NORTH-HOLLAND

Some Convex and Monotone Matrix Functions

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ABSTRACT

We prove some matrix monotonicity and matrix convexity properties for functions derived from operator connections. As a consequence, we prove that a certain product of matrix monotone functions is a matrix monotone function. Similar results are proved for matrix convex functions. These results include some known results for matrix monotone and matrix convex functions.

1. INTRODUCTION

Throughout, M_n will denote the set of all complex hermitian $n \times n$ matrices, and S_n the cone of positive semidefinite matrices in M_n, while P_n denotes the cone of positive definite matrices in M_n. Furthermore, I and J will be intervals in R. We shall let $S(I; M_n)$ denote the totality of all members in M_n whose spectra are contained in I.

Let f be a real valued function defined on I, and $A \in S(I; M_n)$. Then $f(A)$ is defined by familiar functional calculi. If f is a real valued function

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defined on $I \times J$ and if $A \in S(I; M_m)$ and $B \in S(J; M_n)$ have spectral resolutions $A = \sum_{i=1}^{m'} \lambda_i P_i$ and $B = \sum_{j=1}^{n'} \mu_j Q_j$ respectively, then $f(A, B) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} f(\lambda_i, \mu_j) P_i \otimes Q_j$ is an $mn \times mn$ matrix in the tensor product space $M_m \otimes M_n$ of M_m and M_n [7, 16]. If $f(x, y) = g(x)h(y)$ then $f(A, B) = g(A) \otimes h(B)$.

Let $A, B, C, \ldots$ denote the elements of S_n. (Here n is arbitrary.) A binary operation σ defined on and with values in S_n is called a *connection* if

(i) $A \leqslant B$, $C \leqslant D$ implies $A \sigma C \leqslant B \sigma D$.
(ii) $C(A \sigma B)C \leqslant (CAC) \sigma (CBC)$.
(iii) $A_k \downarrow A$ and $B_k \downarrow B$ imply $(A_k \sigma B_k) \downarrow (A \sigma B)$.

A *mean* is a connection with normalization condition

(iv) $I_n \sigma I_n = I_n$. (Here I_n denotes the identity element in M_n.)

There exists an affine order isomorphism between the class of connections and the class of positive operator monotone functions on R_+. This isomorphism $\sigma \leftrightarrow f$ is characterized by the relation

$$A \sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

for $A, B \in P_n$ [10].

For $A, B \in P_n$, the adjoint σ^* of a nonzero connection σ is defined to be $A \sigma^* B = (A^{-1} \sigma B^{-1})^{-1}$.

For $A, B \in S_n$, the arithmetic mean $\triangledown$ is defined by $A \triangledown B = (A + B)/2$, and for $A, B \in P_n$, the harmonic mean $!$ is defined by $A ! B = (A^{-1} \triangledown B^{-1})^{-1}$ and the geometric mean # by

$$A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

For $A, B \in S_n$, the harmonic mean and the geometric mean are defined by continuity. The harmonic mean and the geometric mean are the most important examples of operator means.

Let g and h be positive functions on I. Then every connection σ gives rise to a function $f_\sigma(x) = g(x) \sigma h(x)$, $x \in I$, of one variable. If h is defined on J, then every connection σ gives rise to a function $F_\sigma(x, y) = g(x) \sigma h(y)$, $(x, y) \in I \times J$, of two variables. We shall study some matrix monotonicity and matrix convexity properties of these functions. As a consequence, we shall prove that a certain product of matrix monotone functions is a matrix monotone function. Similar results are proved for matrix convex functions.

2. FUNCTIONS OF ONE VARIABLE

Let f be a real valued function defined on I. Then f is called matrix monotone on I of order n if $A, B \in S(I; M_n)$ with $A \geqslant B$ implies

$$f(A) \geqslant f(B).$$

Likewise f is called matrix convex on I of order n if for $A, B \in S(I; M_n)$ and $0 \leqslant \lambda \leqslant 1$, the inequality

$$f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda) f(B)$$

holds. The function f is called matrix concave on I of order n if f is matrix convex on I of order n. A function f which is matrix monotone (convex, concave) on I of all orders n is called operator monotone (convex, concave) on I. For properties of matrix monotone functions, the reader may refer to Löwner [12], Donoghue [5], Kwong [11], and for properties of matrix convex functions, useful references are Kraus [8], Davis [4], Ando [1, 2].

LEMMA 2.1. *Let σ be a connection. Let g and h be positive functions on I, and $f_\sigma(x) = g(x)\,\sigma\, h(x)$, $x \in I$. Then for $A \in S(I; M_n)$, one has $f_\sigma(A) = g(A)\,\sigma\, h(A)$.*

Proof. Let $A \in S(I; M_n)$ have spectral resolution $A = \sum_{i=1}^{n'} \lambda_i P_i$. Let f be the operator monotone function representing the operator connection σ. Then

$$\begin{aligned} g(A)\,\sigma\, h(A) &= [g(A)]^{1/2} f\big([g(A)]^{-1/2} h(A) [g(A)]^{-1/2}\big)[g(A)]^{1/2} \\ &= g(A) f\big([g(A)]^{-1} h(A)\big) \\ &= \left(\sum_{i=1}^{n'} g(\lambda_i) P_i\right) f\left(\left(\sum_{i=1}^{n'} [g(\lambda_i)]^{-1} P_i\right)\left(\sum_{i=1}^{n'} h(\lambda_i) P_i\right)\right) \\ &= \left(\sum_{i=1}^{n'} g(\lambda_i) P_i\right) f\left(\sum_{i=1}^{n'} [g(\lambda_i)]^{-1} h(\lambda_i) P_i\right) \end{aligned}$$

$$= \left(\sum_{i=1}^{n'} g(\lambda_i) P_i \right) \left(\sum_{i=1}^{n'} f\big([g(\lambda_i)]^{-1} h(\lambda_i) \big) P_i \right)$$

$$= \sum_{i=1}^{n'} g(\lambda_i) f\big([g(\lambda_i)]^{-1} h(\lambda_i) \big) P_i$$

$$= \sum_{i=1}^{n'} [g(\lambda_i)\, \sigma\, h(\lambda_i)] P_i$$

$$= \sum_{i=1}^{n'} f_\sigma(\lambda_i) P_i$$

$$= f_\sigma(A). \qquad \blacksquare$$

The following lemma is Theorem 2.4 in [6], but we restate it for our purpose.

LEMMA 2.2. *Let f be a function on $I = [0, \infty)$ with $f(0) = 0$ and $g(x) = f(x)/x$, $x \in (0, \infty)$. If f is matrix convex on I of order $2n$, then g is matrix monotone on $(0, \infty)$ of order n. Conversely, if g is matrix monotone on $(0, \infty)$ of order $2n$, then f is matrix convex on I of order n.*

THEOREM 2.3. *Let g and h be positive functions on I, let σ be a connection, and let $f_\sigma(x) = g(x)\, \sigma\, h(x)$, $x \in I$.*

(a) *If g and h are matrix monotone on I of order n, then so is $f_\sigma(x)$. If f_σ is matrix monotone on I of order n for every connection σ, then so are g and h.*

(b) *If g and h are matrix concave on I of order n, then so is $f_\sigma(x)$. If f_σ is matrix concave on I of order n for every connection σ, then so are g and h. Further, if σ is nonzero, then $\tilde{f}_\sigma(x) = [g(x)]^{-1} \sigma [h(x)]^{-1}$ is matrix convex on I of order n.*

(c) *If $I = [0, \infty)$ and $g(0) = 0 = h(0)$ and if g and h are matrix convex on I of order $4n$, then $f_\sigma(x)$ is matrix convex on I of order n.*

Proof. (a): Let $A, B \in S(I; M_n)$ be such that $A \geqslant B$. Thus $g(A) \geqslant g(B)$ and $h(A) \geqslant h(B)$. Then the definition of σ implies

$$f_\sigma(A) = g(A)\, \sigma\, h(A) \geqslant g(B)\, \sigma\, h(B) = f_\sigma(B),$$

using Lemma 2.1. The second part follows on choosing σ appropriately.

(b): Let $A, B \in S(I; M_n)$ and $0 \leqslant \lambda \leqslant 1$. Then

$$g(\lambda A + (1-\lambda)B)\ \sigma\ h(\lambda A + (1-\lambda)B)$$

$$\geqslant [\lambda g(A) + (1-\lambda)g(B)]\ \sigma\ [\lambda h(A) + (1-\lambda)h(B)]$$

$$\geqslant \lambda g(A)\ \sigma\ h(A) + (1-\lambda)g(B)\ \sigma\ h(B),$$

using Theorem 3.5 (I′) in [10]. The second part follows on choosing σ appropriately. The third part follows on replacing σ by σ^* and using that $x \to -x^{-1}$ is operator monotone on $(0,\infty)$ and that $x \to x^{-1}$ is operator convex on $(0,\infty)$ [14].

(c): Consider the functions $g_1(x) = g(x)/x$, $h_1(x) = h(x)/x$ defined on $(0,\infty)$. Then by Lemma 2.2, g_1 and h_1 are matrix monotone on $(0,\infty)$ of order $2n$. Consequently, the function $x \to g_1(x)\ \sigma\ h_1(x)$ is matrix monotone on $(0,\infty)$ of order $2n$, using part (a). Now the result follows using Lemma 2.2 once again. ■

COROLLARY 2.4. *Let g and h be positive matrix monotone functions on $I = (0,\infty)$ of order $2n$. Then the function $j_\sigma(x) = g(x^{-1})\ \sigma\ h(x^{-1})$ is matrix convex on I of order n for all nonzero connections σ.*

Proof. The proof is given by the second part of the above Theorem 2.3(b), since $[g(x^{-1})]^{-1}$ and $[h(x^{-1})]^{-1}$ are positive matrix monotone on I of order $2n$ and hence, by Theorem 6 in [13], are matrix concave on I of order n. ■

COROLLARY 2.5. *If f is a positive operator monotone function on $(0,\infty)$, then the function $x^{-1}f(x)$ is operator convex on $(0,\infty)$.*

Proof. There exists a connection σ such that $f(x) = 1\ \sigma\ x$. Therefore $x^{-1}f(x) = x^{-1}\ \sigma\ 1$ is operator convex, using Theorem 2.3(b). ■

COROLLARY 2.6. *If f is a positive operator convex function on $[0,\infty)$ with $f(0) = 0$, then the function $[f(x^{-1})]^{-1}$ is operator convex on $(0,\infty)$.*

Proof. By Lemma 2.2, the function $f(x)/x$ is operator monotone on $(0,\infty)$. Let σ be the operator connection such that $f(x)/x = 1\ \sigma\ x$. Then $[f(x^{-1})]^{-1} = x\ \sigma^*\ x^2$ is operator convex, using Theorem 2.3(c). ■

We give below two proofs of Corollary 2.7. The first one has been suggested by a referee, and the second has its germs in [15].

COROLLARY 2.7. *Let g and h be positive functions on I and $f(x) = [g(x)]^{\alpha}[h(x)]^{\beta}$, where $0 \leqslant \alpha, \beta$ and $\alpha + \beta \leqslant 1$.*

(a) *If g and h are matrix monotone on I of order n, then so is $f(x)$.*

(b) *If g and h are matrix concave on I of order n, then so is $f(x)$.*

(c) *If $I = [0, \infty)$, $g(0) = 0 = h(0)$, $\alpha + \beta = 1$ and if g and h are matrix convex on I of order $4n$, then $f(x)$ is matrix convex on I of order n.*

Proof. (a): When $\alpha + \beta = 1$, the monotonicity of f follows on considering the connection corresponding to x^{β} in Theorem 2.3(a). When $\alpha + \beta < 1$, it follows on considering the functions $g(x)$ and $[h(x)]^{\gamma}$ with suitable $0 < \gamma < 1$ and the connection corresponding to x^{β} in Theorem 2.3(a). The proofs of (b) and (c) follow from similar considerations. ■

Alternate proof. (a): Let E be the set of all (α, β), $\alpha, \beta \geqslant 0$, for which f is matrix monotone on I of order n. We shall prove that E is closed and convex. Observe that (0, 0), (1, 0), and (0, 1) are in E. It is also clear that E is closed. Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in E$. We shall prove that $((\alpha_1 + \alpha_2)/2, (\beta_1 + \beta_2)/2) \in E$. For this it is enough to show that if the functions f_1 and f_2 are positive matrix monotone on I of order n, then so is the function $x \to [f_1(x)f_2(x)]^{1/2}$. However, this is a special case of Theorem 2.3(a). This completes the proof.

The proofs of (b) and (c) are no different from the one given for (a) and are therefore not included. However, in these cases we need to use Theorem 2.3(b) and Theorem 2.3(c) respectively. ■

3. FUNCTIONS OF TWO VARIABLES

Let f be a real valued function defined on $I \times J$. Then f is called matrix monotone on $I \times J$ of order (m, n) if for $A, B \in S(I; M_m)$ and $C, D \in S(J; M_n)$ with $A \geqslant B$ and $C \geqslant D$, the inequality

$$f(A,C) - f(B,C) - f(A,D) + f(B,D) \geqslant 0$$

holds [7, 16]. If $J = I$ and $m = n$, we say that f is diagonally matrix monotone on $I \times I$ of order (n, n) if $A, B \in S(I; M_n)$ with $A \geqslant B$ implies

$$f(A,A) - f(B,A) - f(A,B) + f(B,B) \geqslant 0.$$

A real valued function f defined on $I \times J$ is called matrix convex on $I \times J$ of order (m, n) if for $A, B \in S(I; M_m)$, $C, D \in S(J; M_n)$, and $0 \leqslant \lambda \leqslant 1$, the inequality

$$f(\lambda A + (1-\lambda)B, \lambda C + (1-\lambda)D) \leqslant \lambda f(A, C) + (1-\lambda) f(B, D)$$

holds [3]. The corresponding notion of matrix concavity is defined as usual.

Note that every matrix monotone function f on $I \times I$ of order (n, n) is diagonally matrix monotone on $I \times I$ of order (n, n). The function $f(x, y) = x^2 y^2$ is diagonally matrix monotone on $I \times I$ of order (1, 1), but is not matrix monotone on $I \times I$ of order (1, 1). However, we have the following theorem.

THEOREM 3.1. *Every diagonally matrix monotone function on $I \times I$ of order $(m+n, m+n)$ is matrix monotone on $I \times I$ of order (m, n).*

Proof. Let $A, B \in S(I; M_m)$ and $C, D \in S(I; M_n)$ be such that $A \geqslant B$ and $C \geqslant D$. Then $\operatorname{diag}(A, C) \geqslant \operatorname{diag}(B, D)$, where $\operatorname{diag}(A, C)$ is the 2×2 matrix with diagonal entries A and C. Since f is diagonally matrix monotone on $I \times I$ of order $(m+n, m+n)$, we have

$$f(\operatorname{diag}(A, C), \operatorname{diag}(A, C)) - f(\operatorname{diag}(A, C), \operatorname{diag}(B, D))$$
$$- f(\operatorname{diag}(B, D), \operatorname{diag}(A, C)) + f(\operatorname{diag}(B, D), \operatorname{diag}(B, D)) \geqslant 0.$$

We observe—on multiplying the above inequality on the left by $E_{11} \otimes F_{12}$ and on the right by $E_{11} \otimes F_{21}$, where

$$E_{11} = \operatorname{diag}(I_m, 0), \qquad F_{12} = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix},$$

and F_{21} is the transpose of F_{12}, and using the spectral theorem—that

$$f(A, C) - f(A, D) - f(B, C) + f(B, D) \geqslant 0. \qquad \blacksquare$$

The following lemma in the case $f(x) = x = g(x)$ and $I = [0, \infty) = J$ is the main result in [9]. But here we give a simple proof. The proof is similar to that of Lemma 2.1.

LEMMA 3.2. *Let σ be a connection. Let g and h be positive functions on I and J respectively, and $F_\sigma(x, y) = g(x)\,\sigma\, h(y)$, $(x, y) \in I \times J$. Then $F_\sigma(A, B) = [g(A) \otimes I_n]\,\sigma\,[I_m \otimes h(B)]$ for $A \in S(I; M_m)$ and $B \in S(J; M_n)$.*

Proof. Let $A \in S(I; M_m)$ and $B \in S(J; M_n)$ have spectral resolutions $A = \sum_{i=1}^{m'} \lambda_i P_i$ and $B = \sum_{j=1}^{n'} \mu_j Q_j$ respectively. Let f be the operator monotone function corresponding to the operator connection σ. Then

$$\begin{aligned}
&[g(A) \otimes I_n]\,\sigma\,[I_m \otimes h(B)] \\
&\quad = [g(A) \otimes I_n] f\big(g(A)^{-1} \otimes h(B)\big) \\
&\quad = \left(\sum_{i=1}^{m'} g(\lambda_i) P_i \otimes I_n\right) f\left(\sum_{i=1}^{m'} \sum_{j=1}^{n'} [g(\lambda_i)]^{-1} h(\mu_j) P_i \otimes Q_j\right) \\
&\quad = \left(\sum_{i=1}^{m'} g(\lambda_i) P_i \otimes I_n\right) \sum_{i=1}^{m'} \sum_{j=1}^{n'} f\big([g(\lambda_i)]^{-1} h(\mu_j)\big) P_i \otimes Q_j \\
&\quad = \sum_{i=1}^{m'} \sum_{j=1}^{n'} g(\lambda_i) f\big([g(\lambda_i)]^{-1} h(\mu_j)\big) P_i \otimes Q_j \\
&\quad = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \big[g(\lambda_i)\,\sigma\, h(\mu_j)\big] P_i \otimes Q_j \\
&\quad = F_\sigma(A, B).
\end{aligned}$$

■

LEMMA 3.3. *Let functions g, h be matrix monotone on I and J of order m and n respectively [or $-g$, $-h$ be matrix monotone on I and J of order m and n respectively]. Let f be a function defined on* Range$(g) \times$ Range(h) *and matrix monotone on* Range$(g) \times$ Range(h) *of order (m, n). Then the function $F(x, y) = f(g(x), h(y))$, $(x, y) \in I \times J$, is matrix monotone on $I \times J$ of order (m, n).*

Now we examine a function $F_\sigma(x, y) = g(x)\,\sigma\, h(y)$ for positive operator monotone functions g and h on I and J respectively. Then for the geometric (arithmetic) mean $\#$ ($\triangledown$),

$$F_\#(A, B) = \sqrt{g(A)} \otimes \sqrt{h(B)}$$

and

$$F_{\triangledown}(A, B) = [g(A) \otimes I_n] \triangledown [I_m \otimes h(B)]$$

for $A \in S(I; M_m)$, $B \in S(J; M_n)$. Thereby, in both cases, F_σ is operator monotone, and moreover so is $F_\sigma^{(2)}(x, y) = [g(x)]^2 \, \sigma \, [h(y)]^2$. Though one might conjecture that F_σ is matrix monotone for all connections, that fails for the harmonic mean, as the following example shows: For

$$A = \begin{pmatrix} 4 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

the function $F_!(x, y) = x \,!\, y$ fails to be diagonally matrix monotone, i.e., the inequality

$$F_!(A, A) - F_!(A, B) - F_!(B, A) + F_!(B, B) \geqslant 0$$

does not hold.

Theorem 3.4 gives sufficient conditions which ensure the monotonicity of F_σ.

THEOREM 3.4. *Let g and h be positive functions on I and J respectively. If both g^2 and h^2 are operator monotone on I and J, then the function $F_\sigma(x, y) = g(x) \, \sigma \, h(y)$ is matrix monotone on $I \times J$ of all orders (m, n) for all connections σ.*

Proof. In view of Lemma 3.3, it follows that we need only to prove that the function $G_\sigma(x, y) = x^{1/2} \, \sigma \, y^{1/2}$ is matrix monotone on $(0, \infty) \times (0, \infty)$ of all orders (m, n). Using Theorem 3.4 in [10], we have

$$G_\sigma(x, y) = a x^{1/2} + b y^{1/2} + \int_0^\infty \frac{1 + t}{t} \left[(t x^{1/2}) \,!\, y^{1/2} \right] d\nu(t),$$

where ν is a positive Radon measure and $a, b \geqslant 0$. Since the sum of matrix monotone functions is matrix monotone, we need only prove that the function $(x, y) \to (t x^{1/2}) \,!\, y^{1/2}$ is matrix monotone, and in view of Lemma 3.3, it is enough to prove that the function $(x, y) \to [x^{1/2} + y^{1/2}]^{-1}$ is matrix monotone. Use of Theorem 3.1 reduces that to proving that the function $(x, y) \to [x^{1/2} + y^{1/2}]^{-1}$ is diagonally matrix monotone. This is proved as follows:

Let $A, B \in P_n$ be such that $A \geqslant B$. If necessary by replacing A with $A + \varepsilon I_n$, we may assume that $A > B$ and hence $A^{1/2} > B^{1/2}$. Then the function $(x, y) \to [x^{1/2} + y^{1/2}]^{-1}$ is diagonally matrix monotone iff

$$\left[A^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1} - \left[A^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1}$$
$$-\left[B^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1} + \left[B^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1} \geqslant 0$$

iff

$$\left[A^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1} - \left[A^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1}$$
$$\geqslant \left[B^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1} - \left[B^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1}$$

iff

$$\left[A^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1}\left[\left(A^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right)\right.$$
$$\left.-\left(A^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right)\right]\left[A^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1}$$
$$\geqslant \left[B^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1}\left[\left(B^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right)\right.$$
$$\left.-\left(B^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right)\right]\left[B^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1}$$

iff

$$\left[A^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1}\left[I_n \otimes (B^{1/2} - A^{1/2})\right]$$
$$\times\left[A^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1}$$
$$\geqslant \left[B^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]^{-1}\left[I_n \otimes (B^{1/2} - A^{1/2})\right]$$
$$\times\left[B^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]^{-1}.$$

Since for $X, Y \in M_n$ we have $0 > X \geqslant Y$ iff $0 > Y^{-1} \geqslant X^{-1}$, the above inequality holds iff

$$\begin{aligned}&\left[B^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]\left[I_n \otimes (B^{1/2} - A^{1/2})^{-1}\right]\\&\times\left[B^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]\\\geqslant &\left[A^{1/2} \otimes I_n + I_n \otimes B^{1/2}\right]\left[I_n \otimes (B^{1/2} - A^{1/2})^{-1}\right]\\&\times\left[A^{1/2} \otimes I_n + I_n \otimes A^{1/2}\right]\end{aligned}$$

iff

$$\begin{aligned}&B \otimes [B^{1/2} - A^{1/2}]^{-1} + B^{1/2} \otimes \left(B^{1/2}[B^{1/2} - A^{1/2}]^{-1}\right)\\&\quad + B^{1/2} \otimes [B^{1/2} - A^{1/2}]^{-1} A^{1/2} + I_n \otimes \left(B^{1/2}[B^{1/2} - A^{1/2}]^{-1} A^{1/2}\right)\\&\quad \geqslant A \otimes [B^{1/2} - A^{1/2}]^{-1} + A^{1/2} \otimes \left(B^{1/2}[B^{1/2} - A^{1/2}]^{-1}\right)\\&\qquad + A^{1/2} \otimes [B^{1/2} - A^{1/2}]^{-1} A^{1/2} + I_n \otimes \left(B^{1/2}[B^{1/2} - A^{1/2}]^{-1} A^{1/2}\right)\end{aligned}$$

iff

$$\begin{aligned}&(B - A) \otimes [B^{1/2} - A^{1/2}]^{-1} + (B^{1/2} - A^{1/2}) \otimes \left(B^{1/2}[B^{1/2} - A^{1/2}]^{-1}\right)\\&\qquad + (B^{1/2} - A^{1/2}) \otimes \left([B^{1/2} - A^{1/2}]^{-1} A^{1/2}\right) \geqslant 0\end{aligned}$$

iff

$$(B - A) \otimes (B^{1/2} - A^{1/2}) + (B^{1/2} - A^{1/2}) \otimes (B - A) \geqslant 0,$$

which is true. ■

COROLLARY 3.5. *A function* $H_\sigma(x, y) = x^\alpha \,\sigma\, y^\beta$ *is matrix monotone on* $(0,\infty) \times (0, \infty)$ *of all orders* (m, n) *for all connections* σ *if and only if* $-\frac{1}{2} \leqslant \alpha, \beta \leqslant 0$ *or* $0 \leqslant \alpha, \beta \leqslant \frac{1}{2}$.

The proof of the following corollary follows on using Theorem 3.4 above and Theorem 4 in [7].

COROLLARY 3.6. *Let σ be a connection. Then the function* $G_\sigma(x, y) = x^{1/2}\,\sigma\, y^{1/2}$, $(x, y) \in (0, \infty) \times (0, \infty)$, *is analytic and can be continued analytically for all nonreal values of the two variables to a function* g_σ *satisfying the following conditions*:

(i) $g_\sigma(\bar{z}_1, \bar{z}_2) = \bar{g}_\sigma(z_1, z_2)$ *for all* $\operatorname{Im} z_1, \operatorname{Im} z_2 \neq 0$;

(ii) $g_\sigma(z_1, z_2) - g_\sigma(\bar{z}_1, z_2) - g_\sigma(z_1, \bar{z}_2) + g_\sigma(\bar{z}_1, \bar{z}_2)$
$= \operatorname{Re}[g_\sigma(z_1, z_2) - g_\sigma(\bar{z}_1, z_2)] \leqslant$
0 *for* $\operatorname{Im} z_1, \operatorname{Im} z_2 > 0$;

(iii) *for all* ϕ $(0 < \phi \leqslant \pi/2)$ *there exists a constant* $M(\phi)$ *such that* $|g_\sigma(z_1, z_2)/z_1 z_2| \leqslant M(\phi)$ *for all* z_1, z_2 *in* $\{z : \phi \leqslant \arg z \leqslant \pi - \phi\}$.

Our next theorems are the two variable analogues of Theorem 2.3(b) and Corollary 2.7(b). The proofs once again are similar to the one variable case and are therefore not included.

THEOREM 3.7. *Let g and h be positive matrix concave functions on I and J of order m and n respectively. Then the function* $F_\sigma(x, y) = g(x)\,\sigma\, h(y)$ *is matrix concave on* $I \times J$ *of order* (m, n) *for all connections* σ. *Further if* σ *is nonzero, then* $\tilde{F}_\sigma(x, y) = [g(x)]^{-1}\,\sigma\,[h(y)]^{-1}$ *is matrix convex on* $I \times J$ *of order* (m, n).

COROLLARY 3.8. *Let g and h be positive matrix monotone functions on* $I = (0, \infty) = J$ *of order* $2m$ *and* $2n$ *respectively. Then the function* $\Psi_\sigma(x, y) = g(x^{-1})\,\sigma\, h(y^{-1})$ *is matrix convex on* $I \times J$ *of order* (m, n) *for all nonzero connections* σ.

THEOREM 3.9. *Let g and h be positive matrix concave functions on I and J of order m and n respectively. Then the function* $f(x, y) = [g(x)]^\alpha [h(y)]^\beta$, $0 \leqslant \alpha, \beta$ *and* $\alpha + \beta \leqslant 1$, *is matrix convex on* $I \times J$ *of order* (m, n).

On taking $g(x) = x = h(x)$ and $I = (0, \infty) = J$ in the above Theorem 3.9, we get the following result proved in [3].

COROLLARY 3.10. *The function* $f(x, y) = x^\alpha y^\beta$ *is matrix concave on* $(0, \infty) \times (0, \infty)$ *for* $0 \leqslant 2$, β *and* $\alpha + \beta \leqslant 1$.

REMARKS.

(i) It is easy to see that if g and h are matrix monotone functions on I and J of order m and n respectively, then the function $f(x, y) = g(x)h(y)$ is matrix monotone on $I \times J$ of order (m, n). Consequently, the two variable analogue of Corollary 2.7(a) is true.

(ii) It may be pointed out that if g and h are positive matrix convex functions on I and J of order m and n respectively, then it is not necessarily true that the function $f(x, y) = [g(x)]^{\alpha}[h(y)]^{1-\alpha}$, $0 \leqslant \alpha \leqslant 1$, is matrix convex on $I \times J$ of order (m, n). Indeed, if $g(x) = x^2 = h(x)$ and $I = [0, \infty) = J$, then g and h are matrix convex on I of order m and n respectively. However, $f(x, y) = x^{2\alpha}y^{2-2\alpha}$, where $\alpha = \frac{1}{2}$, is not matrix convex on $I \times J$ of any order (m, n).

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