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Note

Complexity of the hamiltonian cycle in regular graph problem

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Abstract

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The problem of deciding whether a 3-regular graph has a hamiltonian cycle (or path) was proved NP-complete. In this paper, we prove that for any fixed $k \geqslant 3$, deciding whether a k-regular graph has a hamiltonian cycle (or path) is a NP-complete problem. When the k-regular graph is planar, deciding whether the graph has a hamiltonian cycle (or path) was proved NP-complete for $k = 3$ and polynomial for $k \geqslant 6$. We prove that for $k = 4$ and $k = 5$ the problem is NP-complete.

1. Introduction

Garey et al. [2] proved deciding whether a 3-regular planar graph has a hamiltonian cycle is a NP-complete problem. When the graph is not constrained to be planar, for 4-regular graph, the problem was conjectured to be NP-complete. In this paper, we first prove that for any fixed $k \geqslant 3$, deciding whether a k-regular graph has a hamiltonian cycle (or path) is a NP-complete problem. Secondly, we will return to the subproblem of planar k-regular graph. In this case, the problem is obviously polynomial for $k \geqslant 6$ since a planar graph cannot have a vertex with degree larger than

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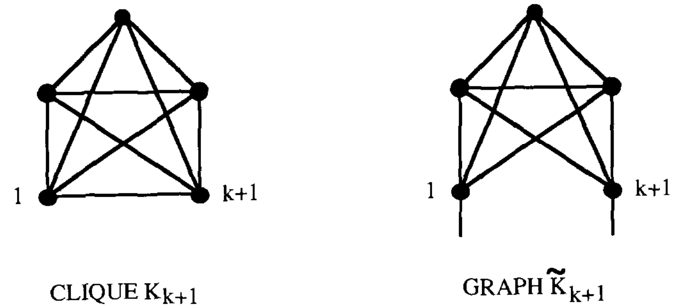


Fig. 1. The k-regular graph $\tilde{K}_{k+1}$.

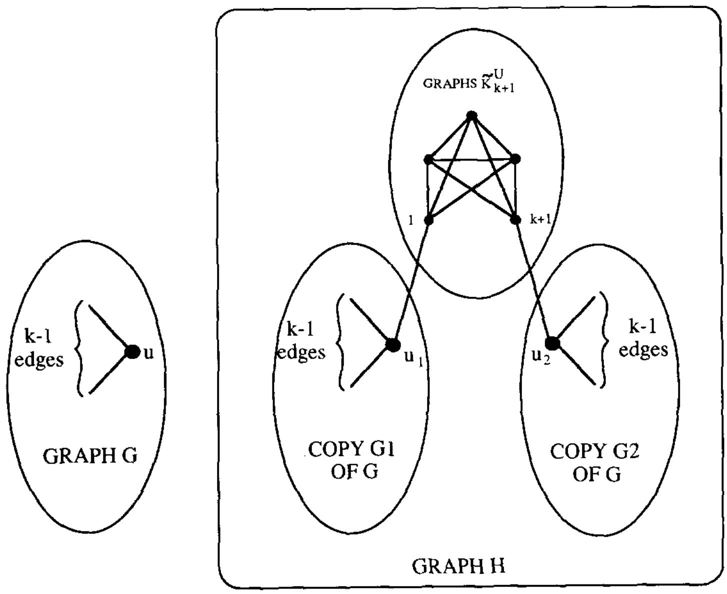


Fig. 2. The polynomial transformation of G into H.

5 [3]. For $k=4$ and $k=5$, we prove that deciding whether a 4-regular planar graph or a 5-regular planar graph has a hamiltonian cycle (or path) are two NP-complete problems.

2. The HC-k-regular problem

The HC-k-regular problem (hamiltonian cycle in a k-regular graph) is polynomial for $k=0$, $k=1$ and $k=2$. We know from [2] that the HC-3-regular problem is

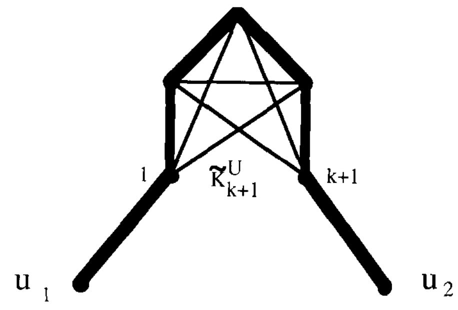


Fig. 3. The chain associated with vertex u.

NP-complete. We will now make use of this last result to show that the problem is NP-complete for any fixed $k \geqslant 3$.

First we will show by induction on k that the special case of HC-k-regular problem with an even number of vertices (denoted by HC-k-regular-(n-even)) is NP-complete for any fixed $k \geqslant 3$. Let us first begin the induction with the following lemma.

Lemma 2.1. HC-3-*regular*-(*n-even*) *is* NP-*complete.*

Proof. The result is obvious since the number of edges of a 3-regular graph is $\frac{3}{2}n$. □

Using the lemma, we come to the following result.

Theorem 2.2. *For any fixed* $k \geqslant 3$, HC-*k-regular*-(*n-even*) *is* NP-*complete.*

Proof. We prove the theorem by induction.

The theorem is true for $k=3$.

Assume now that the theorem is true for $k-1$, $k \geqslant 4$.

Let G be a $(k-1)$-regular graph with an even number of vertices and let $\tilde{K}_{k+1}$ be the variant of the clique K_{k+1} shown in Fig. 1. The following polynomial transformation is used to build a k-regular graph H with an even number of vertices.

First two copies, G_1 and G_2, of G are built. Then for any vertex u of G the corresponding vertices u_1 and u_2 are linked by a component $\tilde{K}^u_{k+1}$ (see Fig. 2). The resulting graph H is k-regular and has an even number of vertices since the number of vertices of G is even.

Assume that G has a hamiltonian cycle. For each vertex u of G, the chain of H with endpoints u_1 and u_2 passing through each vertex of $\tilde{K}^u_{k+1}$ as shown by Fig. 3 is selected.

Since the number of vertices of G is even, we can colour in red and black alternately the edges of the hamiltonian cycle and colour in the same way the corresponding

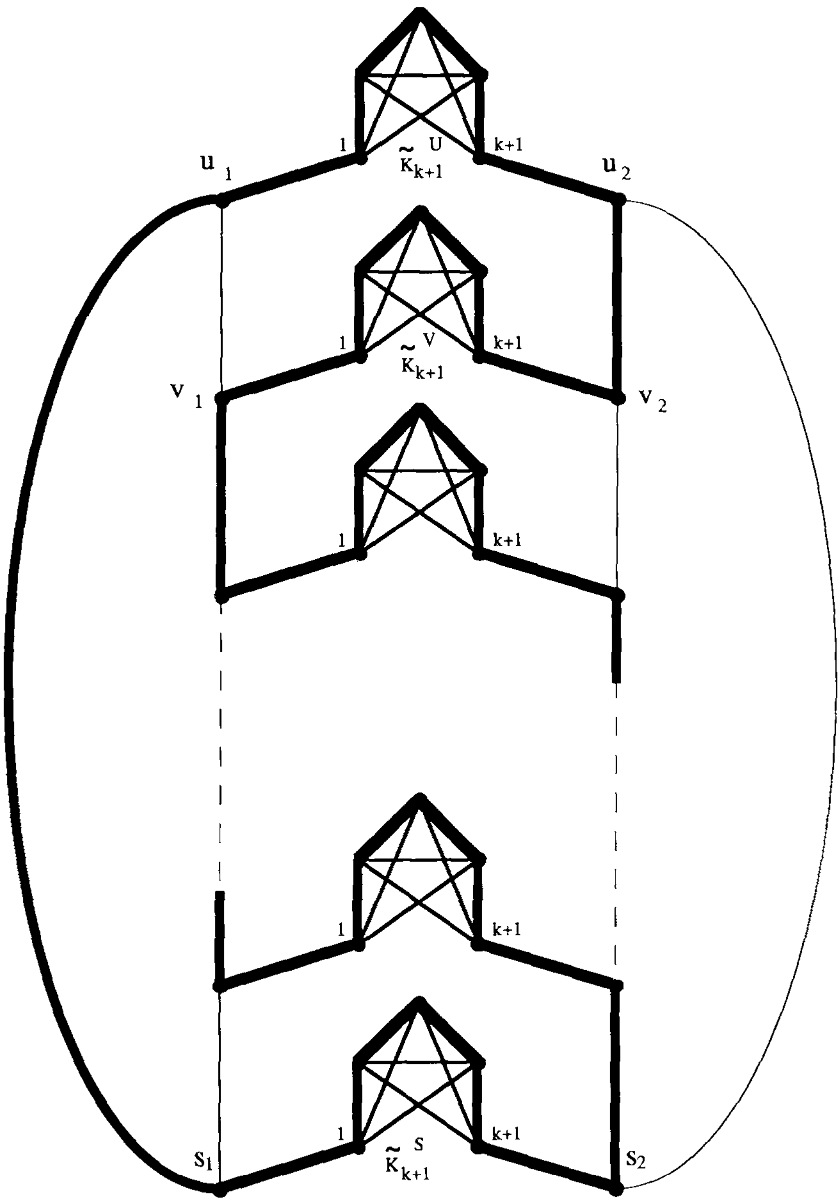


Fig. 4. The hamiltonian cycle in H.

edges of G_1 and G_2. By further selecting the black edges in G_1 and the red edges in G_2, the set of all selected edges make a hamiltonian cycle in H (see Fig. 4).

Assume now that H has a hamiltonian cycle. This cycle passes through the vertices of each component $\tilde{K}^u_{k+1}$. Since a component $\tilde{K}^u_{k+1}$ is linked with only two edges to

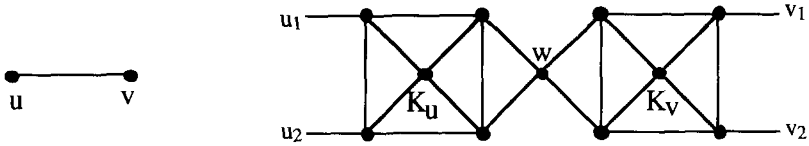


Fig. 5. The planar subgraph $H_{u,v}$ of H associated with edge (u,v) of G.

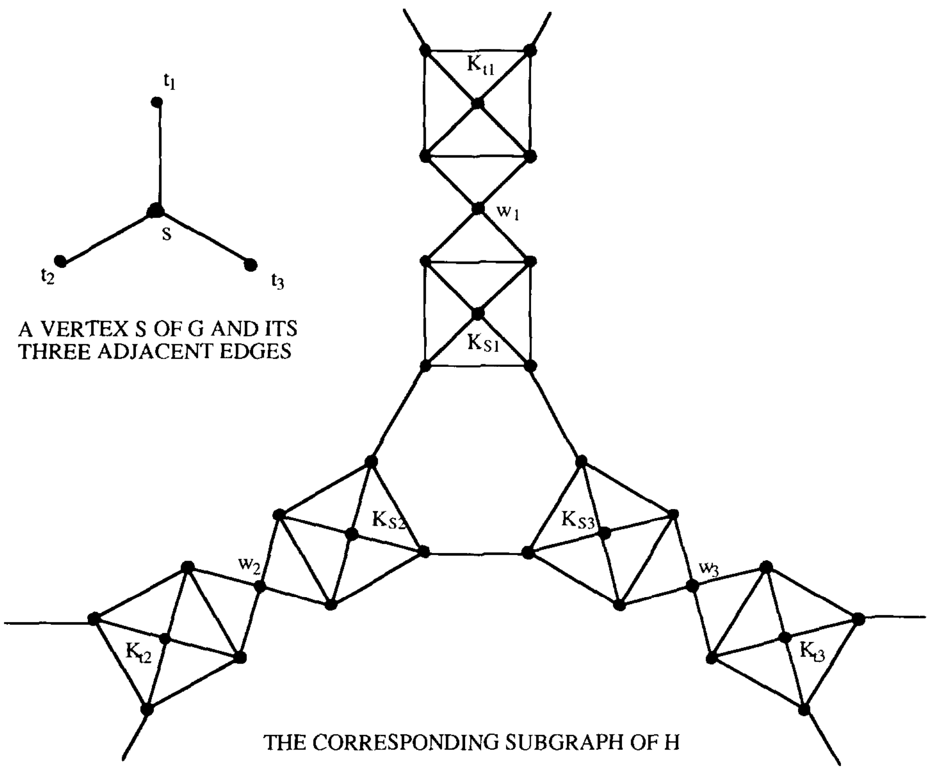


Fig. 6. The transformation of HC3*r* into HC4*r*.

the remaining part of H, one with an endpoint in G_1, the other with an endpoint in G_2, for each $\tilde{K}^u_{k+1}$, the hamiltonian cycle passes necessarily through the vertex u_1, then through the nodes of $\tilde{K}^u_{k+1}$ and finally through u_2. Furthermore, the other neighbour of u_1 (resp. u_2) in the hamiltonian cycle must be a vertex v_1 of G_1 (resp. a vertex v_2 of G_2) such that the corresponding edge (u,v) is in G (see Fig. 2). Thus, a hamiltonian cycle in H is a sequence of distinct nodes $\langle u_1 v_1 \langle \tilde{K}^v_{k+1} \rangle v_2 w_2 \langle \tilde{K}^w_{k+1} \rangle w_1 \cdots u_2 \langle \tilde{K}^u_{k+1} \rangle u_1 \rangle$ where the subsequences $\langle \tilde{K}^t_{k+1} \rangle$ are the hamiltonian paths passing through the components $\tilde{K}^t_{k+1}$. By replacing each distinct subsequence

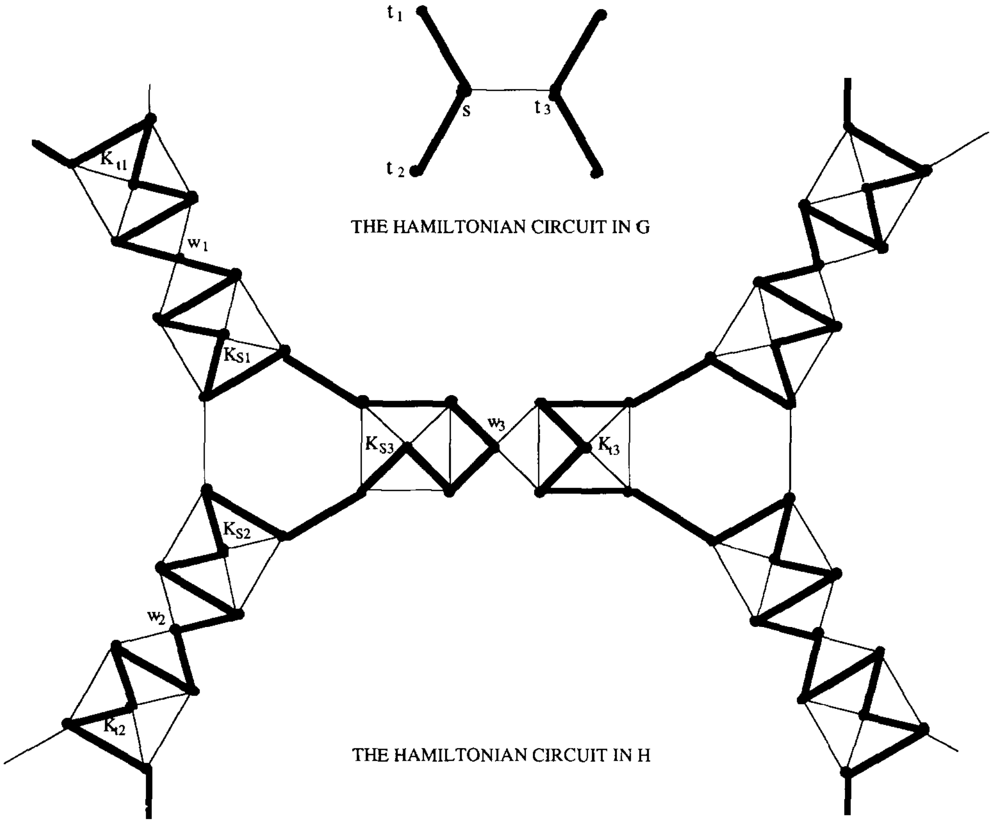


Fig. 7. The building of a hamiltonian circuit in HC4*r*.

$\langle s_1 \langle \tilde{K}^s_{k+1} \rangle s_2 \rangle$ or $\langle s_2 \langle \tilde{K}^s_{k+1} \rangle s_1 \rangle$ by $\langle s \rangle$, we get a sequence $\langle uvw \cdots u \rangle$ of distinct nodes of G which is a hamiltonian cycle. □

The following result is an immediate consequence of the preceding theorem.

Corollary 2.3. *For any fixed* $k \geqslant 3$, HC-*k-regular is* NP-*complete.*

3. The HC-*k*-regular-planar problem

We will now study the special case of planar *k*-regular graph. The HC-*k*-regular-planar problem (hamiltonian cycle in a planar *k*-regular graph) is obviously polynomial for $k=0$, $k=1$ and $k=2$. We know from [2] that the HC-3-regular-planar problem is NP-complete. For any $k \geqslant 6$, a *k*-regular graph cannot be planar (see [3]), then the problem is obviously polynomial. We will use the result of HC-3-regular-

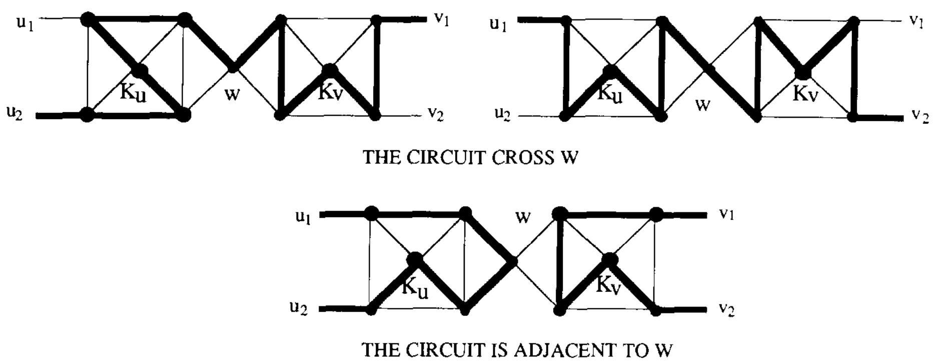
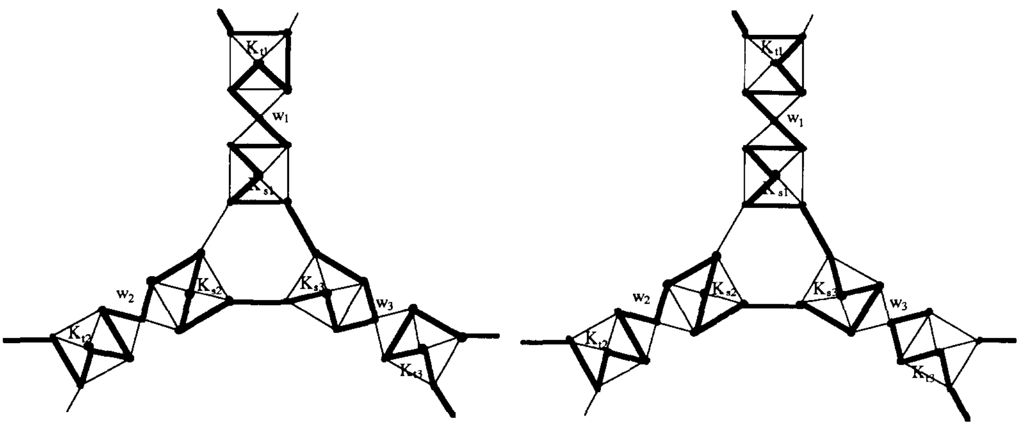


Fig. 8. How a hamiltonian circuit passes through a vertex w.



THE HAMILTONIAN CIRCUIT CROSS W1 AND W2 AND IS ADJACENT TO W3

Fig. 9. A hamiltonian circuit crosses two vertices w.

planar problem to show that the HC-4-regular-planar and the HC-5-regular-planar problems are NP-complete.

Theorem 3.1. *The* HC-4-*regular-planar problem is* NP-*complete.*

Proof. We note by HC3r the hamiltonian circuit in a 3-regular-planar graph problem and by HC4r the hamiltonian circuit in a 4-regular-planar graph problem.

We show that HC3$r \propto$ HC4r.

The reader can easily verify that HC4r is in NP.

Let $G=(V, E)$ be any instance of HC3r. We build an instance $H=(U, F)$ of HC4r by the following polynomial transformation.

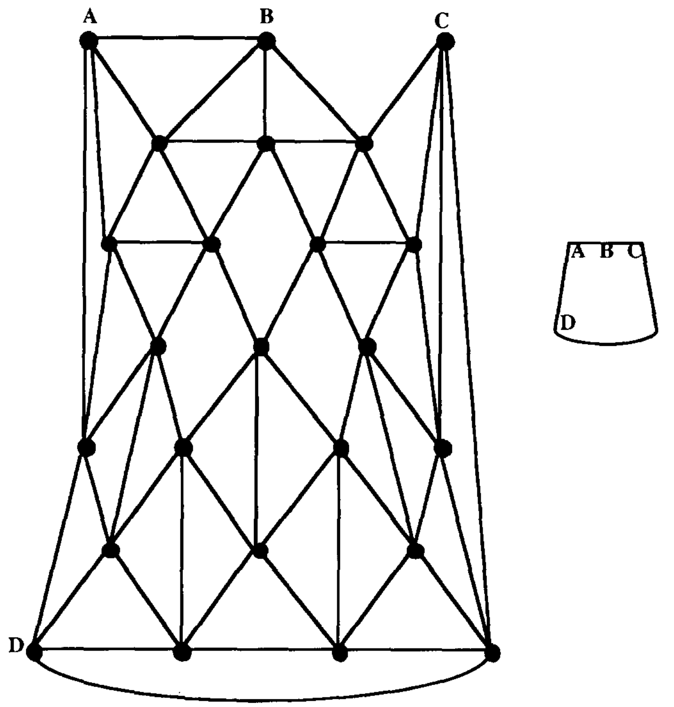


Fig. 10. The subgraph K and its abbreviation.

With each edge $(u,v)\in E$ is associated the planar subgraph $H_{u,v}$ of H made with two planar components with five vertices K_u and K_v linked with the vertex w (see Fig. 5).

In G, each vertex s having three adjacent edges (t_1,s), (t_2,s), (t_3,s), we link the three subgraphs $H_{t_1,s}, H_{t_2,s}, H_{t_3,s}$ of H in the manner shown in Fig. 6.

We can easily verify that H is 4-regular and planar.

We show that if G has a hamiltonian circuit then there is a hamiltonian circuit in H.

Assume that G has a hamiltonian circuit. Each vertex s being in the hamiltonian circuit, exactly two of its three adjacent edges (s,t_1), (s,t_2), (s,t_3) are in the hamiltonian circuit. Assume that (s,t_1) and (s,t_2) are these two edges. We associate at these two edges the single path of H shown in Fig. 7.

The edge (s,t_3) not being in the hamiltonian circuit, its two other adjacent edges in vertex t_3 are necessary in the hamiltonian circuit. We associate at these two edges the single path of H shown in Fig. 7. So the path obtained in H passes exactly one time through each vertex of the subgraph. In this way we obtain a hamiltonian circuit in H.

We show that if H has a hamiltonian circuit then there is a hamiltonian circuit in G.

Assume that H has a hamiltonian circuit.

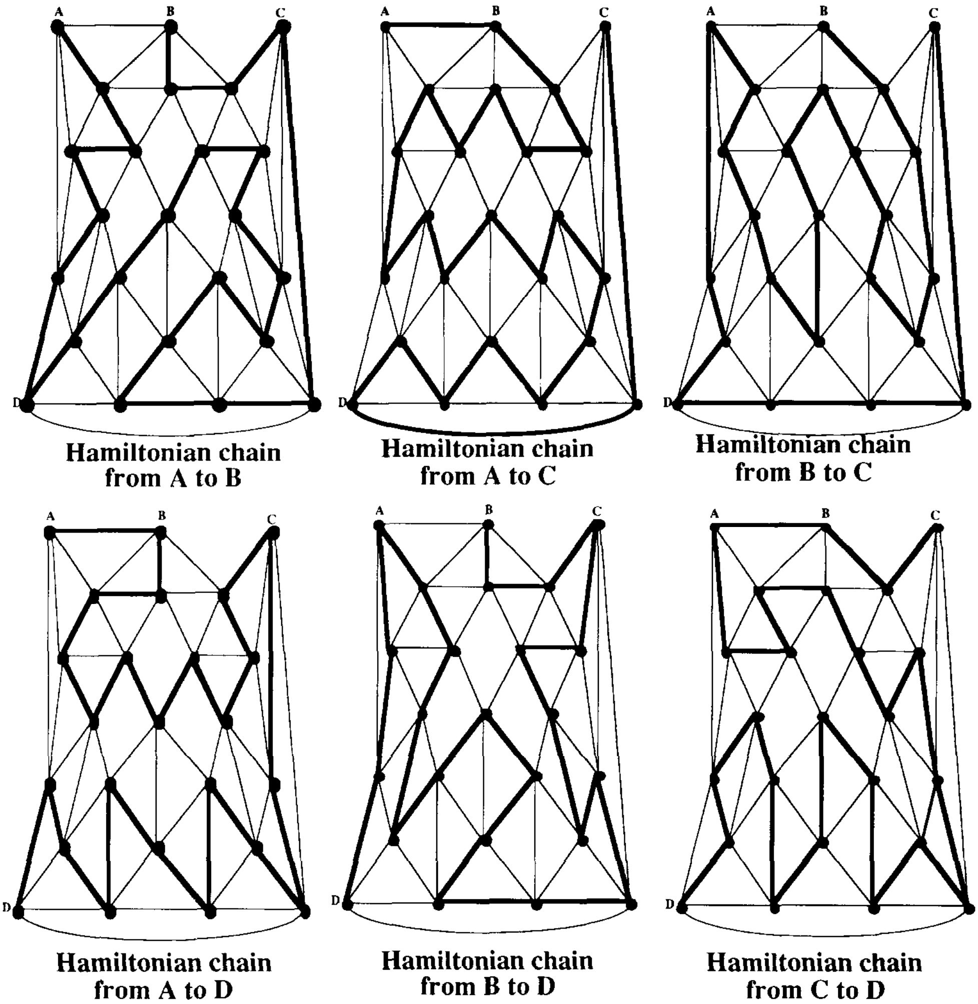


Fig. 11. The hamiltonian chains into K.

We say that the circuit crosses a vertex w if its two neighbours in the hamiltonian circuit are in the two distinct subgraphs K_u and K_v (see Fig. 8); otherwise, if its two neighbours are in the same subgraph K_u, we say that the hamiltonian circuit is adjacent to the vertex w (see Fig. 8).

The reader can easily verify that for each subgraph $(K_{t_1,s_1}, K_{t_2,s_2}, K_{t_3,s_3})$, a hamiltonian circuit does necessarily cross exactly two vertices w (see Fig. 9).

By keeping only the edges corresponding to the crossed vertices w, we obtain a hamiltonian circuit in G. Indeed, for each vertex of G exactly two of its three adjacent edges are selected. The hamiltonian circuit of H passes successively through each

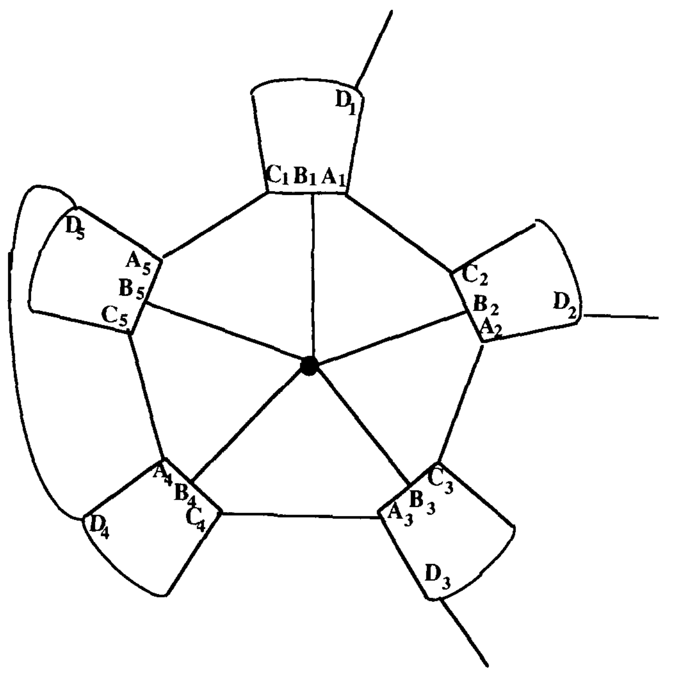


Fig. 12. The planar 5-regular subgraph L.

subgraph corresponding to each vertex of G, thus we have a hamiltonian circuit in G. □

Theorem 3.2. *The* HC-5-*regular-planar problem is* NP-*complete.*

Proof. We refer the subgraph shown in Fig. 10 as K. It is planar and has four distinguished vertices A, B, C and D with degree 4, the other vertices have a degree 5. Moreover, K has a hamiltonian chain between each pair of its four distinguished vertices (see Fig. 11).

With five copies of K, we build the subgraph called L shown in Fig. 12. The graph L is planar, 5-regular and has three outputs. Moreover, L has a hamiltonian chain between each pair of its three outputs (see Fig. 13).

Let G be a 3-regular-planar graph. The following polynomial transformation is used to build a 5-regular-planar graph H.

With each node of G we associate a subgraph L of H. Since G is 3-regular and a subgraph L has 3 outputs, we connect the outputs of the subgraphs L in the same manner as their corresponding vertices are connected in G. Since the subgraphs L are planar and 5-regular, G being planar, the graph H is planar and 5-regular.

Assume that G has a hamiltonian cycle. For each subgraph L of H corresponding to vertex v of G, we select the hamiltonian chain showed in Fig. 13 linking the two

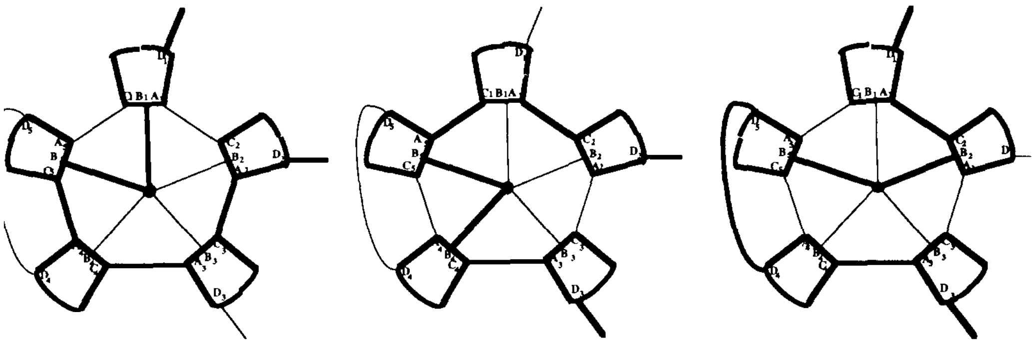


Fig. 13. The hamiltonian chains into L.

subgraphs L associated to the two neighbours of v in the hamiltonian cycle of G. Thus, we obtain a hamiltonian cycle in H.

Assume now that H has a hamiltonian cycle. The subgraphs L having three outputs, a hamiltonian cycle in H passes necessarily through one output and then through the remaining vertices of L before passing through the second output. Thus, by selecting successively the nodes of G corresponding to the subgraphs L successively crossed by the hamiltonian cycle, we obtain a hamiltonian cycle in G. □

Remark

We obtain the same result for the problem to decide whether a k-regular graph has a hamiltonian path. Indeed, deciding whether a 3-regular planar graph has a hamiltonian path is a NP-complete problem [2]. Using the same proofs as above, with similar arguments, we prove that deciding whether, for any fixed $k \geqslant 3$, a k-regular graph has a hamiltonian path or whether a 4-regular planar graph or a 5-regular planar graph has a hamiltonian path, are NP-complete problems.

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