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P-critical integral quadratic forms and positive unit forms: An algorithmic approach ☆

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ABSTRACT

We introduce the class of P-critical integral unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$ containing the critical forms in the sense of Ovsienko [13]. Several characterisations of P-critical forms are given. In particular, it is proved that q is P-critical if and only if there is a uniquely determined extended Dynkin diagram $\Delta \in \{\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and a special group isomorphism $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $q \circ T$ is the quadratic form $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}, n = |\Delta_0|$, of the diagram Δ. A correspondence between positive forms $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ with a sincere root and P-critical forms $q : \mathbb{Z}^n \to \mathbb{Z}$ is described and efficient linear algebra algorithms for computing P-critical unit forms and positive forms are constructed.

1. Introduction

Throughout, we denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{Z}$ the ring of integers. We view $\mathbb{Z}^n$, with $n \geqslant 1$, as a free abelian group. We denote by $e_1, \ldots, e_n$ the standard $\mathbb{Z}$-basis of $\mathbb{Z}^n$. By an **integral quadratic form** (more precisely, a homogeneous $\mathbb{Z}$-quadratic mapping) we mean a map $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}, \quad n \geqslant 1$, defined by the formula

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$$q(x) = q(x_1, \dots, x_n) = q_{11}x_1^2 + \cdots + q_{nn}x_n^2 + \sum_{i<j} q_{ij}x_ix_j, \tag{1.1}$$

where $q_{ij} \in \mathbb{Z}$, for $i, j \in \{1, \dots, n\}$. If $q_{11} = \cdots = q_{nn} = 1$, we call q a **unit** form. Given $j \in \{1, \dots, n\}$, we denote by $q^{(j)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ the jth restriction of q defined by the formula

$$q^{(j)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = q(x)|_{x_j=0} = q(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n).$$

Following [13], we call q **positive** (resp. **weakly positive**), if $q(v) > 0$, for all non-zero vectors $v \in \mathbb{Z}^n$ (resp. for all non-zero vectors $v \in \mathbb{N}^n$). We call q **non-negative** (resp. **weakly non-negative**), if $q(v) \geqslant 0$, for all vectors $v \in \mathbb{Z}^n$ (resp. for all vectors $v \in \mathbb{N}^n$).

A vector $v \in \mathbb{Z}^n$ is said to be a ***q*-root** (of unity), if $q(v) = 1$. The root v is called **positive** if the coordinates of $v_1, \dots, v_n$ are non-negative. We denote by

$$\mathcal{R}_q = \{v \in \mathbb{Z}^n;\ q(v) = 1\} \supset \mathcal{R}_q^+ = \{v \in \mathbb{N}^n;\ q(v) = 1\}$$

the set of q-roots and positive q-roots, respectively, see also [1, Section VII.3]. A vector $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$ is said to be **sincere**, if $v_1 \neq 0, \dots, v_n \neq 0$. We say that v is **positive**, if $v \neq 0$ and $v_1 \geqslant 0, \dots, v_n \geqslant 0$.

We recall from [13], that q is defined to be **critical** if q is not weakly positive, and each of the restrictions $q^{(1)}, \dots, q^{(n)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ of q, is weakly positive, see also [21, Section XIV.1]. Following Bondarenko and Polishchuk [3], we introduce the contept of a P-critical unit form, where P means, “with respect to the positivity".

Definition 1.2. The unit form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z},\ n \geqslant 1$, (1.1) is defined to be ***P*-critical** if q is not positive, and each of the restrictions $q^{(1)}, \dots, q^{(n)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ of q is positive.

It is clear that the class of critical forms is a subclass of the class of P-critical forms. The inverse inclusion does not hold, because there is a lot of weakly positive unit forms that are P-critical. For example, the weakly positive form $q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 - x_2x_3$ is P-critical, its kernel is generated by the sincere vector $\mathbf{h} = (-1, 1, 1)$, and is not critical.

The aim of the paper is to give several characterisations of P-critical unit forms. In particular, we show in Theorem 2.3 that the classification of P-critical unit forms reduces to the classification of the critical ones, and we prove that, given a unit form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$, with $n \geqslant 3$, the following four conditions are equivalent.

(a) q is P-critical.

(b) q is non-negative and the free abelian group $\operatorname{Ker} q = \{v \in \mathbb{Z}^n, q(v) = 0\}$ is infinite cyclic and is generated by a sincere vector $\mathbf{h} = (h_1, \dots, h_n)$, such that $1 \leqslant |h_j| \leqslant 6$, for all $j \in \{1, \dots, n\}$, and $|h_s| = 1$, for some $s \in \{1, \dots, n\}$.

(c) The set of roots of q is infinite, and each of the restrictions $q^{(1)}, \dots, q^{(n)} : \mathbb{Z}^n \to \mathbb{Z}$ of q has only finitely many roots.

(d) There exist an extended Dynkin diagram $\Delta \in \{\widetilde{\mathbb{A}}_n, n \geqslant 1, \widetilde{\mathbb{D}}_n, n \geqslant 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ (see [1, p. 252], [18,21]) and a group isomorphism $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $q \circ T$ is the quadratic form $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}, n = |\Delta_0|$, of the diagram Δ and T carries a sincere vector $\mathbf{h}' \in \operatorname{Ker} q_\Delta$ to a sincere vector.

For $n \geqslant 3$, a correspondence between positive forms $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ with a sincere root and P-critical forms $q : \mathbb{Z}^n \to \mathbb{Z}$ is described in Section 3. The P-critical form q constructed from p may be viewed as a one-point extension of p, compare with [15] and [21]. The correspondence is successfully applied in producing a class of P-critical unit forms.

In Section 4, we present two algorithms that compute positive unit forms and P-critical unit forms, for any $n \geqslant 3$. We describe in Corollaries 4.9 and 4.10 all positive unit forms for $n = 2, 3, 4, 5$, and all P-critical unit forms for $n = 3, 4, 5$, up to permutation of variables and up to the operation $q(x_1, \dots, x_n) \mapsto q(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$, with $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$.

In the proof of our main results we follow the ideas of Ovsienko [13] and von Höhne [12] applied in the study of critical forms and their classification. If we apply our theorem and its proof to the forms that are not weakly positive, we get the result of Ovsienko [13] extended by some useful equivalent conditions, see Corollary 2.7.

Using main results of this paper and recent results by Bondarenko-Polishchuk [3] and Bondarenko-Styopochkina [4,5], we will prove in a subsequent paper [14] that if I is a one-peak finite posets such that the Tits quadratic form $\widehat{q}_I$ [19] is P-critical (resp. positive) then there exists an extended Dynkin quiver Q (resp. Dynkin quiver Q) such that the $\mathbb{Z}$-bilinear Tits form $\hat{b}_I : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ of I (see [19]) is $\mathbb{Z}$-bilinear equivalent to the $\mathbb{Z}$-bilinear Euler form $b_Q : \mathbb{Z}^Q \times \mathbb{Z}^Q \to \mathbb{Z}$ of Q.

Our motivation for the study the P-critical forms comes from the fact that critical forms are P-critical and the critical forms (and their positive roots) have a lot of important applications in the study of tame algebras, tame vector space categories and tame bimodule matrix problems, see [6,8,11,15–17,21]. It follows from the results of Happel [9,10] that positive unit forms, the P-critical unit forms and their roots (not necessarily positive) provide with useful combinatorial tools for the study of tame derived categories $\mathcal{D}^b(A)$ of finite dimensional algebras A and their Auslander-Reiten quivers, see also [18]. We shall discuss the problem in a subsequent paper.

2. Main results

The quadratic form q (1.1) is uniquely determined by the symmetric Gram matrix $G_q = \frac{1}{2}[\check{G}_q + \check{G}_q^{tr}]$ of q, where

$$\check{G}_q = \begin{bmatrix} q_{1\,1} & q_{1\,2} & \cdots & q_{1\,n} \\ 0 & q_{2\,2} & \cdots & q_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n\,n} \end{bmatrix} \in \mathbb{M}_n(\mathbb{Z}) \tag{2.1}$$

is the non-symmetric Gram matrix of q (see [18]) and $\check{G}_q^{tr}$ means the transpose of $\check{G}_q$. Note that $q(x) = x \cdot \check{G}_q \cdot x^{tr} = x \cdot G_q \cdot x^{tr}$. We often use the symmetric $\mathbb{Z}$-bilinear **polar form**

$$b_q : \mathbb{Z}^n \times \mathbb{Z}^n \to \frac{1}{2} \cdot \mathbb{Z}$$

of q defined by the formula $b_q(x, y) = x \cdot G_q \cdot y^{tr} = \frac{1}{2}[q(x + y) - q(x) - q(y)]$, where the vectors $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ are viewed as one-row matrices. We recall that the kernel $\operatorname{Ker} q = \{v \in \mathbb{Z}^n;\ q(v) = 0\}$ of q is a subgroup of $\mathbb{Z}^n$, if q is non-negative. Following [20], we call the form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ **principal** if q is non-negative and the subgroup $\operatorname{Ker} q$ of $\mathbb{Z}^n$ is infinite cyclic.

In the proof of our main results we use the following key lemma.

Lemma 2.2. *Assume that $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is a unit form, $n \geqslant 3$, and $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ is a non-zero vector such that $q(\mathbf{h}) \leqslant 0$, and the norm $||\mathbf{h}|| := |h_1| + \cdots + |h_n|$ is minimal.*

(a) *If q is P-critical or q is critical and $\mathbf{h}$ is positive, the form q is non-negative, $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$ and $\mathbf{h}$ is sincere. The vector $\mathbf{h}$ is positive, if q is critical.*

(b) *If q is P-critical, the following three conditions are equivalent:*

(b1) *q is weakly positive,*

(b2) *the sincere vectors $\mathbf{h}$ and $-\mathbf{h}$ are not positive,*

(b3) *q is not critical.*

Proof. Here we follow an idea of Ovsienko in [13], see also [15] and [21, Section XIV.1]. Throughout the proof, given a vector $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ and any $j \in \{1, \ldots, n\}$, we set $v^{(j)} = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) \in \mathbb{Z}^{n-1}$. Let b_q be the bilinear polar form of q.

Assume that $n \geqslant 3$, q is the unit form (1.1), $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ is non-zero such that $q(\mathbf{h}) \leqslant 0$, and the norm $||\mathbf{h}||$ is minimal. Since $q(\mathbf{h}) = q(-\mathbf{h})$, without loss of generality, we can assume that there exits $s \in \{1, \ldots, n\}$ such that $h_1 \geqslant 1, \ldots, h_s \geqslant 1$ and $h_{s+1} \leqslant 0, \ldots, h_n \leqslant 0$.

1°: To show that the vector $\mathbf{h}$ is sincere we assume, to the contrary, that $h_j = 0$, for some $j \leqslant n$. If q is P-critical, the forms $q^{(1)}, \ldots, q^{(n)}$ are positive, $n \geqslant 3$, $\mathbf{h} \neq 0$ and we get the contradiction $0 < q^{(j)}(\mathbf{h}^{(j)}) = q(\mathbf{h}) \leqslant 0$. If q is critical and $\mathbf{h}$ is positive, the forms $q^{(1)}, \ldots, q^{(n)}$ are weakly positive, $s = n$, and we get the contradiction $0 < q^{(j)}(\mathbf{h}^{(j)}) = q(\mathbf{h}) \leqslant 0$.

It follows that $h_1 \geqslant 1, \ldots, h_s \geqslant 1, h_{s+1} \leqslant -1, \ldots, h_n \leqslant -1, ||\mathbf{h}|| = h_1 + \cdots + h_s - (h_{s+1} + \cdots + h_n)$, and $s = n$, if $\mathbf{h}$ is positive.

2°: Now we show that $q(\mathbf{h} - e_i) \geqslant 1$ and $q(\mathbf{h} + e_j) \geqslant 1$, for all $i \in \{1, \ldots, s\}$ and $j \in \{s+1, \ldots, n\}$.

Assume that q is P-critical. If $h_i = 1$ or $h_j = -1$, we have $\mathbf{h} - e_i \neq 0, \mathbf{h} + e_j \neq 0, q(\mathbf{h} - e_i) = q^{(i)}(\mathbf{h}^{(i)}) > 0$ and $q(\mathbf{h} + e_j) = q^{(j)}(\mathbf{h}^{(j)}) > 0$, because the forms $q^{(1)}, \ldots, q^{(n)}$ are positive. If $h_i > 1$ or $h_j < -1$, we have $||\mathbf{h} - e_i|| < ||\mathbf{h}||, ||\mathbf{h} + e_j|| < ||\mathbf{h}||$ and, hence $q(\mathbf{h} - e_i) > 0$ and $q(\mathbf{h} + e_j) > 0$, by the minimality of $||\mathbf{h}||$.

Assume that q is critical and $\mathbf{h}$ is positive. Then $s = n$, and, if $h_i = 1$ we get $q(\mathbf{h} - e_i) = q^{(i)}(\mathbf{h}^{(i)}) > 0$, because $\mathbf{h}^{(i)}$ is positive and $q^{(i)}$ is weakly positive. If $h_i > 1$, we have $q(\mathbf{h} - e_i) > 0$, because of $||\mathbf{h} - e_i|| < ||\mathbf{h}||$ and the minimality of $||\mathbf{h}||$.

3°: Next we show that $\mathbf{h} \in \operatorname{Ker} q$. For, we note that 2° yields

$$\begin{aligned} 1 &\leqslant q(\mathbf{h} - e_i) = q(\mathbf{h}) + q(e_i) - 2b_q(\mathbf{h}, e_i), \quad \text{for any } i \in \{1, \ldots, s\}, \\ 1 &\leqslant q(\mathbf{h} + e_j) = q(\mathbf{h}) + q(e_j) + 2b_q(\mathbf{h}, e_j), \quad \text{for any } j \in \{s+1, \ldots, n\}. \end{aligned}$$

Since $q(e_i) = 1$, we get $2b_q(\mathbf{h}, e_i) < q(\mathbf{h}) + 1$, for $i \leqslant s$, $-2b_q(\mathbf{h}, e_j) < q(\mathbf{h}) + 1$, for $j \leqslant s + 1$, and consequently we have

$$2b_q(\mathbf{h}, e_i) \leqslant q(\mathbf{h}) \leqslant 0, \text{ if } h_i \geqslant 1, \quad \text{and} \quad -2b_q(\mathbf{h}, e_j) \leqslant q(\mathbf{h}) \leqslant 0, \text{ if } h_j \geqslant 1, \tag{$*$}$$

because $2b_q(\mathbf{h}, e_1), \ldots, 2b_q(\mathbf{h}, e_n), q(\mathbf{h})$ are integers. Since $h_1 \geqslant 1, \ldots, h_s \geqslant 1$ and $h_{s+1} \leqslant -1, \ldots, h_n \leqslant -1$, the formula $(*)$ yields

$$2h_j \cdot b_q(\mathbf{h}, e_j) \leqslant h_j \cdot q(\mathbf{h}), \text{if } h_j > 0, \quad \text{and} \quad 2h_j \cdot b_q(\mathbf{h}, e_j) \leqslant -h_j \cdot q(\mathbf{h}), \text{if } h_j \leqslant 0, \tag{$**$}$$

and we get the inequalities

$$\begin{aligned} 2q(\mathbf{h}) &= 2b_q(\mathbf{h}, \mathbf{h}) = 2b_q(\mathbf{h}, \textstyle\sum_{j=1}^n h_j \cdot e_j) = \sum_{j=1}^n 2h_j \cdot b_q(\mathbf{h}, e_j) \\ &\leqslant h_1 \cdot q(\mathbf{h}) + \cdots + h_s \cdot q(\mathbf{h}) - h_{s+1} \cdot q(\mathbf{h}) - \cdots - h_n \cdot q(\mathbf{h}) \\ &= (h_1 + \cdots + h_s - h_{s+1} - \cdots - h_n) \cdot q(\mathbf{h}) = ||\mathbf{h}|| \cdot q(\mathbf{h}). \end{aligned}$$

Consequently, we have $2q(\mathbf{h}) \leqslant ||\mathbf{h}|| \cdot q(\mathbf{h})$. It follows that $q(\mathbf{h}) = 0$, because $q(\mathbf{h}) \leqslant 0$ and the inequalities $2q(\mathbf{h}) \leqslant ||\mathbf{h}|| \cdot q(\mathbf{h}), q(\mathbf{h}) < 0$ yield the contradiction $2 \geqslant ||\mathbf{h}|| \geqslant n \geqslant 3$.

4°: Next we show that $b_q(\mathbf{h}, -) = 0$. For, we note that 3° yields $0 = q(\mathbf{h}) = b_q(\mathbf{h}, \mathbf{h}) = h_1 \cdot b_q(\mathbf{h}, e_1) + h_2 \cdot b_q(\mathbf{h}, e_2) + \cdots + h_n \cdot b_q(\mathbf{h}, e_n) \leqslant 0$, because $q(\mathbf{h}) = 0$ and we have $h_r \cdot b_q(\mathbf{h}, e_r) \leqslant 0$, for $r \in \{1, \ldots, n\}$, by $(**)$. Since $h_1 \neq 0, \ldots, h_n \neq 0$, we get the equalities $b_q(\mathbf{h}, e_1) = 0, \ldots, b_q(\mathbf{h}, e_n) = 0$, and consequently $b_q(\mathbf{h}, -) = 0$.

5°: Finally we show that q is non-negative and $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$ by proving that any vector $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ satisfying $q(w) \leqslant 0$ lies in $\mathbb{Z} \cdot \mathbf{h}$. Assume that q is P-critical. Then $q^{(1)}$ is positive and, given $w \in \mathbb{Z}^n$ such that $q(w) \leqslant 0$, the vector $v := h_1 w - w_1 \mathbf{h}$ has $v_1 = 0$ and we have $0 \leqslant q^{(1)}(v^{(1)}) = q(v) = q(h_1 \cdot w - w_1 \cdot \mathbf{h}) = q(h_1 \cdot w) + q(w_1 \cdot \mathbf{h}) - 2 \cdot h_1 \cdot w_1 \cdot b_q(w, \mathbf{h}) = h_1^2 \cdot q(w) + w_1^2 \cdot q(\mathbf{h}) - 2 \cdot h_1 \cdot w_1 \cdot b_q(z, \mathbf{h}) = h_1^2 \cdot q(w) \leqslant 0$. Hence $q^{(1)}(v^{(1)}) = 0$ and the positivity of $q^{(1)}$ yields $v^{(1)} = 0$ and $v = 0$, that is, $w = \frac{w_1}{h_1} \cdot \mathbf{h}$. Hence, if q is P-critical, the statement 5° follows, because one easily checks that $\frac{w_1}{h_1}$ is an integer, see [21, p. 230]. Since the proof of 5° is analogous in case q is critical, the proof of (a) is complete.

(b) By (a), q is non-negative and $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$. Hence (b) follows. □

Given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and a unit form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$(1.1), we define the unit for $q * \varepsilon : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ by the formula $(q * \varepsilon)(x_1, \ldots, x_n) = q(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$.

Now we are able to prove the main result of the paper.

Theorem 2.3. *Let $n \geqslant 2$ and let $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ be a unit form* (1.1).

1°*: The following conditions are equivalent.*

(a) *The form q is P-critical.*
(b) *q is either critical, or it is P-critical and weakly positive.*

(c) *There exists* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, *with* $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, *such that the form* $q * \varepsilon$ *is critical.*
(d) *One of the following two exclusive conditions is satisfied*:

(d1) q *is not weakly positive, the set* $\mathcal{R}_q^+$ *is infinite, and* $\mathcal{R}^+_{q^{(1)}}, \ldots, \mathcal{R}^+_{q^{(n)}}$ *are finite,*
(d2) q *is weakly positive, the set* $\mathcal{R}_q$ *is infinite, and* $\mathcal{R}_{q^{(1)}}, \ldots, \mathcal{R}_{q^{(n)}}$ *are finite.*

2°: *If* $n = 2$ *then each of the conditions* (a)–(d) *is equivalent to the following one*:

(a′) *Either* q *is not weakly positive and* $q_{12} \leqslant -2$, *or* q *is weakly positive and* $q_{12} \geqslant 2$.
3°: *If* $n \geqslant 3$ *then each of the conditions* (a)–(d) *is equivalent to each of the following four equivalent conditions*.
(e) *The form* q *is non-negative and the group* Ker q *is infinite cyclic generated by a sincere vector.*
(e′) *The form* q *is principal and there exist a sincere vector* $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ *and* $s \in \{1, \ldots, n\}$ *such that* $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, $h_s \in \{-1, 1\}$ *and* $-6 \leqslant h_j \leqslant 6$, *for all* $j \in \{1, \ldots, n\}$.
(e″) *The form* q *is non-negative, there exists a sincere vector* $\mathbf{h}$ *such that* $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, *with* $h_1 \geqslant 1$, *and we have*

- q *is not weakly positive (i.e.* q *is critical) if and only if* $\mathbf{h}$ *is positive, and*
- q *is weakly positive (i.e.* q *is not critical) if and only if* $\mathbf{h}$ *is not positive.*

(f) *There exist an extended Dynkin diagram* $\Delta \in \{\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ *(see* [1, p. 252], [18,21]) *and a group isomorphism* $T : \mathbb{Z}^n \to \mathbb{Z}^n$ *such that* $q \circ T$ *is the quadratic form* $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}$, $n = |\Delta_0|$, *of the diagram* Δ *and* T *carries a sincere vector* $\mathbf{h}' \in \operatorname{Ker} q_\Delta$ *to a sincere one.*

If q *is non-negative and* $\mathbf{h} = (h_1, \ldots, h_n)$ *is a sincere vector such that* $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$ *then* $1 \leqslant |h_j| \leqslant 6$, *for all* $j \in \{1, \ldots, n\}$, *and* $h_s \in \{-1, 1\}$, *for some* $s \in \{1, \ldots, n\}$.

Proof. First we assume that $n = 2$. Then $q(x_1, x_2) = x_1^2 + x_2^2 + q_{12}x_1x_2$, the forms $q^{(1)}(x_2) = x_2^2$, $q^{(2)}(x_1) = x_1^2$ are obviously positive and $|\mathcal{R}_{q^{(1)}}| = |\mathcal{R}_{q^{(2)}}| = 2$, $|\mathcal{R}^+_{q^{(1)}}| = |\mathcal{R}^+_{q^{(2)}}| = 1$. Hence easily follows that
(i) q is P-critical if and only if $|q_{12}| \geqslant 2$, and
(ii) q is critical if and only if $q_{12} \leqslant -2$.
Hence the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (a′) $\Leftrightarrow$ (c) easily follow.
To prove the implication (b) $\Rightarrow$ (d) for $n = 2$, assume that q is P-critical and weakly positive. By (i), we have $|q_{12}| \geqslant 2$ and it follows that the set $\mathcal{R}_q$ contains the infinite sequence $w^{(0)}, w^{(1)}, w^{(2)}, \ldots, w^{(m)}, \ldots$ defined by the recursive formula:

$$w^{(m)} = \begin{cases} (1, 0), & \text{for } m = 0, \\ (q_{12}, -1), & \text{for } m = 1, \\ q_{12} \cdot w^{(m-1)} - w^{(m-2)}, & \text{for } m \geqslant 2. \end{cases}$$

Indeed, a simple calculation shows that $q(w^{(0)}) = q(1, 0) = 1$, $q(w^{(1)}) = q(q_{12}, -1) = 1$ and $q(w^{(2)}) = 1$, because $w^{(2)} = (q_{12}^2 - 1, -q_{12})$. Hence, by induction on $m \geqslant 3$, easily follows that $q(w^{(m)}) = 1$, for any $m \geqslant 3$. Since the vectors $w^{(0)}, w^{(1)}, w^{(2)}, \ldots, w^{(m)}, \ldots$ are pairwise different, the set $\mathcal{R}_q$ is infinite and (d2) follows.
Now assume that q is critical and weakly positive. By (ii), we have $q_{12} \leqslant -2$ and it follows that the set $\mathcal{R}_q^+$ contains the infinite sequence $u^{(0)}, u^{(1)}, u^{(2)}, \ldots, u^{(m)}, \ldots$ defined by the formula $u^{(m)} = w^{(2m)}$, for $m \geqslant 0$, that is, we take for $\{u^{(m)}\}$ the even part of the infinite sequence $w^{(0)}, w^{(1)}, w^{(2)}, \ldots, w^{(m)}, \ldots$ defined earlier. Note that the vectors $u^{(0)} = (1, 0)$ and $u^{(1)} = w^{(2)} = (q_{12}^2 - 1, -q_{12})$ are positive, because $q_{12} \leqslant -2$. Hence, by the induction on $m \geqslant 2$, follows that the vector $u^{(m)}$ is positive, for any $m \geqslant 2$. This shows that the set $\mathcal{R}_q^+$ is infinite, and (b1) follows.

To prove the implication (d) $\Rightarrow$ (b) for $n = 2$, assume that (d2) holds, that is, the set $\mathcal{R}_q$ is infinite. It follows that $|q_{12}| \geqslant 2$ (and hence (b) follows), because otherwise $q_{12} \in \{-1, 0, 1\}$ and a direct calculation shows that $|\mathcal{R}_q| = 4$, if $q_{12} = 0$, $|\mathcal{R}_q| = 6$, if $|q_{12}| = 1$, and we get a contradiction. Since the implication (d1) $\Rightarrow$ (b) follows in a similar way, the implication (d) $\Rightarrow$ (b) is proved, for $n = 2$.

Next we assume that $n \geqslant 3$. To prove the implication (a) $\Rightarrow$ (e), assume that q is P-critical. Then q is not positive, there is a non-zero vector $\mathbf{h}$ such that $q(\mathbf{h}) \leqslant 0$ and (e) follows by applying Lemma 2.2.

(e) $\Rightarrow$ (a) Assume that q is non-negative and $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, where $\mathbf{h}$ is sincere. It follows that the forms $q^{(1)}, \ldots, q^{(n)}$ are non-negative and q is not positive, because $q(\mathbf{h}) = 0$. To show that the forms are positive, it remains to prove that $\operatorname{Ker} q^{(j)} = 0$, for any $j \in \{1, \ldots, n\}$.

Assume that $(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_n) \in \mathbb{Z}^{n-1}$ and $q^{(j)}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_n) = 0$. Then $0 = q^{(j)}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_n) = q(\widehat{v})$, where $\widehat{v} = (v_1, \ldots, v_{j-1}, 0, v_{j+1}, \ldots v_n)$, and hence $\widehat{v} \in \operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$. It follows that $\widehat{v} = 0$ and $(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_n) = 0$, because $\mathbf{h}$ is sincere. Consequently, the forms $q^{(1)}, \ldots, q^{(n)}$ are positive and q is P-critical.

(e) $\Rightarrow$ (b) Assume that q is non-negative and $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, where $\mathbf{h}$ is sincere and $h_1 > 0$. If $\mathbf{h}$ is positive then q is not weakly positive and, by the arguments applied in the proof of (e) $\Rightarrow$ (a), the forms $q^{(1)}, \ldots, q^{(n)}$ are weakly positive and q is critical.

If $\mathbf{h}$ is not positive, there is $s \geqslant 2$ such that $h_s < 0$. To prove that q is weakly positive, assume that there is a non-zero vector $v \in \mathbb{N}^n$ such that $q(v) = 0$. Then $v \in \operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$ and we get a contradiction, because $h_1 > 0$ and $h_s < 0$. This finishes the proof of (e) $\Rightarrow$ (b).

(b) $\Rightarrow$ (e) Assume that q is critical. Then q is not weakly positive, there is a positive vector $\mathbf{h}$ such that $q(\mathbf{h}) \leqslant 0$ and (e) follows by applying Lemma 2.2. If q is P-critical, the implication (a) $\Rightarrow$ (e) yields (b) $\Rightarrow$ (e).

The implication (c) $\Rightarrow$ (e) is obvious. To prove the implication (e) $\Rightarrow$ (c), assume that q is non-negative and $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, where $\mathbf{h} \neq 0$ is sincere. We define $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ by setting $\varepsilon_j = 1$, if $h_j > 0$, and $\varepsilon_j = -1$, if $h_j < 0$. Obviously the form $q * \varepsilon$ is non-negative and $\operatorname{Ker}(q * \varepsilon)$ is generated by the sincere positive vector $\mathbf{h} * \varepsilon = (\varepsilon_1 h_1, \ldots, \varepsilon_n h_n)$. One easily shows, as in the proof of the implication (e) $\Rightarrow$ (b), that the form $q * \varepsilon$ is critical.

(e) $\Rightarrow$ (e$'$) Obviously (e) implies that q is principal and there exists a sincere vector $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ such that $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$. It is shown in the proof of the implication (e) $\Rightarrow$ (c) that there exists $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that the form $q * \varepsilon$ is critical and $\operatorname{Ker}(q * \varepsilon)$ is generated by the sincere positive vector $\mathbf{h} * \varepsilon = (\varepsilon_1 h_1, \ldots, \varepsilon_n h_n)$. By [12, Corollary 1.3] and [13], we have $1 \leqslant \varepsilon_1 h_1 \leqslant 6, \ldots, 1 \leqslant \varepsilon_n h_n \leqslant 6$. Moreover, by [12, Remark 2] and [13], there exists s such that $\varepsilon_s h_s = 1$. Hence, $h_s \in \{-1, 1\}$, $-6 \leqslant h_j \leqslant 6$, for all $j \in \{1, \ldots, n\}$, and (e$'$) follows.

The implication (e$'$) $\Rightarrow$ (e) is obvious.

(e$'$) $\Rightarrow$ (e$''$) Apply the equivalences (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (e$'$) and their proofs.

(e) $\Leftrightarrow$ (f). Assume that q is non-negative and $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, where $\mathbf{h}$ is sincere. It is shown in the proof of the implication (e)$\Rightarrow$(c) that there exists $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that the form $q * \varepsilon$ is critical and $\operatorname{Ker}(q * \varepsilon)$ is generated by the sincere positive vector $\mathbf{h} * \varepsilon = (\varepsilon_1 h_1, \ldots, \varepsilon_n h_n)$. By [12, Theorem 1.2] and [13], there exist an extended Dynkin diagram $\Delta \in \{\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and a group isomorphism $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that the unit form $(q * \varepsilon) \circ T$ is the quadratic form $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}$, $n = |\Delta_0|$, of the diagram Δ. Since $\operatorname{Ker} q_\Delta$ is generated by a positive sincere vector $\mathbf{h}' \in \operatorname{Ker} q_\Delta$ and $\operatorname{Ker}(q * \varepsilon) = \mathbb{Z} \cdot (\mathbf{h} * \varepsilon)$, where $\mathbf{h}$ is sincere, the automorphism T carries the vector $\mathbf{h}'$ to a sincere vector in $\operatorname{Ker}(q * \varepsilon)$. Since the converse implication (e) $\Leftarrow$ (f) follows in a similar way, the statements (a)–(f) are equivalent, and it remains to prove the equivalence of (a) and (d).

(a) $\Rightarrow$ (d) Assume that q is P-critical. By (b) $\Rightarrow$ (e), q is non-negative and $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, where $\mathbf{h}$ is sincere. Moreover, if q is critical, the vector $\mathbf{h}$ is positive. We recall from [1, p. 261], [15, p. 3], [20, Proposition 2.7] that $\operatorname{Ker} q$ coincides with the kernel of the gradient group homomorphism

$$Dq : \mathbb{Z}^n \to \mathbb{Z}^n, \quad v \mapsto Dq(v) = \left(\frac{\partial q}{\partial x_1}(v), \ldots, \frac{\partial q}{\partial x_n}(v) \right),$$

if q is non-negative. Since $\mathbf{h} \in \operatorname{Ker} q$ and $2b_q(x, e_j) = \frac{\partial q(x)}{\partial x_j}$, for $j = 1, \ldots, n$, we have $b_q(\mathbf{h}, -) = 0$. Then, given $\lambda \in \mathbb{Z}$ and $s \in \{1, \ldots, n\}$, we get

$$\begin{aligned} q(e_s + \lambda \cdot \mathbf{h}) &= b_q(e_s + \lambda \cdot \mathbf{h},\ e_s + \lambda \cdot \mathbf{h}) \\ &= b_q(e_s, e_s) + 2\lambda \cdot b_q(e_s,\ \mathbf{h}_q) + \lambda^2 \cdot b_q(\mathbf{h}_q,\ \mathbf{h}_q) \\ &= b_q(e_s, e_s) = q(e_s) = 1. \end{aligned} \tag{2.4}$$

This shows that q has infinitely many roots, if q is P-critical. Moreover, if q is critical then $\mathbf{h}$ is positive and $q(e_1 + \lambda \cdot \mathbf{h}) = 1$, for any $\lambda \in \mathbb{N}$, that is, q has infinitely many positive roots.

(d) $\Rightarrow$ (a) Assume that $n \geqslant 3$ and q is a unit form (1.1) such that the set $\mathcal{R}_q$ is infinite, and the sets $\mathcal{R}_{q^{(1)}}, \ldots, \mathcal{R}_{q^{(n)}}$ are finite.

First we prove that $-1 \leqslant q_{ij} \leqslant 1$, for all $i \leqslant j$. Assume, to the contrary, that $|q_{ij}| \geqslant 2$, for some $i < j$. Then the restriction $q_{[i,j]} : \mathbb{Z}^2 \to \mathbb{Z}$ of q given by the formula $q_{[i,j]}(x_i, x_j) = x_i^2 + x_j^2 + q_{ij}x_ix_j$ has only finitely many roots, because $n \geqslant 3$ and for any root (u_i, u_j) of $q_{[i,j]}$, the vector $(0, \ldots, 0, u_i, 0, \ldots, 0, u_j, 0, \ldots, 0) \in \mathbb{Z}^{n-1}$ belongs to the finite set $\mathcal{R}_{q^{(1)}} \cup \cdots \cup \mathcal{R}_{q^{(n)}}$. On the other hand, by 2°, the form $q_{[i,j]}$ has infinitely many roots, if $|q_{ij}| \geqslant 2$; a contradiction.

Next we prove by induction on $n \geqslant 2$ the following fact we use in the proof of (d) $\Rightarrow$ (a).

Claim 1. *Let $n \geqslant 2$ and $g : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form.*

(a) *The set $\mathcal{R}_g$ is finite if and only if g is positive.*

(b) *The set $\mathcal{R}_g^+$ is finite if and only if g is weakly positive.*

If g is positive, the set $\mathcal{R}_g$ is finite, by [7] and [18, Proposition 4.1]. Conversely, assume that $\mathcal{R}_g$ is finite. If $n = 2$ then $g_{12} \in \{-1, 0, 1\}$, by the observation made earlier (in the proof of the equivalence (b) $\Leftrightarrow$ (d)). Hence easily follows that g is positive.

Assume that $n \geqslant 3$, $g : \mathbb{Z}^n \to \mathbb{Z}$ is a unit form such that the set $\mathcal{R}_g$ is finite and our claim is proved for unit forms $\mathbb{Z}^{n-1} \to \mathbb{Z}$. Since $\mathcal{R}_g$ is finite, by the observation made earlier, the sets $\mathcal{R}_{g^{(1)}}, \ldots, \mathcal{R}_{g^{(n)}}$ are also finite and hence the forms $g^{(1)}, \ldots, g^{(n)}$ are positive, by the induction hypothesis. It follows that g is positive, because otherwise g is P-critical and, by the implication (a) $\Rightarrow$ (d) proved earlier, the set $\mathcal{R}_g$ is infinite and we get a contradiction. This finishes the proof of the statement (a) of Claim. The proof of (b) is similar and we leave it to the reader.

Now we complete the proof of (d) $\Rightarrow$ (a) by applying induction on $n \geqslant 3$. We recall that $-1 \leqslant q_{ij} \leqslant 1$, for all $i < j$. If $n = 3$, by Claim applied to $n = 2$, the forms $q^{(1)}, q^{(2)}, q^{(3)}$ are positive. Since the set $\mathcal{R}_q$ is infinite, by Claim, the form q is not positive. It follows that q is P-critical and we are done.

Assume that $n \geqslant 4$, q is a unit form (1.1) such that the set $\mathcal{R}_q$ is infinite and the sets $\mathcal{R}_{q^{(1)}}, \ldots, \mathcal{R}_{q^{(n)}}$ are finite. Assume that the implication (d) $\Rightarrow$ (b) is proved for unit forms of $n - 1$ variables. Hence the forms $q^{(1)}, \ldots, q^{(n)}$ are positive. Since the set $\mathcal{R}_q$ is infinite, by Claim, the form q is not positive, that is, q is P-critical. This finishes the proof of the implication (d) $\Rightarrow$ (a) and completes the proof the theorem. □

Corollary 2.5. *Let $n \geqslant 2$ and $p : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form.*

(a) *If p is positive then $|p_{ij}| \leqslant 1$, for all $i < j$, and $p^{(1)}, \ldots, p^{(n)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ are positive.*

(b) *The set $\mathcal{R}_p$ is finite if and only if p is positive.*

(c) *The set $\mathcal{R}_p^+$ is finite if and only if p is weakly positive.*

Proof. Apply Claim, its proof, [7] and [18, Proposition 4.1]. □

Corollary 2.6. (a) *For $n \geqslant 3$, a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ (1.1) is P-critical if and only if q is non-negative and* Ker q *is generated by a sincere vector.*

(b) *The classification of P-critical unit forms reduces to the classification of critical unit forms discussed in* [12].

Proof. Apply the equivalences (e) $\Leftrightarrow$ (a) $\Leftrightarrow$ (c) of Theorem 2.3. $\Box$

A consequence of Theorem 2.3, is the following extension of Ovsienko's theorem [13].

Corollary 2.7. *Let* $n \geqslant 2$ *and let* $q : \mathbb{Z}^n \to \mathbb{Z}$ *be unit form* (1.1).

1°*: If* $n = 2$, *q is critical if and only if* $q_{12} \leqslant -2$, *or equivalently, if and only if the set* $\mathcal{R}_q^+$ *is infinite.*

2°*: If* $n \geqslant 3$, *the following statements are equivalent.*

(a) *The form q is critical.*

(a′) *The form q is not weakly positive, and the restrictions* $q^{(1)}, \ldots, q^{(n)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ *of q are positive.*

(b) *The form q is non-negative and there exists a sincere positive vector* $\mathbf{h} \in \mathbb{Z}^n$ *such that* $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$.

(c) *The form q is principal and there exist a sincere vector* $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ *and* $s \in \{1, \ldots, n\}$ *such that* $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, $h_s = 1$ *and* $1 \leqslant h_j \leqslant 6$, *for all* $j \in \{1, \ldots, n\}$.

(d) *The set* $\mathcal{R}_q^+$ *is infinite and the sets* $\mathcal{R}_{q^{(1)}}^+, \ldots, \mathcal{R}_{q^{(n)}}^+$ *are finite.*

Corollary 2.8. *If* $q : \mathbb{Z}^n \to \mathbb{Z}$ *is positive unit form* (1.1), *with* $n \geqslant 2$, *and* $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ *is a root of q then* $-6 \leqslant v_j \leqslant 6$, *for all* $j \in \{1, \ldots, n\}$.

Proof. Assume that $q : \mathbb{Z}^n \to \mathbb{Z}$ is positive and $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ is a root of q, i.e. $q(v) = 1$. We define $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ by setting $\varepsilon_j = 1$, if $v_j > 0$, and $\varepsilon_j = -1$, if $v_j < 0$. Obviously the form $q * \varepsilon$ is positive (in particular, weakly positive) and the vector $v * \varepsilon = (\varepsilon_1 v_1, \ldots, \varepsilon_n v_n)$ is positive (because v is non-zero), and we have $1 = q(v) = (q * \varepsilon)(v * \varepsilon)$, that is, the positive vector $v * \varepsilon$ is a root of the weakly positive unit form $q * \varepsilon$. By theorem of Ovsienko [13], we have $1 \leqslant \varepsilon_j v_j \leqslant 6$, for all $j \in \{1, \ldots, n\}$ and hence $-6 \leqslant v_j \leqslant 6$, for all $j \in \{1, \ldots, n\}$. $\Box$

3. Positive unit forms versus *P*-critical ones

In this section we present a useful correspondence (3.10) between positive unit forms $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ with sincere roots and P-critical forms $q : \mathbb{Z}^n \to \mathbb{Z}$. The P-critical form q constructed from p may be viewed as a one-point extension of p, compare with [15] and [21]. The correspondence (3.10) can be successfully applied in producing a class of P-critical unit forms.

Following [20, Section 2], in the classification of P-critical forms $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ (1.1) we use the finite subgroup

$$\widehat{\mathbf{S}_n\mathbf{C}}_2 = \widehat{\mathbf{S}}_n \rtimes \widehat{\mathbf{C}}_2^n \subseteq \mathrm{O}(n, \mathbb{Z}); \tag{3.1}$$

of order $n! \cdot 2^n$ of the group $\mathrm{O}(n, \mathbb{Z})$ of orthogonal matrices in $\mathbb{M}_n(\mathbb{Z})$ generated by the two sets of matrices:

- the matrices $\hat{\varepsilon} = \varepsilon \cdot E$, where $E \in \mathbb{M}_n(\mathbb{Z})$ is the unity matrix and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbf{C}_2^n$ runs through all vectors with coefficients $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{C}_2 = \{-1, 1\}$, the cyclic group of order two, and
- the matrices $\widehat{\sigma} = M_\sigma$ of the group homomorphisms $\sigma : \mathbb{Z}^n \to \mathbb{Z}^n$ given by the permutation $\sigma \in \mathbf{S}_n$ and defined by $\sigma(x) = x \cdot M_\sigma^{tr} = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, where $\mathbf{S}_n$ is the symmetric group of order $n!$.

Following [18] and [20], we introduce the following definition.

Definition 3.2. (a) Given two matrices $A, B \in \mathbb{M}_n(\mathbb{Z})$, we set $A * B = B^{tr} \cdot A \cdot B$. We associate with A the quadratic form $q_A : \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula $q_A(x) = x \cdot A \cdot x^{tr}$.

(b) For $n \geqslant 2$, we denote by $\mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ the set of all unit forms $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ (1.1), and by

$$\begin{gathered}\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z}) \subseteq \mathbf{nneg}(\mathbb{Z}^n, \mathbb{Z}) \subseteq \mathcal{U}(\mathbb{Z}^n, \mathbb{Z}) \\ \mathbf{crit}(\mathbb{Z}^n, \mathbb{Z}) \subseteq P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z}) \subseteq \mathbf{princ}(\mathbb{Z}^n, \mathbb{Z}) \subseteq \mathcal{U}(\mathbb{Z}^n, \mathbb{Z})\end{gathered} \tag{3.3}$$

the subsets of $\mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ consisting of the positive, non-negative, critical, P-critical, and principal forms, respectively.

(c) For $n \geqslant 2$, we define a right action

$$* \ : \mathcal{U}(\mathbb{Z}^n, \mathbb{Z}) \times \widehat{\mathbf{S}_n\mathbf{C}}_2 \longrightarrow \mathcal{U}(\mathbb{Z}^n, \mathbb{Z}) \tag{3.4}$$

of the group $\widehat{\mathbf{S}_n\mathbf{C}}_2$ on the set $\mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ of unit forms $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ (1.1) by associating to $q \in \mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ and a matrix $B \in \widehat{\mathbf{S}_n\mathbf{C}}_2 = \widehat{\mathbf{S}}_n \ltimes \widehat{\mathbf{C}}_2^n$ the unit form $q * B : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ by setting $(q * B)(x) = q(x \cdot B^{tr})$, for $x \in \mathbb{Z}^n$ (see [20]).

Remark 3.5. (i) In general, $\check{G}_q * B$ is not the non-symmetric Gram matrix of $q * B$.

(ii) The numbers of zero coefficients of the matrices $\check{G}_q * B$ and $\check{G}_q$ are equal, because $(q * \widehat{\sigma})(x) = q(x \cdot \widehat{\sigma}^{tr}) = q(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is obtained from $q(x)$ by permuting the coordinates $x_1, \dots, x_n$ of the vector $x = (x_1, \dots, x_n)$ under $\sigma \in \mathbf{S}_n$; and we have $(q * \hat{\varepsilon})(x) = q(x \cdot \hat{\varepsilon}) = q(\varepsilon_1 \cdot x_1, \dots, \varepsilon_n \cdot x_n)$, for $\varepsilon \in \mathbf{C}_2^n$, with $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$.

(iii) $\det B \in \{-1, 1\}$, the form $q * B$ is $\mathbb{Z}$-equivalent with q and we have

$$G_{q * B} = G_q * B \quad \text{and} \quad q_A * B = q_{A * B}, \tag{3.6}$$

for any $B \in \widehat{\mathbf{S}_n\mathbf{C}}_2$ and $A \in \mathbb{M}_n(\mathbb{Z})$.

(iv) It is easy to see that $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$, $\mathbf{nneg}(\mathbb{Z}^n, \mathbb{Z})$, $\mathbf{princ}(\mathbb{Z}^n, \mathbb{Z})$, and P-$\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})$ are $\widehat{\mathbf{S}_n\mathbf{C}}_2$-invariant subsets of $\mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$.

A relation between the P-critical forms and the positive ones is described as follows.

Proposition 3.7. *Assume that $q : \mathbb{Z}^n \to \mathbb{Z}$ is a unit form* (1.1), *with $n \geqslant 3$.*

(a) *If q is P-critical,* $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$ *and $s \in \{1, \dots, n\}$ is such that $h_s \in \{-1, 1\}$ (see Theorem 2.3* (e$'$)) *then*

(a1) *the vector $\mathbf{h}^{(s)} := (h_1, \dots, h_{s-1}, h_{s+1}, \dots, h_n) \in \mathbb{Z}^{n-1}$ is a sincere root of the positive (connected) unit form $q^{(s)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$, and*

(a2) *the form q can be reconstructed from the triple $(q^{(s)}, s, \mathbf{h}^{(s)})$ by the formula*

$$q(x) = q^{(s)}(x^{(s)}) + x_s^2 - 2 \cdot b_{q^{(s)}}(x^{(s)}, \mathbf{h}^{(s)}) \cdot h_s \cdot x_s, \tag{3.8}$$

where $b_{q^{(s)}}$ is the symmetric bilinear polar form of $q^{(s)}(x^{(s)}) = q^{(s)}(x_1, \dots, x_{s-1}, x_{s+1}, \dots x_n)$.

(b) *Given $s \in \{0, 1, \dots, n\}$, $\varepsilon_s \in \{-1, 1\}$, a positive (connected) unit form $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$, with $n \geqslant 2$, and a sincere root $w = (w_1, \dots, w_{n-1}) \in \mathbb{Z}^{n-1}$ of p, the unit form $q := q_{p,s,w,\varepsilon_s} : \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula*

$$q(x_1, \dots, x_n) = p(x^{(s)}) + x_s^2 - 2 \cdot b_p(x^{(s)}, w) \cdot \varepsilon_s \cdot x_s \tag{3.9}$$

is P-critical and $\operatorname{Ker} q = \mathbb{Z} \cdot \widehat{w}^{\varepsilon_s}$, *where $\widehat{w}^{\varepsilon_s} := (w_1, \dots, w_{s-1}, \varepsilon_s, w_s, \dots w_{n-1}) \in \mathbb{Z}^n$.*

(c) *The set*

$$\mathcal{Z}_{n-1} = \{(p, w);\ \ p \in \mathbf{posit}(\mathbb{Z}^{n-1}, \mathbb{Z}),\ w \in \mathbb{Z}^{n-1} \text{ a sincere root of } p\} \subseteq \mathbf{posit}(\mathbb{Z}^{n-1}, \mathbb{Z}) \times \mathbb{Z}^{n-1}$$

is an $\widehat{\mathbf{S}_{n-1}\mathbf{C}}_2$-invariant subset of $\mathbf{posit}(\mathbb{Z}^{n-1}, \mathbb{Z}) \times \mathbb{Z}^{n-1}$ under the action $(p, w) * B := (p * B, w \cdot B^{tr})$. The map $(p, w) \mapsto \operatorname{ind}(p, w) := q_{p,s,w,\varepsilon_s}$ described in (3.9) *defines a surjection*

$$\widehat{\mathbf{S}_{n-1}\mathbf{C}}_2 - \mathcal{O}rb(\mathcal{Z}_{n-1}) \xrightarrow{\ \text{ind}\ } \widehat{\mathbf{S}_n\mathbf{C}}_2 - \mathcal{O}rb(P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})) \tag{3.10}$$

between the set of $\widehat{\mathbf{S}_{n-1}\mathbf{C}}_2$-orbits of $\mathcal{Z}_{n-1}$ and the set of $\widehat{\mathbf{S}_n\mathbf{C}}_2$-orbits of P-$\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})$. A right inverse of ind *is given by the formula $q \mapsto \operatorname{res}_s(q) := (q^{(s)}, \mathbf{h}^{(s)})$ defined in* (a), *that associates to any P-critical form q, with* $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$ *and $h_s \in \{-1, 1\}$, the pair $(q^{(s)}, \mathbf{h}^{(s)}) \in \mathcal{Z}_{n-1}$.*

Proof. (a) Fix $s \in \{1, \dots, n\}$ and assume that $h_s \in \{-1, 1\}$. By (2.4), the vector $\check{\mathbf{h}}^{h_s} := \mathbf{h} - h_s e_s$ is a root of the P-critical form q. Hence $1 = q(\check{\mathbf{h}}^{h_s}) = q^{(s)}(\mathbf{h}^{(s)})$, that is, the sincere vector $\mathbf{h}^{(s)}$ is a root of the positive form $q^{(s)}$ and (a1) follows. It is easy to see that the form $q^{(s)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$ is connected, because it has a sincere root.

(a2) First we recall that $b_q(-, \mathbf{h}) = 0$, because q is non-negative and $\mathbf{h} \in \operatorname{Ker} q$. It follows that, given $j \neq s$, we have $0 = b_q(e_j, \mathbf{h}) = b_q(e_j, \mathbf{h} - h_s e_s + h_s e_s) = b_q(e_j, \check{\mathbf{h}}^{h_s}) + h_s \cdot b_q(e_j, e_s)$, that is,

$$b_q(e_j, e_s) = -h_s^{-1} \cdot b_q(e_j, \check{\mathbf{h}}^{h_s}) = -h_s \cdot b_{q^{(s)}}(e_j, \mathbf{h}^{(s)}),$$

because $h_s^{-1} = h_s$. Hence we get the equalities

$$\begin{aligned} q(x) &= q((x_1, \ldots, x_{s-1}, 0, x_{s+1}, \ldots x_n) + x_s e_s) \\ &= q^{(s)}(x^{(s)}) + x_s^2 + 2 \cdot b_q((x_1, \ldots, x_{s-1}, 0, x_{s+1}, \ldots x_n), x_s e_s) \\ &= q^{(s)}(x^{(s)}) + x_s^2 + \sum_{j \neq s} 2 \cdot b_q(e_j, e_s) \cdot x_j \cdot x_s \\ &= q^{(s)}(x^{(s)}) + x_s^2 - \sum_{j \neq s} 2 \cdot h_s \cdot b_{q^{(s)}}(e_j, \mathbf{h}^{(s)}) \cdot x_j \cdot x_s \\ &= q^{(s)}(x^{(s)}) + x_s^2 - 2 \cdot b_{q^{(s)}}(x^{(s)}, \mathbf{h}^{(s)}) \cdot h_s \cdot x_s \end{aligned}$$

and (3.8) follows.

(b) Assume that $(p, w) \in \mathcal{Z}_{n-1}$ and $q := q_{p,s,w,\varepsilon_s}$ is defined by the formula (3.9). For simplicity of the presentation, we assume that $s = 0$. Then the Gram matrix $G_q = \frac{1}{2}[\check{G}_q + \check{G}_q^{tr}]$ of q and the non-symmetric Gram matrix $\check{G}_q$ (2.1) of q have the forms

$$G_q = \left[\begin{array}{c|c} 1 & -w \cdot \epsilon_0 \cdot G_p \\ \hline -G_p \cdot w^{tr} \cdot \epsilon_0 \cdot & G_p \end{array}\right] \quad \text{and} \quad \check{G}_q = \left[\begin{array}{c|c} 1 & -w \cdot 2\epsilon_0 \cdot G_p \\ \hline 0 & \check{G}_p \end{array}\right], \tag{3.11}$$

where G_p and $\check{G}_p$ is the Gram matrix and the non-symmetric Gram matrix of the positive form p. Hence $\det G_p \neq 0$ and a simple calculation shows that $\det G_q = (1 - w \cdot G_p \cdot w^{tr}) \cdot \det G_p = (1 - p(w)) \cdot \det G_p = 0$, because w is a root of p. It follows that the matrix G_q is of corank one, q is non-negative and $\operatorname{Ker} q = \{v \in \mathbb{Z}^n;\ G_q \cdot v^{tr} = 0\}$ is an infinite cyclic group. Since the vector $\widehat{w}^{\varepsilon_s} = (w_1, \ldots, w_{s-1}, \varepsilon_s, w_s, \ldots, w_{n-1}) \in \mathbb{Z}^n$ is sincere and $q(\widehat{w}^{\varepsilon_s}) = p(w) + 1 - 2 \cdot b_p(w, w) \cdot \varepsilon_s \cdot \varepsilon_s = 0$, the kernel $\operatorname{Ker} q$ of q is generated by the sincere vector $\widehat{w}^{\varepsilon_s}$, and q is P-critical, by Corollary 2.6. Hence (b) follows.

Since (c) is an immediate consequence of (a) and (b), the proof is complete. □

In the following section we show how P-critical forms $q : \mathbb{Z}^n \to \mathbb{Z}$ can be constructed from positive forms $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$, with a sincere root, by applying the correspondence ind in (3.10) described by the formula (3.9).

4. Positive unit forms: algorithms

It follows from Proposition 3.7 that a description of P-critical unit forms $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$ reduces to a description of positive unit forms $p : \mathbb{Z}^n \to \mathbb{Z}$ with a sincere root, for $n \geqslant 3$. We recall that $v * B := (v_1, \ldots, v_n) \cdot B^{tr}$, for $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ and $B \in \widehat{\mathbf{S}_n \mathbf{C}}_2$.

The main aim of this section is to find an algorithmic procedure that constructs all positive unit forms $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$ from the set $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z}) \subseteq \mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ of positive forms $p : \mathbb{Z}^n \to \mathbb{Z}$, for $n \geqslant 2$. The procedure is described in the following theorem and its proof.

Theorem 4.1. *Assume that $n \geqslant 2$ and $\widehat{\mathbf{S}_n \mathbf{C}}_2 \subseteq \mathrm{O}(n, \mathbb{Z})$ is the finite group* (3.1). *Let $\mathbf{posit}_n^\bullet$ be a fixed set of pairwise different representatives $p : \mathbb{Z}^n \to \mathbb{Z}$ of all $\widehat{\mathbf{S}_n \mathbf{C}}_2$-orbits in $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ and define $\check{\mathcal{W}}_n \subseteq \mathcal{W}_n \subseteq \mathbb{Z}^n$ to be the finite sets*

$$\check{\mathcal{W}}_n := \{\mu = (\mu_1, \ldots, \mu_s, 1, \ldots, 1) \in \mathbb{Z}^n, \text{ with } \mu_1 = \ldots = \mu_s = 0 \text{ and } 0 \leqslant s \leqslant n\}$$

$$\mathcal{W}_n := \{w = (w_1, \ldots, w_n) \in \mathbb{Z}^n, \text{ with } w_1, \ldots, w_n \in \{-1, 0, 1\} \text{ and } w \neq 0\} \tag{4.2}$$

of cardinality $|\check{\mathcal{W}}_n| = n$ *and* $|\mathcal{W}_n| = 3^n - 1$. *Moreover, given* $p \in \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$, *we set*

$$\mathcal{W}^+_{n,p} = \left\{\mu \in \mathcal{W}_n, \ \det \begin{bmatrix} 2G_p & \mu^{tr} \\ \mu & 2 \end{bmatrix} > 0\right\} \subseteq \mathcal{W}_n, \tag{4.3}$$

where G_p *is the Gram matrix of* p.

(a) *The subset* $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ *of* $\mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ *is finite and* $\widehat{\mathbf{S}_n\mathbf{C}}_2$*-invariant. The subset* $\mathcal{W}_n$ *of* $\mathbb{Z}^n$ *is* $\widehat{\mathbf{S}_n\mathbf{C}}_2$*-invariant and the equality* $\mathcal{W}_n = \check{\mathcal{W}}_n * \widehat{\mathbf{S}_n\mathbf{C}}_2$ *holds.*

(b) *Given* $p \in \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ *and* $\mu \in \mathcal{W}^+_{n,p}$, *the unit form*

$$\hat{p}^{\mu} : \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \tag{4.4}$$

defined by the formula $\hat{p}^{\mu}(x) = x \cdot \begin{bmatrix} \check{G}_p & \mu^{tr} \\ 0 & 1 \end{bmatrix} \cdot x^{tr}$, *is positive and* $\check{G}_{\hat{p}^{\mu}} = \left[\begin{array}{c|c} \check{G}_p & \mu^{tr} \\ \hline 0 & 1 \end{array}\right]$ *is its non-symmetric Gram matrix, where* $\check{G}_p$ *is the non-symmetric Gram matrix* (2.1) *of* p.

(c) *If* $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ *is a sincere root of* $p \in \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ *such that* $w \cdot \mu^{tr} := w_1\mu_1 + \cdots + w_n\mu_n \neq 0$, *then* $\widehat{w}^{\mu} = (w, -w \cdot \mu^{tr}) \in \mathbb{Z}^{n+1}$ *is a sincere root of* $\hat{p}^{\mu}$.

Conversely, if $v = (v_1, \ldots, v_n, v_{n+1}) \in \mathbb{Z}^{n+1}$ *is a sincere root of* $\hat{p}^{\mu}$ *and* $-v_{n+1} = \check{v} \cdot \mu^{tr} := v_1\mu_1 + \cdots + v_n\mu_n \neq 0$, *then the vector* $\check{v} = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ *is a sincere root of* p.

(d) *For any unit form* $q \in \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ *there exist* $p \in \mathbf{posit}^{\bullet}_n \subseteq \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$, $\mu \in \mathcal{W}^+_{n,p}$, *and* $B \in \widehat{\mathbf{S}_n\mathbf{C}}_2$ *such that*

$$q * \overline{B} = \hat{p}^{\mu},$$

where $\overline{B} = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array}\right] \in \widehat{\mathbf{S}_{n+1}\mathbf{C}}_2$.

(e) *Every* $q \in \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ *is of the form* $\hat{p}^{\mu}$ (4.4), *where* $p \in \mathbf{posit}^{\bullet}_n$ *and* $\mu \in \mathcal{W}^+_{n,p}$, *up to the action* $* : \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z}) \times \widehat{\mathbf{S}_{n+1}\mathbf{C}}_2 \longrightarrow \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ (3.4) *of the group* $\widehat{\mathbf{S}_{n+1}\mathbf{C}}_2$.

Proof. (a) Assume that $n \geqslant 2$ and $p : \mathbb{Z}^n \to \mathbb{Z}$ is a positive form defined by the formula $p(x_1, \ldots, x_n) = x_1^2 + \cdots + x_1^2 + \sum_{i<j} p_{ij}x_ix_j$, where $p_{ij} \in \mathbb{Z}$. Then, given $i < j$, the restriction $p_{[i,j]} : \mathbb{Z}^2 \to \mathbb{Z}$ of p given by the formula $p_{[i,j]}(x_i, x_j) = x_i^2 + x_j^2 + p_{ij}x_ix_j$ is also positive and $|p_{ij}| \leqslant 1$, by the observation (i) in the proof of Theorem 2.3. It follows that $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ is a finite set, for any $n \geqslant 2$.

It is clear that each of the coordinates of the vector $v * B := (v_1, \ldots, v_n) \cdot B^{tr}$ lies in $\{-1, 0, 1\}$, if $v = (v_1, \ldots, v_n) \in \mathcal{W}_n$ and $B \in \widehat{\mathbf{S}_n\mathbf{C}}_2$. It follows that $\mathcal{W}_n$ is a $\widehat{\mathbf{S}_n\mathbf{C}}_2$-invariant subset of $\mathbb{Z}^n$ and the inclusion $\mathcal{W}_n \supseteq \check{\mathcal{W}}_n * \widehat{\mathbf{S}_n\mathbf{C}}_2$ holds. To prove the inverse inclusion, assume that $v = (v_1, \ldots, v_n) \in \mathcal{W}_n$. Define the vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbf{C}_2^n$ by setting $\varepsilon_j = -1$, if $v_j = -1$, and $\varepsilon_j = 1$, if $v_j \geqslant 0$. Then $\hat{\varepsilon} \in \widehat{\mathbf{S}_n\mathbf{C}}_2$, the vector $\eta = (\eta_1, \ldots, \eta_n) := v * \hat{\varepsilon}$ is non-zero and $\eta_1, \ldots, \eta_n \in \{0, 1\}$. It follows that there exist a permutation $\sigma \in \mathbf{S}_n$ such that the vector $\mu = \eta * \widehat{\sigma}$ lies in $\check{\mathcal{W}}_n$. Hence $\mu = \eta * \widehat{\sigma} = (v * \hat{\varepsilon}) * \widehat{\sigma} = v * B$, where $B = \hat{\varepsilon} * \widehat{\sigma} \in \widehat{\mathbf{S}_n\mathbf{C}}_2$, and the inclusion $\mathcal{W}_n \subseteq \check{\mathcal{W}}_n * \widehat{\mathbf{S}_n\mathbf{C}}_2$ holds. This completes the proof of (a).

(b) Fix $p \in \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ and $\mu \in \mathcal{W}^+_{n,p}$. If $\check{G}_p \in \mathbb{M}_n(\mathbb{Z})$ is the non-symmetric Gram matrix (2.1) of p then $2G_{\hat{p}^{\mu}} = \check{G}_{\hat{p}^{\mu}} + \check{G}^{tr}_{\hat{p}^{\mu}} = \left[\begin{array}{c|c} 2G_p & \mu^{tr} \\ \hline \mu & 2 \end{array}\right]$ and obviously $\check{G}_{\hat{p}^{\mu}} = \left[\begin{array}{c|c} \check{G}_p & \mu^{tr} \\ \hline 0 & 1 \end{array}\right]$ is the non-symmetric Gram matrix of the form $\hat{p}^{\mu}$. Since p is positive, in view of the determinant Sylvester criterion, the form $\hat{p}^{\mu} : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is positive if and only if $\det 2G_{\hat{p}^{\mu}} > 0$, or equivalently, if and only if $\det \begin{bmatrix} 2G_p & \mu^{tr} \\ \mu & 2 \end{bmatrix} > 0$. It follows that the form $\hat{p}^{\mu} : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is positive, because we assume that $\mu \in \mathcal{W}^+_{n,p}$.

(c) Assume that $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ is a sincere root of p such that $w \cdot \mu^{tr} \neq 0$, and let $\widehat{w}^{\mu} = (w, -w \cdot \mu^{tr})$. Then $1 = p(w) = w \cdot \check{G}_p \cdot w^{tr}$ and we have $\hat{p}^{\mu}(\widehat{w}^{\mu}) = \widehat{w}^{\mu} \cdot \begin{bmatrix} \check{G}_p & \mu^{tr} \\ 0 & 1 \end{bmatrix} \cdot (\widehat{w}^{\mu})^{tr} =$

$(w \cdot \check{G}_p, 0) \cdot (\widehat{w}^{\mu})^{tr} = w \cdot \check{G}_p \cdot w^{tr} = 1$, that is, $\widehat{w}^{\mu}$ is a sincere root of $\hat{p}^{\mu}$. The converse implication follows in a similar way.

(d) Assume that $q \in \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$. Then the restriction $p_0 := q^{(n+1)} : \mathbb{Z}^n \to \mathbb{Z}$ is positive and $|q_{ij}| \leqslant 1$, for all $i < j$, by Corollary 2.5(a). If G_q is the Gram matrix of q then the matrix $2G_q$ has the form $2G_q = \left[\begin{array}{c|c} 2G_{p_0} & w^{tr} \\ \hline w & 2 \end{array}\right]$, where $w = (q_{1n+1}, \ldots, q_{nn+1}) \in \mathbb{Z}^n$. It follows that q has the form $q = \hat{p}_0^w$. Since $|q_{jn+1}| \leqslant 1$, for all $j = 1, \ldots, n$, the vector w lies in $\mathcal{W}_n = \check{\mathcal{W}}_n * \widehat{\mathbf{S}_n\mathbf{C}}_2$, by (a). Hence, there is a matrix $B_1 = \hat{\varepsilon} \cdot \widehat{\sigma} \in \widehat{\mathbf{S}_n\mathbf{C}}_2$, with $\varepsilon \in \mathbf{C}_2^n$ and $\sigma \in \mathbf{S}_n$, such that the vector $\eta := w * B_1$ lies in the set $\check{\mathcal{W}}_n$. Let $\overline{B}_1 = \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & 1 \end{array}\right] \in \mathbf{S}_{n+1} \subseteq \widehat{\mathbf{S}_{n+1}\mathbf{C}}_2$.

If $\check{G}_q$ is the non-symmetric Gram matrix of q then the matrix

$$A_1 := \check{G}_q * \overline{B}_1 = \left[\begin{array}{c|c} \check{G}_{p_0} * B_1 & (w * B_1)^{tr} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \check{G}_{p_0} * B_1 & \eta^{tr} \\ \hline 0 & 1 \end{array}\right]$$

defines the quadratic form $q_1 := q * \overline{B}_1 : \mathbb{Z}^{n+1} \to \mathbb{Z}$ such that $2G_{q_1} = \left[\begin{array}{c|c} 2\check{G}_{p_0} * B_1 & \eta^{tr} \\ \hline \eta & 2 \end{array}\right]$ and $q_1 = q_{\check{G}_{p_0} * B_1}$, see (3.6). It is easy to see that q_1 is positive. Hence, the restriction $p_1 := q_1^{(n+1)} : \mathbb{Z}^n \to \mathbb{Z}$ is also positive, $q_1 = q_{A_1}$, and we have $p_1 = p_0 * B_1, q_1 = \widehat{p_0 * B_1}^{\eta}$ and $\eta \in \mathcal{W}^+_{n,p_1}$.

Since p_1 is positive, there exists $B_2 \in \widehat{\mathbf{S}_n\mathbf{C}}_2$ such that the unit form $p := p_1 * B_2$ lies in $\mathbf{posit}^\bullet_n$. If we set $\mu := \eta * B_2$ and $\overline{B}_2 = \left[\begin{array}{c|c} B_2 & 0 \\ \hline 0 & 1 \end{array}\right] \in \widehat{\mathbf{S}_{n+1}\mathbf{C}}_2$ then $q_2 \coloneqq q_1 * \overline{B}_2 = q_{A_1 * \overline{B}_2}$, by (3.6),

$$2G_{q_2} = 2G_{q_1} * \overline{B}_2 = \left[\begin{array}{c|c} 2G_{p_1} * B_2 & (\eta * B_2)^{tr} \\ \hline \eta * B_2 & 2 \end{array}\right] = \left[\begin{array}{c|c} 2G_p & \mu^{tr} \\ \hline \mu & 2 \end{array}\right]$$

is the symmetric Gram matrix of q_2, and hence $2G_{q_2} = \left[\begin{array}{c|c} 2G_p & \mu^{tr} \\ \hline \mu & 2 \end{array}\right]$. Since q_2 is positive, we have $\det \begin{bmatrix} 2G_p & \mu^{tr} \\ \mu & 2 \end{bmatrix} > 0$, that is, the vector μ lies in $\mathcal{W}^+_{n,p}$, $q_2 = \hat{p}^{\mu}$, $p = p_0 * B$ and $\hat{p}^{\mu} = q_2 = q * \overline{B}$, where $B = B_1 \cdot B_2 \in \widehat{\mathbf{S}_n\mathbf{C}}_2$ and $\overline{B} = \overline{B}_1 \cdot \overline{B}_2 \in \widehat{\mathbf{S}_{n+1}\mathbf{C}}_2$. This finishes the proof of (d). Since (e) is an immediate consequence of (d), the proof is complete. □

As a consequence of Theorem 4.1 and its proof we get the following algorithm producing the set $\mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ from the set $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$, for $n \geqslant 2$.

Algorithm 4.5. Input: An integer $n \geqslant 2$, the finite sets of matrices $\widehat{\mathbf{S}_n\mathbf{C}}_2 \subseteq \mathbb{M}_n(\mathbb{Z})$ and $\widehat{\mathbf{S}_{n+1}\mathbf{C}}_2 \subseteq \mathbb{M}_{n+1}(\mathbb{Z})$ (see (3.1)), and a fixed finite set $\mathbf{posit}^\bullet_n \subseteq \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ of representatives $p : \mathbb{Z}^n \to \mathbb{Z}$ of all $\widehat{\mathbf{S}_n\mathbf{C}}_2$-orbits in $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$.

Output: The finite family $\{\mathcal{W}^+_{n,p}\}_{p \in \mathbf{posit}^\bullet_n}$ of the finite sets $\mathcal{W}^+_{n,p}$ (4.3), and a finite set $\mathbf{posit}^\bullet_{n+1} \subseteq \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ of pairwise different representatives $\hat{p}^{\mu} : \mathbb{Z}^{n+1} \to \mathbb{Z}$ of all $\widehat{\mathbf{S}_{n+1}\mathbf{C}}_2$-orbits in $\mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$.

Step 1°: Construct the finite sets $\check{\mathcal{W}}_n \subseteq \mathcal{W}_n$ (4.2) as lists of vectors in $\mathbb{Z}^n$.

Step 2°: Given $p \in \mathbf{posit}^\bullet_n$, construct the matrices $\check{G}_p$, $2G_p = \check{G}_p + \check{G}_p^{tr}$, and the finite set $\mathcal{W}^+_{n,p}$ (4.3) as the sublist of the list $\mathcal{W}_n$, by selecting the vectors $\mu \in \mathcal{W}_n$ satisfying the inequality $\det \begin{bmatrix} 2G_p & \mu^{tr} \\ \mu & 2 \end{bmatrix} > 0$.

Step 3°: Given $p \in \mathbf{posit}^\bullet_n$ and $\mu \in \mathcal{W}^+_{n,p}$, construct the matrix $G_{p,\mu} := \begin{bmatrix} G_p & \mu^{tr} \\ 0 & 1 \end{bmatrix} \in \mathbb{M}_{n+1}(\mathbb{Z})$ and the positive unit form $\hat{p}^{\mu}(x) = x \cdot G_{p,\mu} \cdot x^{tr}$, where $x = (x_1, \ldots, x_n, x_{n+1})$.

Step 4°: Construct the list $\mathcal{P}_{n+1} = \{\hat{p}^{\mu}\} \subseteq \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$, where p and μ run through all $p \in \mathbf{posit}^\bullet_n$ and $\mu \in \mathcal{W}^+_{n,p}$.

Step 5°: Construct a finite set $\mathbf{posit}^{\bullet}_{n+1} \subseteq \mathcal{P}_{n+1}$ by selecting pairwise different representatives $\hat{p}^{\mu}$ of all $\widehat{\mathbf{S}_{n+1}\mathbf{C}_2}$-orbits of the vectors in $\mathcal{P}_{n+1}$.

Hint. By Theorem 4.1 (e), we have $\mathcal{P}_{n+1} * \widehat{\mathbf{S}_{n+1}\mathbf{C}_2} = \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ and therefore we can take for $\mathbf{posit}^{\bullet}_{n+1}$ the subset of $\mathcal{P}_{n+1}$constructed in Step 5°.

By applying Proposition 3.7 and the correspondence (3.10), we get the following algorithm producing the set P-$\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ of P-critical forms $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$ from the unit forms $p \in \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$, with a sincere roots w, for $n \geqslant 3$.

Algorithm 4.6. Input: An integer $n \geqslant 3$, the finite sets of matrices $\widehat{\mathbf{S}_n\mathbf{C}_2} \subseteq \mathbb{M}_n(\mathbb{Z})$ and $\widehat{\mathbf{S}_{n+1}\mathbf{C}_2} \subseteq \mathbb{M}_{n+1}(\mathbb{Z})$ (see (3.1)), and a finite set $\mathbf{posit}^{\bullet}_n \subseteq \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ of pairwise different representatives $p : \mathbb{Z}^n \to \mathbb{Z}$ of all $\widehat{\mathbf{S}_n\mathbf{C}_2}$-orbits in $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$.

Output: A finite set P-$\mathbf{crit}^{\bullet}_{n+1} \subseteq P$-$\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ of pairwise different representatives of all $\widehat{\mathbf{S}_{n+1}\mathbf{C}_2}$-orbits in P-$\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$.

Step 1°: Construct a finite set $\mathbf{posit}^{\bullet}_n \subseteq \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ of pairwise different representatives of all $\widehat{\mathbf{S}_n\mathbf{C}_2}$-orbits in $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$, by applying Algorithm 4.5.

Step 2°: Given a unit form $p \in \mathbf{posit}^{\bullet}_n$, construct the set $\mathcal{R}_p = \{w \in \mathbb{Z}^n;\ p(w) = 1\}$ of roots of p, and then form the list $\mathcal{SR}_p$ of all sincere vectors in $\mathcal{R}_p$. Here we can apply the restrictively counting algorithm [18, Algorithm 4.2].

Step 3°: Construct a finite set $\mathcal{Z}^{-}_n$ of pairwise different representatives of all $\widehat{\mathbf{S}_n\mathbf{C}_2}$-orbits in the finite set $\mathcal{Z}^{\bullet}_n = \{(p, w);\ \ p \in \mathbf{posit}^{\bullet}_n,\ w \in \mathcal{SR}_p\}$, see Proposition 3.7 (c). Note that (p, w) and $(p, -w)$ lie in the same $\widehat{\mathbf{S}_n\mathbf{C}_2}$-orbit.

Step 4°: Given $(p, w) \in \mathcal{Z}^{-}_n$, construct the Gram matrices $\check{G}_p$ and G_p of p, then construct the matrix $G_{p,w} = \left[\begin{array}{c|c} 1 & -w \cdot 2 \cdot G_p \\ \hline 0 & \check{G}_p \end{array}\right]$, and finally construct the unit form $q_{p,w} : \mathbb{Z}^{n+1} \to \mathbb{Z}$ by the formula $q_{p,w}(x) = x \cdot G_{p,w} \cdot x^{tr}$.

Hint. It follows from (3.11) and Proposition 3.7 (b) (with $s = 0, \varepsilon_0 = 1$, and with n and $n + 1$ interchanged) that the form $q_{p,w}$ is P-critical such that $\mathrm{ind}(p, w) = q_{p,w}$, see (3.10).

Step 5°: Define P-$\mathbf{crit}^{\bullet}_{n+1}$ to be the finite set $\{q_{p,w}\}_{(p,w)\in\mathcal{Z}^{-}_n}$.

Hint. The choice of P-$\mathbf{crit}^{\bullet}_{n+1}$ in Step 5° is a proper one, because ever $\widehat{\mathbf{S}_{n+1}\mathbf{C}_2}$-orbit in P-$\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ is represented by a P-critical form $q_{p,w}$, with $(p, w) \in \mathcal{Z}^{-}_n$, by Proposition 3.7 (c).

We recall from [20] the following definition, see also [18] and [19].

Definition 4.7. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form and let $\check{G}_q$ be its non-symmetric Gram matrix (2.1).

(a) The **Coxeter-Gram polynomial** of q is the characteristic polynomial

$$\mathrm{cox}_q(t) := \det(t \cdot E - \mathrm{Cox}_q) \in \mathbb{Z}[t] \tag{4.8}$$

of the **Coxeter-Gram matrix** $\mathrm{Cox}_q := -\, \check{G}_q \cdot \check{G}_q^{-tr}$ of q.

(b) If Δ is a Dynkin diagram, we say that q is of **Coxeter-Gram type** Δ, if the Coxeter-Gram polynomial $\mathrm{cox}_q(t)$is the Coxeter polynomial $\mathrm{cox}_A(t)$ of a morsification $b_A : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ of the quadratic form q_Δ of Δ, see [19, 3.12] and compare with [2].

We recall from [12] (see also [2]), that a unit form is said to be connected if its bigraph is connected.

By applying Algorithm 4.5 implemented in MAPLE, we get the following classification of positive unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$, for $n = 2, 3, 4, 5$.

Corollary 4.9. *Let $n \geqslant 2$. Up to the action $* : \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z}) \times \widehat{\mathbf{S}_n\mathbf{C}_2} \longrightarrow \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ (3.4) of the group $\widehat{\mathbf{S}_n\mathbf{C}_2}$ (3.1), the connected positive unit forms $p : \mathbb{Z}^n \to \mathbb{Z}$ that admit a sincere root w are the following.*

(a) *If $n = 2$ then $p(x) = x_1^2 + x_2^2 - x_1x_2 = q_{\mathbb{A}_2}(x)$, $w = (1, 1)$ is a sincere root of p, and $\mathrm{cox}_p(t) = F_{\mathbb{A}_2}(t) = t^2 + t + 1$ is the Coxeter-Gram polynomial (4.8) of p.*

(b) *If* $n = 3$ *then* $p(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 = q_{\mathbb{A}_3}(x)$, $w = (1, 1, 1)$ *is a sincere root of* p, *and* $\mathrm{cox}_p(t) = F_{\mathbb{A}_3}(t) = t^3 + t^2 + t + 1$, *see* [19, 3.12].

(c) *If* $n = 4$ *then* $p(x)$ *is one of the following three unit forms*

- (c1) $p_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_2x_3 - x_3x_4 = q_{\mathbb{A}_4}(x)$, *the vector* $w = (1, 1, 1, 1)$ *is a sincere root of* p_1, *and* $\mathrm{cox}_{p_1}(t) = F_{\mathbb{A}_4}(t) = t^4 + t^3 + t^2 + t + 1$.
- (c2) $p_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - (x_1 + x_2 + x_3)x_4 = q_{\mathbb{D}_4}(x)$, *the vectors* $w = (1, 1, 1, 1)$ *and* $u = (1, 1, 1, 2)$ *are sincere root of* p_2, *and* $\mathrm{cox}_{p_2}(t) = F_{\mathbb{D}_4}(t) = t^4 + t^3 + t + 1$.
- (c3) $p_3(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 + x_1x_3 - x_2x_3 + x_2x_4 - x_3x_4$, *the vector* $v = (1, 1, -1, -1)$ *is a sincere root of* p_3, *and* $\mathrm{cox}_{p_3}(t) = F_{\mathbb{D}_4}(t) = t^4 + t^3 + t + 1$.

(d) *If* $n = 5$ *then* $p(x)$ *is one of the following seven unit forms having a sincere root*.

- (d1) $p_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 = q_{\mathbb{A}_5}(x)$, $w = (1, 1, 1, 1, 1)$ *is a sincere root of* p_1, *and* $\mathrm{cox}_{p_1}(t) = F_{\mathbb{A}_5}(t) = t^5 + t^4 + t^3 + t^2 + t + 1$,
- (d2) $p_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1x_3 - x_2x_3 - x_3x_4 - x_4x_5 = q_{\mathbb{D}_5}(x)$, $w = (1, 1, 1, 1, 1)$, $u = (1, 1, 2, 1, 1)$ *and* $v = (1, 1, 2, 2, 1)$ *are sincere roots of* p_2, *and* $\mathrm{cox}_{p_2}(t) = F_{\mathbb{D}_5}(t) = t^5 + t^4 + t + 1$, *see* [19, 3.12],
- (d3) $p_3(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1(x_2 + x_3 + x_4 - x_5) + x_3x_4$, $u = (2, 1, 1, 1, -1)$ *is a sincere root of* p_3, *and* $\mathrm{cox}_{p_3}(t) = F_{\mathbb{D}_5}(t) = t^5 + t^4 + t + 1$,
- (d4) $p_4(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1(x_2 + x_3 + x_4 - x_5) + x_2x_4 + x_3x_4$, $u = (1, 1, 1, -1, -1)$ *is a sincere root of* p_4, *and* $\mathrm{cox}_{p_4}(t) = F_{\mathbb{D}_5}(t) = t^5 + t^4 + t + 1$,
- (d5) $p_5(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1(x_2 + x_3 + x_4) + x_2x_4 + x_2x_5 + x_3x_4$, $u = (1, 1, 1, -1, -1)$ *and* $w = (1, 2, 1, -1, -1)$ *are sincere roots of* p_5, *and* $\mathrm{cox}_{p_5}(t) = t^5 + t^4 + t + 1$,
- (d6) $p_6(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1(x_2 + x_3 + x_4) + x_2x_5 + x_3x_4 - x_4x_5$, $u = (1, 1, 1, -1, -1)$ *is a sincere root of* p_6, *and* $\mathrm{cox}_{p_6}(t) = t^5 + t^3 + t^2 + 1$,
- (d7) $p_7(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1(x_2 + x_3 + x_4 - x_5) + x_2x_3 + x_2x_4 + x_3x_4 - x_4x_5$, $u = (1, 1, 1, -1, -1)$ *is a sincere root of* p_7, *and* $\mathrm{cox}_{p_7}(t) = t^5 + t^3 + t^2 + 1$.

Proof. (a) Since p has a sincere root, we have $|p_{12}| = 1$ and (a) easily follows.

(b) By applying Theorem 4.1 (in particular the formula (4.4)), we construct $\hat{p}^{\mu} : \mathbb{Z}^3 \to \mathbb{Z}$, with $p(x) = q_{\mathbb{A}_2}(x) = x_1^2 + x_2^2 - x_1x_2$ and $\mu = (1, 1)$, see (a). It is easy to see that there are only two $\widehat{\mathbf{S}_3\mathbf{C}_2}$-orbits in $\mathbf{posit}(\mathbb{Z}^3, \mathbb{Z})$ represented by the forms $p(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 = q_{\mathbb{A}_3}(x)$ and $p_0(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3 - x_2x_3$. Observe that the form $p_0(x)$has no sincere root.

(c) By applying the proof of (b) to the form $p(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3$ and to the vector $\mu = (1, 1, 1)$, we get the form (c1) with the sincere root $w = (1, 1, 1)$. To construct the remaining forms (c2) and (c3), we apply Algorithm 4.5 (implemented in MAPLE) to $n = 3$ and the form $p(x)$. Note that $\check{\mathcal{W}}_3 = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ and the set $\mathcal{W}_3 = \check{\mathcal{W}}_3 * \widehat{\mathbf{S}_3\mathbf{C}_2}$ consists of all non-zero vectors $v = (v_1, v_2, v_3)$, with $v_1, v_2, v_3 \in \{-1, 0, 1\}$. One can check (using MAPLE) that there are only three $\widehat{\mathbf{S}_3\mathbf{C}_2}$-orbits in $\mathcal{W}_{3,p}$; they are represented by the vectors $w = (1, 1, 1)$, $u = (0, -1, 0)$ and $v = (1, -1, -1)$. It is easy to see that the unit form $\hat{p}^{\mu}(x)$ (4.4), with $\mu \in \{w, u, v\}$, equals $p_1(x)$, $p_2 * \hat{\tau}(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - (x_1 + x_4 + x_3)x_2$, with $\tau = (2, 4) \in \mathbf{S}_4$, and $p_3(x)$, if $\mu = w$, $\mu = u$, and $\mu = v$, respectively. Hence, in view of Theorem 4.1, (c) follows. Note also that $p_2(x) = \hat{p}_0^{\mu}(x)$, where $p_0(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3 - x_2x_3$ and $\mu = (0, -1, -1) \in \mathcal{W}_{3,p_0}$.

The proof of (d) is analogous to that of (c) and we leave it to the reader. □

By applying Algorithms 4.5 and 4.6 implemented in MAPLE, we get the following classification of P-critical unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$, for $n = 3, 4, 5$.

Corollary 4.10. *If* $n \in \{3, 4, 5\}$ *and* $\widehat{\mathbf{S}_n\mathbf{C}_2}$ *is the group* (3.1) *then, up to the action* $* : P{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z}) \times \widehat{\mathbf{S}_n\mathbf{C}_2} \longrightarrow P{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})$ (3.4), *the P-critical unit forms* $q : \mathbb{Z}^n \to \mathbb{Z}$, *with* $\mathrm{Ker} q = \mathbb{Z} \cdot \mathbf{h}$, *are the following*.

(a) *If* $n = 3$ *then* $q(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3 = q_\Delta(x)$, $\mathbf{h} = (1, 1, 1)$, *and* $\mathrm{cox}_q(t) = t^3 - t^2 - t + 1 = F_\Delta(t)$, *where* $\Delta = \widetilde{\mathbb{A}}_{1,2}$ (*see* [19, 3.12] *and* [21, p. 146]).

(b) *If* $n = 4$ *then* $q(x)$ *is one of the following two unit forms*

(b*1*) $q_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_4 = q_{\widetilde{\mathbb{A}}_{2,2}}(x)$, $\mathbf{h} = (1, 1, 1, 1)$, *and* $\mathrm{cox}_{q_1}(t) = t^4 - 2t^2 + 1 = F_\Delta(t)$, *where* $\Delta = \widetilde{\mathbb{A}}_{2,2}$.

(b*2*) $q_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_1x_4 - x_2x_3 - x_3x_4 = q_{\widetilde{\mathbb{A}}_{1,3}}(x)$, $\mathbf{h} = (1, 1, 1, 1)$, *and* $\mathrm{cox}_{q_2}(t) = t^4 - t^3 - t + 1 = F_\Delta(t)$, *where* $\Delta = \widetilde{\mathbb{A}}_{1,3}$,

Observe that $\mathrm{cox}_{q_1}(t) \neq \mathrm{cox}_{q_2}(t)$ *and* $q_1 = q_2 * \hat{\tau}$, *where* $\tau = (3, 4)$.

(c) *If* $n = 5$ *then* $q(x)$ *is one of the following four unit forms*

(c*1*) $q_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1x_2 - x_1x_4 - x_2x_3 - x_3x_5 - x_4x_5$, $\mathbf{h} = (1, 1, 1, 1, 1)$, *and* $\mathrm{cox}_{q_1}(t) = t^5 - t^3 - t^2 + 1 = F_\Delta(t)$, *where* $\Delta = \widetilde{\mathbb{A}}_{2,3}$.

(c*2*) $q_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1x_2 - x_1x_5 - x_2x_3 - x_3x_4 - x_4x_5$, $\mathbf{h} = (1, 1, 1, 1, 1)$, *and* $\mathrm{cox}_{q_2}(t) = t^5 - t^4 - t + 1 = F_\Delta(t)$, *where* $\Delta = \widetilde{\mathbb{A}}_{1,4}$.

(c*3*) $q_3(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1(x_2 + x_3 + x_4 - x_5) - (x_2 + x_3 + x_4)x_5$, $\mathbf{h} = (1, 1, 1, 1, 1)$, *and* $\mathrm{cox}_{q_3}(t) = t^5 + t^4 - 2t^3 - 2t^2 + t + 1 = F_\Delta(t)$, *where* $\Delta = \widetilde{\mathbb{D}}_4$,

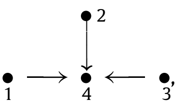
(c*4*) $q_4(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - (x_1 + x_2 + x_3 + x_4)x_5 = q_\Delta(x)$, $\mathbf{h} = (1, 1, 1, 1, 2)$, *and* $\mathrm{cox}_{q_4}(t) = t^5 + t^4 - 2t^3 - 2t^2 + t + 1 = F_\Delta(t)$, *where* $\Delta = \widetilde{\mathbb{D}}_4$,

Observe that $\mathrm{cox}_{q_1}(t) \neq \mathrm{cox}_{q_2}(t)$ *and* $q_1 = q_2 * \hat{\tau}$, *where* $\tau = (4, 5)$.

Proof. For $n = 3, 4, 5$, we construct the P-critical forms $q : \mathbb{Z}^n \to \mathbb{Z}$ from positive forms $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$, with a sincere root w, by applying the correspondence ind in (3.10) described by the formula (3.9).

(c) Assume that $n = 5$. To prove (c3) and (c4), assume that $p = p_2 : \mathbb{Z}^4 \to \mathbb{Z}$ is the positive form p_2 in Corollary 4.9 (c2), that is, $p_2 = q_{\mathbb{D}_4} : \mathbb{Z}^4 \to \mathbb{Z}$ is the positive Euler form of the Dynkin quiver

$\mathbb{D}_4$:

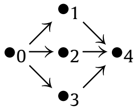


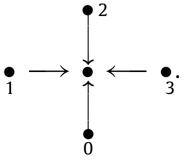
i.e. $p(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - (x_1 + x_2 + x_3)x_4$. The form p has precisely four sincere roots: two positive roots $w = (1, 1, 1, 1)$ and $u = (1, 1, 1, 2)$, and two negative roots $-w$ and $-u$. By applying the map ind in (3.9), with $s = 0$ and $\varepsilon_0 = 1$, we construct two P-critical unit forms $q_3 = \mathrm{ind}(p, w)$ and $q_4 = \mathrm{ind}(p, u)$ as follows. First we note that $-2 \cdot G_p = \begin{bmatrix} -2 & 0 & 0 & -1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & -1 \\ -1 & -1 & -1 & -2 \end{bmatrix}$,

$w \cdot (-2 \cdot G_p) = (-1, -1, -1, 1)$, and $u \cdot (-2 \cdot G_p) = (0, 0, 0, -1)$. Hence, by (3.11), we have $\check{G}_{q_3} = \left[\begin{array}{c|cccc} 1 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$, $\check{G}_{q_4} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & -1 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$, and

$$q_3(x) = x \cdot \check{G}_{q_3} \cdot x^{tr} = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0(x_1 + x_2 + x_3 - x_4) - (x_1 + x_2 + x_3)x_4,$$
$$q_4(x) = x \cdot \check{G}_{q_4} \cdot x^{tr} = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - (x_0 + x_1 + x_2 + x_3)x_4,$$

where $x = (x_0, x_1, x_2, x_3, x_4)$. By Proposition 3.7 (b), the forms $q_3, q_4 : \mathbb{Z}^5 \to \mathbb{Z}$ are P-critical, $\mathrm{Ker}\, q_3 = \mathbb{Z} \cdot \widehat{w}^{\varepsilon_0}$ and $\mathrm{Ker}\, q_4 = \mathbb{Z} \cdot \widehat{u}^{\varepsilon_0}$, where $\varepsilon_0 = 1$, $\widehat{w}^{\varepsilon_0} = (\varepsilon_0, 1, 1, 1, 1) = (1, 1, 1, 1, 1)$, and $\widehat{u}^{\varepsilon_0} = (\varepsilon_0, 1, 1, 1, 2) = (1, 1, 1, 1, 2)$. Note that $q_3, q_4 : \mathbb{Z}^5 \to \mathbb{Z}$ are the Euler forms of the canonical algebra $C(2, 2, 2) = KQ(2, 2, 2)/\mathcal{I}$ (see [21, Section XII.1]) and the tame hereditary algebra of the extended Dynkin quiver $\widetilde{\mathbb{D}}_4$, respectively, where

$Q(2,2,2)$: 

and $\widetilde{\mathbb{D}}_4$: 

The proof in remaining cases is analogous to that one for (c3) and (c4), and we leave it to the reader. Note that the statement (a) follows from Theorem 2.3 and [20, Example 2.6(b)]. The Coxeter-Gram polynomials $\mathrm{cox}_q(t)$ (4.8) are obtained by a direct case by case calculation. □

Remark 4.11. For $n = 5$, the surjective correspondence (3.10) is not bijective, because

- the pairs $(p_2, w), (p_3, v) \in \mathcal{Z}_4$ lie in different $\widehat{\mathbf{S}_4\mathbf{C}}_2$-orbits, where (p_2, w) and (p_3, v) are as in (c2) and (c3) of Corollary 4.9,
- $\mathrm{ind}(p_2, w)$ is the P-critical form q_3 of Corollary 4.10(c3),
- $\mathrm{ind}(p_3, v)$ is the P-critical form $q_5(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1(x_3 - x_4) - x_2x_3 + x_2x_4 - x_3x_4 + x_3x_5 - x_4x_5$, with $\mathrm{Ker}\, q_5 = \mathbb{Z} \cdot (1, 1, 1, -1, -1)$, and $\mathrm{cox}_{q_5}(t) = F_\Delta(t) = t^5 + t^4 - 2t^3 - 2t^2 + t + 1$, where $\Delta = \widetilde{\mathbb{D}}_4$, and
- the forms q_3 and q_5 lie in the same $\widehat{\mathbf{S}_5\mathbf{C}}_2$-orbit, because $q_5 = q_3 * B$, where $B = \hat{\sigma} \cdot \hat{\varepsilon} \in \widehat{\mathbf{S}_5\mathbf{C}}_2$, with $\sigma = (1.4) \cdot (3, 5)$ and $\varepsilon = (1, 1, 1, -1, -1)$.

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