for any set $Z$, the power series semiring $S(\langle \langle Z^* \rangle \rangle)$ is also a continuous semiring, and thus a $\Sigma\Sigma$ a. The 0-algebraic objects are the rational power series and the algebraic objects the algebraic power series. See [22] and [16] for generalizations involving tree series. Below we will mention some further examples, where at least one of the algebras involved is not ordered, and the general categorical structure is essential.
• Trees and words. Let $A = T^\omega_Z$, the category of trees, and $B = W_\Sigma$, the category of words on the alphabet $Z$; the signature $\Sigma$ has $\Sigma_2 = \{\cdot\}$, and $\Sigma_0 = Z$. The binary operation $t_0 \cdot t_1$ on trees was defined in (5) and on words in (6). Last, $h$ is the yield function (see Section 4.2). In fact, the 0-algebraic trees and words are the regular such objects, see [12,8], and this case of the Mezei–Wright theorem is known [12]. The proof is a special case of our Theorem 5.3. The regular words in $W_\Sigma$ have been described in several ways and their equational properties axiomatized in [8]. The algebraic trees have been characterized in [13] in terms of their “branch languages”, and the algebraic words in [9]: a word is algebraic iff it is a deterministic context-free word.

• Trees and synchronization trees. Suppose that $A$ is an alphabet and let $\Sigma$ consist of the binary symbols $+$ and $\cdot$ and the letters in $A$ as constants. Consider the ordered cc$\Sigma$A $T^\omega_{\Sigma}$ and the the cc$\Sigma$A ST(A). Since $T^\omega_{\Sigma}$ is initial, there is a unique cc$\Sigma$A morphism $T^\omega_{\Sigma} \to$ ST(A). It follows from Theorem 5.3 that a synchronization tree in ST(A) is (0-)algebraic iff it is the image of a 0-algebraic tree in $T^\omega_{\Sigma}$. Using the fact that the 0-algebraic trees in $T^\omega_{\Sigma}$ are the regular trees having finitely many subtrees, we obtain the characterization of the 0-algebraic synchronization trees in ST(A). We are not aware of any characterization of algebraic synchronization trees, but expect that a characterization can be derived from the characterization of algebraic trees mentioned above.

• Synchronization trees and bisimilarity. There is an obvious cc$\Sigma$A morphism ST(A) $\to$ BST(A) that maps a synchronization tree to its bisimilarity equivalence class, cf. Section 4. Accordingly, the 0-algebraic objects in BST(A) are the bisimilarity equivalence classes of regular synchronization trees, and the algebraic synchronization trees are the bisimilarity equivalence classes of algebraic synchronization trees. We are not aware of any characterization of algebraic synchronization trees up to bisimilarity.

• Synchronization trees and traces. Regarding trace semantics, there is the obvious cc$\Sigma$A morphism trace : ST(A) $\to$ Tr$A$, and in fact a cc$\Sigma$A morphism BST(A) $\to$ Tr$A$, see Section 4. Indeed, for any tree $t$ in ST(A), we have defined trace($t$) = $(X_t, Y_t)$ in Section 4.5. When $f$ is a morphism $t \to t'$ and trace($t'$) = $(X_t'$, $Y_t'$), then $X_t \subseteq X_t'$. We thus define trace($f$) as the binary relation $Y_t \to (X_t \cup Y_t')$ such that $(u, v) \in$ trace($f$) for $u \in Y_t$ and $v \in X_t \cup Y_t$; if $u = v$ is an $\omega$-word, or there is a maximal path $p$ in $t$ whose label sequence is $u$ and a maximal path $p'$ or a path $p'$ to an exit vertex in $t'$ whose label sequence is $v$, and the image of $p$ under $f$ is a prefix of $p'$. It can be seen that trace is a morphism of cc$\Sigma$A. (Note that if $p$ is a maximal path in $t$ whose label sequence is the $\omega$-word $u$, then the image of $p$ under $f$ is such a path in $t'$.)

Thus, by Theorem 5.3, (0-)algebraic synchronization trees correspond to (0-)algebraic traces. However, there is no known characterization of these traces.

The last two examples may be enlarged by allowing more operations, such as (synchronous) parallel product, hiding, etc.

9. Summary

We have shown how to generalize the well-known Mezei–Wright theorem from continuous ordered $\Sigma$-algebras to continuous categorical $\Sigma$-algebras, capturing many more natural settings in which one wishes to obtain solutions of fixed point equations. Further, the result sometimes helps to characterize the algebraic elements in such cc$\Sigma$A’s.

Our proof of the Mezei–Wright theorem in the setting of cc$\Sigma$A’s is certainly more difficult than the one for ordered algebras, since the categories involved are more complex than those modeling ordered algebras, but the extra work pays off in that more examples are captured.

One of the referees asked us to compare the setting of cc$\Sigma$A’s with that considered by [25], and many papers by Adamek, Milius and Verebil (see e.g., [2–4] and the references therein). In the setting of [25], one works in a category $\mathcal{A}$ equipped with an endofunctor $H : \mathcal{A} \to \mathcal{A}$. Further, it is assumed that $\mathcal{A}$ has finite coproducts and, for every object $c$, the functor $x \mapsto Hx + c$ has a terminal coalgebra. (The present authors have frequently considered the case that the category $\mathcal{A}$ is a Lawvere theory.) As the authors put it, “the price we pay for working in such a general setting is that our theory takes somewhat more effort to build”. Indeed, in [25], it takes about 35 pages even before the definition of a recursive program scheme is given. This is not meant as a disparaging remark. There are payoffs for their general treatment, in that even more examples are covered. However, the Mezei–Wright result sought here does not seem to require such generality. Indeed, in the setting of Elgot algebras [4], morphisms of Elgot algebras preserve solutions of recursive program schemes, so that a version of the Mezei–Wright theorem holds by definition.

There are many connections between our setting and the one in [25]. We obtain “least solutions” in categorical algebras with colimits of $\omega$-diagrams. The existence of all of these terminal coalgebras in [25] can be guaranteed by the existence of enough colimits (see [2]).

Acknowledgments

The authors would like to thank each of the referees who made many helpful suggestions and corrected several slips. The second author was partially supported by grant OTKA K 75249.

References