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Non-equilibrium time evolution of higher order cumulants of conserved charges and event-by-event analysis



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ABSTRACT

We investigate the time evolution of higher order cumulants of conserved charges in a volume with the diffusion master equation. Applying the result to the diffusion of non-Gaussian fluctuations in the hadronic stage of relativistic heavy ion collisions, we show that the fourth-order cumulant of net-electric charge at LHC energy is suppressed compared with the recently observed second-order cumulant at ALICE, if the higher order cumulants at hadronization are suppressed compared with their values in the hadron phase in equilibrium. The significance of the experimental information on the rapidity window dependence of various cumulants in investigating the history of the dynamical evolution of the hot medium created in relativistic heavy ion collisions is emphasized.

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1. Introduction

Statistical mechanics tells us that observables are fluctuating even in equilibrated medium. Because the fluctuations are determined by the microscopic nature of the medium and sensitive to critical phenomena, they can be exploited to reveal and characterize properties of the medium. In experimental attempts to map the global nature of QCD phase transition at nonzero baryon density in relativistic heavy ion collisions, fluctuation observables, especially those of conserved charges, are believed to be promising observables to diagnose the property of the hot medium [1–8]. Active investigation in heavy ion collisions by event-by-event analyses has recently been performed at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) [9–12]. Numerical analyses of higher order cumulants in equilibrium have been also carried out in lattice QCD Monte Carlo simulations [13].

As an experiment in which fluctuations are measured, heavy ion collisions have several notable features. First, higher order cumulants, which characterize the non-Gaussian nature of fluctuations, have been experimentally observed with good statistics up to the fourth order [10,11]. The measurement is possible because the system observed in the experiments is not large; the observed

particle number is at most of order 10^3 , while the event-by-event statistics exceed 10^7 . Second, the fluctuations observed in experiments, especially those of conserved charges, are not necessarily the same as those in an equilibrated medium, because of the dynamical nature of the hot medium created by heavy ion collisions. Because of these properties, an appropriate description of the dynamical evolution of non-Gaussian fluctuations is required in order to understand the experimental results on higher order cumulants.

Concerning the second point, we remark that the recent experimental result on the net-electric charge fluctuation, $\langle(\delta N_Q^{(\text{net})})^2\rangle$, by ALICE Collaboration at LHC [12] supports the non-thermal nature of the observed fluctuation. The value of $\langle(\delta N_Q^{(\text{net})})^2\rangle$ in this experiment is suppressed compared with the one in the equilibrated hadronic medium which has been calculated by lattice QCD simulations [13] and the hadron resonance gas (HRG) model [14]. Moreover, the dependence of $\langle(\delta N_Q^{(\text{net})})^2\rangle$ on the size of the rapidity window, $\Delta\eta$, shows that the suppression of $\langle(\delta N_Q^{(\text{net})})^2\rangle$ is more pronounced for larger $\Delta\eta$. These experimental results are reasonably explained if one attributes the suppression to the survival of fluctuations generated in the primordial deconfined medium [1,3,4,15]. In Refs. [3,4], it is argued that $\langle(\delta N_Q^{(\text{net})})^2\rangle$ per unit rapidity normalized by an extensive conserved quantity, such as entropy, takes 2–3 times smaller value in the deconfined medium than the hadronic one. After hadronization, this small fluctuation approaches the equilibrated value in the hadronic medium. Since the variation of the local density of a conserved charge is achieved only through diffusion, the approach of the fluctuation to the equilibrated value becomes slower as the volume where the charge is counted becomes larger. $\langle(\delta N_Q^{(\text{net})})^2\rangle$ thus takes a smaller value as

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$\Delta\eta$ becomes larger, which is consistent with the experimental result at ALICE.

On the other hand, the value of $\langle(\delta N_Q^{(\text{net})})^2\rangle$ observed at RHIC energy is consistent with the one in the equilibrated hadronic medium [22,23]. The difference between RHIC and LHC energies indicates that the evolution of the fluctuation in the hot medium is qualitatively different between these energies.

In order to confirm the validity of the above explanation on the fluctuation measured at ALICE and clarify the origin of the qualitative difference between the experimental results at RHIC and LHC energies, it should be instructive to measure the $\Delta\eta$ dependences of various other fluctuation observables in addition to $\langle(\delta N_Q^{(\text{net})})^2\rangle$ in these experiments. For example, the net-baryon number fluctuation, $\langle(\delta N_B^{(\text{net})})^2\rangle$, is an experimentally-observable conserved charge fluctuation [16,17], although neutral baryons are not directly observable. Because the diffusion of the baryon number in the hadronic phase is slower than that of the electric charge due to the large mass of its carriers, baryons, if the origin of the suppression of $\langle(\delta N_Q^{(\text{net})})^2\rangle$ at ALICE is indeed traced back to the smallness of the primordial fluctuations, $\langle(\delta N_B^{(\text{net})})^2\rangle$ must have steeper suppression as a function of $\Delta\eta$ than $\langle(\delta N_Q^{(\text{net})})^2\rangle$.

In event-by-event analyses, one can also measure higher order cumulants of conserved charges [10,11] such as the fourth-order ones $\langle(N_Q^{(\text{net})})^4\rangle_c$ and $\langle(N_B^{(\text{net})})^4\rangle_c$. The experimental analysis of these observables as functions of $\Delta\eta$ can obviously provide us more information on the time evolution of fluctuations in the hot medium. So far, however, systematic studies on the dynamical evolution of higher order cumulants in heavy ion collisions, whose results can be compared with their experimental observation, have not been carried out to the best of the authors' knowledge. The purpose of the present Letter is to make the first investigation on this issue using a simple but theoretically lucid model, and to make a prediction on the $\Delta\eta$ dependence of higher order cumulants in relativistic heavy ion collisions.

2. Stochastic formalism to describe diffusive systems

In relativistic heavy ion collisions with sufficiently large collision energy per nucleon, $\sqrt{s_{NN}}$, the hot medium created at mid-rapidity has an approximate boost invariance. Useful coordinates to describe such a system are the coordinate-space rapidity η and proper time τ . We denote the net number of a conserved charge per unit coordinate-space rapidity as $n(\eta, \tau)$. In a class of experiments, event-by-event fluctuations of the charge at kinetic freezeout in a phase space corresponding to the rapidity window determined by the experiment are observed. The phase space approximately corresponds to a finite coordinate-space rapidity interval [1]. Assuming that the kinetic freezeout takes place at a certain proper time τ_{f0} , the experimentally-observed conserved charge number at mid-rapidity at RHIC and LHC is given by

$$Q(\tau) = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta n(\eta, \tau) \quad (1)$$

at $\tau = \tau_{f0}$ with the rapidity window to count the particle number $\Delta\eta$. In the following, we investigate the time evolution of higher order cumulants of $Q(\tau)$.

In a sufficiently large space-time scale where hydrodynamic equations at first order are applicable, the average of $n(\eta, \tau)$ follows the diffusion equation

$$\partial_\tau n(\eta, \tau) = D \partial_\eta^2 n(\eta, \tau), \quad (2)$$

where D is the diffusion constant in this coordinate system. In a boost-invariantly expanding system, D receives a factor τ^{-2} compared with the diffusion constant in Cartesian coordinate. In order to describe fluctuations around the solution of Eq. (2), one may employ a stochastic model, in which the time evolution of the deterministic part satisfies Eq. (2).

A choice of such stochastic models is the theory of hydrodynamic fluctuations [18,19], in which the hydrodynamic equations are promoted to Langevin equations with stochastic terms representing fast random forces arising from microscopic interactions. In the equation corresponding to the conservation law of a charge, Eq. (2), the derivative of the stochastic force, $\partial_\eta \xi(\eta, \tau)$, is added to the right-hand side of Eq. (2) [15]. The equation is referred to as *stochastic diffusion equation*. Up to Gaussian fluctuations, property of $\xi(\eta, \tau)$ is completely determined by the fluctuation–dissipation relation, which is obtained from the locality of $\xi(\eta, \tau)$ and large time behavior of $n(\eta, \tau)$ [18]. It is known that the stochastic equation determined in this way well describes Gaussian fluctuations in fluids [18]. However, extension of this formalism to treat higher order fluctuations is nontrivial. There is no unique generalization of the fluctuation–dissipation relation to higher orders, or no a priori justification of such extensions.

Concerning the difficulty in the description of non-Gaussian fluctuations, it is worthwhile to note a theorem on Markov process, which states that stochastic forces in a Langevin equation for Markov process are of Gaussian when the equation describes stochastic variables which are continuous and vary continuously¹ [20]. Since the standard theory of hydrodynamic fluctuations describes a Markov process and the hydrodynamic variables are continuous, the theorem demands that $\xi(\eta, \tau)$ be of Gaussian; to allow for nonzero higher order correlations of $\xi(\eta, \tau)$, one must relax at least one of the two conditions, i.e. Markovian and the continuity.

Without the higher order correlation of $\xi(\eta, \tau)$, all higher order cumulants of $Q(\tau)$ vanish in equilibrium unless D explicitly depends on n . Even in such a formalism, the relaxation of non-Gaussianity starting from a particular initial condition can be described. In the physics of fluctuations in relativistic heavy ion collisions, however, nonzero higher order cumulants in equilibrium play a crucial role. First, the experimental results obtained so far report nonzero higher order cumulants near the equilibrated values [10–12]. Second, higher order cumulants are expected to *increase* toward the equilibrated values in the hadronic medium [3, 4]. To reproduce these features, the stochastic model must obviously have nonzero higher order cumulants in equilibrium.

In the present study, instead of directly extending the theory of hydrodynamic fluctuations, we investigate the time evolution of higher order cumulants starting from a microscopic model. In this exploratory analysis, as such a model we consider a simple one-dimensional system composed of Brownian particles. Instead of tracking the motion of each Brownian particle separately, however, we represent the system as follows (see, Fig. 1). First, the coordinate η is divided into discrete cells with an equal length a . Second, we consider a single species of particle for the moment, and denote the number of particles in each cell, labeled by an integer m , as n_m , and the probability that each cell contains n_m

¹ This condition needs a brief explanation. For simplicity, we here assume that the stochastic variable is one-dimensional, denoted by x and y . Let $P(x, t + \Delta t | y, t)$ be the conditional probability of x at time $t + \Delta t$ given the system was at y at time t . When the stochastic variable is continuous, it is shown that $P(x, t + \Delta t | y, t)$ satisfies the Lindeberg condition $\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{|x-y|>\varepsilon} dx P(x, t + \Delta t | y, t) = 0$, for arbitrary $\varepsilon > 0$ [20]. This condition is used effectively in the proof of the Gaussianity. Note that, when the stochastic variables are discrete, the Lindeberg condition is obviously violated.

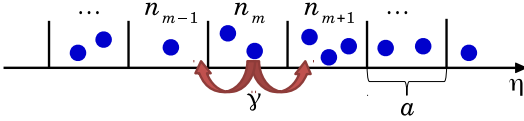


Fig. 1. System described by the diffusion master equation (3).

particles as $P(\mathbf{n}, \tau)$ with $\mathbf{n} = (\dots, n_{m-1}, n_m, n_{m+1}, \dots)$. Finally, we assume that each particle moves to adjacent cells with a probability γ per unit proper time, as a result of microscopic interactions and random motion. The probability $P(\mathbf{n}, \tau)$ then follows the differential equation

$$\partial_\tau P(\mathbf{n}, \tau) = \gamma \sum_m [(n_m + 1) \{P(\mathbf{n} + \mathbf{e}_m - \mathbf{e}_{m+1}, \tau) + P(\mathbf{n} + \mathbf{e}_m - \mathbf{e}_{m-1}, \tau)\} - 2n_m P(\mathbf{n}, \tau)], \quad (3)$$

which is referred to as *diffusion master equation* in the literature [20], where \mathbf{e}_m is the vector that all components are zero except for m th one, which takes unity. We will see later that in the continuum limit, $a \rightarrow 0$, the average density and Gaussian fluctuation of $n(\eta, \tau)$ in Eq. (3) agree with those in the stochastic diffusion equation with $D = \gamma a^2$.

Before solving Eq. (3), some remarks are in order here. First, Eq. (3) is often solved with an approximation that n_m are sufficiently large so that they can be regarded as continuous [20]. For the present purpose, however, the equation should be solved without this approximation, because once n_m are set continuous, using an argument similar to the proof of the above-mentioned theorem on Markov processes one can show that all higher order cumulants become zero in equilibrium [20]. Discrete nature of n_m should be kept to give rise to non-Gaussianity in equilibrium. Second, particles described by Eq. (3) behave as Brownian particles [21] without correlations with one another. From this property, it is immediately concluded that the distribution of Q in the $\tau \rightarrow \infty$ limit becomes of Poissonian, in which all cumulants take the same value $\langle Q^n \rangle_c = \langle Q \rangle$, provided that $\Delta\eta$ is sufficiently narrower than the total length of the system. Note that this behavior of cumulants is consistent with the hadronic fluctuations in the HRG model, owing to the non-interacting nature of hadrons in this model. Finally, our model would not be suitable to describe the non-Gaussian fluctuations enhanced near the critical point. To deal with this problem, one may, for example, introduce explicit n dependence of the diffusion constant and makes it bring about non-Gaussian fluctuations without altering the Gaussianity of the random force. The existence of a slow critical mode near the critical point, which may result in temporal correlations of the stochastic effects, also needs to be considered. In the experimental results obtained so far, however, ratios of cumulants are not far from the value in the hadronic medium [10–12], whereas an interesting suppression of higher order cumulants, which might be a signal of a critical phenomenon, is reported in the beam energy scan program at RHIC. This result implies that critical phenomena do not come into play in the diffusion in the hot medium at least in the late stage in heavy ion collisions. In spite of the simplicity of the model Eq. (3), it is thus expected that the model qualitatively well describes the time evolution of $\langle Q^n \rangle_c$ in the hadronic stage.

3. Solving diffusion master equation

Now, let us determine the time evolution of cumulants for the stochastic process Eq. (3). We first consider the time evolution of the probability $P(\mathbf{n}, \tau)$ with a fixed initial condition

$$P(\mathbf{n}, 0) = \prod_m \delta_{n_m, M_m}, \quad (4)$$

i.e. the initial particle numbers are fixed as $n_m = M_m$ for all m without fluctuation. By introducing the factorial generating function,

$$G_f(\mathbf{s}, \tau) = \sum_{\mathbf{n}} \prod_m s_m^{n_m} P(\mathbf{n}, \tau), \quad (5)$$

one obtains

$$\partial_\tau G_f(\mathbf{s}, \tau) = \gamma \sum_m (s_{m+1} - 2s_m + s_{m-1}) \frac{\partial}{\partial s_m} G_f(\mathbf{s}, \tau). \quad (6)$$

Eq. (6) is a first-order partial differential equation, and solved with the method of characteristics. The solution with the initial condition Eq. (4) with the periodic boundary condition is given by

$$G_f(\mathbf{s}, \tau) = \prod_m \left(\sum_j r_j e^{-\omega_j \tau} e^{-2\pi j m i / N} \right)^{M_m}, \quad (7)$$

with

$$r_j = \frac{1}{N} \sum_{m'} s_{m'} e^{2\pi j m' i / N}, \quad (8)$$

$$\omega_j = -\gamma (e^{2\pi j i / N} + e^{-2\pi j i / N} - 2), \quad (9)$$

where N denotes the total number of cells and i is the imaginary unit. The factorial cumulants of the Fourier transform of n_m , $\tilde{n}_k = \sum_m n_m e^{-2\pi k m i / N}$, are given by

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \cdots \tilde{n}_{k_l} \rangle_{fc} = \left. \frac{\partial^l K_f}{\partial r_{k_1} \partial r_{k_2} \cdots \partial r_{k_l}} \right|_{\mathbf{s}=\mathbf{1}}, \quad (10)$$

with $K_f(\mathbf{s}, \tau) = \log G_f(\mathbf{s}, \tau)$. Using Eqs. (10) and (7), the factorial cumulants up to the fourth order are calculated to be

$$\langle \tilde{n}_k \rangle_{fc} = \tilde{M}_k e^{-\omega_k \tau}, \quad (11)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \rangle_{fc} = -\tilde{M}_{k_1+k_2} e^{-(\omega_{k_1} + \omega_{k_2})\tau}, \quad (12)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \rangle_{fc} = 2\tilde{M}_{k_1+k_2+k_3} e^{-(\omega_{k_1} + \omega_{k_2} + \omega_{k_3})\tau}, \quad (13)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \tilde{n}_{k_4} \rangle_{fc} = -6\tilde{M}_{k_1+k_2+k_3+k_4} e^{-(\omega_{k_1} + \omega_{k_2} + \omega_{k_3} + \omega_{k_4})\tau}, \quad (14)$$

with $\tilde{M}_k = \sum_m M_m e^{-2\pi k m i / N}$.

The cumulants of n_m are given by

$$\langle n_{m_1} n_{m_2} \cdots n_{m_l} \rangle_c = \left. \frac{\partial^l K}{\partial \theta_1 \cdots \partial \theta_l} \right|_{\theta=0} \quad (15)$$

with $K(\theta, \tau) = K_f(\mathbf{s}, \tau)|_{s_m=e^{\theta m}}$. Using Eq. (15), they are related to Eqs. (11)–(14) as

$$\langle \tilde{n}_k \rangle_c = \langle \tilde{n}_k \rangle_{fc}, \quad (16)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \rangle_c = \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \rangle_{fc} + \langle \tilde{n}_{k_1+k_2} \rangle_{fc}, \quad (17)$$

and so forth.

Since we are interested in the solution of Eq. (3) in the continuum limit, $a \rightarrow 0$, here we introduce the shorthand notation for this limit: the particle density per unit rapidity $n(\eta) = n_m/a$, with the rapidity of the m th cell $\eta = ma$, $\omega_p = \gamma a^2 p^2$ with the conjugate momentum $p = 2\pi k/Na$. Also, the probability $P(\mathbf{n}, \tau)$ is promoted to a functional, which we denote as $P[n(\eta), \tau]$. Note, however, that these notations are conceptual; In actual applications, the functional $P[n(\eta), \tau]$ is understood as the limit of the function $P(\mathbf{n}, \tau)$ with small but finite a . One finds from Eq. (11) that the continuum limit has to be taken with $D = \gamma a^2$ fixed, so that the deterministic part of Eq. (3) follows Eq. (2). The factorial

cumulants of $\tilde{n}(p)$ in the continuum limit are obtained by simply replacing k with p in Eqs. (11)–(14).

In the following, we consider an infinitely long system without boundaries. With the fixed initial condition $n(\eta, 0) = M(\eta)$, the cumulants of Eq. (1) at proper time τ are calculated to be

$$\langle (Q(\tau))^n \rangle_c = \int_{-\infty}^{\infty} d\eta M(\eta) H_X^{(n)}(\eta), \quad (18)$$

with

$$H_X^{(1)}(z) = I_X(z/\Delta\eta), \quad (19)$$

$$H_X^{(2)}(z) = I_X(z/\Delta\eta) - I_X(z/\Delta\eta)^2, \quad (20)$$

$$H_X^{(3)}(z) = I_X(z/\Delta\eta) - 3I_X(z/\Delta\eta)^2 + 2I_X(z/\Delta\eta)^3, \quad (21)$$

$$H_X^{(4)}(z) = I_X(z/\Delta\eta) - 7I_X(z/\Delta\eta)^2 + 12I_X(z/\Delta\eta)^3 - 6I_X(z/\Delta\eta)^4, \quad (22)$$

and

$$I_X(z) = \int_{-1/2}^{1/2} dx \int \frac{dq}{2\pi} e^{-X^2 q^2} e^{iq(x+z)}, \quad (23)$$

where $\Delta\eta$ and τ dependences are encoded in the dimensionless parameter

$$X = \frac{\sqrt{D\tau}}{\Delta\eta}. \quad (24)$$

Next, we extend this result to general initial conditions containing fluctuations. We also extend the result to systems with two particle species with densities $n_1(\eta, \tau)$ and $n_2(\eta, \tau)$, and consider cumulants of the difference

$$Q_{(\text{net})}(\tau) = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta (n_1(\eta, \tau) - n_2(\eta, \tau)), \quad (25)$$

in order to compare the results with the cumulants of the net-electric charge and baryon numbers, which are given by the difference of particle numbers. When the two particle species separately follow Eq. (3), the probability with the initial condition $P[n_1(\eta), n_2(\eta), 0] = F[M_1(\eta), M_2(\eta)]$ is given by the superposition of the solutions of fixed initial conditions

$$P[n_1(\eta), n_2(\eta), \tau] = \sum_{\{M_1(\eta), M_2(\eta)\}} F[M_1(\eta), M_2(\eta)] \times P_{M_1}[n_1(\eta), \tau] P_{M_2}[n_2(\eta), \tau], \quad (26)$$

where $P_M[n(\eta), \tau]$ is the solution of Eq. (3) with the fixed initial condition $n(\eta, 0) = M(\eta)$, and the sum runs over the function spaces of $M_1(\eta)$ and $M_2(\eta)$. From Eq. (26), one finds that the cumulants of Eq. (25) are obtained by using the cumulants of $P_M[n(\eta), \tau]$ and $F[M_1(\eta), M_2(\eta)]$ using the superposition formula given in Refs. [24,25]. The general results will be presented in Ref. [25].

In the present Letter, we focus on the results for initial conditions satisfying spatial uniformity and locality, i.e.

$$\langle M_{i_1}(\eta_1) M_{i_2}(\eta_2) \cdots M_{i_l}(\eta_l) \rangle_c = [M_{i_1} M_{i_2} \cdots M_{i_l}]_c \delta(\eta_1 - \eta_2) \cdots \delta(\eta_1 - \eta_l), \quad (27)$$

where this equation defines $[M_{i_1} M_{i_2} \cdots M_{i_l}]_c$ on the right-hand side, which are generalized susceptibilities to higher orders and

off-diagonal components. The condition Eq. (27) is satisfied, for example, in free gas, as well as systems well described by hydrodynamic equations, in equilibrium. With this initial condition, cumulants of $Q_{(\text{net})}(\tau)$ are given by

$$\langle Q_{(\text{net})} \rangle_c = \Delta\eta [M_{(\text{net})}]_c, \quad (28)$$

$$\langle Q_{(\text{net})}^2 \rangle_c = \Delta\eta \{ [M_{(\text{net})}^2]_c F_X^{(2)} + [M_{(\text{tot})}]_c (1 - F_X^{(2)}) \}, \quad (29)$$

$$\langle Q_{(\text{net})}^3 \rangle_c = \Delta\eta \{ [M_{(\text{net})}^3]_c F_X^{(3)} + 3[M_{(\text{net})} M_{(\text{tot})}]_c (F_X^{(2)} - F_X^{(3)}) + [M_{(\text{net})}]_c (1 - 3F_X^{(2)} + 2F_X^{(3)}) \}, \quad (30)$$

$$\langle Q_{(\text{net})}^4 \rangle_c = \Delta\eta \{ [M_{(\text{net})}^4]_c F_X^{(4)} + 6[M_{(\text{net})}^2 M_{(\text{tot})}]_c (F_X^{(3)} - F_X^{(4)}) + 3[M_{(\text{tot})}^2]_c (F_X^{(2)} - 2F_X^{(3)} + F_X^{(4)}) + 4[M_{(\text{net})}^2]_c (F_X^{(2)} - 3F_X^{(3)} + 2F_X^{(4)}) + [M_{(\text{tot})}]_c (1 - 7F_X^{(2)} + 12F_X^{(3)} - 6F_X^{(4)}) \}, \quad (31)$$

with

$$F_X^{(n)} = \int_{-\infty}^{\infty} dz [I_X(z)]^n, \quad (32)$$

and $M_{(\text{net}),(\text{tot})}(\eta) = M_1(\eta) \mp M_2(\eta)$, respectively. $F_X^{(n)}$ with $n \geq 2$ are monotonically decreasing functions of X and satisfy $F_0^{(n)} = 1$ and $\lim_{X \rightarrow \infty} F_X^{(n)} = 0$. From this property of $F_X^{(n)}$ and Eqs. (28)–(31), one easily finds that the distribution of $Q_{(\text{net})}$ approaches a Skellam one with

$$\lim_{\tau \rightarrow \infty} \langle Q_{(\text{net})}^{2n+1} \rangle_c = \Delta\eta [M_{(\text{net})}]_c, \quad (33)$$

$$\lim_{\tau \rightarrow \infty} \langle Q_{(\text{net})}^{2n} \rangle_c = \Delta\eta [M_{(\text{tot})}]_c. \quad (34)$$

From Eq. (29), one also finds that the time evolution of the Gaussian fluctuation with the initial condition Eq. (27) is equivalent with the one in the stochastic diffusion equation [15] with the fluctuation in equilibrium Eq. (34).

The details of the procedure to deal with Eq. (3) omitted in this Letter will be elucidated in the forthcoming publication [25].

4. Diffusion in hadronic medium

Next, let us investigate the diffusion of higher order cumulants in the hadronic medium in relativistic heavy ion collisions using the above results. To make the argument simple, we assume that a boost invariant system with local equilibration is realized just above the critical temperature of the deconfinement transition. Then, the fluctuations at this time satisfy the locality condition Eq. (27). The hot medium then undergoes hadronization and chemical freezeout, which take place at almost the same time, $\tau = \tau_0$, for large $\sqrt{s_{\text{NN}}}$. Because of the local charge conservations, fluctuations of local densities of the conserved charges are unchanged during these processes [1]. The fluctuations of conserved charges just after hadronization thus also satisfy Eq. (27) to a good approximation, with cumulants in the equilibrated deconfined medium, which are significantly smaller than the ones in the equilibrated hadronic medium [3,4,6]. We take this configuration at $\tau = \tau_0$ as the initial condition.

Due to the diffusion in the hadronic phase, the fluctuations approach the equilibrated values in the hadronic medium until kinetic freezeout at $\tau = \tau_{f0}$. Provided that the τ dependence of the

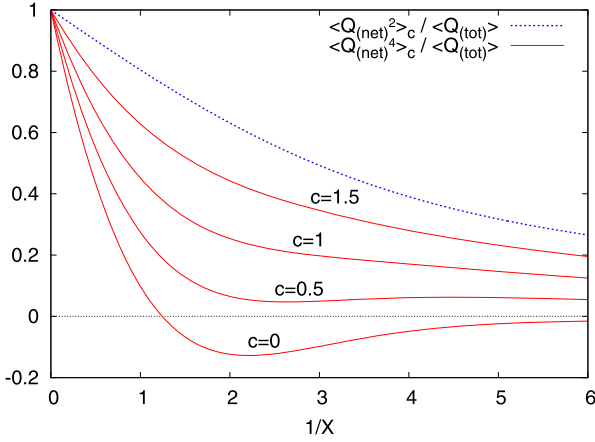


Fig. 2. Second and fourth order cumulants of $Q_{(\text{net})}$ as functions of $1/X$ with the initial condition $[M_{(\text{net})}^2]_c = [M_{(\text{net})}^4]_c = [M_{(\text{net})}^2 M_{(\text{tot})}]_c = 0$.

diffusion constant D in Eq. (2) is weak, this diffusion process is approximately described by Eq. (3).

Now we consider the diffusion of the net-electric charge and baryon numbers, $N_Q^{(\text{net})}$ and $N_B^{(\text{net})}$, as conserved charges. Since experimental measurements of odd order cumulants of these charges are difficult at LHC energy because of their smallness, we also limit our attention to the second- and fourth-order cumulants. Because these charges in the hadron phase are predominantly carried by charged pions and nucleons, respectively, the cumulants of $N_Q^{(\text{net})}$ and $N_B^{(\text{net})}$ at kinetic freezeout are approximately given by Eqs. (29) and (31) with $\tau = \tau_{\text{fo}} - \tau_0$, where $n_1(\eta, \tau)$ and $n_2(\eta, \tau)$ are the densities of positive and negative pions, and nucleons and anti-nucleons, respectively.

To see the behavior of the cumulants determined by Eqs. (29) and (31), one must fix the parameters for the initial condition in these equations. As discussed above, the cumulants of the conserved charges at $\tau = \tau_0$ are suppressed compared with the equilibrated values. For even n , this condition is represented as $[M_{(\text{net})}^n]_c \ll [M_{(\text{tot})}]_c$, because the cumulants in equilibrium are given by Eq. (34). Because of the suppression of $[M_{(\text{net})}^2]_c$, unless $(\delta M_{(\text{net})})^2$ and $M_{(\text{tot})}$ have a strong positive correlation $[M_{(\text{net})}^2 M_{(\text{tot})}]_c \ll [M_{(\text{tot})}]_c$ should also be satisfied. On the other hand, the value of $[M_{(\text{tot})}^2]_c$ in Eq. (31) is not constrained by the conservation laws and strongly depends on the hadronization mechanism. In the following, we treat this quantity as a parameter that characterizes the hadronization mechanism, and propose to utilize this parameter to constrain the ambiguity in it.

In Fig. 2, we show the $1/X$ dependence of $\langle Q_{(\text{net})}^2 \rangle_c$ and $\langle Q_{(\text{net})}^4 \rangle_c$ given by Eqs. (29) and (31) normalized by the equilibrated value Eq. (34) with the initial condition that the fluctuations of conserved charges do not exist at all,

$$[M_{(\text{net})}^2]_c = [M_{(\text{net})}^4]_c = [M_{(\text{net})}^2 M_{(\text{tot})}]_c = 0. \quad (35)$$

Since the value of $[M_{(\text{tot})}^2]_c$ is not constrained by the local charge conservation at hadronization as we discussed above, we regard the ratio

$$c = \frac{[M_{(\text{tot})}^2]_c}{[M_{(\text{tot})}]_c}, \quad (36)$$

as a free parameter and vary c in the range $0 \leq c \leq 1.5$. The result on the second order, $\langle Q_{(\text{net})}^2 \rangle_c$, reproduces the one obtained by the stochastic diffusion equation [15].

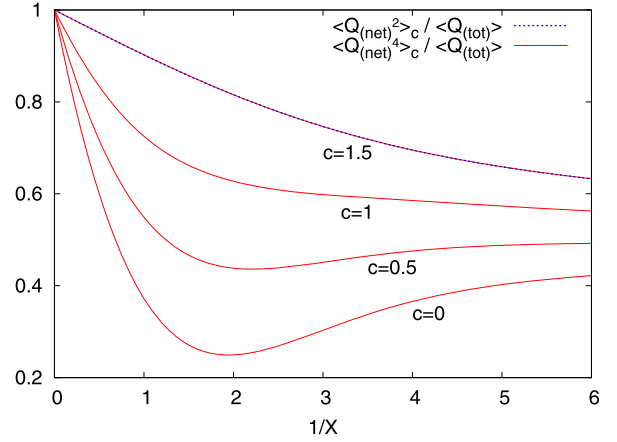


Fig. 3. Second and fourth order cumulants of $Q_{(\text{net})}$ as functions of $1/X$ with $[M_{(\text{net})}^2]_c = [M_{(\text{net})}^4]_c = [M_{(\text{net})}^2 M_{(\text{tot})}]_c = 0.5[M_{(\text{tot})}]_c$.

In the figure, one finds that the behaviors of $\langle Q_{(\text{net})}^4 \rangle_c$ and $\langle Q_{(\text{net})}^2 \rangle_c$ as functions of $1/X$ are qualitatively different. While the behavior of $\langle Q_{(\text{net})}^4 \rangle_c$ depends on the value of c , $\langle Q_{(\text{net})}^4 \rangle_c$ is suppressed compared with $\langle Q_{(\text{net})}^2 \rangle_c$ in the parameter range $c \lesssim 1.5$. Note that $\langle Q_{(\text{net})}^4 \rangle_c$ can become negative in a range of $1/X$, although the fourth-order cumulants at the initial time and in equilibrium are both non-negative. It is interesting that the sign of the fourth-order cumulant can be flipped owing to the non-equilibrium effect.

Since $1/X$ is proportional to $\Delta\eta$, the result in Fig. 2 can directly be compared with the experimental result on the $\Delta\eta$ dependence of the net-electric charge cumulants $\langle (N_Q^{(\text{net})})^2 \rangle_c = \langle (\delta N_Q^{(\text{net})})^2 \rangle$ and $\langle (N_Q^{(\text{net})})^4 \rangle_c$. In particular, Fig. 2 indicates that the $\langle (N_Q^{(\text{net})})^4 \rangle_c$ at ALICE, which has not been measured yet, will be suppressed compared with the $\langle (N_Q^{(\text{net})})^2 \rangle_c$ which has been already measured [12]. This statement is, however, altered for $c \gtrsim 2$. The same conclusion is also anticipated for the relation between the baryon number cumulants, $\langle (N_B^{(\text{net})})^2 \rangle_c$ and $\langle (N_B^{(\text{net})})^4 \rangle_c$.

The suppression of the fourth-order cumulant compared with the second-order one can be intuitively understood as follows. In our analysis, we consider time evolution with the initial condition with small fluctuation. With this initial condition, the probability distribution becomes wider as τ becomes larger, and thus longer time is required until the probability in the tail is equilibrated. Because the higher order cumulants are sensitive to the tail behavior of the probability distribution, the approach of the cumulants to the equilibrated values is slower for higher order ones.

The result in Fig. 2 is obtained with the idealized initial condition Eq. (35), where fluctuations of the conserved charges completely vanish at $\tau = \tau_0$. Next, we see how this result is modified when these cumulants have small but nonzero values at $\tau = \tau_0$. In Fig. 3 we show $1/X$ dependences of $\langle Q_{(\text{net})}^2 \rangle_c$ and $\langle Q_{(\text{net})}^4 \rangle_c$ with the initial condition,

$$[M_{(\text{net})}^2]_c = [M_{(\text{net})}^4]_c = [M_{(\text{net})}^2 M_{(\text{tot})}]_c = \frac{1}{2}[M_{(\text{tot})}]_c. \quad (37)$$

The figure shows that the suppression of $\langle Q_{(\text{net})}^4 \rangle_c$ compared with $\langle Q_{(\text{net})}^2 \rangle_c$ is not as clear as the result in Fig. 2, but $\langle Q_{(\text{net})}^4 \rangle_c < \langle Q_{(\text{net})}^2 \rangle_c$ is still satisfied for a rather wide range of c . The suppression of $\langle Q_{(\text{net})}^4 \rangle_c$ thus is a rather robust feature reflecting the small initial fluctuations. Therefore, this behavior of the cumulants at LHC energy can be used as an experimental probe to confirm the suppression of the fluctuations at hadronization.

The results in Figs. 2 and 3 also show that the cumulants $\langle Q_{(\text{net})}^2 \rangle_c$ and $\langle Q_{(\text{net})}^4 \rangle_c$ have characteristic behaviors as functions of $\Delta\eta$ depending on the initial condition. These results indicate that experimental measurements of not only the magnitudes of various cumulants at a fixed $\Delta\eta$ but also their $\Delta\eta$ dependence enable us to explore various information on the time evolution of the hot medium and the hadronization mechanism in the experiments. In particular, these analyses would enable us to estimate the magnitude of the parameter c . Because this parameter is sensitive to the hadronization mechanism, such experimental information will be used as an important clue to it.

5. Discussions

In this study, we employed a simple model, Eq. (3), to investigate the time evolution of non-Gaussian fluctuations. The model is expected to describe well the qualitative feature of the diffusion of higher order cumulants in the hadronic stage as already discussed. In order to explore the diffusion in each stage in heavy ion collisions more quantitatively, however, one must take various effects into account. In the present model, for example, fluctuations in equilibrium are given by the Poisson or Skellam distribution as a result of the absence of interactions between each Brownian particle. The model thus is not suitable to describe systems where equilibrated fluctuations do not follow either of these distributions. The easiest way to treat non-Poissonian fluctuations is to consider a system composed of several particle species having different charges which separately follow Eq. (3). The fluctuations of total charge of all particle species then become neither Poissonian nor Skellam distribution. This modification would be used as a first approximation to model the non-Skellam behavior of the net-electric charge fluctuations. When the non-Poissonian behavior comes from interaction between particles, terms describing the interaction should be introduced in Eq. (3). Next, the model Eq. (3) describes a Markov process and the stochastic effect does not have temporal correlations. To take account of nonzero temporal correlations, the model should be extended to non-Markov ones. Such a treatment would be required in, for example, dealing with the non-Gaussianity associated with critical phenomena and considering phenomena of the time scale of microscopic interaction. In the present study, it is also assumed that the numbers of two particle species $n_1(\eta, \tau)$ and $n_2(\eta, \tau)$ which carry opposite charges are separately conserved. While this assumption would be well justified for baryon numbers after chemical freezeout, effects of pair-creation and annihilation of electric charges would modify the time evolution of the net-electric charge fluctuations qualitatively. Finally, whereas the system investigated in the present study corresponds to the one with fixed diffusion constant D , τ dependence of this parameter should also be taken into account especially when one investigates expanding systems [19].

Besides these technical issues on the model, several assumptions also have been introduced on the geometry of the system and initial condition. To simplify the arguments, in the present Letter we limited our attention to the system which is infinitely long and the event-by-event configuration of the initial distribution is uniform and local, and thus satisfy Eq. (27). The hot medium created by heavy ion collisions, on the other hand, is a finite system, and the effects of global charge conservation and the violation of boost invariance modify the values of the fluctuation observables [1] and render them dependent on the position of the rapidity window. The validity of locality condition in Eq. (27) should also be investigated carefully. To investigate these effects, the model Eq. (3) has to be solved with appropriate boundary conditions and various initial conditions. Part of these issues will be investigated in the forthcoming paper [25].

In the present study, we limited our attention to sufficiently large $\sqrt{s_{\text{NN}}}$, where the measurements of odd order cumulants are difficult owing to their smallness. On the other hand, in the range of $\sqrt{s_{\text{NN}}}$ where the beam energy scan program at RHIC is exploring, the third-order cumulants are also observable [10,11] and will carry significant information on the QCD phase transition [7]. The time evolution and $\Delta\eta$ dependence of the third-order cumulants will also be discussed in Ref. [25].

In the present Letter, we investigated the time evolution of non-Gaussian fluctuations of conserved charges in a volume through diffusion using the diffusion master equation (3), which is intended to model the time evolution of higher order cumulants of conserved charges $N_Q^{(\text{net})}$ and $N_B^{(\text{net})}$ in the hadronic stage of relativistic heavy ion collisions. Our analysis shows that the fourth-order cumulant $\langle (N_Q^{(\text{net})})^4 \rangle_c$ is suppressed compared with the second-order one $\langle (N_Q^{(\text{net})})^2 \rangle_c$, provided that the suppression of $\langle (N_Q^{(\text{net})})^2 \rangle_c$ at ALICE [12] is a consequence of the survival of the smallness of the fluctuation at the primordial stage. The same conclusion is also anticipated for baryon number cumulants. It should, however, be noted that this result qualitatively depends on the parameter c in Eq. (36), which is not constrained only by the local charge conservation and sensitive to hadronization mechanism. We emphasize that the fourth-order cumulants of net-electric charge and baryon numbers, $\langle (N_Q^{(\text{net})})^4 \rangle_c$ and $\langle (N_B^{(\text{net})})^4 \rangle_c$, can be observed as functions of $\Delta\eta$ in the mid-rapidity region in heavy ion collisions at LHC and RHIC. The comparison of the experimental data on various cumulants with the present analysis will reveal the dynamical evolution of fluctuations and hadronization mechanism in heavy ion collisions.

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