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Bounds on graph eigenvalues II

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Abstract

We prove three results about the spectral radius $\mu(G)$ of a graph G:

(a) Let $T_r(n)$ be the r-partite Turán graph of order n. If G is a K_{r+1}-free graph of order n, then

$$\mu(G) < \mu(T_r(n))$$

unless $G = T_r(n)$.

(b) For most irregular graphs G of order n and size m,

$$\mu(G) - 2m/n > 1/(2m + 2n).$$

(c) Let $0 \leqslant k \leqslant l$. If G is a graph of order n with no $K_2 + \overline{K}_{k+1}$ and no $K_{2,l+1}$, then

$$\mu(G) \leqslant \min\left\{\Delta(G), \left(k - l + 1 + \sqrt{(k - l + 1)^2 + 4l(n - 1)}\right) \Big/ 2\right\}.$$

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1. Introduction

Our notation follows [1]; thus, we write $G(n)$ for a graph of order n and $\mu(G)$ for the maximum eigenvalue of the adjacency matrix of G.

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Write $T_r(n)$ for the r-partite Turán graph of order n and let $G = G(n)$. In [7] it is shown that if G is r-partite, then $\mu(G) < \mu(T_r(n))$ unless $G = T_r(n)$. On the other hand, Wilf [13] showed that if G is K_{r+1}-free, then $\mu(G) \leqslant (1 - 1/r)n$. We strengthen these two results as follows.

Theorem 1. *If $G = G(n)$ is a K_{r+1}-free graph, then $\mu(G) < \mu(T_r(n))$ unless $G = T_r(n)$.*

Next, let G be a graph of order n, size m, and maximum degree $\Delta(G) = \Delta$. One of the best known facts about $\mu(G)$ is the inequality $\mu(G) \geqslant 2m/n$, due to Collatz and Sinogowitz [4]. In [11] we gave upper and lower bounds on $\mu(G) - 2m/n$ in terms of degree deviation. In turn, Cioabă and Gregory [3] showed that, if G is irregular and $n \geqslant 4$, then $\mu - 2m/n > 1/(n\Delta + 2n)$. In this note we give another proof of this bound and improve it for most graphs.

Call a graph *subregular* if $\Delta(G) - \delta(G) = 1$ and all but one vertices have the same degree.

Theorem 2. *If G is an irregular graph of order $n \geqslant 4$ and size m, then*

$$\mu(G) - 2m/n > 1/(2m + 2n) \tag{1}$$

unless G is subregular. If G is subregular with $\Delta(G) = \Delta$, then

$$\mu(G) - 2m/n > 1/(n\Delta + 2n). \tag{2}$$

Finally, let $B_k = K_2 + \overline{K}_k$, i.e., the graph B_k consists of k triangles sharing an edge.

Let $0 \leqslant k \leqslant l \leqslant \Delta$. Shi and Song [12] showed that if $G = G(n)$ is a connected graph with $\Delta(G) = \Delta$, with no B_{k+1} and no $K_{2,l+1}$, then

$$\mu(G) \leqslant \left(k - l + \sqrt{(k-l)^2 + 4\Delta + 4l(n-1)}\right)\Big/ 2. \tag{3}$$

We extend this result as follows.

Theorem 3. *Let $0 \leqslant k \leqslant l$. If $G = G(n)$ is a graph with $\Delta(G) = \Delta$, with no B_{k+1} and no $K_{2,l+1}$, then*

$$\mu(G) \leqslant \min\left\{\Delta, \left(k - l + 1 + \sqrt{(k-l+1)^2 + 4l(n-1)}\right)\Big/ 2\right\}. \tag{4}$$

If G is connected, equality holds if and only if one of the following conditions holds:

(i) $\Delta^2 - \Delta(k - l + 1) \leqslant l(n-1)$ *and G is Δ-regular;*

(ii) $\Delta^2 - \Delta(k - l + 1) > l(n-1)$ *and every two vertices of G have k common neighbors if they are adjacent, and l common neighbors otherwise.*

We note without a proof that (4) implies (3).

2. Proofs

Proof of Theorem 1. Write $k_r(G)$ for the number of r-cliques of G. The following result is given in [10]: if G is K_{r+1}-free graph, then

$$\mu^r(G) \leqslant \sum_{s=2}^{r} (s-1) k_s(G) \mu^{r-s}(G). \tag{5}$$

According to a result of Zykov [14] (see also Erdős [5]), if the clique number of a graph G is r, then $k_s(G) < k_s(T_r(n))$ for every $2 \leqslant s \leqslant r$, unless $G = T_r(n)$. Assuming that $G \neq T_r(n)$, Zykov's theorem implies that $k_s(G) < k_s(T_r(n))$ for every $2 \leqslant s \leqslant r$. Hence, in view of (5), we have

$$\mu^r(G) < \sum_{s=2}^{r} (s-1)k_s(T_r(n))\mu^{r-s}(G).$$

This implies that $\mu(G) < x$, where x is the largest root of the equation

$$x^r = \sum_{s=2}^{r} (s-1)k_s(T_r(n))x^{r-s}. \tag{6}$$

It is known (see, e.g., [2, p. 74]) that (6) is the characteristic equation of the Turán graph; so, $\mu(G) < x = \mu(T_r(n))$, completing the proof. □

To simplify the proof of Theorem 2, we first prove inequality (1) for two special graphs.

Proposition 4. *Inequality* (1) *holds if G has $n-2$ vertices of degree $n-1$ and 2 vertices of degree $n-2$.*

Proof. Clearly, G is the complete graph of order n with one edge removed. Using the theorem of Finck and Grohmann [6] (see also [2, Theorem 2.8]), we find that

$$\mu(G) = \frac{n-3+\sqrt{n^2+2n-7}}{2}.$$

Hence, in view of $2m = n^2 - n - 2$, we obtain,

$$\mu(G) - \frac{2m}{n} = \frac{\sqrt{n^2+2n-7} - \left(n+1-\frac{4}{n}\right)}{2} = \frac{4n-8}{n^2\left(\sqrt{n^2+2n-7} + \left(n+1-\frac{4}{n}\right)\right)}$$
$$> \frac{4n-8}{n^2\left(n+1+\left(n+1-\frac{4}{n}\right)\right)} \geqslant \frac{2n-4}{n(n^2+n-2)} \geqslant \frac{1}{n^2+n-2} = \frac{1}{2m+2n},$$

completing the proof. □

Proposition 5. *Inequality* (1) *holds if G has $n-2$ vertices of degree $n-2$ and 2 vertices of degree $n-1$.*

Proof. We easily deduce that n is even, say $n = 2k$, and that G is the complement of a $(k-1)$-matching. Using the theorem of Finck and Grohmann, we find that

$$\mu(G) = \frac{n-3+\sqrt{n^2-2n+9}}{2}.$$

Hence, in view of $2m = n^2 - 2n + 2$, we obtain

$$\mu(G) - \frac{2m}{n} = \frac{\sqrt{n^2-2n+9} - \left(n-1+\frac{4}{n}\right)}{2} = \frac{4n-8}{n^2\left(\sqrt{n^2-2n+9} + \left(n-1+\frac{4}{n}\right)\right)}$$
$$> \frac{4n-8}{n^2\left(n+1+\left(n-1+\frac{4}{n}\right)\right)} = \frac{2n-4}{n(n^2+2)} \geqslant \frac{1}{n^2+2} = \frac{1}{2m+2n},$$

completing the proof. □

Proof of Theorem 2. Set $V = V(G)$, $\mu = \mu(G)$, and $\delta = \delta(G)$. Assume first that G is not subregular.

Proof of inequality (1)

Our proof is based on Hofmeister's inequality [9]: $\mu^2 \geqslant (1/n) \sum_{u \in V} d^2(u)$.

Case: $\Delta - \delta \geqslant 2$

In this case we easily see that

$$\sum_{u \in V} \left(d(u) - \frac{2m}{n} \right)^2 \geqslant 2 > \frac{2m}{m+n} + \frac{n}{4(m+n)^2},$$

and so,

$$\mu \geqslant \sqrt{\frac{1}{n} \sum_{u \in V} d^2(u)} = \sqrt{\frac{1}{n} \sum_{u \in V} \left(d(u) - \frac{2m}{n} \right)^2 + \frac{4m^2}{n^2}} > \frac{2m}{n} + \frac{1}{2m+2n},$$

as claimed. Thus, hereafter we shall assume that $\Delta - \delta = 1$.

Case: $\Delta - \delta = 1$

Letting k be the number of vertices of degree $\Delta = \delta + 1$, we have $2m/n = \delta + k/n$, and so,

$$\frac{1}{n} \sum_{u \in V} \left(d(u) - \frac{2m}{n} \right)^2 = \frac{n-k}{n} \left(\frac{k}{n} \right)^2 + \frac{k}{n} \left(\frac{n-k}{n} \right)^2 = \frac{k(n-k)}{n^2}.$$

Hence, if

$$\frac{k(n-k)}{n^2} > \frac{2m}{n(m+n)} + \frac{1}{4(m+n)^2}, \tag{7}$$

then inequality (1) follows as above. Assume for contradiction that (7) fails.

Suppose first that either $k = 2$ or $n - k = 2$. Since (7) fails, we see that

$$\begin{aligned} 2 - \frac{4}{n} = \frac{(n-2)2}{n} \leqslant \frac{k(n-k)}{n} \leqslant \frac{2m}{m+n} + \frac{n}{4(m+n)^2} \\ = 2 - \frac{2n}{m+n} + \frac{n}{4(m+n)^2}. \end{aligned} \tag{8}$$

In view of Propositions 4 and 5, we may assume that $\delta \leqslant n - 3$, and so,

$$2m = \delta n + k \leqslant \delta n + n - 2 \leqslant n^2 - 2n - 2.$$

Noting that (8) increases in m, we obtain

$$-\frac{4}{n^2} \leqslant -\frac{4}{n^2 - 2} + \frac{1}{(n^2-2)^2},$$

a contradiction for $n \geqslant 4$.

Finally, let $k \geqslant 3$ and $n - k \geqslant 3$; thus, $n \geqslant 6$. We have

$$2m = \delta n + k \leqslant \delta n + n - 3 \leqslant (n-2)n + n - 3.$$

By assumption inequality (7) fails; hence,

$$3 - \frac{9}{n} \leqslant \frac{(n-k)k}{n} \leqslant 2 - \frac{2n}{m+n} + \frac{n}{4(m+n)^2} \leqslant 2 - \frac{4n}{n^2+n-3} + \frac{n}{(n^2+n-3)^2}.$$

This inequality is a contradiction for $n \geqslant 6$, completing the proof of (7).

Proof of inequality (2) *when G is subregular*

As mentioned in the introduction, inequality (2) was first proved by Cioabă and Gregory [3]. For reader's convenience we present here a more direct proof, based on the same ideas – equitable partitions and interlacing.

Since G is subregular, it has either a single vertex of degree Δ or a single vertex of degree δ. Clearly, $\delta \geqslant 1$, and so, $m > n/2$.

Case: *G has a single vertex of degree* Δ

Setting $\Delta = k + 1$ and

$$c = \frac{nk+1}{n} + \frac{1}{n(k+3)} = k + \frac{k+4}{n(k+3)},$$

in view of $2m = nk + 1$, inequality (2) amounts to $\mu > c$.

Select a vertex $u \in V$ with $d(u) = k + 1$; partition V as $V = \{u\} \cup V\backslash\{u\}$ and let B be the quotient matrix of this partition (see, e.g. [8, chapter 9]), i.e.,

$$B = \begin{pmatrix} 0 & \frac{k+1}{n-1} \\ k+1 & k - \frac{k+1}{n-1} \end{pmatrix}.$$

Writing $P(x)$ for the characteristic polynomial of B and observing that $k \leqslant n - 2$, we have

$$\begin{aligned} P(c) &= \left(k + \frac{k+4}{n(k+3)}\right)\left(k + \frac{k+4}{n(k+3)} - \left(k - \frac{k+1}{n-1}\right)\right) - \frac{(k+1)^2}{n-1} \\ &= k\frac{k+4}{n(k+3)} + \frac{1}{n^2}\left(\frac{k+4}{k+3}\right)^2 + \left(\frac{k+4}{n(k+3)}\right)\frac{k+1}{n-1} - \frac{k+1}{n-1} \\ &= -\frac{3}{n(k+3)} + \frac{1}{n^2}\left(\frac{k+4}{k+3}\right)^2 + \frac{(k+1)}{n(n-1)(k+3)} \\ &= \frac{1}{n^2(k+3)}\left(-3n + 2k + 6 + \frac{1}{k+3} + \frac{k+1}{n-1}\right) \\ &\leqslant \frac{1}{n^2(k+3)}\left(-3n + 2(n-2) + 6 + \frac{1}{4} + 1\right) < 0. \end{aligned}$$

By interlacing, $P(\mu) \geqslant 0 > P(c)$, and so $\mu > c$, completing the proof of (2) in this case.

Case: *G has a single vertex of degree* δ

Setting $\Delta = k$ and

$$c = \frac{nk-1}{n} + \frac{1}{n(k+3)} = k - \frac{k+1}{n(k+2)},$$

in view of $2m = nk - 1$, inequality (2) amounts to $\mu > c$.

Select $u \in V$ with $d(u) = k - 1$; partition V as $V = \{u\} \cup V\backslash\{u\}$ and let B be the quotient matrix of this partition, i.e.,

$$B = \begin{pmatrix} 0 & \frac{k-1}{n-1} \\ k-1 & k - \frac{k-1}{n-1} \end{pmatrix}.$$

Writing $P(x)$ for the characteristic polynomial of B and observing that $k \leqslant n - 2$, we have

$$P(c) = \left(k - \frac{k+1}{n(k+2)}\right)\left(k - \frac{k+1}{n(k+2)} - \left(k - \frac{k-1}{n-1}\right)\right) - \frac{(k-1)^2}{n-1}$$

$$
\begin{aligned}
&= -\frac{k(k+1)}{n(k+2)} + \frac{1}{n^2}\left(\frac{k+1}{k+2}\right)^2 + \frac{k-1}{n(n-1)(k+2)} + \frac{k-1}{n}\\
&= -\frac{2}{n(k+2)} + \frac{1}{n^2}\left(\frac{k+1}{k+2}\right)^2 + \frac{k-1}{n(n-1)(k+2)}\\
&= \frac{1}{n^2(k+2)}\left(-2n+2k+1+\frac{1}{k+2}+\frac{k-1}{(n-1)}\right)\\
&< \frac{1}{n^2(k+2)}\left(-2n+2(n-2)+1+\frac{1}{1+2}+1\right) < 0.
\end{aligned}
$$

By interlacing, $P(\mu) \geqslant 0 > P(c)$, completing the proof of (2). □

Proof of Theorem 3. Set $V = V(G)$ and $\mu = \mu(G)$; given $u \in V$, write $\Gamma(u)$ for the set of neighbors of u. Select $u \in V$; let $A = \Gamma(u)$, $B = V\backslash(\Gamma(u) \cup \{u\})$, and $e(A, B)$ be the number of $A - B$ edges. Since G contains no B_{k+1} and no $K_{2,l+1}$, we see that

$$\sum_{v\in A}(d(v)-k-1) \leqslant \sum_{v\in A}|\Gamma(v)\cap B| = e(A,B) = \sum_{v\in A}|\Gamma(v)\cap A| \leqslant (n-d(u)-1)\,l. \qquad (9)$$

Letting A be the adjacency matrix of G, note that the uth row sum of the matrix

$$C = A^2 - (k+1-l)A - (n-1)lI_n$$

is equal to

$$\sum_{v\in A}(d(v)-k-1) - (n-1-d(u))l;$$

consequently, all row sums C are nonpositive. Letting $\mathbf{x} = (x_1, \ldots, x_n)$ be an eigenvector of A to μ, we see that the value

$$\lambda = \mu^2 - (k+1-l)\mu - (n-1)l$$

is an eigenvalue of C with eigenvector $\mathbf{x}$. Therefore, $\lambda \leqslant 0$, and so,

$$\mu \leqslant \left(k-l+1+\sqrt{(k-l+1)^2+4l(n-1)}\right)\Big/ 2,$$

completing the proof of inequality (4).

Let equality hold in (4) and G be connected; thus, the eigenvector $\mathbf{x} = (x_1, \ldots, x_n)$ to μ is positive. We shall prove the necessity of conditions (i) and (ii). If

$$\mu = \Delta \leqslant \left(k-l+1+\sqrt{(k-l+1)^2+4l(n-1)}\right)\Big/ 2,$$

then $\Delta^2 - \Delta(k-l+1) \leqslant l(n-1)$ and G is Δ-regular.

On the other hand, if

$$\mu = \left(k-l+1+\sqrt{(k-l+1)^2+4l(n-1)}\right)\Big/ 2 < \Delta,$$

then $\Delta^2 - \Delta(k-l+1) > l(n-1)$ and $\lambda = 0$. Scaling $\mathbf{x}$ so that $x_1 + \cdots + x_n = 1$, we see that λ is a convex combination of the row sums of C which are nonpositive; thus, all row sums of C are 0. Since equality holds in (9) for every $u \in [n]$, every two vertices have exactly k common neighbors if they are adjacent, and exactly l common neighbors otherwise. This completes the proof. □

3. Concluding remarks

Finding tight bounds on the spectral radius of subregular graphs is a challenging problem. Specifically, we cannot determine for which subregular graphs G one has

$$\mu(G) > \frac{2m}{n} + \frac{1}{2m+2n}.$$

Note that strongly regular graphs satisfy condition (ii) for equality in (4), but irregular graphs can satisfy this condition as well, e.g., the star $K_{1,n-1}$ and the friendship graph.

Finally, setting $l = \Delta$ or $k = 0$, Theorem 3 implies assertions that strengthen Corollaries 1 and 2 of [12].

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References

[1] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, Springer-Verlag, New York, 1998, xiv+394 pp.
[2] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980, 368 pp.
[3] S.M. Cioabă, D. Gregory, Large matchings from eigenvalues, Linear Algebra Appl. 422 (2007) 308–317.
[4] L. Collatz, U. Sinogowitz, Spektren endlicher Grafen, Abh. Math. Sem. Univ. Hamburg 21 (1957) 63–77.
[5] P. Erdős, On the number of complete subgraphs contained in certain graphs, Publ. Math. Inst. Hung. Acad. Sci. VII (Ser. A3) (1962) 459–464.
[6] H.J. Finck, G. Grohmann, Vollständiges Produkt, chromatische Zahl und charakteristisches Polynom regulärer Graphen. I. (German), Wiss. Z. Techn. Hochsch. Ilmenau 11 (1965) 1–3.
[7] L. Feng, Q. Li, X.-D. Zhang, Spectral radii of graphs with given chromatic number, Appl. Math. Lett. 20 (2007) 158–162.
[8] C. Godsil, G. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics, Springer-Verlag, New York, 2001, xx+439 pp.
[9] M. Hofmeister, Spectral radius and degree sequence, Math. Nachr. 139 (1988) 37–44.
[10] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, Combin. Probab. Comput. 11 (2002) 179–189.
[11] V. Nikiforov, Eigenvalues and degree deviation in graphs, Linear Algebra Appl. 414 (2006) 347–360.
[12] L. Shi, Z. Song, Upper bounds on the spectral radius of book-free and/or $K_{2,l+1}$ graphs, Linear Algebra Appl. 420 (2007) 526–529.
[13] H. Wilf, Spectral bounds for the clique and independence numbers of graphs, J. Combin. Theory Ser. B 40 (1986) 113–117.
[14] A.A. Zykov, On some properties of linear complexes (in Russian), Mat. Sbornik N.S. 24 (66) (1949) 163–188.