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On the Recursive Sequence $x_{n+1} = \alpha + x_{n-1}/x_n$

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We study the global stability, the boundedness character, and the periodic nature of the positive solutions of the difference equation $x_{n+1} = \alpha + x_{n-1}/x_n$, where $\alpha \in [0, \infty)$, and where the initial conditions x_{-1} and x_0 are arbitrary positive real numbers. © 1999 Academic Press

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1. INTRODUCTION AND SOME BASIC OBSERVATIONS

We study the global stability, the boundedness character, and the periodic nature of the positive solutions of the recursive sequence

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \qquad n = 0, 1, \ldots, \tag{1}$$

where $\alpha \in [0, \infty)$, and where the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

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The results in this paper confirm Conjecture x.y.4 in [3].

Clearly, the only equilibrium point of Eq. (1) is $\bar{x} = \alpha + 1$.

We show that a necessary and sufficient condition that every positive solution of (1) be bounded is $\alpha \geq 1$. Furthermore, we show that if $\alpha = 1$, then every positive solution of (1) converges to a two-cycle, while if $\alpha > 1$, then $\bar{x} = \alpha + 1$ is a globally asymptotically stable equilibrium point of Eq. (1).

The linearized equation of Eq. (1) about the equilibrium point $\bar{x} = \alpha + 1$ is

$$y_{n+1} + \frac{1}{\alpha + 1} y_n - \frac{1}{\alpha + 1} y_{n-1} = 0, \qquad n = 0, 1, \ldots . \tag{2}$$

LEMMA 1.1. *The following statements are true.*

1. *The equilibrium point* $\bar{x} = \alpha + 1$ *of Eq.* (1) *is locally asymptotically stable if* $\alpha > 1$.
2. *The equilibrium point* $\bar{x} = \alpha + 1$ *of Eq.* (1) *is unstable* (*and in fact is a saddle point*) *if* $0 \leq \alpha < 1$.

Proof. The proof is a simple consequence of the so-called Linearized Stability Theorem. (See [1, p. 11].) ∎

The proofs of the following three lemmas follow from simple computations and will be omitted.

LEMMA 1.2. *The following statements are true.*

1. *Equation* (1) *has solutions of prime period* 2 *if and only if* $\alpha = 1$.
2. *Suppose* $\alpha = 1$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of* (1). *Then* $\{x_n\}_{n=-1}^{\infty}$ *is periodic with period* 2 *if and only if* $x_{-1} \neq 1$ *and* $x_0 = x_{-1}/(x_{-1} - 1)$.

LEMMA 1.3. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.* (1) *which is eventually constant. Then* $\{x_n\}_{n=-1}^{\infty}$ *is the trivial solution*

$$x_n = \alpha + 1, \qquad n = -1, 0, \ldots .$$

LEMMA 1.4. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.* (1), *and let* $L > \alpha$. *Then the following statements are true.*

1. $\lim_{n\to\infty} x_{2n} = L$ *if and only if* $\lim_{n\to\infty} x_{2n+1} = L/(L - \alpha)$.
2. $\lim_{n\to\infty} x_{2n+1} = L$ *if and only if* $\lim_{n\to\infty} x_{2n} = L/(L - \alpha)$.

2. ANALYSIS OF THE SEMI-CYCLES OF (1)

In this section, we give some results about the semi-cycles of (1) which shall be useful in the sequel.

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (1). A *positive semi-cycle* of $\{x_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either } l = -1 \quad \text{or} \quad l > -1 \text{ and } x_{l-1} < \bar{x}$$

and

$$\text{either } m = \infty \quad \text{or} \quad m < \infty \text{ and } x_{m+1} < \bar{x}.$$

A *negative semi-cycle* of $\{x_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either } l = -1 \quad \text{or} \quad l > -1 \text{ and } x_{l-1} \geq \bar{x}$$

and

$$\text{either } m = \infty \quad \text{or} \quad m < \infty \text{ and } x_{m+1} \geq \bar{x}.$$

A solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1) is called *nonoscillatory* if there exists $N \geq -1$ such that either

$$x_n > \bar{x} \quad \text{for all } n \geq N$$

or

$$x_n < \bar{x} \quad \text{for all } n \geq N.$$

$\{x_n\}_{n=-1}^{\infty}$ is called *oscillatory* if it is not nonoscillatory.

LEMMA 2.1. *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (1) which consists of a single semi-cycle. Then $\{x_n\}_{n=-1}^{\infty}$ converges monotonically to $\bar{x} = \alpha + 1$.*

Proof. Suppose $0 < x_{n-1} < \alpha + 1$ for all $n \geq 0$. The case where $x_{n-1} \geq \alpha + 1$ for all $n \geq 0$ is similar and will be omitted. Note that for $n \geq 0$,

$$0 < \alpha + \frac{x_{n-1}}{x_n} = x_{n+1} < \alpha + 1$$

and so

$$0 < x_{n-1} < x_n < \alpha + 1,$$

from which the result follows. ∎

LEMMA 2.2. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.* (1) *which consists of at least two semi-cycles. Then* $\{x_n\}_{n=-1}^{\infty}$ *is oscillatory. Moreover, with the possible exception of the first semi-cycle, every semi-cycle has length* 1 *and every term of* $\{x_n\}_{n=-1}^{\infty}$ *is strictly greater than* α, *and with the possible exception of the first two semi-cycles, no term of* $\{x_n\}_{n=-1}^{\infty}$ *is ever equal to* $\alpha + 1$.

Proof. It suffices to consider the following two cases.

Case 1. Suppose $x_{-1} < \alpha + 1 \leq x_0$. Then

$$x_1 = \alpha + \frac{x_{-1}}{x_0} < \alpha + 1 \quad \text{and} \quad x_2 = \alpha + \frac{x_0}{x_1} > \alpha + 1.$$

Case 2. Suppose $x_0 < \alpha + 1 \leq x_{-1}$. Then

$$x_1 = \alpha + \frac{x_{-1}}{x_0} > \alpha + 1 \quad \text{and} \quad x_2 = \alpha + \frac{x_0}{x_1} < \alpha + 1.$$

∎

The next lemma will be useful in the sequel in determining the limiting behavior of positive solutions of Eq. (1).

LEMMA 2.3. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.* (1), *and let* $N \geq 0$ *be a nonnegative integer. Then the following statements are true.*

1. $x_{N+1} > x_{N-1}$ *if and only if* $x_{N-1} + \alpha x_N - x_{N-1}x_N > 0$.
2. $x_{N+1} = x_{N-1}$ *if and only if* $x_{N-1} + \alpha x_N - x_{N-1}x_N = 0$.
3. $x_{N+1} < x_{N-1}$ *if and only if* $x_{N-1} + \alpha x_N - x_{N-1}x_N < 0$.

Proof. The proof follows from the computation

$$x_{N+1} - x_{N-1} = \left(\alpha + \frac{x_{N-1}}{x_N}\right) - x_{N-1} = \frac{\alpha x_N + x_{N-1} - x_{N-1}x_N}{x_{N-1}}.$$

∎

3. THE CASE $0 \leq \alpha < 1$

In this section, we consider the case where $0 \leq \alpha < 1$, and we show that there exist positive solutions of Eq. (1) which are unbounded.

THEOREM 3.1. *Let* $0 \leq \alpha < 1$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.* (1) *such that* $0 < x_{-1} \leq 1$ *and* $x_0 \geq 1/(1-\alpha)$. *Then the following statements are true.*

1. $\lim_{n\to\infty} x_{2n} = \infty$.
2. $\lim_{n\to\infty} x_{2n+1} = \alpha$.

Proof. Note that $1/(1-\alpha) > \alpha + 1$, and so $x_0 > \alpha + 1$. It suffices to show that

$$x_1 \in (\alpha, 1] \quad \text{and} \quad x_2 \geq \alpha + x_0.$$

Indeed, $x_1 = \alpha + x_{-1}/x_0 > \alpha$. Also,

$$x_1 = \alpha + \frac{x_{-1}}{x_0} \leq \alpha + \frac{1}{x_0} \leq \alpha + (1-\alpha) = 1,$$

and so $x_1 \in (\alpha, 1]$. Hence $x_2 = \alpha + x_0/x_1 \geq \alpha + x_0$. ∎

4. THE CASE $\alpha = 1$

In this section, we consider the case where $\alpha = 1$, and we show that every positive solution of Eq. (1) converges to a two-cycle.

Clearly, if $\alpha = 1$, then the unique equilibrium point of Eq. (1) is $\bar{x} = 2$.

THEOREM 4.1. *Let* $\alpha = 1$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.* (1). *Then the following statements are true.*

1. *Suppose* $\{x_n\}_{n=-1}^{\infty}$ *consists of a single semi-cycle. Then* $\{x_n\}_{n=-1}^{\infty}$ *converges monotonically to* $\bar{x} = 2$.
2. *Suppose* $\{x_n\}_{n=-1}^{\infty}$ *consists of at least two semi-cycles. Then* $\{x_n\}_{n=-1}^{\infty}$ *converges to a prime period*-2 *solution of Eq.* (1).

Proof. We know by Lemma 2.1 that if $\{x_n\}_{n=-1}^{\infty}$ consists of a single semi-cycle, then $\{x_n\}_{n=-1}^{\infty}$ converges monotonically to $\bar{x}$. So it suffices to consider the case where $\{x_n\}_{n=-1}^{\infty}$ consists of at least two semi-cycles.

So assume that $\{x_n\}_{n=-1}^{\infty}$ consists of at least two semi-cycles. We know by Lemma 2.2 that $\{x_n\}_{n=-1}^{\infty}$ is oscillatory, and that except for possibly the first semi-cycle, every semi-cycle has length 1 and every term of $\{x_n\}_{n=-1}^{\infty}$ is greater than $\alpha = 1$.

Now observe that for $n \geq 0$,

$$x_n + x_{n+1} - x_n x_{n+1} = \frac{x_{n-1} + x_n - x_{n-1}x_n}{x_n},$$

and so by Lemma 2.3, the following three statements are true:

(a) Suppose $x_{-1} < x_1$. Then

$$x_{-1} < x_1 < x_3 < \cdots$$

and

$$x_0 < x_2 < x_4 < \cdots .$$

(b) Suppose $x_{-1} = x_1$. Then

$$x_{-1} = x_1 = x_3 = \cdots$$

and

$$x_0 = x_2 = x_4 = \cdots .$$

(c) Suppose $x_{-1} > x_1$. Then

$$x_{-1} > x_1 > x_3 > \cdots$$

and

$$x_0 > x_2 > x_4 > \cdots .$$

The proof of the theorem follows from Lemma 1.4 and statements (a), (b), and (c) above. ∎

5. THE CASE $\alpha > 1$

In this section, we consider the case where $\alpha > 1$, and we show in Theorem 5.2 that the equilibrium point $\bar{x} = \alpha + 1$ of Eq. (1) is globally asymptotically stable. We first give a lemma which shall be useful in the sequel.

LEMMA 5.1. *Let* $\alpha > 1$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.* (1). *Then*

$$\alpha + \frac{\alpha - 1}{\alpha} \leq \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \leq \frac{\alpha^2}{\alpha - 1}.$$

Proof. It follows by Lemmas 2.1 and 2.2 that we may assume that every semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length 1, that $\alpha < x_n$ for all $n \geq -1$, and that $\alpha < x_0 < \alpha + 1 < x_{-1}$.

We shall first show that $\limsup_{n \to \infty} x_n \leq \alpha^2/(\alpha - 1)$. Note that for $n \geq 0$,

$$x_{2n+1} < \alpha + \frac{x_{2n-1}}{\alpha}.$$

So as every solution of the difference equation

$$y_{m+1} = \alpha + \frac{1}{\alpha} y_m, \qquad m = 0, 1, \ldots$$

converges to $\alpha^2/(\alpha - 1)$, it follows that $\limsup_{n\to\infty} x_n \leq \alpha^2/(\alpha - 1)$.

We shall next show that $\alpha + (\alpha - 1)/\alpha \leq \liminf_{n\to\infty} x_n$. Let $\varepsilon > 0$. There clearly exists $N \geq 0$ such that for all $n \geq N$,

$$x_{2n-1} < \frac{\alpha^2 + \varepsilon}{\alpha - 1}.$$

Let $n \geq N$. Then

$$x_{2n} = \alpha + \frac{x_{2n-2}}{x_{2n-1}} > \alpha + \alpha\left(\frac{\alpha - 1}{\alpha^2 + \varepsilon}\right) = \frac{\alpha^3 + \alpha\varepsilon + \alpha(\alpha - 1)}{\alpha^2 + \varepsilon},$$

and so

$$\liminf_{n\to\infty} x_n \geq \frac{\alpha^3 + \alpha\varepsilon + \alpha(\alpha - 1)}{\alpha^2 + \varepsilon}.$$

So as ε is arbitrary, we have

$$\liminf_{n\to\infty} x_n \geq \frac{\alpha^3 + \alpha(\alpha - 1)}{\alpha^2} = \alpha + \frac{\alpha - 1}{\alpha}.$$

∎

We next state the following theorem, a minor modification of Theorem 5.2 in [2], which provides the key step in proving Theorem 5.2.

THEOREM A. *Let* $f\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ *be a continuous function, and consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, \ldots, \tag{3}$$

where $x_{-1}, x_0 \in (0,\infty)$. *Suppose f satisfies the following conditions*:

(a) *There exist positive numbers a and b with* $a < b$ *such that*

$$a \leq f(x, y) \leq b \quad \text{for all } x, y \in [a, b];$$

(b) $f(x, y)$ *is nonincreasing in* $x \in [a, b]$ *for each* $y \in [a, b]$, *and* $f(x, y)$ *is nondecreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$;

(c) *Equation* (3) *has no solutions of prime period* 2 *in* $[a, b]$.

Then there exists exactly one equilibrium $\bar{x}$ *of Eq.* (3) *which lies in* $[a, b]$. *Moreover, every solution of Eq.* (3) *which lies in* $[a, b]$ *converges to* $\bar{x}$.

We are now ready for the main result of this section.

THEOREM 5.2. *Let* $\alpha > 1$. *Then* $\bar{x} = \alpha + 1$ *is a globally asymptotically stable equilibrium point of Eq.* (1).

Proof. We know by Lemma 1.1 that $\bar{x} = \alpha + 1$ is a locally asymptotically stable equilibrium point of Eq. (1). So let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (1). It suffices to show that

$$\lim_{n \to \infty} x_n = \alpha + 1.$$

For $x, y \in (0, \infty)$, set

$$f(x, y) = \alpha + \frac{y}{x}.$$

Then f: $(0, \infty) \times (0, \infty) \to (0, \infty)$ is a continuous function, f decreasing in $x \in (0, \infty)$ for each $y \in (0, \infty)$, and f increasing in $y \in (0, \infty)$ for each $x \in (0, \infty)$. Recall that by Lemma 1.2, there exist no solutions of Eq. (1) with prime period 2. Let $\varepsilon > 0$, and set

$$a = \alpha \quad \text{and} \quad b = \frac{\alpha^2 + \varepsilon}{\alpha - 1}.$$

Note that

$$f\left(\frac{\alpha^2 + \varepsilon}{\alpha - 1}, \alpha\right) = \alpha + \alpha\left(\frac{\alpha - 1}{\alpha^2 + \varepsilon}\right) > \alpha$$

and

$$f\left(\alpha, \frac{\alpha^2 + \varepsilon}{\alpha - 1}\right) = \alpha + \frac{1}{\alpha} \cdot \frac{\alpha^2 + \varepsilon}{\alpha - 1}$$
$$= \frac{\alpha^3 + \varepsilon}{\alpha^2 - \alpha} < \frac{\alpha^3 + \varepsilon \cdot \alpha}{\alpha^2 - \alpha} = \frac{\alpha^2 + \varepsilon}{\alpha - 1}.$$

Hence

$$\alpha < f(x, y) < \frac{\alpha^2 + \varepsilon}{\alpha - 1} \quad \text{for all } x, y \in \left[\alpha, \frac{\alpha^2 + \varepsilon}{\alpha - 1}\right].$$

Finally, note that by Lemma 5.1,

$$\alpha < \alpha + \frac{\alpha - 1}{\alpha} \leq \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \leq \frac{\alpha^2}{\alpha - 1} < \frac{\alpha^2 + \varepsilon}{\alpha - 1}$$

and so by Theorem A,

$$\lim_{n\to\infty} x_n = \alpha + 1.$$

∎

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