PERGAMON

Applied Mathematics Letters 14 (2001) 13–16

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Control of Nonlinear Systems That Are Subjected to Kinematic Inequalities

Y. YAVIN
Laboratory for Decision and Control
Department of Electrical and Electronic Engineering
University of Pretoria, Pretoria, South Africa

(Received January 2000; accepted February 2000)

Abstract—This letter deals with the Lagrange equations concerning nonlinear systems which are subjected to a class of kinematic inequalities. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Lagrange equations, Control of nonlinear systems, A class of kinematic inequalities.

1. INTRODUCTION

Let $\mathbf{q} = (q_1, q_2, \dots, q_n)^\top$ be a vector of generalized coordinates describing the motion of a dynamical system, and denote

$$\mathbf{p} = \left(\frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots, \frac{dq_n}{dt}\right)^\top = (p_1, p_2, \dots, p_n)^\top.$$

Nonholonomic constraints encountered in mechanics can usually be expressed in the following form:

$$\sum_{j=1}^{n} a_{ij}(\mathbf{q}; t)p_j + b_i(\mathbf{q}; t) = 0, \qquad i = 1, \dots, m, \tag{1}$$

see, for example, [1–4]. This letter deals with control problems of mechanical systems subjected to kinematic constraints given by

$$f_i(q_1, \dots, q_n, p_1, \dots, p_n) \le 0, \qquad i = 1, \dots, m, \tag{2}$$

$m \le n$, where $f_i(\mathbf{q}, \mathbf{p})$, $i = 1, \dots, m$ are given smooth functions on $\Re^{2n}$. This letter is a sequel to [5] where the case of constraints of the form

$$f_i(q_1, \dots, q_n, p_1, \dots, p_n) = 0, \qquad i = 1, \dots, m, \tag{3}$$

is considered.

PII: S0893-9659(00)00104-X

2. THE LAGRANGE EQUATIONS

Let $T(\mathbf{q}, \mathbf{p})$ denote the kinetic energy and let $V_T(\mathbf{q}, t)$ denote the potential energy of the system. It is assumed here that

$$V_T(\mathbf{q}(t), t) = V(\mathbf{q}(t)) - \mathbf{q}^\top(t)\mathbf{B}\mathbf{u}(t), \tag{4}$$

where $V(\mathbf{q})$ is the potential energy due to the conservative forces, $\mathbf{u}(t) \in \Re^p$, $p \leq n$, $\mathbf{B} \in \Re^{n\times p}$, $\operatorname{rank} \mathbf{B} = p$, and $-\mathbf{q}^\top(t)\mathbf{B}\mathbf{u}(t)$ is the "potential energy" due to the applied control force $\mathbf{u}(t)$. Denote $\mathcal{L}_O = T - V_T$, $\mathcal{L} = T - V$, and define the following functional:

$$J = \int_{t_1}^{t_2} \mathcal{L}_O(\mathbf{q}(t), \mathbf{p}(t), t)\, dt. \tag{5}$$

The following are assumed here.

1. The functional J given by (5) has an extremum on the set of the $(\mathbf{q}, \mathbf{p})$ elements in $\Re^{2n}$ that satisfy inequalities (2). By this, it is tacitly assumed that $(\mathbf{q}(t), \mathbf{p}(t))$ satisfy inequalities (2) for all $t \in [t_1, t_2]$.
2. There is a nonvanishing Jacobian of order m, for instance,

$$\frac{\partial(f_1, f_2, \dots, f_m)}{\partial(p_1, p_2, \dots, p_m)} \neq 0, \qquad \text{in the domain defined by (2).} \tag{6}$$

By using the Calculus of Variations (see, for example, [6]), the first assumption leads to

$$\int_{t_1}^{t_2} \sum_{j=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}_O}{\partial p_j}\right) - \frac{\partial \mathcal{L}_O}{\partial q_j} \right] \delta q_j\, dt = 0, \tag{7}$$

where the variations δq_j, $j = 1, \dots, n$ satisfy

$$f_i(q_1 + \delta q_1, \dots, q_n + \delta q_n, p_1 + \delta p_1, \dots, p_n + \delta p_n) \leq 0, \qquad i = 1, \dots, m,$$

$$\delta q_j(t_1) = \delta q_j(t_2) = 0, \qquad j = 1, \dots, n. \tag{8}$$

Note that

$$\delta p_j = \delta\left(\frac{dq_j}{dt}\right) = \frac{d}{dt}\delta q_j, \qquad j = 1, \dots, n. \tag{9}$$

Thus, the variations $\{\delta q_j\}$ are not independent, and one cannot deduce the Euler-Lagrange equations from (7). By introducing the Valentine Variables $\{\psi_i\}_{i=1}^m$ (see [7]), the inequality constraints (2) are transformed into the following equality constraints:

$$F_i(\mathbf{q}(t), \mathbf{p}(t), \psi_i(t)) = f_i(\mathbf{q}(t), \mathbf{p}(t)) + \psi_i^2(t) = 0, \qquad i = 1, \dots, m, \tag{10}$$

where $\{\psi_i(t)\}_{i=1}^m$ are real-valued nonnegative continuous functions of time which are determined later. Now, the variations $\{\delta q_j\}_{j=1}^n$ and $\{\delta \psi_i\}_{i=1}^m$ must satisfy the following relations, obtained by varying the constraints $F_i = 0$, $i = 1, \dots, m$:

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\delta q_j + \sum_{j=1}^{n} \frac{\partial f_i}{\partial p_j}\frac{d}{dt}\delta q_j + 2\psi_i \delta\psi_i = 0, \qquad i = 1, \dots, m. \tag{11}$$

By multiplying successively each of these equations by a Lagrange multiplier and then integrating from t_1 to t_2, the following equations are obtained:

$$\int_{t_1}^{t_2} \lambda_i(t) \left[\sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\delta q_j + \sum_{j=1}^{n} \frac{\partial f_i}{\partial p_j}\frac{d}{dt}\delta q_j + 2\psi_i \delta\psi_i \right] dt = 0, \qquad i = 1, \dots, m. \tag{12}$$

But

$$\int_{t_1}^{t_2} \lambda_i(t) \sum_{j=1}^{n} \frac{\partial f_i}{\partial p_j} \frac{d}{dt} \delta q_j \, dt$$

$$= - \int_{t_1}^{t_2} \left[\frac{d\lambda_i(t)}{dt} \sum_{j=1}^{n} \frac{\partial f_i}{\partial p_j} \delta q_j + \lambda_i(t) \sum_{j=1}^{n} \frac{d}{dt} \left(\frac{\partial f_i}{\partial p_j} \right) \delta q_j \right] dt + \left[\lambda_i(t) \sum_{j=1}^{n} \frac{\partial f_i}{\partial p_j} \delta q_j \right]_{t_1}^{t_2} . \tag{13}$$

Also, by using (8), it follows that $[\lambda_i(t) \sum_{j=1}^{n} \frac{\partial f_i}{\partial p_j} \delta q_j]_{t_1}^{t_2} = 0$. Thus, inserting (13) into (12) yields

$$\begin{aligned} 0 = C_i(f_i, \lambda_i, \psi_i) = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left[\lambda_i(t) \frac{\partial f_i}{\partial q_j} - \lambda_i(t) \frac{d}{dt} \left(\frac{\partial f_i}{\partial p_j} \right) - \frac{d\lambda_i(t)}{dt} \frac{\partial f_i}{\partial p_j} \right] \delta q_j \, dt \\ + 2 \int_{t_1}^{t_2} \lambda_i(t) \psi_i(t) \delta \psi_i(t) \, dt, \qquad i = 1, \ldots, m. \end{aligned} \tag{14}$$

Hence, by subtracting the expression $\sum_{i=1}^{m} C_i(f_i, \lambda_i, \psi_i)$ from (7), the following equation is obtained:

$$\int_{t_1}^{t_2} \sum_{j=1}^{n} E_j(\mathbf{q}(t), \mathbf{p}(t), t) \delta q_j \, dt - 2 \int_{t_1}^{t_2} \sum_{i=1}^{m} \lambda_i(t) \psi_i(t) \delta \psi_i(t) \, dt = 0, \tag{15}$$

where

$$\begin{aligned} E_j(\mathbf{q}, \mathbf{p}, t) = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial p_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} + \sum_{i=1}^{m} \lambda_i(t) \left[\frac{d}{dt} \left(\frac{\partial f_i}{\partial p_j} \right) - \frac{\partial f_i}{\partial q_j} \right] \\ + \sum_{i=1}^{m} \frac{d\lambda_i(t)}{dt} \frac{\partial f_i}{\partial p_j} - (\mathbf{Bu}(t))_j, \qquad j = 1, \ldots, n. \end{aligned} \tag{16}$$

Denote

$$b_j = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial p_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} - (\mathbf{Bu}(t))_j, \qquad j = 1, \ldots, n. \tag{17}$$

By using the assumption given by (6), it follows that the following set of differential equations:

$$\sum_{i=1}^{m} \frac{d\lambda_i(t)}{dt} \frac{\partial f_i}{\partial p_j} + \sum_{i=1}^{m} \lambda_i(t) \left[\frac{d}{dt} \left(\frac{\partial f_i}{\partial p_j} \right) - \frac{\partial f_i}{\partial q_j} \right] + b_j = 0, \qquad j = 1, \ldots, m, \tag{18}$$

has a solution $(\lambda_1(t), \ldots, \lambda_m(t)$. Substituting these $\{\lambda_i\}_{i=1}^{m}$ into (15) yields

$$\int_{t_1}^{t_2} \sum_{j=m+1}^{n} E_j(\mathbf{q}(t), \mathbf{p}(t), t) \delta q_j \, dt - 2 \int_{t_1}^{t_2} \sum_{i=1}^{m} \lambda_i(t) \psi_i(t) \delta \psi_i(t) \, dt = 0, \tag{19}$$

where the variations δq_j, $j = m+1, m+2, \ldots, n$, $\delta \psi_i$, $i = 1, \ldots, m$ are independent. Hence, equations (18) and (19) yield the following set of equations:

$$\begin{aligned} (\mathbf{Bu}(t))_j = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial p_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} + \sum_{i=1}^{m} \lambda_i(t) \left[\frac{d}{dt} \left(\frac{\partial f_i}{\partial p_j} \right) - \frac{\partial f_i}{\partial q_j} \right] \\ + \sum_{i=1}^{m} \frac{d\lambda_i(t)}{dt} \frac{\partial f_i}{\partial p_j}, \qquad j = 1, \ldots, n, \end{aligned} \tag{20}$$

$$\lambda_i(t) \psi_i(t) = 0, \quad \psi_i(t) \geq 0, \qquad i = 1, \ldots, m, \tag{21}$$

which have to be solved together with

$$f_i(\mathbf{q}(t), \mathbf{p}(t)) + \psi_i^2(t) = 0, \qquad i = 1, \ldots, m. \tag{22}$$

Hence, equations (20)–(22) constitute $n + 2m$ equations for the solution of $q_j(t)$, $j = 1, \dots, n$; $\lambda_i(t)$, $i = 1, \dots, m$; and $\psi_i(t)$, $i = 1, \dots, m$. These equations are necessary conditions for the functional J, (5), to have an extremum on the set

$$\left\{(\mathbf{q}, \mathbf{p}) \in \Re^{2n} : f_i(\mathbf{q}, \mathbf{p}) \leq 0, \ i = 1, \dots, m\right\}. \tag{23}$$

REMARK 1. The conditions given above were derived for a case of *virtual unilateral constraints*, i.e., without incorporating any physical rule or restitution mapping at the contact. For more information on impact problems, see [8].

REMARK 2. Assume that for some time interval $[t_3, t_4] \subset [t_1, t_2]$, $f_i(\mathbf{q}(t), \mathbf{p}(t)) < 0$, $i = 1, \dots, m$. Then, (22) yields $\psi_i(t) > 0$, $i = 1, \dots, m$, $t \in [t_3, t_4]$. Consequently, (21) yields $\lambda_i(t) = 0$, $i = 1, \dots, m$, $t \in [t_3, t_4]$. Thus, in this case, equations (20)–(22) reduce to

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = (\mathbf{B}\mathbf{u}(t))_j, \qquad j = 1, \dots, n, \tag{24}$$

together with

$$f_i(\mathbf{q}(t), \mathbf{p}(t)) < 0, \qquad i = 1, \dots, m, \tag{25}$$

for all $t \in [t_3, t_4]$.

REMARK 3. Assume that for some time interval $[t_3, t_4] \subset [t_1, t_2]$, $f_i(\mathbf{q}(t), \mathbf{p}(t)) = 0$, $i = 1, \dots, m$. Then, (22) yields $\psi_i(t) = 0$, $i = 1, \dots, m$, $t \in [t_3, t_4]$. Consequently, (21) implies $\sum_{i=1}^{m} \lambda_i^2(t)$ is not necessarily zero. Hence, in this case, equations (20)–(22) reduce to (20) and

$$f_i(\mathbf{q}(t), \mathbf{p}(t)) = 0, \qquad i = 1, \dots, m, \quad t \in [t_3, t_4], \tag{26}$$

which is the result obtained in [5].

REFERENCES

1. E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, U.K., (1917).
2. Ju.I. Neimark and N.A. Fufaev, *Dynamics of Nonholonomic Systems*, American Mathematical Society, Providence, RI, (1972).
3. R.M. Rosenberg, *Analytical Dynamics of Discrete Systems*, Plenum Press, New York, (1977).
4. B.J. Torby, *Advanced Dynamics for Engineers*, Holt-Saunders International Editions, New York, (1984).
5. Y. Yavin, Control of nonlinear systems with general nonholonomic constraints, *Appl. Math. Lett.* **13** (6), 73–75 (2000).
6. L.E. Elsgolc, *Calculus of Variations*, Pergamon Press, Oxford, U.K., (1963).
7. M.R. Hestenes, Variational theory and optimal control theory, In *Computing Methods in Optimization Problems*, (Edited by A.V. Balakrishnan and L.W. Neustadt), Academic Press, New York, (1964).
8. B. Brogliato, *Nonsmooth Mechanics*, Second Edition, Springer-Verlag, London, (1999).