Available online at www.sciencedirect.com

ScienceDirect

Journal of Computational and Applied Mathematics 222 (2008) 690–700

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Gauss–Lobatto to Bernstein polynomials transformation

Loredana Coluccio, Alfredo Eisinberg, Giuseppe Fedele*

Dip. Elettronica Informatica e Sistemistica, Università degli Studi della Calabria, 87036, Rende (Cs), Italy

Received 2 July 2007; received in revised form 3 September 2007

Abstract

The aim of this paper is to transform a polynomial expressed as a weighted sum of discrete orthogonal polynomials on Gauss–Lobatto nodes into Bernstein form and vice versa. Explicit formulas and recursion expressions are derived. Moreover, an efficient algorithm for the transformation from Gauss–Lobatto to Bernstein is proposed. Finally, in order to show the robustness of the proposed algorithm, experimental results are reported.

Keywords: Orthogonal polynomials; Gauss–Lobatto nodes; Bernstein polynomials

1. Introduction

Frequently, in many application fields it is necessary to model an unknown function $f(t)$, only available on a finite grid $G = \{x_i\}_{i=1}^{n}$ of distinct points, by a linear combination of basis functions $\left\{\phi_j(t)\right\}_{j=1}^{m}$:

$$p(t) = \sum_{j=1}^{m} c_j \phi_j(t). \tag{1}$$

In many practical situations, the measurements of $f(t)$ should be affected by noise. In these cases a high degree of $p(t)$ is not convenient, then $m \ll n$ is chosen. It is well known that this choice effectively reduces the influence of the random errors in measurements [1], but requires an appropriate selection of m and n. For example, a large value of n guarantees that, over the grid of measurement points, the variance of the smoothed function values is $\gamma^2 m/n$, where $\gamma^2 I$ is the covariance matrix of the estimated c_j^* [1]. The case of $m = n$ corresponds to the "interpolation problem" in which it is required that a $(n-1)$ degree polynomial satisfies

$$p(x_i) = \sum_{j=1}^{n} c_j \phi(x_j) = f(x_i), \quad i = 1, 2, \ldots, n. \tag{2}$$

* Corresponding author. Tel.: +39 0984 494720; fax: +39 0984 494713.
E-mail address: fedele@si.deis.unical.it (G. Fedele).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.12.007

Usually, discrete orthogonal polynomials are used to find coefficients c_j in the least-squares sense, since they are easy to manipulate, have good convergence properties, and give a well-conditioned representation of a function, [1]. Every type of polynomial basis, such as the power, Bernstein, Jacobi, Hermite, Legendre, has its strength and advantages and sometimes it has some disadvantages; by the appropriate choice of the basis many problems can be solved and many difficulties can be removed. The choice of the basis to be used depends on the problem to be solved. The polynomials determined in the Bernstein basis enjoy considerable popularity in many different applications. For example in computer-aided design (CAD) applications [2,3], in finding the roots of a transcendental function $f(x)$ on an interval $[a, b]$ [4], in the approximation theory of functions defined over finite domains, in finding solutions of systems of polynomial equations [5], in many interesting control system designs, analysis problems and in robust analysis of linear dynamic systems [6–10]. Since Bernstein polynomials are not orthogonal, they are not convenient to use in the least-squares approximation. For that reason, it could be interesting to combine the superior performance of the least-squares of some orthogonal polynomials with the geometrical insight of the Bernstein polynomial basis. With this aim, many authors have proposed different papers in relation to the transformation of one basis into Bernstein polynomial basis, [11–14]. Nevertheless, it is well known that mapping from one basis to another and vice versa, is usually an ill-conditioned problem, therefore there is a growing interest in finding explicit formulas for such transformations and developing numerically robust and stable algorithms. We would like to stress that it is not possible to derive a closed form for the transformation matrix from Legendre to Bernstein [11], from Chebyshev to Bernstein [12], and from Jacobi to Bernstein [13]. In fact, the expression of the entries of the transformation matrices involves a summation that has a closure in terms of Hypergeometric functions, [15], not easy to manage. Vice versa for the orthogonal polynomials on Gauss–Lobatto nodes, [16–18], explicit transformation formulas, expressed as the product of matrices whose entries are characterized by closed expressions, will be presented. In this paper, transformation matrices, mapping the Gauss–Lobatto and Bernstein forms of a degree $(m - 1)$ polynomial into each other, are derived and examined. The choice of the Gauss–Lobatto nodes is due to the high use in numerical applications such as in polynomial interpolation [17], approximation theory [16,18], spectral methods [19,20], method for estimating the length of a parametric curve using only samples of points [21], method for estimating surface area [22], quadrature formulas, etc. Their nonuniform distribution (with highest density towards the end-points) gives the least-interpolation error in the L^2-norm. Considering the high use of the Gauss–Lobatto nodes it is seemed natural to work on the discrete orthogonal polynomials defined on such nodes proposed in [16]. Another good reason to use them is that the condition number of this transformation grows at a significantly slower rate than the Chebyshev–Bernstein conversion basis, as it will be showed in what follows. The rest of this paper is organized as follows: in Section 2 a brief review on the Bernstein polynomials is reported; in Section 3 the discrete orthogonal polynomials on Gauss–Lobatto nodes are introduced; in Section 4 the main results are summarized and an algorithm for an efficient forward mapping is proposed; numerical tests and comparison are shown in Section 5; summary and conclusions are reported in Section 6; finally, an Appendix section contains the proofs of some results.

2. Bernstein polynomials

In this section some definitions and formulas for Bernstein polynomials are summarized. For every natural number m, the Bernstein polynomials of degree $(m - 1)$ on [0, 1] are defined by

$$B_k^m(x) = \binom{m-1}{k-1} x^{k-1}(1-x)^{m-k}, \quad k = 1, \ldots, m. \tag{3}$$

Let M_B be the coefficient matrix of polynomials $B_k^m(x)$, $k = 1, \ldots, m$, where $M_B(i, j)$ is the coefficient of the polynomial $B_i^m(x)$ with respect to the monomial x^{j-1}, then by Eq. (3) it is possible to obtain[1]

$$M_B(i, j) = (-1)^{i+j} \binom{m-1}{i-1} \binom{m-i}{j-i}, \quad i = 1, \ldots, m, \ j = i, \ldots, m. \tag{4}$$

[1] For the sake of brevity in the notation, the non-specified entries in a matrix will be assumed to be zero.

By the transformation of a polynomial from its power form into its Bernstein form [7], results that :

$$\begin{cases} x^k = \displaystyle\sum_{i=k}^{m} \frac{\binom{i-1}{k}}{\binom{m-1}{k}} B_i^m(x), \quad k = 1, \ldots, m-1, \\ 1 = \displaystyle\sum_{i=1}^{m} B_i^m(x) \end{cases} \tag{5}$$

then the generic entry of the inverse of M_B is

$$M_B^{-1}(i, j) = \frac{\binom{j-1}{i-1}}{\binom{m-1}{i-1}}, \quad i = 1, \ldots, m, \ j = i, \ldots, m. \tag{6}$$

3. Gauss–Lobatto polynomials

Gauss–Lobatto polynomials [16] are discrete orthogonal polynomials over the so-called Gauss–Lobatto–Chebyshev points [17]:

$$x_k = -\cos\left(\frac{k-1}{n-1}\pi\right), \quad k = 1, 2, \ldots, n, \tag{7}$$

with respect to the inner product

$$\langle f, g \rangle = \sum_{k=1}^{n} f(x_k) g(x_k). \tag{8}$$

The generic Gauss–Lobatto polynomial $P(n, k, x)$ of degree $(k-1)$ on the interval $[-1, 1]$ has the following explicit expression

$$\begin{aligned} P(n, k, x) = (n + k - 3)x^{k-1} + \sum_{q=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^q \frac{1}{q 2^{2q}} \binom{k-q-2}{q-1} \\ \times \left[(k-1)n + (k-1)(k-3) + 2q\right] x^{k-2q-1}. \end{aligned} \tag{9}$$

And thus the first three polynomials are given by

$$\begin{aligned} &P(n, 1, x) = n - 2, \\ &P(n, 2, x) = (n - 1)x, \\ &P(n, 3, x) = nx^2 - \frac{n+1}{2}. \end{aligned} \tag{10}$$

Gauss–Lobatto polynomials also satisfy the orthogonality conditions

$$\begin{aligned} &\langle P(n, 1, x), P(n, 1, x) \rangle = n(n-2)^2, \\ &\langle P(n, k, x), P(n, k, x) \rangle = \frac{(n-1)(n+k-1)(n+k-3)}{2^{2k-3}}, \quad k = 2, 3, \ldots, n-1, \\ &\langle P(n, n, x), P(n, n, x) \rangle = \frac{(n-1)^2(2n-3)}{2^{2n-5}}, \end{aligned} \tag{11}$$

and the three-terms relation

$$P(n, k, x) = \alpha_k x P(n, k-1, x) + \gamma_k P(n, k-2, x), \quad k = 4, 5, \ldots, n, \tag{12}$$

where

$$\begin{aligned} \alpha_k &= \frac{n+k-3}{n+k-4}, \qquad k = 4, 5, \ldots, n, \\ \gamma_k &= -\frac{n+k-2}{4(n+k-4)}, \;\; k = 4, 5, \ldots, n. \end{aligned} \tag{13}$$

Let $C_P^{[-1,1]}$ be the coefficient matrix of polynomials $P(n, i, x),\ i = 1, \ldots, m$ with $m \le n$, where $C_P^{[-1,1]}(i, j)$ is the coefficient of the polynomial $P(n, i, x)$ with respect to the monomial x^{j-1}, then by Eq. (9) it follows that

$$\begin{cases} C_P^{[-1,1]}(2i, 2j) = \dfrac{(-1)^{i+j}}{(i-j)2^{2i-2j}} \dbinom{i+j-2}{i-j-1} \\ \quad \times [(2i-1)n + (2i-1)(2i-3) + 2(i-j)], \quad i = 1, \ldots, \left\lfloor \dfrac{m}{2} \right\rfloor, \; j = 1, \ldots, i-1, \\ C_P^{[-1,1]}(2i-1, 2j-1) = \dfrac{(-1)^{i+j}}{(i-j)2^{2i-2j}} \dbinom{i+j-3}{i-j-1} \\ \quad \times [(2i-2)n + (2i-2)(2i-4) + 2(i-j)], \quad i = 1, \ldots, \left\lceil \dfrac{m}{2} \right\rceil, \; j = 1, \ldots, i-1, \\ C_P^{[-1,1]}(i, i) = n + i - 3, \quad i = 1, \ldots, m. \end{cases} \tag{14}$$

Proposition 1. *The matrix $C_P^{[-1,1]}$ can be factorized as the product of two matrices C_1 and C_2 where*

$$\begin{cases} C_1(2i, 2i) = n + 2i - 3, & i = 1, \ldots, \left\lfloor \dfrac{m}{2} \right\rfloor, \\ C_1(2i-1, 2i-1) = n + 2i - 4, & i = 1, \ldots, \left\lceil \dfrac{m}{2} \right\rceil, \\ C_1(2i, 2j) = -\dfrac{1}{2^{2i-2j-1}}, & i = 2, \ldots, \left\lfloor \dfrac{m}{2} \right\rfloor, \; j = 1, \ldots, i-1, \\ C_1(2i-1, 2j-1) = -\dfrac{1}{2^{2i-2j-1}}, & i = 2, \ldots, \left\lceil \dfrac{m}{2} \right\rceil, \; j = 1, \ldots, i-1 \end{cases} \tag{15}$$

and

$$\begin{cases} C_2(1, 1) = 1, & \\ C_2(2i, 2j) = (-1)^{i+j} \dfrac{(2i-1)}{2^{2i-2j}(2j-1)} \dbinom{i+j-2}{i-j}, & i = 1, \ldots, \left\lfloor \dfrac{m}{2} \right\rfloor, \; j = 1, \ldots, i, \\ C_2(2i-1, 1) = (-1)^{i+1} \dfrac{1}{2^{2i-3}}, & i = 2, \ldots, \left\lceil \dfrac{m}{2} \right\rceil \\ C_2(2i-1, 2j-1) = (-1)^{i+j} \dfrac{(2i-2)}{2^{2i-2j}(2j-2)} \dbinom{i+j-3}{i-j}, & i = 2, \ldots, \left\lceil \dfrac{m}{2} \right\rceil, \; j = 2, \ldots, i. \end{cases} \tag{16}$$

Proof. For the sake of brevity, we will consider the proof only for the elements in even positions. Let Γ be the product of C_1 and C_2, then $\Gamma(2i, 2j)$ can be expressed as

$$\begin{aligned} \Gamma(2i, 2j) &= \sum_{k=1}^{i} C_1(2i, 2k) C_2(2k, 2j) \\ &= \sum_{k=1}^{i-1} C_1(2i, 2k) C_2(2k, 2j) + C_1(2i, 2i) C_2(2i, 2j). \end{aligned} \tag{17}$$

We will show that Eq. (17) reduces to the first expression in Eq. (14). By considering the explicit expressions of $C_1(2i, 2k)$ and $C_2(2k, 2j)$ involved in Eq. (17) and using standard arguments it follows that

$$\Gamma(2i,2j) = \frac{(-1)^j}{2^{2i-2j}(2j-1)} \times \left[(-1)^i(n+2i-3)(2i-1)\binom{i+j-2}{i-j} - 2\sum_{k=1}^{i-1}(-1)^k(2k-1)\binom{k-j-2}{k-j}\right]. \quad (18)$$

The summation involved in Eq. (18) has an explicit closure, according to Lemma 1 (see Appendix A), then Eq. (18) becomes

$$\Gamma(2i,2j) = \frac{(-1)^{i+j}}{2^{2i-2j}}\left[\frac{(2i-1)(n+2i-3)}{2j-1}\binom{i+j-2}{i-j} + 2\binom{i+j-2}{2j-1}\right]. \quad (19)$$

Finally, by rearranging Eq. (19), it is effortless to obtain $\Gamma(2i,2j) = C_P^{[-1,1]}(2i,2j)$. The same approach can be used for the proof of elements in odd positions. ■

The explicit expressions of the entries of C_1^{-1} and C_2^{-1} are also available (see Proposition 3 in Appendix A).

Another result, which will be useful in the following, is the matrix transformation, namely T, that maps the Gauss–Lobatto polynomials from the interval $[-1, 1]$ in $[0, 1]$.

Proposition 2. *Let $C_P^{[0,1]}$ be the coefficient matrix of polynomials $P(n,i,2x-1)$, $i = 1,2,\ldots,m$, then*

$$C_P^{[0,1]} = C_P^{[-1,1]}T, \quad (20)$$

where

$$T(i,j) = (-1)^{i+j}2^{j-1}\binom{i-1}{i-j}, \quad i = 1,\ldots,m, \; j = 1,\ldots,i. \quad (21)$$

Proof. Let us consider, for brevity, the polynomials $P(n,2k,2x-1)$, $k = 1,\ldots,\lfloor \frac{m}{2} \rfloor$ of degree $(2k-1)$. Since that polynomials are obtained from the Gauss–Lobatto polynomials shifted on the interval $[0, 1]$, obviously they can be written in terms of the elements of the matrix $C_P^{[-1,1]}$ as:

$$P(n,2k,2x-1) = \sum_{q=1}^{k} C_P^{[-1,1]}(2k,2q)(2x-1)^{2q-1}. \quad (22)$$

Expanding the term $(2x-1)^{2q-1}$ by using the binomial theorem [23]

$$(2x-1)^{2q-1} = \sum_{s=1}^{2q}(-1)^s 2^{s-1}\binom{2q-1}{s-1}x^{s-1}, \quad (23)$$

the polynomial $P(n,2k,2x-1)$ can be expressed as:

$$P(n,2k,2x-1) = \sum_{q=1}^{k}\sum_{s=1}^{2q}(-1)^s 2^{s-1}\binom{2q-1}{s-1} C_P^{[-1,1]}(2k,2q)x^{s-1}. \quad (24)$$

Observing Eqs. (21) and (24), it is evident that the coefficient of the polynomial $P(n,2k,2x-1)$ with respect to the monomial x^{s-1}, namely $C_P^{[0,1]}(2k,s)$, can be rewritten as:

$$C_P^{[0,1]}(2k,s) = \sum_{q=1}^{k} C_P^{[-1,1]}(2k,2q)T(2q,s). \quad ■ \quad (25)$$

Remark 1. By using the properties of binomial coefficients [23], the inverse of T is

$$T^{-1}(i,j) = \frac{\binom{i-1}{j-1}}{2^{i-1}}, \quad i = 1,\ldots,m, \; j = 1,\ldots,i. \quad (26)$$

Since $P(n,k,x),\ k=1,\ldots,m$ is a set of orthogonal polynomials over the set of nodes (7), under an affine transformation $x \to 2x-1$ they remain orthogonal over the new set of nodes $\hat{x}_k = \frac{x_k+1}{2},\ k=1,\ldots,n$. Under such transformation the first three polynomials become

$$\begin{aligned} P(n,1,2x-1) &= n-2, \\ P(n,2,2x-1) &= 2(n-1)x-(n-1), \\ P(n,3,2x-1) &= 4nx^2-4nx+\frac{n-1}{2}, \end{aligned} \tag{27}$$

and they satisfy the three-terms recurrence relation

$$P(n,k,2x-1) = \alpha_k(2x-1)P(n,k-1,2x-1)+\gamma_k P(n,k-2,2x-1), \quad k=4,5,\ldots,n. \tag{28}$$

For details on the Gauss–Lobatto polynomials and accurate algorithms for the least-square problem on the Gauss–Lobatto nodes we refer the reader to [16,18].

4. Main results

Our aim is to transform a polynomial expressed as a weighted sum of discrete orthogonal polynomials on the Gauss–Lobatto nodes into Bernstein form and vice versa:

$$\sum_{k=1}^{m} g_k P(n,k,2x-1) = \sum_{k=1}^{m} b_k B_k^m(x). \tag{29}$$

The following Theorem gives a factorized form of the transformation matrices $T_{GL\to B}$ from the Gauss–Lobatto polynomials $P(n,k,2x-1),\ k=1,\ldots,m$ into Bernstein polynomial basis and its inverse, $T_{B\to GL}$.

Theorem 1.

$$T_{GL\to B} = C_1C_2TM_B^{-1}, \tag{30}$$

$$T_{B\to GL} = M_BT^{-1}C_2^{-1}C_1^{-1}. \tag{31}$$

Proof. By using the formula (5), it is straightforward to express the ith Gauss–Lobatto polynomial $P(n,i,2x-1)$ in Bernstein basis form $B_q^m(x),\ q=1,2,\ldots,m$ as

$$P(n,i,2x-1) = \sum_{q=1}^{m} \rho_{i,q} B_q^m(x), \quad i=1,2,\ldots,m, \tag{32}$$

where

$$\rho_{i,q} = \sum_{j=1}^{m} v_i(j) M_B^{-1}(j,q) \tag{33}$$

and

$$v_i(j) = C_P^{[0,1]}(i,j), \quad j=1,\ldots,m. \tag{34}$$

Taking into account Propositions 1 and 2, Eqs. (30) and (31) follow. ■

Since the problem of mapping from one basis to another is ill-conditioned, an effort to develop numerically robust and stable algorithm, was done looking for: explicit formulas, recursion expressions for a simple construction of the matrices involved in the transformation operations and cunning solutions to avoid severe numerical errors. In what follows an algorithm for an efficient mapping is proposed, but first of all some properties on the involved matrices in such mapping are reported.

Observing Eq. (30), the only involved matrices are $C_P^{[0,1]}$ and M_B^{-1}. The matrix $C_P^{[0,1]}$ can be decomposed as

$$C_P^{[0,1]} = nG_1 + G_2, \tag{35}$$

where

$$\begin{aligned} &G_1 = C_2 T, \\ &G_2 = C_1|_{n=0} C_2 T. \end{aligned} \tag{36}$$

The matrices G_1 and G_2 can be effectively constructed by using the three-terms recurrence relations satisfied by the Gauss–Lobatto polynomials. The first four rows of G_1 are set by inspection as

$$\begin{cases} G_1(1,1) = 1, \\ G_1(2,1) = -1, \quad G_1(2,2) = 2, \\ G_1(3,1) = \dfrac{1}{2}, \quad G_1(3,2) = -4, \quad G_1(3,3) = 4, \\ G_1(4,1) = -\dfrac{1}{4}, \quad G_1(4,2) = \dfrac{9}{2}, \quad G_1(4,3) = -12, \quad G_1(4,4) = 8, \end{cases} \tag{37}$$

and

$$\begin{cases} G_1(i,1) = -\dfrac{1}{4} G_1(i-2,1) - G_1(i-1,1), & i = 5, \dots, m, \\ G_1(i,i) = 2G_1(i-1,i-1), & i = 5, \dots, m, \\ G_1(i,j) = 2G_1(i-1,j-1) - G_1(i-1,j) - \dfrac{1}{4} G_1(i-2,j), & i = 5, \dots, m, \\ & j = 2, \dots, i-1. \end{cases} \tag{38}$$

Following the same line for the construction of the matrix G_1, G_2 can be obtained by

$$\begin{cases} G_2(1,1) = -2, \\ G_2(2,1) = 1, \quad G_2(2,2) = -2, \\ G_2(3,1) = -\dfrac{1}{2}, \\ G_2(4,1) = \dfrac{1}{4}, \quad G_2(4,2) = \dfrac{7}{2}, \quad G_2(4,3) = -12, \quad G_2(4,4) = 8, \end{cases} \tag{39}$$

and

$$\begin{cases} G_2(i,1) = -\dfrac{i-2}{4(i-4)} G_2(i-2,1) - \dfrac{i-3}{i-4} G_2(i-1,1), & i = 5, \dots, m, \\ G_2(i,i) = \dfrac{2(i-3)}{i-4} G_2(i-1,i-1), & i = 5, \dots, m, \\ G_2(i,j) = \dfrac{i-3}{i-4} \left(2G_2(i-1,j-1) - G_2(i-1,j)\right) - \dfrac{i-2}{4(i-4)} G_2(i-2,j), & i = 5, \dots, m, \\ & j = 2, \dots, i-1. \end{cases} \tag{40}$$

The matrices G_1 and G_2, can be made integer through pre-multiplication by the diagonal matrix D_1 defined as:

$$\begin{cases} D_1(1,1) = 1, \quad D_1(2,2) = 1, \\ D_1(i,i) = 2^{i-2}, \quad i = 3, \dots, m. \end{cases} \tag{41}$$

Moreover, the matrix M_B^{-1} can be factorized as the product of an integer matrix M_{B_1}, and a diagonal matrix M_{B_2},

$$M_{B_1}(i,j) = \binom{m-i}{j-i}, \quad i = 1, \dots, m, \ j = i, \dots, m, \tag{42}$$

$$M_{B_2}(i,j) = \frac{1}{\binom{m-1}{i-1}} \delta_{i,j}, \quad i, j = 1, \dots, m, \tag{43}$$

Table 1
Maximum and mean value of ϵ, 100 000 experiments, $n = 10\,000$, $g \in [-1, 1]$

m	$\bar{T}_{GL\to B}$		$T_{GL\to B}$	
	$\epsilon_{\max}$	ϵ_{mean}	$\epsilon_{\max}$	ϵ_{mean}
20	3.46−10	1.29−14	4.35−04	1.20−07
19	4.20−10	1.14−14	2.97−03	8.94−08
18	2.51−11	7.71−15	2.33−04	1.72−08
17	1.22−10	1.00−14	2.59−04	9.43−09
16	8.14−11	9.91−15	2.70−04	5.11−09
15	2.92−11	6.51−15	5.73−05	1.24−09
14	6.17−09	6.92−14	2.93−06	2.47−10
13	4.61−11	6.03−15	8.39−07	8.79−11
12	7.46−11	7.72−15	5.59−07	3.63−11
11	1.25−10	6.24−15	8.91−08	1.08−11
10	5.10−11	5.13−15	7.02−08	5.60−12

where

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}. \end{cases}$$

Note that M_{B_2} can be made integer multiplying by $(m-1)!$. Let $\hat{M}_{B_2}$ be the matrix defined as

$$\hat{M}_{B_2} = (m-1)!\,M_{B_2}. \tag{44}$$

On the base of the previous observations, Eq. (30) assumes the form

$$\bar{T}_{GL\to B} = \frac{1}{(m-1)!} D_1^{-1} \left(n D_1 G_1 + D_1 G_2\right) M_{B_1} \hat{M}_{B_2}. \tag{45}$$

5. Numerical properties

To stress the effectiveness of the proposed formula (45), 100 000 experiments have been run, with $n = 10\,000$ for different values of $m \in [10, 20]$. In each iteration, coefficients g_k in Eq. (29), were generated with uniform distribution in the interval $[-1, 1]$. We have used package Matlab [24] to compute the numerical solutions $\hat{b}^{\mathrm{T}} = g^{\mathrm{T}} T_{GL\to B}$, $\hat{b}^{\mathrm{T}} = g^{\mathrm{T}} \bar{T}_{GL\to B}$, and the package Mathematica [25] (using extended precision) for the exact one and for the error

$$\epsilon = \max_{1\le i\le m} \frac{|\hat{b}_i - b_i|}{|b_i|}. \tag{46}$$

We would like to underline that the numerical results are extremely sensitive to the execution order of operations involved in Eq. (45). Our results were obtained by using the following three steps:

1. build matrices $Q_1 = D_1 G_1$, $Q_2 = D_1 G_2$ and the vector $q^{\mathrm{T}} = g^{\mathrm{T}} D_1^{-1}$,
2. build the matrix $G = n Q_1 M_{B_1} + Q_2 M_{B_1}$,
3. form the vector $\hat{b} = \frac{1}{(m-1)!}\left[\left(q^{\mathrm{T}} G\right) \hat{M}_{B_2}\right]$.

Table 1 reports the maximum and the mean value of (46) obtained by calculating the coefficients $\hat{b}$ by using Eq. (30) and those which were obtained by Eq. (45) following the suggested execution order. By comparing the results, it is possible to observe how much improvement in roundoff error is obtained by the proposed procedure. The required conversion from the Gauss–Lobatto form to the Bernstein one is ill-conditioned for large value of m. However, for $m \le 20$, the proposed algorithm seems to be accurate and robust as it is evident from results in Table 1. In Table 2, we report the maximum and the mean value of (46) for the interpolation problem on the Gauss–Lobatto nodes ($m = n \in [5, 10]$). Also in this case, comparisons between the numerical results obtained by $\hat{b}^{\mathrm{T}} = g^{\mathrm{T}} T_{GL\to B}$ and $\hat{b}^{\mathrm{T}} = g^{\mathrm{T}} \bar{T}_{GL\to B}$ put in evidence the effectiveness of the proposed algorithm. It is well known that the conversion

Table 2
Maximum and mean value of ϵ, 100 000 experiments, $m = n$, $g \in [-1, 1]$

m	$\bar{T}_{GL\to B}$		$T_{GL\to B}$	
	$\epsilon_{\max}$	ϵ_{mean}	$\epsilon_{\max}$	ϵ_{mean}
10	1.04−10	4.88−15	6.37−08	7.02−12
9	7.61−12	2.93−15	4.69−06	4.89−11
8	4.32−11	3.83−15	2.28−08	9.41−13
7	8.63−11	3.77−15	6.04−09	3.63−13
6	1.42−11	2.21−15	4.64−10	9.72−14
5	4.72−11	1.66−15	1.34−10	2.69−14

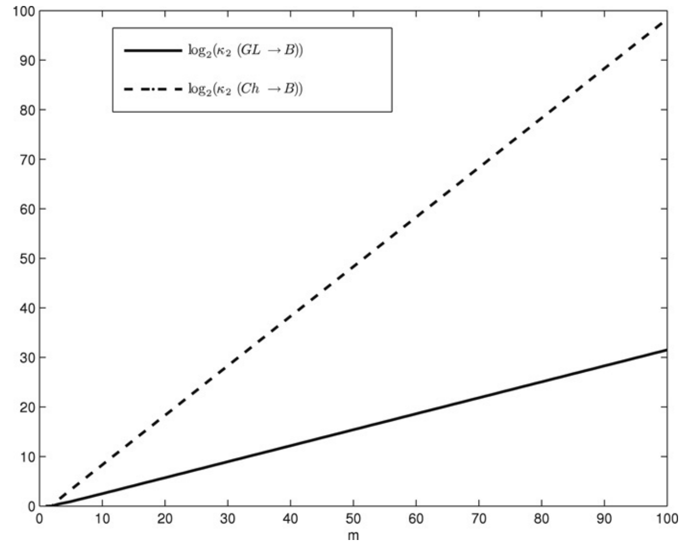

Fig. 1. $\kappa_2(GL \to B)$ and $\kappa_2(Ch \to B)$ on base-2 logarithmic scale, for $n = 10\,000$ and $m \in [2, 100]$.

matrix from coefficients of a polynomial in the usual monomial or power form to Bernstein coefficients is ill-conditioned [26]. The condition number, κ, associated with a problem is a measure of how numerically well-posed the problem is. In the transformation problem, for a transformation matrix A, the condition number can be defined, in any consistent norm, as:

$$\kappa(A) = \|A\| \left\|A^{-1}\right\|. \tag{47}$$

If the L^2-norm is considered then

$$\kappa_2(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}, \tag{48}$$

where $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are the maximal and minimal singular values of A respectively. Here we report the numerical comparison between the condition number of the Chebyshev to Bernstein conversion matrix, $\kappa_2(Ch \to B)$, and the Gauss–Lobatto to Bernstein one, $\kappa_2(GL \to B)$. Fig. 1 shows the curves for both the condition numbers on base-2 logarithmic scale for $m \in [2, 100]$ and $n = 10\,000$. As is evident, $\kappa_2(GL \to B)$ is considerably smaller than $\kappa_2(Ch \to B)$, i.e. the transformation from Gauss–Lobatto to Bernstein form is sensitively well-conditioned with respect to the Chebyshev–Bernstein one.

6. Conclusions

In this paper, explicit transformation matrices to map a polynomial expressed as a weighted sum of discrete orthogonal polynomials on Gauss–Lobatto nodes into Bernstein form and vice versa were derived. A useful explicit factorization for the coefficient matrix of Gauss–Lobatto polynomials and its inverse was gained. An effort to rearrange the conversion matrix $T_{GL\to B}$ as a product of dense matrices, with integer coefficients to avoid floating point error, followed by divisions with diagonal matrix, was done. An advise on the execution order of operations involved in the algorithm was given. Finally, to show that the proposed forward transformation basis is less ill-conditioned with respect to the Chebyshev one, an experimental comparison between the condition number of the Chebyshev to Bernstein conversion basis and the Gauss–Lobatto to Bernstein one is reported. From a practical point of view the proposed formulas give an algorithm that seems to be accurate and robust as it is confirmed by numerical experiments.

Appendix

Lemma 1.

$$\sum_{k=1}^{i-1}(-1)^k(2k-1)\binom{k+j-2}{k-j} = (-1)^{i+1}(2j-1)\binom{i+j-2}{2j-1}, \quad i \geq 0,\ j = 1, 2, \ldots, i. \tag{A.1}$$

Proof. The proof of this Lemma can be easily obtained by induction on the variable i and by standard algebraic manipulations. Since $j > -1$ Eq. (A.1) for $i = 0$ is true. Suppose that Eq. (A.1) for $i = i^*$ is true, i.e.,

$$\sum_{k=1}^{i^*-1}(-1)^k(2k-1)\binom{k+j-2}{k-j} = (-1)^{i^*+1}(2j-1)\binom{i^*+j-2}{2j-1}, \tag{A.2}$$

then for $i = i^* + 1$ it must be

$$\sum_{k=1}^{i^*-1}(-1)^k(2k-1)\binom{k+j-2}{k-j} + (-1)^{i^*}(2i^*-1)\binom{i^*+j-2}{i^*-j} = (-1)^{i^*}(2j-1)\binom{i^*+j-1}{2j-1}. \tag{A.3}$$

By substituting Eq. (A.2) in Eq. (A.3) and by taking into account standard properties of binomial coefficients [23], the proof follows. ■

Proposition 3.

$$\begin{cases} C_1^{-1}(2i,2i) = \dfrac{1}{n+2i-3}, & i = 1,\ldots,\left\lfloor \dfrac{m}{2} \right\rfloor, \\ C_1^{-1}(2i-1,2i-1) = \dfrac{1}{n+2i-4}, & i = 1,\ldots,\left\lceil \dfrac{m}{2} \right\rceil, \\ C_1^{-1}(2i,2j) = \dfrac{1}{2^{2i-2j-1}(n+2j-3)(n+2j-1)}, & i = 1,\ldots,\left\lfloor \dfrac{m}{2} \right\rfloor,\ j = 1,\ldots,i-1, \\ C_1^{-1}(2i-1,2j-1) = \dfrac{1}{2^{2i-2j-1}(n+2j-4)(n+2j-2)}, & i = 1,\ldots,\left\lceil \dfrac{m}{2} \right\rceil,\ j = 1,\ldots,i-1. \end{cases} \tag{A.4}$$

$$\begin{cases} C_2^{-1}(2i,2j) = \dfrac{1}{2^{2i-2j}}\dbinom{2i-1}{i-j}, & i = 1,\ldots,\left\lfloor \dfrac{m}{2} \right\rfloor,\ j = 1,\ldots,i, \\ C_2^{-1}(2i-1,2j-1) = \dfrac{1}{2^{2i-2j}}\dbinom{2i-2}{i-j}, & i = 1,\ldots,\left\lceil \dfrac{m}{2} \right\rceil,\ j = 1,\ldots,i. \end{cases} \tag{A.5}$$

Proof. Let $C_{s_{(m-1)}}$ and $C^{-1}_{s_{(m-1)}}$ be the C_1 and C_2 matrices with their inverses, respectively for $s = 1, 2$, of dimensions $(m-1)\times(m-1)$. Observing the structure of matrices C_1, C_2 and their inverses we have

$$C_{s_{(m)}} = \left[\begin{array}{c|c} C_{s_{(m-1)}} & O \\ \hline u^{\mathrm{T}}_{s_{(m)}} & \alpha_{s_{(m)}} \end{array}\right], \tag{A.6}$$

$$C^{-1}_{s_{(m)}} = \left[\begin{array}{c|c} C^{-1}_{s_{(m-1)}} & O \\ \hline v^{\mathrm{T}}_{s_{(m)}} & \beta_{s_{(m)}} \end{array}\right], \tag{A.7}$$

where O is a $(m-1)$ zero-column vector, and the quantities $u^{\mathrm{T}}_{s_{(m)}}$, $\alpha_{s_{(m)}}$, $v^{\mathrm{T}}_{s_{(m)}}$ and $\beta_{s_{(m)}}$ can be effortlessly derived from Eqs. (15), (16), (A.4) and (A.5). The proof can be made by induction on the dimension m of such matrices which leads to the following equalities

$$\begin{cases} u^{\mathrm{T}}_{s_{(m)}} C^{-1}_{s_{(m-1)}} + \alpha_{s_{(m)}} v^{\mathrm{T}}_{s_{(m)}} = O^{\mathrm{T}}, \\ \alpha_{s_{(m)}} \beta_{s_{(m)}} = 1, \qquad\qquad s = 1, 2, \end{cases} \tag{A.8}$$

simply verified by inspection. ■

References

[1] A. Bjorck, Numerical Methods for Least Squares Problems, SIAM, 1996.

[2] R. Winkel, Generalized Bernstein polynomials and Bézier curves: An application of umbral calculus to computer aided geometric design, Adv. Appl. Math. 27 (1) (2001) 51–81.

[3] R.T. Farouki, T.N.T. Goodman, T. Sauer, Construction of orthogonal bases for polynomials in Bernstein form on triangular and simplex domains, Comput. Aided Geom. Des. 20 (4) (2003) 209–230.

[4] J.P. Boyd, A test, based on conversion to the Bernstein polynomial basis, for an interval to be free of zeros applicable to polynomials in Chebyshev form and to transcendental functions approximated by Chebyshev series, Appl. Math. Comput. 188 (2) (2007) 1780–1789.

[5] A.P. Smith, J. Garloff, in: G. Alefeld, S. Rump, J. Rohn, T. Yamamoto (Eds.), Solution of Systems of Polynomial Equations by Using Bernstein Expansion, in Symbolic Algebraic Methods and Verification Methods, Springer, 2001, pp. 89–97.

[6] M.B. Egerstedt, C.F. Martin, A note on the connection between Bézier curves and linear optimal control, IEEE Trans. Automat. Control 49 (10) (2004) 1728–1732.

[7] M. Zettler, J. Garloff, Robustness Analysis of polynomials with polynomial parameter dependency using Bernstein expansions, IEEE Trans. Automat. Control 43 (3) (1998) 425–431.

[8] J. Garloff, B. Graf, M. Zettler, Speeding up an algorithm for checking robust stability of polynomials, in: Proc. 2nd IFAC Symp. Robust Control Design, 1998, pp. 183–188.

[9] J. Garloff, Application of Bernstein expansion to the solution of control problems, in: Proceedings of MISC'99 - Workshop on Applications of Interval Analysis to Systems and Control, 1999, pp. 421–430.

[10] J. Garloff, B. Graf, Solving strict polynomial inequalities by Bernstein expansion, The Use of Symbolic Methods in Control System Analysis and Design (1999) 339–352.

[11] R.T. Farouki, Legendre-Bernstein basis transformations, J. Comput. Appl. Math. 119 (2000) 145–160.

[12] A. Rababah, Transformation of Chebyshev–Bernstein polynomial basis, Comput. Methods Appl. Math. 3 (4) (2003) 608–622.

[13] A. Rababah, Jacobi-Bernstein basis transformation, Comput. Methods Appl. Math. 4 (2) (2004) 206–214.

[14] S.R. Jiang, G.J. Wang, Conversion and evaluation for two types of parametric surfaces constructed by NTP bases, Math. Comput. Model. 49 (2005) 321–329.

[15] Y.L. Luke, The special functions and their approximations, vol. 1, Academic Press, 1969.

[16] A. Eisinberg, G. Fedele, Discrete orthogonal polynomials on Gauss–Lobatto Chebyshev nodes, J. Approx. Theory 144 (2007) 238–246.

[17] A. Eisinberg, G. Fedele, Vandermonde systems on Gauss–Lobatto Chebyshev nodes, Appl. Math. Comput. 170 (1) (2005) 633–647.

[18] A. Eisinberg, G. Fedele, Explicit solution of the polynomial least-squares approximation on Chebyshev extrema nodes, Linear Algebra Appl. 422 (2007) 553–562.

[19] G.N. Elnagar, M. Razzaghi, A collocation-type method for linear quadratic optimal control problems, Optim. Control Appl. Methods 18 (1997) 227–235.

[20] L. Valdettaro, M. Rieutord, T. Braconnier, V. Frayssé, Convergence and round-off errors in a two-dimensional eigenvalue problem using spectral methods and Arnoldi–Chebyshev algorithm, J. Comput Appl. Math. 205 (1) (2007) 382–393.

[21] M.S. Floater, A.F. Rasmussen, Point-based methods for estimating the length of a parametric curve, J. Comput. Appl. Math. 196 (2) (2006) 512–522.

[22] A.F. Rasmussen, M.S. Floater, A point-based method for estimating surface area, SIAM Conf. on Geom. Design and Comp. in Phoenix (2005).

[23] D.E. Knuth, The Art of Computer Programming, second ed., vol. 1, Addison-Wesley, Reading, MA, 1973.

[24] The MathWorks Inc., MATLAB Reference Guide, 1994.

[25] S. Wolfram, Mathematica: A System for Doing Mathematics by Computers, second ed., Addison-Wesley, 1991.

[26] R.T. Farouki, On the stability of transformations between power and Bernstein polynomial forms, Comput. Aided Geom. Des. 8 (1) (1991) 29–36.