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Spectra of coronae

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ABSTRACT

We introduce a new invariant, the *coronal* of a graph, and use it to compute the spectrum of the corona $G \circ H$ of two graphs G and H. In particular, we show that this spectrum is completely determined by the spectra of G and H and the coronal of H. Previous work has computed the spectrum of a corona only in the case that H is regular. We then explicitly compute the coronals for several families of graphs, including regular graphs, complete n-partite graphs, and paths. Finally, we use the corona construction to generate many infinite families of pairs of cospectral graphs.

1. Introduction

Let G and H be (finite, simple, non-empty) graphs. The *corona* $G \circ H$ of G and H is constructed as follows: Choose a labeling of the vertices of G with labels $1, 2, \ldots, m$. Take one copy of G and m disjoint copies of H, labeled $H_1, \ldots, H_m$, and connect each vertex of H_i to vertex i of G. This construction was introduced by Frucht and Harary [4] with the (achieved) goal of constructing a graph whose automorphism group is the wreath product of the two component automorphism groups. Since then, a variety of papers have appeared investigating a wide range of graph-theoretic properties of coronas, such as the bandwidth [2], the minimum sum [15], applications to Ramsey theory [10], etc. Further, the spectral properties of coronas are significant in the study of invertible graphs. Briefly, a graph G is invertible if the inverse of the graph's adjacency matrix is diagonally similar to the adjacency matrix of another graph G^+, dubbed the dual of G (see [5]). Motivated by applications to quantum chemistry, Godsil [5] studies invertible bipartite graphs with unique perfect matching. In response to his question

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asking for a characterization of such graphs with the additional property that $G \cong G^+$, Simion and Cao [13] determine the answer to be exactly the coronas of bipartite graphs with the single-vertex graph K_1.

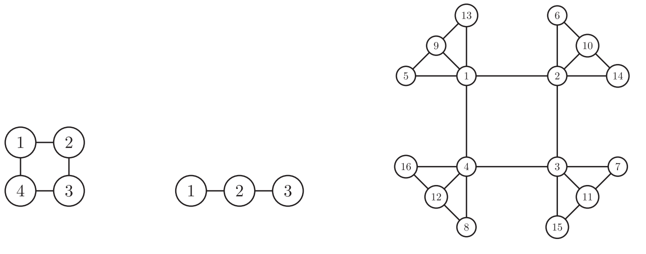
The study of spectral properties of coronas was continued by Barik et al. in [1], who found the spectrum of the corona $G \circ H$ in the special case that H is regular. In Section 2, we remove the regularity hypothesis on H and compute the spectrum of the corona of any pair of graphs using a new graph invariant called the *coronal*. In Section 3, we introduce two different techniques for evaluating coronals, and apply these techniques to several important families of graphs, including complete n-partite graphs, paths, and (as in [1]), regular graphs. In each of these instances, the coronal has a concise explicit form. Finally, in Section 4, we see that the computation of the spectrum of coronas lends itself to finding many large families of cospectral graph pairs.

1.1. Notation

The symbols $\mathbf{0}_n$ and $\mathbf{1}_n$ (resp., $\mathbf{0}_{mn}$ and $\mathbf{1}_{mn}$) will stand for the length-n column vectors (resp. $m \times n$ matrices) consisting entirely of 0's and 1's. For two matrices A and B, the matrix $A \otimes B$ is the Kronecker (or tensor) product of A and B. For a graph G with adjacency matrix A, the characteristic polynomial of G is $f_G(\lambda) := \det(\lambda I - A)$. We use the standard notations P_n, C_n, S_n, and K_n for the path, cycle, star, and complete graph on n vertices.

2. The main theorem

Let G and H be finite simple graphs on m and n vertices, respectively, and let A and B denote their respective adjacency matrices. We begin by choosing a convenient labeling of the vertices of $G \circ H$. Recall that $G \circ H$ is comprised of the m vertices of G, which we label arbitrarily using the symbols $\{1, 2, \ldots, m\}$, and m copies $H_1, H_2, \ldots, H_m$ of H. Choose an arbitrary ordering $h_1, h_2, \ldots, h_n$ of the vertices of H, and label the vertex in H_i corresponding to h_k by the label $i + mk$. Below is a sample corona with the above labeling procedure:



G $\quad$ H $\quad$ $G \circ H$

Under this labeling the adjacency matrix of $G \circ H$ is given by

$$A \circ B := \begin{bmatrix} A & \mathbf{1}_n^T \otimes I_m \\ \mathbf{1}_n \otimes I_m & B \otimes I_m \end{bmatrix}.$$

The goal now is to compute the eigenvalues of this corona matrix in terms of the spectra of A and B. We introduce one new invariant for this purpose.

Definition 1. Let H be a graph on n vertices, with the adjacency matrix B. Note that, viewed as a matrix over the field of rational functions $\mathbb{C}(\lambda)$, the characteristic matrix $\lambda I - B$ has determinant $\det(\lambda I - B) = f_H(\lambda) \neq 0$, so is invertible. The *coronal* $\chi_H(\lambda) \in \mathbb{C}(\lambda)$ of H is defined to be the sum of the entries of the matrix $(\lambda I - B)^{-1}$. Note this can be calculated as

$$\chi_H(\lambda) = \mathbf{1}_n^T(\lambda I_n - B)^{-1}\mathbf{1}_n.$$

Our main theorem is that, beyond the spectra of G and H, only the coronal of H is needed to compute the spectrum of $G \circ H$.

Theorem 2. *Let G and H be graphs with m and n vertices. Let $\chi_H(\lambda)$ be the coronal of H. Then the characteristic polynomial of $G \circ H$ is*

$$f_{G\circ H}(\lambda) = f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda)).$$

In particular, the spectrum of $G \circ H$ is completely determined by the characteristic polynomials f_G and f_H, and the coronal χ_H of H.

Proof. Let A and B denote the respective adjacency matrices of G and H. We compute the characteristic polynomial of the matrix $A \circ B$. For this, we recall two elementary results from linear algebra on the multiplication of Kronecker products and determinants of block matrices:

- In cases where each multiplication makes sense, we have
$$M_1M_2 \otimes M_3M_4 = (M_1 \otimes M_3)(M_2 \otimes M_4).$$
- If M_4 is invertible, then
$$\det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_4)\det(M_1 - M_2M_4^{-1}M_3).$$

Combining these two facts, we have (as an equality of rational functions)

$$\begin{aligned}
f_{G\circ H}(\lambda) &= \det(\lambda I_{m(n+1)} - A \circ B) \\
&= \det \begin{pmatrix} \lambda I_m - A & -\mathbf{1}_n^T \otimes I_m \\ -\mathbf{1}_n \otimes I_m & \lambda I_{mn} - B \otimes I_m \end{pmatrix} \\
&= \det \begin{pmatrix} \lambda I_m - A & -\mathbf{1}_n^T \otimes I_m \\ -\mathbf{1}_n \otimes I_m & (\lambda I_n - B) \otimes I_m \end{pmatrix} \\
&= \det((\lambda I_n - B) \otimes I_m)\det\left[(\lambda I_m - A) - (\mathbf{1}_n^T \otimes I_m)((\lambda I_n - B) \otimes I_m)^{-1}(\mathbf{1}_n \otimes I_m)\right] \\
&= \det(\lambda I_n - B)^m \det(\lambda I_m - A - (\mathbf{1}_n^T(\lambda I_n - B)^{-1}\mathbf{1}_n) \otimes I_m) \\
&= \det(\lambda I_n - B)^m \det((\lambda - \chi_H(\lambda))I_m - A) \\
&= f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda)). \qquad \square
\end{aligned}$$

Remark 3. A natural question is whether or not the spectrum of $G \circ H$ is determined by the spectra of G and H, i.e., whether knowledge of the coronal is necessary. We find that indeed it is necessary: Computing the coronals of the cospectral graphs S_5 and $C_4 \cup K_1$, we have

$$\chi_{S_5}(\lambda) = \frac{5\lambda + 8}{\lambda^2 - 4} \quad \text{and} \quad \chi_{C_4 \cup K_1}(\lambda) = \frac{5\lambda - 2}{\lambda^2 - 2\lambda}.$$

Thus cospectral graphs need not have identical coronals, and hence for a random graph G, the spectra of $G \circ S_5$ and $G \circ (C_4 \cup K_1)$ will likely be distinct. Note that this stands in stark contrast to the situation for the Cartesian and tensor products of graphs. In both of these cases, the spectrum of the product is determined by the spectra of the components.

The examples in the above remark are representative of a fairly common phenomena: since the coronal $\chi_H(\lambda) = \frac{\widetilde{\chi}_H(\lambda)}{f_H(\lambda)}$ can be computed as the quotient of the sum $\widetilde{\chi}_H(\lambda)$ of the cofactors of $\lambda I - B$ by the characteristic polynomial $f_H(\lambda)$, it is *a priori* the quotient of a degree $n - 1$ polynomial by a degree n polynomial. In practice, however, as in the examples, these two polynomials typically have roots in common, providing for a reduced expression for the coronal. Let us suppose that $g(\lambda) := \gcd(\widetilde{\chi}_H(\lambda), f_H(\lambda))$ has degree $n-d$ (the gcd being taken in $\mathbb{C}[\lambda]$), so that $\chi_H(\lambda)$ in its reduced form is a quotient of a degree $d-1$ polynomial by a degree d polynomial. Moreover, since the denominator of this reduced fraction is a factor of $f_H(\lambda)$, and since f_G is of degree m, each pole of $\chi_H(\lambda)$ is simultaneously a multiplicity-m pole of $f_G(\lambda - \chi_H(\lambda))$ and a multiplicity-m root of $f_H(\lambda)^m$. Since these contributions cancel in the overall determination of the roots of $f_{G\circ H}(\lambda)$ in the expression

$$f_{G\circ H}(\lambda) = f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda))$$

from Theorem 2, we can now more explicitly describe the spectrum of the corona. Namely, let d be the degree of the denominator of $\chi_H(\lambda)$ as a reduced fraction. Then the spectrum of $G \circ H$ consists of:

- Some "old" eigenvalues, i.e., the roots of $f_H(\lambda)$ which are not poles of $\chi_H(\lambda)$ (or equivalently, the roots of $g(\lambda)$), each with multiplicity $|G|$; and
- Some "new" eigenvalues, i.e., the values of λ such that $\lambda - \chi_H(\lambda)$ is an eigenvalue μ of G (with the multiplicity of λ equal to the multiplicity of μ as an eigenvalue of G.)

Since for a given μ, solving $\lambda - \chi_H(\lambda) = \mu$ by clearing denominators amounts to finding the roots of a degree $d+1$ polynomial in λ, the above two bullets combine to respectively provide all $(n-d)m + m(d+1) = m(n+1)$ eigenvalues of $G \circ H$. Table 1, computed using SAGE [14], gives the number of graphs on n vertices whose coronal has a denominator of degree d (as a reduced fraction), as well as the average degree of this denominator, for $1 \leqslant n \leqslant 7$.

Since determining the characteristic polynomial of $G \circ H$ from the spectra of G and H requires only the extra knowledge of the coronal of H, it remains to develop techniques for computing these coronals. In Section 3, we will develop shortcuts for these computations, but we briefly conclude this section with some more computationally-oriented approaches. A first such option is to have a software package with linear algebra capabilities directly compute the inverse of $\lambda I - B$ and sum its entries, as done in the computations for Table 1. This seems to be computationally feasible only for rather small graphs (e.g., $n \leqslant 12$). A second, more graph-theoretic, option relies on a combinatorial result of Schwenk [12] to compute each cofactor of $\lambda I - B$ individually, before summing them to compute the coronal.

Theorem 4 (Schwenk [12]). *For vertices i and j of a graph H with adjacency matrix B, let $\mathcal{P}_{i,j}$ denote the set of paths from i to j. Then*

$$\operatorname{adj}(\lambda I - B)_{i,j} = \sum_{P \in \mathcal{P}_{i,j}} f_{H-P}(\lambda).$$

Table 1
Number of graphs on n vertices whose coronal has denominator of degree d.

$d\backslash n$	1	2	3	4	5	6	7
1	1	2	2	4	3	8	6
2		0	2	5	12	28	44
3			0	2	13	50	138
4				0	6	40	304
5					0	22	246
6						8	214
7							92
Total	1	2	4	11	34	156	1044
Average d	1	1	1.5	1.82	2.65	3.41	4.68
(Average d)/n	1	0.5	0.5	0.45	0.53	0.57	0.66

Again, this approach becomes computationally infeasible fairly quickly without a method for pruning the number of cofactors to calculate. We explore this idea in the next section. Regardless, from Theorem 4, we obtain:

Corollary 5. *The spectrum of the corona $G \circ H$ is determined by the spectrum of G and the spectra of the subgraphs of H (or more economically, only those subgraphs obtained by deleting paths from H).*

3. Computing coronals

In this section, we will compute the coronals $\chi_H(\lambda)$ for several families of graphs, and hence obtain the full spectrum of the corona $G \circ H$ for any G. We produce two relatively independent techniques for computing the coronal of a graph. The first exploits the regularity or near-regularity of a graph in order to greatly reduce the number of cofactor calculations (relative to those required by Theorem 4) needed to compute the coronal. In particular, we use these ideas to compute the coronals of regular graphs, complete bipartite graphs, and paths. The second approach relates the coronal to a previously-studied graph invariant, the *walk generating function*, which culminates in giving the coronal of H in terms of the characteristic polynomials of H and its complement. In cases where the latter is known (e.g., multi-partite graphs), one is led to simple formulas for the coronal.

3.1. Coronals via near-regularity

For graphs that are "nearly regular" in the sense that their degree sequences are almost constant, we can take advantage of linear-algebraic symmetries to compute the coronals. We begin with two concrete computations, regular and complete bipartite graphs, before extracting the underlying heuristic and applying it to the coronal of path graphs. The case of regular graphs, first addressed in [1], is particularly straight-forward from this viewpoint.

Proposition 6 (Regular graphs)**.** *Let H be r-regular on n vertices. Then*

$$\chi_H(\lambda) = \frac{n}{\lambda - r}.$$

Thus for any graph G, the spectrum of $G \circ H$ consists precisely of:

- *Every non-maximum eigenvalue of H, each with multiplicity $|G|$.*
- *Two multiplicity-one eigenvalues*

  $$\frac{\mu + r \pm \sqrt{(r-\mu)^2 + 4n}}{2}$$

  for each eigenvalue μ of G.

Proof. Let B be the adjacency matrix of H. By regularity, we have $B\mathbf{1}_n = r\mathbf{1}_n$, and hence $(\lambda I - B)\mathbf{1}_n = (\lambda - r)\mathbf{1}_n$. Cross-dividing and multiplying by $\mathbf{1}_n^T$,

$$\chi_H(\lambda) = \mathbf{1}_n^T(\lambda I - B)^{-1}\mathbf{1}_n = \frac{\mathbf{1}_n^T\mathbf{1}_n}{\lambda - r} = \frac{n}{\lambda - r}.$$

The only pole of $\chi_H(\lambda)$ is the maximal eigenvalue $\lambda = r$ of H, and the "new" eigenvalues are obtained by solving $\lambda - \frac{n}{\lambda - r} = \mu$ for each eigenvalue μ of G. $\square$

It is noteworthy that all r-regular graphs on n vertices have the same coronal, especially given that the cofactors of the corresponding matrices $(B - \lambda I)^{-1}$ appear to be markedly dissimilar. The simplicity of this scenario, and the easily checked observation that cospectral regular graphs must have the same regularity, lead to the following corollary. We will make use of this corollary in the final section.

Corollary 7. *Cospectral regular graphs have the same coronal.*

As a second class of examples, we compute the coronals of complete bipartite graphs. (Though this calculation will later be subsumed by Proposition 16, the formula given here is more elegant and the proof does not require previous knowledge of the various characteristic polynomials arising in that proof.)

Proposition 8 (Complete bipartite graphs)**.** *Let $H = K_{p,q}$ be a complete bipartite graph on $p + q = n$ vertices. Then*

$$\chi_H(\lambda) = \frac{n\lambda + 2pq}{\lambda^2 - pq}.$$

For any graph G, the spectrum of $G \circ H$ is given by:

- *The eigenvalue* 0 *with multiplicity $m(n-2)$; and*
- *For each eigenvalue μ of G, the roots of the polynomial*

$$x^3 - \mu x^2 - (p + q + pq)x + pq(\mu - 2).$$

Proof. Let $B = \begin{bmatrix} \mathbf{0}_{pp} & \mathbf{1}_{pq} \\ \mathbf{1}_{qp} & \mathbf{0}_{qq} \end{bmatrix}$ be the adjacency matrix of $K_{p,q}$ and let $X = \mathrm{diag}((q+\lambda)I_p, (p+\lambda)I_q)$ be the diagonal matrix with the first p diagonal entries being $(q+\lambda)$ and the last q entries being $(p+\lambda)$. Then $(\lambda I - B)X\mathbf{1}_n = (\lambda^2 - pq)\mathbf{1}_n$, and so

$$\chi_H(\lambda) = \mathbf{1}_n^T(\lambda I - B)^{-1}\mathbf{1}_n = \frac{\mathbf{1}_n^T X \mathbf{1}_n}{\lambda^2 - pq} = \frac{(p+q)\lambda + 2pq}{\lambda^2 - pq}.$$

Thus the coronal has poles at both of the non-zero eigenvalues $\pm\sqrt{pq}$ of $K_{p,q}$, leaving only the eigenvalue 0 with multiplicity $p + q - 2$. Finally, solving $\lambda - \chi_H(\lambda) = \mu$ gives the new eigenvalues in the spectrum as stated in the proposition. $\square$

Remark 9. It might be tempting in light of Propositions 6 and 8 to hope that the degree sequence of a graph determines its coronal. This too, like the analogous conjecture stemming from cospectrality (Remark 3), turns out to be false: The graphs P_5 and $K_2 \cup K_3$ have the same degree sequence, but we find by direct computation that

$$\chi_{P_5}(\lambda) = \frac{5\lambda^2 + 8\lambda - 1}{\lambda^3 - 3\lambda} \quad \chi_{K_2 \cup K_3}(\lambda) = \frac{5\lambda - 7}{\lambda^2 - 3\lambda + 2}.$$

The proof technique for the last two propositions generalizes to "nearly regular" graphs, by which we mean graphs H for which all but a small number of vertices have the same degree r. In this case, we can write

$$(\lambda I - B)\mathbf{1}_n = (\lambda - r)\mathbf{1}_n + \mathbf{v},$$

where $\mathbf{v} = (v_i)$ is a vector consisting mostly of 0's. This gives

$$(\lambda I - B)^{-1}\mathbf{1}_n = \frac{1}{\lambda - r}\left[\mathbf{1}_n - (\lambda I - B)^{-1}\mathbf{v}\right],$$

and thus, using the adjugate formula for the determinant,

$$\chi_H(\lambda) = \mathbf{1}_n^T(\lambda I - B)^{-1}\mathbf{1}_n = \frac{1}{\lambda - r}\left[n - \frac{1}{f_H(\lambda)}\sum_{1 \leqslant i,j \leqslant n} v_i C_{i,j}\right],$$

where $C_{i,j}$ denotes the (i,j)-cofactor of $\lambda I - B$. Since v_i is zero for most values of i, we have an effective technique for computing coronals if we can compute a small number of cofactors (as opposed to, in particular, computing *all* of the cofactors and using Theorem 4). For example, if we let $f_n = f_{P_n}(\lambda)$ be the characteristic polynomial of the path graph P_n on n vertices (by convention, set $f_0 = 1$), we can compute coronals of paths as follows:

Proposition 10 (Path graphs). *Let $H = P_n$. Then*

$$\chi_H(\lambda) = \frac{nf_n - 2\sum_{j=0}^{n-1} f_j}{(\lambda-2)f_n} = \frac{(n(\lambda-2)-2)f_n + 2f_{n-1} + 2}{(\lambda-2)^2 f_n}.$$

Proof. The proof of the first equality will reflect the discussion above, and the second equality is a combinatorial re-writing using a recurrence relation for the functions f_n. In the notation of the discussion preceding the proposition, we take $r = 2$ and $\mathbf{v} = [1\,0\,0\,\cdots\,0\,0\,1]^T$. Further, we note that an easy induction argument using cofactor expansion gives $C_{1,j} = C_{n,j} = f_{j-1}$. Thus we obtain

$$\chi_{P_n}(\lambda) = \frac{1}{\lambda-2}\left[n - \frac{1}{f_n}\sum_{j=1}^{n}(C_{1,j} + C_{n,j})\right] = \frac{1}{\lambda-2}\left[n - \frac{2}{f_n}\sum_{j=0}^{n-1} f_j\right],$$

from which the first equality follows. For the second equality, we recall from [9, Theorem 3] that the path graph polynomials f_n satisfy $f_0 = 1, f_1 = \lambda$, and $f_n = \lambda f_{n-1} - f_{n-2}$ for $n \geqslant 2$. The sum in the previous formula can now be evaluated as follows:

$$\begin{aligned} S := \sum_{j=0}^{n-1} f_j &= 1 + \lambda + \sum_{j=2}^{n-1}(\lambda f_{j-1} - f_{j-2}) \\ &= 1 + \lambda + \lambda(S - 1 - f_{n-1}) - (S - f_{n-1} - f_{n-2}). \end{aligned}$$

Solving gives $(\lambda-2)S = \lambda f_{n-1} - f_{n-2} - f_{n-1} - 1 = f_n - f_{n-1} - 1$, and substituting $S = \frac{f_n - f_{n-1} - 1}{\lambda - 2}$ into the first equality gives the second. □

From this, we easily calculate the coronals for the first few path graphs, as shown in Table 2.

Remark 11. This particular example can also be computed using Theorem 4: For i and j distinct, there is a unique path $[i,j]$ from vertex i to vertex j, so the sum in the theorem reduces to a single term:

$$\operatorname{adj}(\lambda I - B)_{i,j} = f_{P_n - [i,j]} = f_{i-1}f_{n-j}.$$

Similarly, we find $\operatorname{adj}(\lambda I - B)_{i,i} = f_{i-1}f_{n-i}$. Summing over all the cofactors gives

$$\chi_{P_n}(\lambda) = \frac{1}{f_n}\left(\sum_{i=1}^{n} f_{i-1}f_{n-i} + 2\sum_{i,j=1}^{n} f_{i-1}f_{n-j}\right),$$

which reduces to the result computed in Proposition 10 after the application of recurrence identities.

3.2. Coronals via the walk generating function

Here we give a combinatorial interpretation of the coronal of a graph H, and show that it can be computed from the knowledge of the characteristic polynomials of H and its complement, which we

Table 2
The coronals $\chi_{P_n}(\lambda)$ for $1 \leqslant n \leqslant 7$.

n	1	2	3	4	5	6	7
$\chi_{P_n}(\lambda)$	$\frac{1}{\lambda}$	$\frac{2}{\lambda-1}$	$\frac{3\lambda+4}{\lambda^2-2}$	$\frac{4\lambda+2}{\lambda^2-\lambda-1}$	$\frac{5\lambda^2+8\lambda-1}{\lambda^3-3\lambda}$	$\frac{6\lambda^2+4\lambda-4}{\lambda^3-\lambda^2-2\lambda+1}$	$\frac{7\lambda^3+12\lambda^2-6\lambda-8}{\lambda^4-4\lambda^2+2}$

denote by $\overline{H}$. We begin with the simple observation that

$$\begin{aligned}\chi_H(\lambda) = \mathbf{1}_n^T(\lambda I_n - B)^{-1}\mathbf{1}_n &= \lambda^{-1}\mathbf{1}_n^T\left(I_n - \lambda^{-1}B\right)^{-1}\mathbf{1}_n \\ &= \lambda^{-1}\mathbf{1}_n^T\left(\sum_{k=0}^{\infty}\lambda^{-k}B^k\right)\mathbf{1}_n \\ &= \frac{1}{\lambda}\sum_{k=0}^{\infty}(\mathbf{1}_n^T B^k \mathbf{1}_n)\lambda^{-k}.\end{aligned}$$

Of course, the kth power of the adjacency matrix has (i,j)-entry equal to the number of length-k walks in H from i to j, and hence the sum $\mathbf{1}_n^T B^k \mathbf{1}_n$ of all the entries in this matrix is the total number w_k of all walks of length k in H. In particular, if $w_H(t) = \sum_{k=0}^{\infty} w_k t^k$ is the generating function of the sequence w_k, the above calculation has proven that

$$\chi_H(\lambda) = \frac{1}{\lambda}w_H\left(\frac{1}{\lambda}\right).$$

This generating function admits a relation to $f_H(\lambda)$ and $f_{\overline{H}}(\lambda)$ as follows [3, Theorem 1.11]:

$$w_H\left(\frac{1}{\lambda}\right) = -\lambda + (-1)^n\lambda\frac{f_{\overline{H}}(-\lambda-1)}{f_H(\lambda)}.$$

Combining the two equations give the principal result of this section.

Theorem 12. *For a graph H of order n,*

$$\chi_H(\lambda) = -1 + (-1)^n\frac{f_{\overline{H}}(-\lambda-1)}{f_H(\lambda)}.$$

Tangentially, we remark that one deduces from this the following curious property about graphs which are cospectral to their complement (e.g., Paley graphs, or self-complementary graphs in general):

Corollary 13. *If H is a graph on n vertices cospectral with its complement, then $\chi_H(-\frac{1}{2})$ is 0 or -2, depending on whether n is even or odd.*

More significantly, we deduce from Theorem 2 the following alternative calculation of the characteristic polynomial of a corona.

Corollary 14. *The characteristic polynomial of the corona $G \circ H$ is given by:*

$$f_{G\circ H}(\lambda) = f_H(\lambda)^m f_G\left(\lambda + 1 - (-1)^m\frac{f_{\overline{H}}(-\lambda-1)}{f_H(\lambda)}\right).$$

Remark 15. The reviewer notes an alternative proof of the corollary using the identity $G \circ H = G \diamond (H \vee K_1)$, where $\vee$ denotes the join of two graphs (see Proposition 17) and $\diamond$ denotes the rooted product (see [7]).

As a first application, we generalize the first part of Proposition 8 to find the coronal of complete multi-partite graphs.

Proposition 16. *Let H be the complete multi-partite graph $K_{n_1,n_2,\ldots,n_k}$ where $n_1 + n_2 + \cdots + n_k = n$. Then*

$$\chi_H(\lambda) = \left(1 - \sum_{i=1}^{k}\frac{n_i}{\lambda + n_i}\right)^{-1} - 1.$$

Proof. This follows directly from Theorem 12 and well-known formulas for the characteristic polynomials of H and $\overline{H} = K_{n_1} \cup \cdots \cup K_{n_k}$. Namely, we have [3, Section 2.6]

$$f_H(\lambda) = \lambda^{n-k} \left(1 - \sum_{i=1}^{k} \frac{n_i}{\lambda + n_i}\right) \prod_{j=1}^{k} (\lambda + n_j)$$

and

$$f_{\overline{H}}(\lambda) = \prod_{i=1}^{k} f_{K_{n_i}}(\lambda) = \prod_{i=1}^{k} (\lambda + 1)^{n_i - 1}(\lambda - n_i + 1) = (\lambda + 1)^{n-k} \prod_{i=1}^{k} (\lambda - n_i + 1).$$

Substituting, we have $f_{\overline{H}}(-\lambda - 1) = (-1)^n \lambda^{n-k} \prod_{i=1}^{k} (\lambda + n_i)$. Now from Theorem 12,

$$\chi_H(\lambda) = -1 + (-1)^n \frac{f_{\overline{H}}(-\lambda - 1)}{f_H(\lambda)} = -1 + \frac{1}{1 - \sum_{i=1}^{k} \frac{n_i}{\lambda + n_i}},$$

as desired. $\square$

Finally, we include the following proposition and its proof, due to the reviewer. The *join* $G \vee H$ of two graphs is the graph obtained by taking the disjoint union of G and H and adding an edge from each vertex of G to each vertex of H. We note that the complement of the join $G \vee H$ is the disjoint union of $\overline{G}$ and $\overline{H}$.

Proposition 17. *Let H_1 be an r_1-regular graph of order n_1 and H_2 an r_2-regular graph of order n_2. If $H = H_1 \vee H_2$, then*

$$\chi_H(\lambda) = \frac{(\lambda - r_1)n_2 + (\lambda - r_2)n_1 + 2n_1 n_2}{(\lambda - r_1)(\lambda - r_2) - n_1 n_2}.$$

Proof. Theorem 2.6 of [3] relates the characteristic polynomials of a regular graph and its complement: For $i = 1, 2$, we have

$$f_{\overline{H_i}}(\lambda) = (-1)^{n_i} \frac{\lambda - n_i + r_i + 1}{\lambda + r_i + 1} f_{H_i}(-\lambda - 1)$$

Now since $f_{\overline{H}}(\lambda) = f_{\overline{H_1}}(\lambda) f_{\overline{H_2}}(\lambda)$, we have

$$f_{\overline{H}}(-\lambda - 1) = (-1)^{n_1 + n_2} \frac{(r_1 - n_1 - \lambda)(r_2 - n_2 - \lambda)}{(r_1 - \lambda)(r_2 - \lambda)} f_{H_1}(\lambda) f_{H_2}(\lambda).$$

Finally, Theorem 2.8 of [3] computes

$$f_H(\lambda) = \frac{f_{H_1}(\lambda) f_{H_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)} [(\lambda - r_1)(\lambda - r_2) - n_1 n_2].$$

The result now follows easily from Theorem 12. $\square$

4. Cospectrality

At the end of [1], the authors prove that if G_1 and G_2 are cospectral graphs, then $G_1 \circ K_1$ and $G_2 \circ K_1$ are also cospectral, and that (by repeated coronation with K_1) this leads to an infinite collection of cospectral pairs. Armed with the characteristic polynomial

$$f_{G \circ H}(\lambda) = f_H(\lambda)^m f_G(\lambda - \chi_H(\lambda))$$

of the corona (Theorem 2), we can greatly generalize this observation on two fronts.

Corollary 18. *If G_1 and G_2 are cospectral, and H is any graph, then $G_1 \circ H$ and $G_2 \circ H$ are cospectral. Further, if H_1 and H_2 are cospectral and $\chi_{H_1}(\lambda) = \chi_{H_2}(\lambda)$, and G is any graph, then $G \circ H_1$ and $G \circ H_2$ are cospectral.*

We remark that examples of this second type do indeed exist. Define the *switching graph* $\mathrm{Sw}(T)$ of a tree T with adjacency matrix A_T to be the graph with adjacency matrix

$$A_{\mathrm{Sw}(T)} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes A_T + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes A_{\overline{T}},$$

and let T_1 and T_2 be non-isomorphic cospectral trees with cospectral complements (note that by [6], generalizing [11], "almost all" trees admit a cospectral pair with cospectral complement). Then the switching graphs $\mathrm{Sw}(T_1)$ and $\mathrm{Sw}(T_2)$ are non-isomorphic cospectral regular graphs (see [8, Construction 3.7]), and also have the same coronal by Corollary 7. Corollary 18 now implies that for *any* graphs G and H, we have the cospectral pair $G \circ \mathrm{Sw}(T_1)$ and $G \circ \mathrm{Sw}(T_2)$ and the cospectral pair $\mathrm{Sw}(T_1) \circ H$ and $\mathrm{Sw}(T_2) \circ H$. This gives, for example, infinitely many cospectral pairs of graphs with a given graph G as an induced subgraph.

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