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# Cobordism and Functoriality of Colorings

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This article is a continuation of the articles "Geometric Coloring Theory"[1] and "Variations on Coloring: Surfaces and Higher-Dimensional Manifolds"[2] which appeared in *Advances in Mathematics*. All previous definitions, results, and references are to be found there.

Chapter VII studies homotopy and cobordism questions of colorings. The objects of these relations are pairs $(M, \pi)$, where $\pi$ is a local coloring of the boundaryless oriented manifold $M$. The first section constructs some invariants of homotopy and cobordism. In Section 2, it is shown that these invariants suffice to classify homotopy and cobordism classes in one dimension. The invariants are shown to be sufficient in dimension 2 in the next section.

Chapter VIII is the final chapter and studies the global properties of coloring. Section 1 introduces a functor $B$ which assigns an $n$-complex $B(X)$ to an $n$-complex $X$. $B(X)$ is built out of colorings of $X$. Section 2 studies fibrations of colorings. The main result is the calculation of $B(X)$ where $X$ is the "universal coloring bundle" of the circle $P(n)$. Section 3 contains the beautiful result that $B(B(X)) = X$, where $X$ is the circle with $n$ vertices. Section 4 shows that finiteness is important by giving an infinite graph $Z$ such that for all finite subgraphs $W$ we have $B(B(W)) = W$ but it is not true that $B(B(Z)) = Z$. Section 5 studies the relations between the automorphisms of $X$ and $B(X)$, Section 6 looks at the structure of $B(X)$, where $X$ is an even triangulation of the 2-sphere. The last section ends with a beautiful result due to Tutte, and surprising calculation of the colorings of a graph due to coxeter.

At the end of each chapter there are some problems. The unsolved problems are marked with a small circle (e.g., 7°).

## VII

### VII.1. *Introduction*

We observed in the last Chapter that if we have an even triangulation of a manifold, then this even structure induces a local coloring of the boundary.

[1] *Adv. Math.* **24** (1977), 298–340.
[2] *Adv. Math.* **25** (1977), 226–266.



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Let $\pi$ be a local coloring of a boundaryless manifold $M$. We say that the pair $(M, \pi)$ is *homotopic* to another pair $(M', \pi')$ if there is a triangulation $T$ of the topological space $M \times I$ ($I$ is the unit interval) such that these conditions are satisfied:

(1) $T$ is an even triangulation,

(2) The boundary of $T$ is $M \cup M'$,

(3) $T$ induces the local colorings $\pi$ on $M$ and $\pi'$ on $M'$.

We say that the two pairs are *cobordant* if there is a triangulation $T$ which satisfies the three conditions, but $T$ is a triangulation of some manifold $W$, not necessarily $M \times I$.

We consider two questions: (1) When are two pairs homotopic? (2) When are two pairs cobordant?. Necessary conditions are of course the topological requirements. In (1) $M$ abd $M'$ must be homeomorphic, and in (2) $M$ and $M'$ must be "topologically cobordant."

In the homotopy case, each pair determines characteristic maps $\psi$ and $\psi'$: $\pi_1(|M|) \to S_{n+2}$, and it is clear that $\psi$ and $\psi'$ are conjugate. Recall (Section I.4) that this means that there is a permutation $\alpha$ in $S_{n+2}$ such that $\psi = \alpha\psi\alpha^{-1}$. We will show that conjugacy is a sufficient condition for homotopy in dimensions 1 and 2.

In the cobordism case there are also some necessary conditions. We say that an assignment $f$ defined on pairs $(M, \pi)$ is an *invariant* if, whenever $(M, \pi)$ is cobordant to $(M', \pi')$, we have $f((M, \pi)) = f((M', \pi'))$. Let $pair(\pi)$ (resp. $sing(\pi)$) be the parity of the number of nonsingular (resp. singular) codimension 1 simplices of $M$ with respect to $\pi$. We have previously seen pair($\pi$) for dimension 2 in Chapter V. The following lemma gives some invariants:

LEMMA 67. *Let the dimension of M be n. The following are invariants of cobordism*:

(1) sing($\pi$) *if* $n = 1 \pmod 4$,

(2) pair($\pi$) *if* $n = 2 \pmod 4$,

(3) sing($\pi$) *and* pair($\pi$) *if* $n = 3 \pmod 4$.

*Proof.* Recall the degree of a simplex $s$, $\rho(s, M)$, defined as the number of top-dimensional simplices of $M$ containing $s$. Define $P(s, M)$ as the number of codimension 1 simplices of $M$ containing $s$. To show that some assignment is an invariant, it suffices to show that if $T$ is an even $n+1$ manifold, then the assignment is zero on the boundary. In Formula (1) (VI.1) we saw that if $s$ is an $(n-2)$-simplex of the boundary $M$ of $T$, then $s$ is nonsingular iff $\rho(s, T)$ is odd. $s$ is singular iff $P(s, T)$ is odd. Let $X$ be the number of $n$-simplices of $T$, and $Y$ the number of $(n-1)$-simplices of $T$. We have the equations

$$\text{pair}(\pi) = \sum_{s \in \partial T} \rho(s, T) = \sum_{s \in T} \rho(s, T) = \binom{n+2}{2} X,$$

$$\text{sing}(\pi) = \sum_{s \in \partial T} \rho(s, T) = \sum_{s \in T} \rho(s, T) = (n+1) Y.$$

These equations show that pair$(\pi)$ is even when $\binom{n+1}{2}$ is even, and sign$(\pi)$ is even when $n$ is even. It is an exercise to show that in the case of $n$ divisible by 4, neither pair$(\pi)$ nor sing$(\pi)$ can be an invariant. ∎

The formulation of homotopy and cobordism here is new, but there have been a few results that fit into this framework. The one-dimensional case have been studied the most. An example of the type of problem studied is: Is there a triangulation of the sphere with exactly two odd vertices, and they are adjacent? In our framework, we can remove a triangle containing the edge, and ask if a certain local coloring of the circle is homotopic to the identity.

Fleischer and Roy (1974) studied some one-dimensional homotopy problems. Fleischer (1974) showed (in our terminology) that the number of nonsingular edges of a 4-coloring of the 2-sphere is even (see Proposition 45 in Chapter V). Jendrol (1975) studied some homotopy questions in one dimension, see also Malkevitch (1970). Wagner (1936), Lawson (1972), and Dewdney (1973) studied homotopy questions in two dimensions.

## VII.2. *The Situation in One Dimension*

The feature that makes the one-dimensional case especially interesting is that there is not just one result for even triangulations, but there are results for all locally-$n$ triangulations. Recall (Section I.5) that a locally-$n$ triangulation of a surface has all interior vertices of degree divisible by $n$. The universal object is the triangulation $R(n)$ of the sphere (if $n < 6$) or the plane (if $n \geqslant 6$). Just as the sequence of degrees modulo 2 on the boundary of an even triangulation gives a local coloring, we shall say that the sequence of degrees modulo $n$ on the boundary of a locally-$n$ surface gives *locally-$n$ circles*. We give the circles the orientation induced by the surface. We ask the same homotopy and cobordism questions about locally-$n$ circles as we did of even triangulations (Section 1).

We begin with the study of homotopy. Let $\beta$ be a locally-$n$ circle. $\beta$ determines a map $\psi$: $\pi_1(|\beta|) \to \text{Aut}(R(n))$, where $|\beta|$ is the underlying topological space of $\beta$ (a 1-sphere) and $\text{Aut}(R(n))$ is the group of orientation preserving automorphisms of $R(n)$. This characteristic map $\psi$ is defined in the same way as the other characteristic maps (I.4). $\psi$ is well defined up to conjugacy. If $\beta$ is induced by a locally-$n$ triangulation of the disk, then the image of $\psi$ is the identity. Any $\beta$ whose characteristic map maps onto the identity we shall call *trivial.* In case that $n < 6$, triviality and being induced by a disk are equivalent. To show this, we need a preliminary lemma. By map we mean as usual a simplicial map which sends triangles onto triangles.

LEMMA 68. *Let $K$ be a triangulation of a surface and $f$ a map from $P(n)$ to $K$ which maps edges onto edges. If $f$ is also null homotopic then there is a triangulation $D$ of the disk satisfying*

(1) $\partial D = P(n)$,

(2) *there is an* $F$: $D \to K$,

(3) $F$ *restricted to* $\partial D$ *is* $f$.

*Proof.* Since $f$ is null homotopic, there is a continuous map from the disk to $K$ which extends $f$. By the simplicial approximation theorem, we may find a triangulation $D$ of the disk and a map $F$: $D \to K$ which extends $f$. Now this map may not be a map in our sense, for it may map edges onto points. If an edge of $D$ is mapped onto a point, collapse the edge to a point. The triangles containing the edge collapse to lines. The resulting triangulation $D'$ has a map to $K$ and one less edge which is collapsed. Continuing, we get a triangulation of the disk as desired. ∎

COROLLARY 69. *If $n < 6$ and $\beta$ is a locally-$n$ circle with trivial characteristic map, then $\beta$ is induced by a locally-$n$ disk.*

*Proof.* Take a $D$ as given in Lemma 68. Orient $D$ and $R(n)$. Consider the orientations of the triangles of $D$ induced by $f$. It is easy to see that at all interior vertices the sum of the orientations (1 for positive, $-1$ for negative) is zero modulo $n$. Compare Proposition 25 in Section IV.1. Pick a triangle $T$ of $D$ which has negative orientation under $f$. Let $D'$ be $D$ with $T$ removed, and let $R(n)'$ be $R(n)$ with one triangle removed. Join $D'$ and $R(n)'$ along these two triangular holes. The resulting triangulation has a map to $R(n)$, and the new triangles added have positive orientation under this map. Do this for all triangles of $D$ with negative orientation. If we end up with $D^2$ then $D^2$ has a map to $R(n)$ with all triangles of positive orientation. Consequently all vertices have degree divisible by $n$ and so $D^2$ is locally-$n$. ∎

It is not at all obvious that there are any locally-$n$ triangulations of closed surfaces for large $n$. For $n$ equal to 6 we have seen that there are locally-6 triangulations of the torus (Section V.1). We shall derive the existance of locally-$n$ triangulations of closed surfaces from a theorem about covering spaces. Husemoller (Ramified coverings of Riemann surface, *Duke Math. J.* **29** (1962), 167–174) proved a result (Theorem 4) which gives conditions for the existance of ramified coverings of a surface with specified branching. A special case of his result shows that for a set of 12 points $S$ on the torus and any integer $n$, there is a ramified covering $K$ of the torus where the only ramified points lie over $S$ and have ramification indices of order $n$. If we take the triangulation $\Delta^2 \times \partial\Delta^3$ of the torus and lift it to a triangulation of the surface $K$ by this covering, then every vertex of this induced triangulation has degree $6n$. Any triangulation with all vertices of degree $6n$ is of course locally-$n$, so we have the

LEMMA 70. *For any $n$ there are locally-$n$ triangulations of some closed compact orientable surface.*

As a corollary of this lemma, we derive the cobordism version of Lemma 68 for $n \geqslant 6$.

COROLLARY 71. *For any $n$, if $\beta$ is a locally-$n$ circle with trivial characteristic map, then $\beta$ is induced by a locally-$n$ surface.*

*Proof.* Let $Q$ be a locally-$n$ triangulation whose existance is guaranteed by the preceeding lemma. The proof of Lemma 68 now applies, with $R(n)$ replaced by $Q$. ∎

In case that $n < 6$ we get a simple homotopy result.

THEOREM 72. *If $n < 6$ and $\beta$ and $\beta'$ are two locally-$n$ circles with conjugate characteristic maps, then $\beta$ and $\beta'$ are homotopic.*

*Proof.* Let $T$ be a triangulation of the annulus with the following properties: (1) $T = S \cup S'$, (2) $S$ (resp. $S'$) has a neighborhood in $T$ which is locally-$n$ and induces the locally-$n$ circle $\beta$ (resp. $\beta'$). $T$ need not be locally-$n$ itself. Picking basepoints in neighborhoods of $S$ and $S'$, we get characteristic maps $\psi$ and $\psi'$. Since they are conjugate, there is an element $\alpha$ of $\mathrm{Aut}(R(n))$ such that $\psi'(s) = \alpha\psi(s)\alpha^{-1}$. Consequently one can join a triangle of a neighborhood of $S$ to a triangle of a neighborhood of $S'$ by a path $P$ of triangles in $T$ with the following property: if $K$ (resp. $K'$) is the path of triangles which goes once around $S$ (resp. $S'$) in the positive direction, then the composite path $KPK'P^{-1}$ has a nonsingular map to $R(n)$. By Lemma 68 there is a locally-$n$ disk $D$ whose boundary is exactly $KPK'P^{-1}$. If we identify $P$ with $P^{-1}$ in $D$ we get a locally-$n$ triangulation of the annulus whose boundary induces $\beta$ and $\beta'$. ∎

The rest of this section will be devoted to cobordism. Suppose that $K$ is a cobordism between two locally-$n$ circles $\beta$ and $\beta'$. Since the supporting circles $S$ and $S'$ of $\beta$ and $\beta'$ are homologous, as homotopy elements they differ by an element of the commutator subgroup of $\pi_1(K)$. Consequently, if we compute the characteristic map $\psi$ on $S$ and $S'$, we see that $\psi(S)\,\psi(S')^{-1}$ lies in the commutator subgroup of $\mathrm{Aut}(R(n))$. Thus, if $\beta$ and $\beta'$ are cobordant, then $\psi(\beta)=\psi(\beta')$, considered as elements of $\mathrm{Aut}(R(n))$ modulo the commutator subgroup. We can explicitely calculate this group.

LEMMA 73. *$\mathrm{Aut}(R(n))$ modulo the commutator subgroup is isomorphic to $\mathbb{Z}_{(6,n)}$, the cyclic group on $(6,n)$ elements.*

*Proof.* $(6,n)$ is the greatest common divisor of 6 and $n$. From Coxeter and Moser (1957) we know a presentation for $\mathrm{Aut}(R(n))$. The generators are $X$ and $Y$ subject to the relations $X^2=Y^3=(XY)^n=1$. Dividing out by the commutator adds one more relation $XY=YX$. Suppose that $(6,n)=3$. In this case, $1=(XY)^n=X^n=X$, the last step from $(n,2)=1$. Thus the only relation is $Y^3=1$ and the group is $\mathbb{Z}_3$. The remaining cases are similiar. ▮

We shall write the group algebra generated by $\mathbb{Z}_r$ as $\mathbb{Z}(\mathbb{Z}_r)$. We now define the usual construction for cobordism groups. Let $\Omega(n)$ be the free group generated by all oriented locally-$n$ circles modulo the group generated by all sums $\beta+\beta'$, where $\beta$ and $\beta'$ are cobordant. We recall that the locally-$n$ circles $\beta$ are oriented, so we are dealing with oriented cobordism.

Let $\beta$ and $\beta'$ be locally-$n$ circles. A connected sum of $\beta$ and $\beta'$ is a locally-$n$ circle constructed as follows. Picking a basepoint on each of $\beta$ and $\beta'$ we may write the sequence of degrees as $\beta(1),\dots,\beta(r)$ (resp. $\beta'(1),\dots,\beta'(r')$). The sequence of degrees $\beta(1),\dots,\beta(r),\beta'(1),\dots,\beta'(r')$ is a connected sum of $\beta$ and $\beta'$.

THEOREM 74 (one dimensional cobordism). *For any $n\geqslant 2$,*

(1) *$\Omega(n)$ is a ring with connected sum as product.*

(2) $\Omega(n)=\mathbb{Z}(\mathbb{Z}_{(6,n)})$.

(3) *$\Omega(n)$ is periodic with period* 6.

*Proof.* Let us denote by $\theta$ the map from $\Omega(n)$ to $\mathbb{Z}(\mathbb{Z}_{(6,n)})$ which sends a generator $\beta$ of $\Omega(n)$ to the class of $\psi(\beta)$ in $\mathrm{Aut}(R(n))$/commutators. To prove (2) it suffices to show that $\theta$ is $1-1$ and onto. It is easy to construct examples of locally-$n$ circles which map onto the elements of $\mathbb{Z}_{(6,n)}$, so $\theta$ is onto. To show that $\theta$ is $1-1$, it must be shown that if $\beta$ is a locally-$n$ circle with $\theta(\beta)=0$, then there is a locally-$n$ surface whose boundary induces $\beta$. Write $\psi(\beta)$ as a product of commutators: $\prod a_i b_i a_i^{-1} b_i^{-1}$. Consider a triangulation $S$ of the disk with $N$ handles. If the boundary of $S$ is $T$ then we can find curves $A_i$ and $B_i$ $(i=1,\dots,N)$ such that the $A$'s and the $B$'s along with $T$ generate the fundamental group of $S$. See Fig. 34. As elements of the fundamental group, $T=\prod A_i B_i A_i^{-1} B_i^{-1}$. Pick paths of triangles $A_i^\circ$ (resp. $B_i^\circ$) such that $A_i^\circ$ (resp. $B_i^\circ$) represents $A_i$ (resp. $B_i$) and if we compute $\psi$

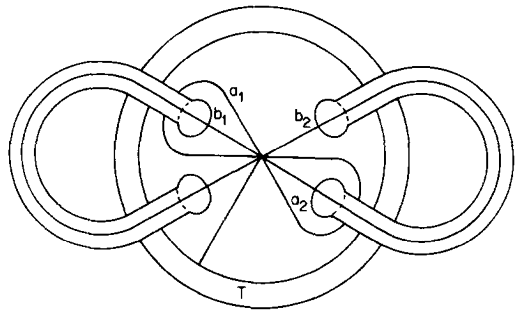



FIG. 34. Simple closed curves on genus two surface.

along $A_i^\circ$ (resp. $B_i^\circ$) we get $a_i$ (resp. $b_i$). Join the common triangle of all these paths to the boundary and triangulate the boundary so that the boundary path represents $T$. We can of course always do this for any $\beta$, but in this case we have that $\psi$ computed over the path $T^{-1} \prod A_i^\circ B_i^\circ A_i^{\circ -1}$ is the identity. Moreover, the complement of the composite path is a disk. We can therefore apply Corollary 71 or Corollary 69 to find a surface $K$ inducing $\beta$. This concludes the proof of (2).

To verify (1), let $\beta^\circ$ be a connected sum of $\beta$ and $\beta'$. $\psi$ computed on $\beta^\circ$ is the product of some conjugate of $\psi(\beta)$ with some conjugate of $\psi(\beta')$. Under the map $\theta$, we have $\theta(\beta) = \theta(\beta')$, so by the preceding paragraph, the cobordism class of $\beta^\circ$ is well defined.

Part (3) is an immediate consequence of (2) and Lemma 73. ∎

How do we compute the isomorphism $\theta$? Theorem 75 will give some simple interpretations of $\theta$. Suppose that $\beta$ is a locally-$n$ circle, where $n$ is even. Define $\alpha^2(\beta)$ to be the sum of the values of $\beta$ plus the number of vertices of the underlying circle of $\beta$. If we define the local degree of a vertex in a graph as the number of edges containing the vertex, then $\alpha^2(\beta)$ is the sum modulo 2 of the local degrees of all the vertices. The fact that the number of vertices of a graph with odd local degree is even becomes the fact that that $\alpha^2(\beta)$ is even when $\beta$ is induced by the boundary of an even triangulation. Thus $\alpha^2$ induces a map from $\Omega(n)$ to $\mathbb{Z}_2$, which we shall denote $\alpha^2$.

Next suppose 3 divides $n$. Define $\alpha^3(\beta)$ as the sum of the values of $\beta$ modulo 3. In case that $\beta$ is zero in $\Omega(n)$, then it is zero in $\Omega(3)$, so we get a map $\alpha^3$ from $\Omega(n)$ to $\mathbb{Z}_3$.

THEOREM 75. *There are maps $w^2$ and $w^3$ such that the following are commuting diagrams of homomorphisms.*

$$\begin{tikzcd} \Omega(2r) \arrow[r, "\theta"] \arrow[dr, "\alpha^2"'] & \mathbb{Z}(\mathbb{Z}_{(6,2r)}) \arrow[d, "w^2"] \\ & \mathbb{Z}_2 \end{tikzcd} \qquad \begin{tikzcd} \Omega(3r) \arrow[r, "\theta"] \arrow[dr, "\alpha^3"'] & \mathbb{Z}(\mathbb{Z}_{(6,3r)}) \arrow[d, "w^3"] \\ & \mathbb{Z}_3 \end{tikzcd}$$

COROLLARY 76. (1) *A locally-2 circle bounds a surface iff the sum of the local degrees is even.*

(2) *A locally-3 circle bounds a surface iff the sum of the degrees is divisible by 3.*

(3) *There exists a triangulation of some orientable surface with exactly two odd vertices, and they are adjacent.*

*Proof of Corollary.* Immediate. In part (3) such a triangulation can actually be found on the torus, the surface of minimal genus on which this is possible.

*Proof of the Theorem.* We shall only do the case of $n = 2r$; the case $n = 3r$ is entirely similar. A locally-$n$ circle with basepoint is a pair $(\beta, p)$, where $p$ is a point of the underlying circle. Given a basepoint, the connected sum of two locally-$n$ circles is now well defined. Let $G(n)$ be the semigroup of all locally-$n$ circles with basepoint. It is easily verified that the map $\psi$ from $G(n)$ to $\mathrm{Aut}(R(n))$ is an antihomomorphism. If $n = 2r$, consider the diagram

$$\begin{tikzcd} G(n) \arrow[r, "\psi"] \arrow[d, "\alpha^2"'] & \mathrm{Aut}(R(n)) \arrow[d] \arrow[dl, dashed] \\ \mathbb{Z}_2 & \mathrm{Aut}(R(n))/\text{commutator} \arrow[l, "w^2"'] \end{tikzcd}$$

If we knew that the diagonal map were well defined, then we could conclude the existance of $w^2$. $w^2$ would be the result of factoring a map to an abelian group through the commutator.

Suppose that $\psi(x) = \psi(y)$, where $x$ and $y$ are elements of $G(n)$. By the last theorem, $x$ and $y$ are cobordant. Hence $\alpha^2(x) + \alpha^2(y) = 0$ so the map $\psi^{-1}\alpha^2$ is well defined. ∎

VII.3. *Two-Dimensional Results*

There are two results in the literature that can be interpreted as results about homotopy. The first is known as Wagner's Theorem (Wagner, 1936). If we take an edge $e$ and a triangulation $M$, we can construct a new triangulation by removing the edge $e$, and adding a new edge $e'$ which joins the other two vertices of the resulant square. Such a change is called a diagonal transformation, and an example is shown in Figure 35A. Wagner

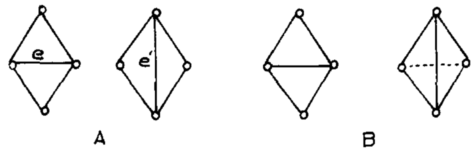



FIG. 35. Diagonal transformations.

proved that any two triangulations of the sphere with the same number of vertices may be transformed into one another by diagonal transformations. (See Lawson (1972) for a metric version of this result.) The second homotopy result is due to Dewdney (1973) who showed that the same result holds for triangulations of the torus.

We can interpretate these results in terms of homotopy as follows. If we take a triangulation $M$ of a surface and an edge $e$ of $M$ and add a tetrahedron to the two triangles containing $e$, we get a space $M'$. Looking at $M'$ from the "inside" we see the triangulation $M$, but looking at $M'$ from the "outside" we see the triangulation which is the result of applying a diagonal transformation at $e$ (see Fig. 35B). If we add a number of tetrahedra then we get some partially thickened surface whose boundary can be thought of as the original triangulation and the final one. Every interior edge has interior degree (there are none) so the thickened surface is an even triangulation. Thus, Wagner's and Dewdney's results imply that any two triangulations of the sphere or torus with the same number of vertices are "homotopic" via some sort of even triangulation.

It is possible to approach homotopy problems for pairs $(M, \beta)$ using diagonal transformations. Our approach will be to use even subdivision (Chapter II).

THEOREM 77. *If $M$ and $M'$ are triangulations of the same surface with local colorings $\pi$ and $\pi'$ and the characteristic homomorphismms of the pairs $(M, \pi)$ and $(M', \pi')$ are conjugate, then the two pairs are homotopic.*

*Proof.* We first observe that if we evenly subdivide $M$ to get $M^\circ$ with $\pi$ inducing $\pi^\circ$, then the pairs $(M, \pi)$ and $(M^\circ, \pi^\circ)$ are homotopic. Recall (Section II.1) that evenly subdividing an edge or triangle can be thought of as adding the boundary of an octahedron with the 3-coloring on it. We get the homotopy between the two pairs by adding a vertex in the interior of the octahedron and joining it to the octahedron.

If the genus of $M$ is $g$, then we can find $2g$ simple closed curves of edges in $M$ (resp. $M'$) $x_i$ (resp. $x_i'$) such that the intersection of any two of the curves is a point $p$ (resp. $p'$). Because of the assumption that the homomorphisms are conjugate, we may assume that the homomorphisms computed along these curves are actually identical. To realize the curves as disjoint may take some even subdivision.

Pick a path $x_i$ and consider a thickening $D$ of the path $x_i - p$ in $M$. $D$ is a disk with boundary $x^+ \cup x^-$. The local coloring of $M$ induces a 4-coloring $f$ on $D$. $\psi$ computed on $x_i$ is equal to $\psi$ computed on $x^+$ or $x^-$. Therefore, computing $\psi$ on the paths $x^+(x_i')^{-1}$ and $x^-(x_i')^{-1}$ gives the identity. By Lemma 57 there are triangulations $D^+$ and $D^-$ of the disk with 4-colorings $f^+$ and $f^-$ such that $f^+ = f$ on $x^+$ and $f^- = f$ on $x^-$. We now apply Lemma 57 to the disk $D = D^+ \cup D^-$ with 4-coloring $f = f_+ \cup f^-$. Let $E$ be the even 3-disk given by the Lemma. Joining $E$ to $M$ along $D$ gives us a new triangulation $A^\circ$ with $A^\circ = M \cup M^\circ$. The local coloring on $M^\circ$ induced by $A$ restricted to the curve $x_i'$ agrees with the local coloring on $x_i'$ determined by $\pi'$. Do this for all the $x_i$, obtaining a triangulation $A$ with boundary $M \cup M^2$. In $M^2$ we have the paths $x_i'$ with the induced local coloring agreeing with the local coloring determined by $\pi'$ on $x_i'$. The complement of the union of the $x_i$ in $M'$ resp. $M^2$) is a disk. By Lemma 57 there is a triangulation $E$ of the sphere whose boundary is $D' \cup D^2$. $E'$ is even and induces the coloring $\pi' \cup \pi^\circ$. Adding $E'$ along $D'$ to $M^2$ gives a triangulation $A \cup E'$. The boundary of $A \cup E'$ is $M \cup M'$, and the induced colorings are $\pi$ and $\pi'$. This is the desired homotopy. ∎

From our general discussion in the first section, we saw one cobordism invariant in two dimensions: the parity of the number of nonsingular edges. We now show that this is the only invariant in dimension 2.

THEOREM 78. *Let $\pi$ be a local coloring of an orientable surface $M$. If $\pi$ has an even number of nonsingular edges then there is an even 3-manifold which induces the pair $(M, \pi)$.*

*Proof.* Using Theorem 77 we can find a triangulation $M^2$ of $M \times I$ such that $M^2$ is even and one end of $M^2$ induces $\pi$. Now since $M$ is orientable, there is a three-dimensional surface $K$ whose boundary is $M$. Join a triangulation of $K$ to the other end of $M^2$ to get a triangulation $N$. $N$ has the property that the boundary is $M$, and a neighborhood of $M$ in $N$ is even and induces $\pi$. The total number of odd edges of $N$ is even, and all the edges of the boundary are even except for an even number of nonsingular edges. Consequently $N$ has an even number of odd edges, and they lie strictly in the interior of $N$.

The proof is in two steps. We first show that we can assume that the odd part consists of disjoint circles, and then we show how to remove the circles.

Suppose that some vertex $p$ of $N$ has more than two odd edges passing through it. Evenly subdividing if necessary, we may assume that the link of $p$ has no odd edges on it. The odd vertices of the link of $p$ correspond to the odd edges at $p$. Subdividing further, we apply Corollary 13 to find an even

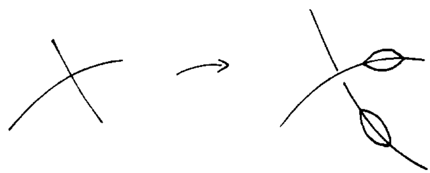


FIG. 36. Simplifying intersections.

subdivision with a 4-coloring $f$ of the link of $p$. Moreover, we may assume that $NS(f)$ consists of free pairs (Fig. 11a). Remove the star of $p$ and fill in the hole with an even triangulation $E$ which induces the 4-coloring $f$ (Lemma 57). The odd edges of the boundary of $E$ are exactly the nonsingular edges of $f$ (Formula 1 of Section VI.1). The intersections are now simplified (see Fig. 36).

We now remove the free pairs. Take an even disk in $N$ whose boundary is the two arcs $a \cup b$. The disk can be choosen so that the 3-coloring of the disk induces a 2-coloring of $a \cup b$. Cut out the disk, obtaining a sphere $S$ whose equator is 2-colored, and with even hemispheres. The odd edges of $S$ in $N$ are exactly $a \cup b$. Take the 4-coloring $g$ of $S$ such that $NS(g) = a \cup b$. Let $E$ be a triangulation of the 3-disk inducing $g$. Joining $E$ to $S$ changes the parity of the arcs $a \cup b$ to even. We have removed a loop. Doing this for all the free pairs, we may assume that the odd part of $N$ consists of disjoint circles.

We have not yet changed the topological type of $N$. In order to kill off the odd circles we are forced to do so. Suppose that we compute $\psi$ on loops of $O(N)$, $L$ and $L'$, and get the identity on each. We show how to remove these two odd circles. After sufficient even subdivision, we can assume that $L$ and $L'$ are contained in the boundary of tori $P$ and $P'$ with the boundary of $P$ and $P'$ 3-colored. Under these 3-colorings, $L$ and $L'$ are necessarily 2-colored. Let $P$ and $P'$ bound solid tori $T$ and $T'$. Remove $T$ and $T'$ from $N$. Give a local coloring $\pi$ (resp. $\pi'$) on $P$ (resp. $P'$) by setting $\pi$ (resp. $\pi'$) to be non singular exactly on $L$ (resp. $L'$). (Since $\psi$ is the identity, this is easily seen to be a local coloring.) By Theorem 77, we can find a triangulation of $P \times I$ whose boundary induces these two local colorings. Adding this triangulation kills off the odd edges of $L$ and $L'$. This surgery changes the homotopy type of $N$.

If there is only one loop $L$ with $\psi$ of it the identity, then we can add a loop somewhere (using Lemma 58) and kill them both off. If $L$ is a loop of odd edges, then $\psi(L)$ must be either the identity, (12), (12)(34), or (34). In these computations, the loop $L$ is always colored with 1 and 2. Notice that $\psi(L)$ is not actually well defined, for we must choose a loop off $L$ to represent it. This representing loop may twist around $L$. Each time it twists around, it add a factor of (34) the characteristic map. Therefore, by choosing the proper number of turns we may assume that we have a representative loop with characteristic map values identity or (12). If the value is (12) then there is an odd number of edges on the loop. Since there are an even number of edges on the interior of $N$, and all our changes have preserved this parity, we see that there are an even number of loops of odd length. Such a pair we may eleminate as in the case of two loops with the identity. Thus, we are done. ∎

### VII.4. *Problems*

*Problem* 1. Show that there exists a locally-6 triangulation with this property: there is a simple closed curve that is a Kempe cycle (with respect to the local coloring of all nonsingular edges) at all but one point.

*Problem* 2. Find a triangulation of the 5-sphere and a coloring $f$ of it such that the number of nonsingular tetrahedra is odd. Find one with an even number of tetrahedra which are nonsingular. What does this say about possible cobordism invariants in dimension 4?

*Problem* 3°. If $M$ and $M^\circ$ are triangulations of the same $n$-manifold, with local colorings $\pi$ and $\pi^\circ$, show that $(M, \pi)$ and $(M^\circ, \pi^\circ)$ are homotopic iff the characteristic maps are conjugate.

*Problem* 4°. Let $\Omega(n)^\circ$ be the group generated by all pairs $(M, \beta)$, with $\beta$ a local coloring of the $n$-manifold $M$, modulo the subgroup generated by all $(M, \beta) + (M^\circ, \beta^\circ)$ whenever the two pairs are cobordant. If $\Omega(n)$ is the $n$-dimensional cobordism group (oriented) (just drop the $\beta$ in the above definition) show that

$$\begin{aligned} \Omega(n)^\circ/\Omega(n) &= 0 \qquad \text{if 4 divides } n, \\ &= \mathbb{Z}_2 \qquad \text{otherwise.} \end{aligned}$$

We have seen this to be true for $n = 1$ and $n = 2$.

## VIII

### VIII.1. *Coloring as a Functor*

In this section we shall show that there is a functor $B$ on a certain category of complexes such that $B(X)$ is built in some way out of the colorings of $X$. We first describe the category and some of its properties. A *pure $n$-complex* is a complex such that every simplex is contained in an $n$-simplex. The objects of our category $C$ are the pure $n$-complexes. A map in $C$ is a simplicial map that sends every $n$-simplex onto an $n$-simplex. This is equivalent to requiring that no edge is mapped onto a vertex. This category has sums and products. The sum of two objects $X$ and $Y$ is their disjoint union $X \sqcup Y$. The product of two objects $X$ and $Y$ is given as follows. The vertices of $X \times Y$ are all pairs $(x, y)$ where $x$ (resp. $y$) is a vertex of $X$ (Resp. $Y$). A set of vertices $(x_0, y_0),\ldots, (x_r, y_r)$ forms an $r$-simplex of the product iff $(x_0,\ldots, x_r)$ (resp. $(y_0,\ldots, y_r)$) is an $r$-simplex of $X$ (resp. $Y$). This is of course just saying that the projections $X \times Y \to X$ and $X \times Y \to Y$ are maps in $C$.

Let $\Delta^n$ be the $n$-simplex. A *coloring* of $X$ is a map $X \to \Delta^n$. We shall first state the properties of the functor $B$. The rest of this section will be devoted to the proof of these assertions.

THEOREM 79. (1) *$B$ is a contavariant functor from $C$ to $C$.*

(2) *$B$ is self-adjoint.*

(3) *There is a natural transformation $\varphi$: $id \to B^2$ such that the following diagram commutes*:

$$\begin{tikzcd} B(X) \arrow[r, "\varphi(B(X))"] \arrow[d, "\approx"'] & B^3(X) \arrow[dl, "B\varphi(x)"] \\ B(X) & \end{tikzcd}$$

(4) *$B(X \sqcup Y) = B(X) \times B(Y)$ ($X$ and $Y$ $n$-complexes).*

(5) *If $X$ and $Y$ are path connected pure $n$-complexes, then $B(X \times Y) = B(X) \sqcup B(Y)$.*

(6) *If $X$ is an $n$-complex, $Y$ an $m$-complex, and $*$ denotes the join then $B(X * Y) = B(X) * B(Y)$.*

(7) *All colorings of $X$ are Kempe related iff $B(X)$ is $(n-2)$ path connected.*

We begin with the definition of $B$. Let $X$ be a pure $n$-complex. A vertex of $B(X)$ is a subset of the vertices of $X$ which are all the points of a coloring that are the same color. That is, if $f$: $X \to \Delta^n$ is a coloring and $p$ is a vertex of $\Delta^n$, then $f^{-1}(p)$ is a vertex of $B(X)$. A set of vertices of $B(X)$ forms an $n$-simplex iff it is of the form $\{f^{-1}(p_0),\ldots, f^{-1}(p_n)\}$, where $f$ is a coloring and $p_0,\ldots, p_n$ are the vertices of $\Delta^n$. All other simplices of $B(X)$ are faces of these simplices. Thus, there is a 1–1 correspondence between the colorings of $X$ and the top-dimensional simplices of $B(X)$.

To help clarify this definition, we compute $B(X)$, where $X$ is the 2-complex given below. Since $X$ is a 2-complex, we consider the 3-colorings of $X$. Figure 37A shows the complex $X$. Figures 37B, C, and D show the three different 3-colorings of $X$. Figure 37E is the complex $B(X)$. Each coloring of $X$ determines three vertices of $B(X)$, but some of the vertices determined by

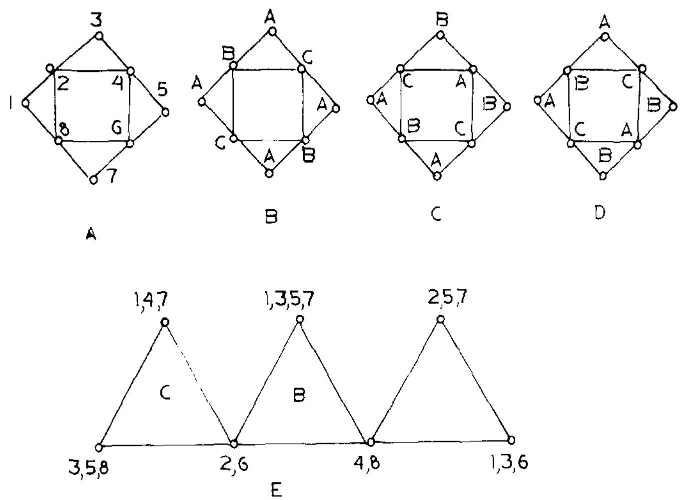



FIG. 37. 3-Coloring the square.

one coloring may be the same vertices determined by another coloring. In this example, $\{2, 6\}$ is all the vertices of coloring (B) colored $B$, and $\{2, 6\}$ is also the set of vertices colored $C$ by the coloring (C). It is in this way that $B(X)$ gets its structure.

*Proof of* 1. Clearly $B(X)$ is an $n$-complex (or empty, which we allow as an $n$-complex). If $g$: $X \to Y$ is a map and $f$ is a coloring of $Y$ then the composite is a coloring of $X$. The map $B(g)$ sends $f^{-1}(p)$ to $g^{-1}f^{-1}(p)$. Since the map $B(g)$ is from $B(Y)$ to $B(X)$, $B$ is contravariant. We leave the relation $B(hg) = B(g)\,B(h)$ to the reader.

*Proof of* 3. By a natural transformation $\varphi$ from *id* to $B^2$ we mean that to every object $X$ of $C$ there is a map $\varphi(X)$ from $X$ to $B(B(X)) = B^2(X)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi(X)"] \arrow[d, "f"] & B^2(X) \arrow[d, "B^2(f)"] \\ Y \arrow[r, "\varphi(Y)"] & B^2(Y) \end{tikzcd}$$

We give a functorial definition of $\varphi$. Let $D$ be a top simplex of $X$. The inclusion $j$ of $D$ in $X$ is a map in $C$, so applying $B$ we get $B(j)$: $B(X) \to B(D)$. Now an $n$-simplex has only one coloring, so $B(D) = \Delta^n$. Therefore, if $p$ is a vertex of $D$, we can identify $p$ with a vertex of $B(D)$, so $B(j)^{-1}(p)$ is a vertex of $B^2(X)$. This is the map $\varphi$. Explicitly, $\varphi(x)(p)$ consists of all the vertices $f^{-1}(q)$ of $B(X)$ such that $p$ is a member of $f^{-1}(q)$.

To prove that the above diagram is commutative, we must do some careful chasing of elements through the diagram. The map $B(f)$ is given by $B(f)(a) = \{x \in X \text{ such that } f(x) \in a\}$. Consequently, $B^2(f)(b) = \{y \in B(Y)$ such that $B(f)(y) \in b\}$. An element $x$ of $X$ is sent by $\varphi$ to the set $\{z \in B(X)$ such that $x \in z\}$. Under $B^2(f)$ this set goes to $W = \{y \in B(Y)$ such that $x \in B(f)(y)\}$. Now $x \in B(f)(y)$ iff $f(x) \in y$, so $W = \{y \in B(Y)$ such that $f(x) \in y\}$ which is exactly $\varphi(f(x))$.

We now show the second property of the natural transformation $\varphi$. Let $z$ be an element of $B(X)$. $\varphi(B(X))z = \{a \in B^2(X)$ such that $z \in a\}$. Now we compute $B(\varphi)(\varphi(B(X))) = \{x \in X$ such that $\varphi(x) \in \varphi(B(X))z\} = \{x \in X$ such that $z \in \varphi(x)\} = \{x \in X$ such that $x \in z\} = z$.

*Proof of* 2. We must first explain what we mean by self-adjoint. Let $X$ and $Y$ be two pure $n$-complexes. By $MOR(X, Y)$ we mean the set of all maps from $X$ to $Y$. $B$ is self-adjoint means that $MOR(X, B(Y))$ is naturally isomorphic to $MOR(Y, B(X))$. The correspondence is as follows. Let $w: X \to B(Y)$. We associate a $w'$: $Y \to B(X)$ by setting $w'$ equal to the composition

$$Y \xrightarrow{\varphi(Y)} B^2(Y) \xrightarrow{B(w)} B(X).$$

We show that if we apply this construction to $w'$ we get back $w$. The construction yields the composition

$$X \xrightarrow{\varphi(X)} B^2(X) \xrightarrow{B^2(w)} B^3(Y) \xrightarrow{B(\varphi(Y))} B(Y).$$

That this composition is $w$ follows from putting the two previous diagrams relating $\varphi$ together:

$$\begin{tikzcd}
X \arrow[r, "w"] \arrow[d, "\varphi(x)"] & B(Y) \arrow[r, "\approx"] \arrow[d, "\varphi(B(x))"] & B(Y) \\
B^2(X) \arrow[r, "B^2(w)"'] & B^3(Y) \arrow[ur, "B(\varphi(Y))"'] &
\end{tikzcd}$$

*Proof of* 4. This is the easiest property. If $f$ is a coloring of $X \sqcup Y$ then it is built out of a coloring of $X$ and one of $Y$. A vertex of $B(X \sqcup Y)$ corresponds to a vertex of $B(X)$ and a vertex of $B(Y)$. Moreover, any vertex of $B(X)$ may be paired with any vertex of $B(Y)$. It is now clear that this is the product.

*Proof of* 5. We shall first show that if $f$ is a map from $\Delta^n \times \Delta^n$ to $\Delta^n$ then $f$ is a projection onto one of the factors composed with some automorphism of $\Delta^n$. Let $p$ be a vertex of $\Delta^n$. We claim that $f^{-1}(p)$ has all elements of one of the two coordinates the same. Indeed, suppose that there were pairs $(a, b)$ and $(c, d)$ of $f^{-1}(p)$ such that $a \neq c$ and $b \neq d$. Then there is an edge joining $(a, b)$ with $(c, d)$ in $\Delta^n \times \Delta^n$. $f$ therefore is not a map in our category, for $f$ colors the ends of an edge the same. By symmetry, assume that $a = c$ and $b \neq d$. Now we claim that all the first coordinates of $f^{-1}(p)$ are equal. If we had some other element $(x, y)$ of $f^{-1}(p)$ such that $x \neq a$, then the above argument shows that $y = d$ and $y = b$, a contradiction.

If $n$ is 1, it is a simple matter to verify the conclusion, so assume that $n$ is at least 2. In that case, $\Delta^n$ has at least three vertices. Assume that there are vertices $p$ and $q$ of $\Delta^n$ such that $f^{-1}(p)$ has all first coordinates equal and $f^{-1}(q)$ has all second coordinates equal. Consequently there are $P$ and $Q$ such that $f^{-1}(p) = P \times \Delta^n$ and $f^{-1}(q) = \Delta^n \times Q$. The vertex $(P, Q)$ belongs to both $f^{-1}(p)$ and $f^{-1}(q)$—contradiction.

By symmetry, we can now say that for every $p$ in $\Delta^n$ there is a $P$ in $\Delta^n$ such that $f^{-1}(p) = P \times \Delta^n$. Thus $f$ is the projection to the first factor followed by the automorphism which sends $p$ to $P$.

Now suppose that we have a map $f$: $X \times Y \to \Delta^n$. Let $D$ and $E$ be $n$-simplices of $X$ and $Y$, respectively. $f$ restricted to $D \times E$ is a projection, say to the first factor. Let $D'$ be an $n$-simplex of $X$ meeting $D$ in a vertex $p$. Since $f$ restricted to $p \times E$ is a projection to the first factor, $f$ is a projection on $D' \times E$ to the first factor. Continuing, $f$ is a projection to the first factor on all of $X \times Y$ because $X$ and $Y$ are path connected.

*Remark.* If $X$ and $Y$ are pure $n$-complexes, $n$ greater than 1, then $X \times Y$ is connected iff both $X$ and $Y$ are. To see this, first suppose that $X$ and $Y$ are connected. If suffices to show that $(p, q)$ and $(p', q)$ are connected in $X \times Y$ for any $p, q, p'$. Let a path from $p$ to $p'$ be $p = p^{\circ}, p^1, p^2, \ldots, p^r = p'$. Since $Y$ is pure, let $q$ belong to the triangle $qst$. Then the following is a sequence of adjacent vertices in $X \times Y$: $(p^{\circ}, q)$, $(p^1, t)$, $(p^2, s)$, $(p^3, t), \ldots, (p^{r-1}, t)$ (or $(p^{r-1}, s)$), $(p^r, q)$. The converse direction is trivial.

This observation is false in case $n$ is 1. Let $u$ denote the edge with two end points. We have $u \times u = u \sqcup u$.

*Proof of* 6. Let $X$ be an $n$-complex and $Y$ an $m$-complex. The join is an $(n + m + 1)$-complex, so we compute the $(n + m + 2)$-colorings of it. Each $n$-simplex of $X$ uses $n + 1$ colors and each $m$-simplex of $Y$ uses $m + 1$ colors. Consequently in an $(n + m + 2)$-coloring $n + 1$ of the colors are used for $X$ and $m + 1$ for $Y$. Any choice of coloring of $X$ may be combined with any choice for $Y$ so we get $B(X * Y) = B(X) * B(Y)$.

*Proof of* 7. By $n - 2$ path connected we mean that given any two $n$-

simplices $D$ and $E$ of $X$, there is a path of $n$-simplices joining $D$ and $E$ such that any two consecuetive $n$-simplices of the path intersect in at least an $(n-2)$-simplex. Let us suppose for a moment that we are considering a manifold $M^n$. If two $(n+2)$-colorings of it are Kempe related, then there are two colors which are changed, and all the others remain fixed. In $B(M)$, the two simplices corresponding to the coloring intersect in $n$ vertices—an $(n-1)$-simplex. Thus we see that Property 7 makes sense for manifolds. In the case of an arbitrary complex, we shall say that two $(n+1)$-colorings are Kempe related iff they have $n-1$ colors in common. In this case, Property 7 becomes tautologous. ∎

The reader may have noticed by now that the coloring functor does not apply to 4-coloring the 2-sphere. The 2-sphere is a pure 2-manifold, so applying $B$ to a 2-sphere is to compute the 3-colorings of it. We shall remedy this in a way that may seem artificial. Given a manifold $M^n$, we form an $n+1$ complex $\hat{M}$ by adding an $(n+1)$-simplex to each $n$-simplex. For example, if $X$ is the circle with four vertices, $P(4)$, then $\hat{X}$ is given in Fig. 37A.

There is a 1–1 correspondence between the 4-colorings of $M$ and $\hat{M}$ where $M$ is a 2-manifold. If $T$ is a tetrahedron which is joined to a triangle of $M$, then a coloring of the triangle uniquely determines a coloring of the remaining vertex of $T$. When we talk of the space of colorings of a manifold, we shall mean $B(\hat{M})$, although we will usually write $B(M)$.

The "hat" is natural in the following sense. We shall show in Section 3 that $B^2(P(n)) = \widehat{P(n)}$.

Another property of $B(X)$ which is easily proved is that if $\beta$ is a codimensional 2-simplex of $B(X)$, then $\rho(\beta, B(X)) = 2^r$ for some integer $r$. $r$ is the number of Kempe cycles minus 1.

## VIII.2. *Coloring Fibrations*

In this section we shall study fibrations in the category of nondegenerate maps and pure simplical complexes. We will first give a motivating example, and then proceed to the general definition.

Suppose that $X$ and $Y$ are complexes, with $X \cap Y = T = \Delta^n$. We would like to compute $B(Z)$, where $Z$ is the union of $X$ and $Y$. Any coloring $f$ of $Z$ is determined by a coloring $f_x$ of $X$ and a coloring $f_y$ of $Y$. Two arbitrary colorings $f$ and $X$ and $g$ of $Y$ do not determine a coloring of $Z$, because $f$ and $g$ must agree on the overlapping portion. Let $P$ and $Q$ be the maps $B(X) \to \Delta^n$ and $B(Y) \to \Delta^n$ determined by the simplex $T$ of their intersection. (That is, $P = \varphi(T)$ and $Q = \varphi(T)$.) A vertex $p$ of $B(X)$ and a vertex $q$ of $B(Y)$ can form a vertex of $B(Z)$ iff $P(p) = Q(q)$. Consequently, $B(Z)$ is a Whitney sum of $B(X)$ and $B(Y)$. We write this as the following diagram:

$$\begin{tikzcd} & B(Z) \arrow[dl, "\pi"'] \arrow[dr] & \\ B(X) \arrow[dr, "P"'] & & B(Y) \arrow[dl, "Q"] \\ & \Delta^n & \end{tikzcd}$$

This is a fiber square. That means that $B(Z)$ is the set of all pairs $(x, y)$, with $x$ in $B(X)$ and $y$ in $B(Y)$ such that $P(x) = Q(y)$. A collection of vertices of $B(Z)$ is a simplex iff their image under each of the projection maps is also a simplex, and the projection maps are nondegenerate.

Consider a top-dimensional simplex $D$ of $B(X)$. Denote by $\pi$ the projection map $B(Z) \to B(X)$. It is clear that $\pi^{-1}(D)$ is isomorphic to $B(Y)$, for any choice of $D$.

The general setup of a fibration is first of all a map $\pi: B \to G$, such that for every top-dimensional simplex $D$ of $G$, $\pi^{-1}(D)$ is isomorphic to a fixed space $F$, called the fiber. This is analagous to a twisted product of spaces, for there is no coherent way the fibers fit together. The fiber $F$ has colorings, for the projection back down $\pi$: $\pi^{-1}(D) \to D$ gives a coloring of $F$. If $p$ is a vertex of $G$, then $\pi^{-1}(p)$ gives a vertex of $B(F)$. We make the fibers coherent by requiring that this vertex of $B(F)$ is independent of the choice of simplex $D$ containing $p$.

Therefore we have the following diagram

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \\ G \arrow[r, "w"] & B(F) \end{tikzcd}$$

This means that for every top-dimensional simplex $D$ of $G$, and vertex $p$ of $D$, $\pi^{-1}(D)$ is isomorphic to $F$, and $\pi^{-1}(p)$ is, under the isomorphism, the set of vertices $w(p)$.

Surprisingly enough, we can find a simple "universal fibration with fiber $F$." The base space of this fibration is $B(F)$—it is coincidental that we are already using the letter $B$ here. The total space is denoted $E(F)$, and has as vertices all pairs $(x, a)$, where $x$ is a vertex of $F$, $a$ is a vertex of $B(F)$, and $x$ belongs to $a$. Clearly the inverse image of a top simplex of $B(F)$ is $F$, and the map $w$ is just the identity.

To see that this is a universal fibration, take $E$, $G$, $F$, and $\pi$ as above. The situation is really trivial, because the definition of fibration includes the map to the base space of the universal fibration. We compute the pull back by $w$ of the universal fibration. The pull back is the set of all $(g, (x, a))$, where $g$ is in $G$, $x$ is in $F$, $a$ is in $B(F)$, $x$ is in $a$, and $w(h) = a$. This set is the same as the set $(g, x)$ such that $x$ is in $w(g)$, but this is isomorphic to $E$.

Observe that all the terms of a fibration have colorings. We have already observed that $F$ did. Since $E$ maps to $G$, and $G$ maps to $B(F)$, and $B(F)$ has colorings, all terms have colorings. The number of top-dimensional simplices of $E$ is the product of the number of $G$ and of $F$. Since all terms have colorings, we shall refer to these fibrations as *coloring fibrations.*

We now describe some coloring fibrations.

EXAMPLE 1. The motivating example is a coloring fibration. The map $w$ is from $B(X)$ to $B(B(Y))$. We think of $Q$ as a simplex of $B^2(Y)$, and map $B(X)$ to it by the map $P$.

EXAMPLE 2. $E(F) \to F$ is a coloring fibration with fiber $B(F)$. The map is the projection of $E(F)$ onto the first coordinate. The map $w$ is the map $\varphi(F)$. To see this consider the diagram:

$$\begin{tikzcd} E(F) \arrow[d] & E(B(F)) \arrow[d] \\ F \arrow[r, "\varphi(F)"] & B(B(F)) \end{tikzcd}$$

The coloring fibration induced by $\varphi(F)$ is the following: the pull back is all pairs $(f, (a, x))$ with $f \in F$, $a \in B(F)$, $x \in B^2(F)$, and $\varphi(f) = x$. Thus, these pairs are just the same as $(f, a)$ with $a \in \varphi(f)$, or $f \in a$. This is just $E(F)$.

EXAMPLE 3. The dual fibration. Given a fibration with fiber $F$, total space $E$ and base $G$, there is another fibration with fiber $G$, total space $E$, and base $F$. Suppose that we have a map $w$: $G \to B(F)$ defining the fibration. We get a map $w'$ by the sequence of maps: $F \to^{\varphi(F)} B^2(F) \to^{B(w)} B(G)$. Let us verify that the total space $E'$ of this fibration is $E$. $E'$ consists of all pairs $(f, g)$, where $w'(f) \in g$. This is equivalent to $f \in w(g)$, which is just $E$.

The reason for the term "dual fibration" is that if we apply this construction once again, we get the original fibration back. In other words, $(w')' = w$. $(w')'$ is the composition $G \to B^2(G) \to B^3(F) \to B(F)$. That this composition is $w$ was shown in the proof of Theorem 79: it was what we needed to show that the functor $B$ was self-adjoint.

EXAMPLE 4 (Products). The last result showed that coloring fibrations have strong symmetry properties, just like products. However, not all products are coloring fibrations. Indeed, if $X$ has no coloring, then $X \times Y \to X$ is not a fibration. If $X$ has a coloring $f$, then we can find a coloring fibration with fiber $Y \times \Delta^n$ and total space $X \times Y$. The structure map is $X \to B(Y \times \Delta^n) = B(Y) \cup \Delta^n$, where we map $X$ to the second factor by $f$. One can check that the total space is $X \times Y$.

Our intuition views a coloring fibration with base $G$ and fiber $F$ as being like a product $G \times F$. It is not a product, for the number of simplices is much smaller that the product $G \times F$. What is the effect of applying $B$ to the total space of a fibration? If the fibration were a product, we should get $B(\text{base}) \cup B(\text{fiber})$. In the exercises we shall give an example where this is not true, but it is true for circles with more than four vertices. Our next project will be to establish this.

For the remainder of this section we shall be concerned only with 2-complexes. $P(n)$ is the circle with $n$ vertices. We assume a result proved in the next section: $\varphi$ is an isomorphism between $P(n)$ and $B^2(P(n))$. We wish to compute $B(E(P(n)))$, for $n$ greater than 4. We begin with the discussion of the general problem.

Suppose that we have a coloring fibration with a coloring $f$ of the total space:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & \Delta^n \\ G \arrow[r, "w"] & B(F) \end{tikzcd}$$

Over each vertex $p$ of $G$, we have a vertex $w(p)$ of $B(F)$, which we also think of as a set of vertices of $E$. The coloring $f$ gives a coloring of this set of vertices. We need to capture how the different colorings of these vertices are related.

Let $X$ be a complex, with a vertex $p$ of $B(X)$. $p$ is thought of as a set of vertices of $X$. Each coloring $f$ of $X$ induces a coloring $f_p$ on the set $p$. We form a complex $T(X)$ by taking as vertices all pairs $(f_p, p)$. A set of pairs forms a simplex if there is a coloring $f$ such that the pairs are of the form $(f_{p_i}, p_i)$. It is possible that there may be two different colorings $f$ and $g$, such that for some $p$, $f_p = g_p$. One case where this happens is when $f$ and $g$ are Kempe related and $p$ is the vertex that they have in common. In this case, all vertices of $p$ are given the same color by both $f$ and $g$. This gives us an inclusion of $B(X)$ in $T(X)$.

Going back to the previous paragraph, we see that the coloring $f$ of $E$ gives rise to a map $f^{\#}: G \to T(F)$. Unfortunately, this is not a 1–1 correspondence, but it is enough to compute $B(E)$ in some special cases.

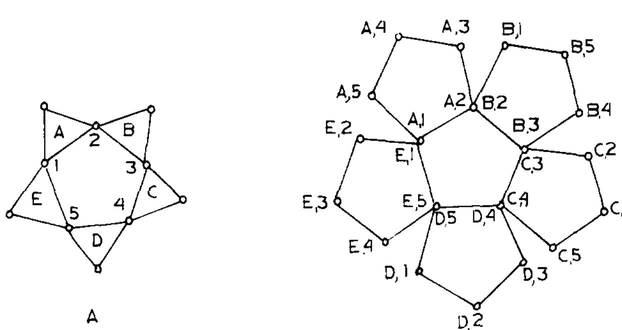
We want to compute $T(B(P(n)))$ for $n$ larger than 4. Let $X$ denote $B(P(n))$. The vertices of $B(X)$ are those of $\widehat{P(n)}$ under the map $\varphi$. The colorings of $X$ are the triangles of $\widehat{P(n)}$ under $\varphi$. The points of $T(X)$ are therefore of the form $(f_x, x)$, where $x$ is a vertex of $\widehat{P(n)}$ and $f_x$ is a triangle of $\widehat{P(n)}$. When is $(f_x, x) = (g_x, x)$? The vertices of $X$ corresponding to $x$ are all those vertices of $X$ containing $x$. Let $f$ be the triangle $(p, q, r)$ and $g$ the triangle $(s, t, u)$. If $x$ is one of $p, q, r, s, t, u$, then $x$ must be a vertex of each of the triangles. Assume that $x$ is none of these vertices. Since $n$ is larger than 4, it is possible to pick vertices of the two triangles, say $p$ and $s$, such that no two of $x, p, s$ are adjacent. Moreover, there are two vertices $A$ and $B$ of $B(P(n))$ such that $A$ contains $x$, $p$ and $s$ while $B$ contains $x$, $p$ and not $s$. Such $A$ and $B$ may not exist if $n$ is 4. Under $f$ both vertices $A$ and $B$ are colored the same, while under $g$ the vertices are colored differently. Thus, $(f_x, x) = (g_x, x)$ iff $f$ and $g$ both contain $x$.

We therefore have an explicit description of $T(B(P(n)))$. There is a central polygon, consisting of the canonical inclusion of $P(n)$ in $T(B(P(n)))$. On each edge is another $n$-gon. Figure 38 gives a representation of $T(B(P(5)))$. For simplicity we leave out all the pendant triangles of $T(B(P(5)))$ and write $E4$ for (E, 4), etc.

We now use this explicit form of $T(B(P(n)))$ to find $B(E(P(n)))$. A coloring $f$: $E(P(n)) \to \Delta^2$ gives rise to a map $f^{\#}$ which makes the following diagram commute

$$\begin{tikzcd} P(n) \arrow[rr, "f^{\#}"] \arrow[dr, "\varphi"'] & & T(B(P(n))) \arrow[dl, "\pi"] \\ & B^2(P(n)) & \end{tikzcd}$$

$\pi$ is the projection on the second coordinate. Because of the nature of the figure, there are only $n+1$ possible maps $f^{\#}$. Suppose that $f^{\#}$ maps onto one of the outer $n$-gons. We clain that there is at most one $f$ inducing it. To see




A B

FIG. 38. A $\widehat{P(5)}$; B $T(B(P(5)))$.

this, note that if two triangles of $T(X)$ meet in a vertex not on the canonical embedding of $B(X)$, then two or three colors are used on this vertex. Hence, there is a unique coloring on the union of the two triangles which induces this $f^{\#}$. Applying $B$ to the map $E(P(n))$ gives $\widehat{P(n)} = B_2(P)n)) \rightarrow B(E(P(n)))$, so each triangle of $\widehat{P(n)}$ induces a coloring of $E(P(n))$, and is seen to be the mapping $f^{\#}$ given above.

Next, suppose that $f^{\#}$ maps to the inner $n$-gon. Any 3-coloring of this inner $n$-gon lifts to a distinct coloring of $E(P(n))$, and all such colorings arise this way. We have shown

THEOREM 80. *If $n \geqslant 5$, then $B(E(P(n)) = B(P(n)) \sqcup \widehat{P(n)}$.*

There is another complex $X$ such that $B(X) = B(E(P(n)))$, namely, $P(n) \times B(P(n))$. This complex has six times as many triangles as $E(P(n))$.

## VIII.3. *3-Coloring*

In the first two sections we investigated general properties of coloring. In this section we study 3-coloring. Three colors is the smallest number of colors for which we get nontrivial results. Some of the results do not generalize to a larger number of colors. For instance, we show below that if $X$ is a graph with a 2-coloring, then $B(X)$ is (Kempe) connected. For 4-coloring, it is necessary to put some restrictions on the topology of $X$. If we choose $X$ to be $\Delta^2 \times \partial\Delta^3$, then $B(X)$ has two Kempe components, one with 38 colorings and the other with 1. $B(X)$ is path connected however. For three colors, Kempe connected and path connected coincide.

We first establish some general results about graphs with a 2-coloring. Such graphs are called *bipartite*.

THEOREM 81. *Let $G$ be a bipartite graph. Then*

(1) *$B(G)$ is connected.*

(2) *There is a bipartite graph $H$ such that $B^2(G) = \hat{H}$.*

*Proof.* Since $G$ is bipartite, we have a decomposition of the vertices of $G$ into two sets $U$ and $V$ such that $U$ is the set of vertices of one color, and $V$ those of the other. Let $f$ be a 3-coloring of $G$ with the three colors inducing a decomposition of two vertices into sets $X, Y, Z$.

We shall show that every triangle of $B(G)$ is connected to the triangle which corresponds to the 2-coloring of $G$. We may thus assume that $X$, $Y$, and $Z$ are all nonempty. Consider the set of vertices colored with $X$ and $Y$. Let $C$ be one of the connected components of the subgraph of $G$ spanned by all vertices colored with $X$ or $Y$. There are two possibilities: either $X \subseteq U$

and $Y \subseteq V$ or $X \subseteq V$ and $Y \subseteq U$. We form a new coloring by interchanging $X$ and $Y$ in those components where $X \subseteq V$ and $Y \subseteq U$. This new coloring $f'$ shares a vertex with $f$ in $B(G)$, namely, the vertex corresponding to the color $Z$. If the new sets of colors are $X'$, $Y'$, $Z'$, then $X' \subseteq U$. This may be a proper inclusion.

Next consider all the vertices in $Y'$ and $Z'$. Change all of the components where we do not have the color of $Y'$ the same as the color of $V$. This gives us a new coloring $f^3$ sharing the vertex $X'$ with $f'$ and having as sets of colors $X^2$, $Y^2$, $Z^2$. Since $X' \subseteq U$, we have that $Y^2 = V$. Consequently, $f^2$ shares the vertex $Y^2$ with the triangle of $B(G)$ corresponding to the 2-coloring, so part (1) is done.

Let $P$ be the vertex of $B(G)$ which is the vertex of the triangle corresponding to the 2-coloring, not equal to $U$ or $V$. We shall show that if $Q$ is a vertex of $B^2(G)$ such that $P \in Q$, then $Q$ lies in exactly one triangle of $B^2(G)$. The graph determined by all the vertices of $B^2(G)$ which do not contain $P$ determines a graph $H$. If $D$ is the triangle of the 2-coloring in $B(G)$, then $\varphi(D)$ determines a coloring of $B^2(G)$ which 2-colors $H$, so $H$ is bipartite.

So let $F$ be a triangle of $B^2(G)$ which we think of as a coloring of $B(G)$. Let the sets of vertices of $B(G)$ determined by $F$ be $X, Y, Z$, with $X$ containing the vertex $P$. The proof in part one shows that the subgraph determined by $Z$ and $Y$ is joined to the triangle of the 2-coloring. Since $Z$ and $Y$ do not contain $P$, the subgraph contains the edge corresponding to $U$ and $V$. Consequently the subgraph determined by $Z$ and $Y$ is connected, and so has at most one 2-coloring. $X$ therefore lies in exactly one triangle of $B(G)$. ∎

This result is interesting in that it says that the functor $B^2$ essentially carries bipartite graphs to bipartite graphs. It would be nice to have a direct description of this assignment. $B^2$ is sometimes a homomorphism. Let $X$ and $Y$ be connected bipartites. Since $B(X)$ and $B(Y)$ are also connected, we have that $B^2(X \sqcup Y) = B(B(X) \times B(Y)) = B^2(X) \sqcup B^2(Y)$. Also, $B^2(X \times Y) = B(B(X) \sqcup B(Y)) = B^2(X) \times B^2(Y)$. Consequently, on the subcategory of connected bipartite graphs, $B^2$ is a multiplicative homomorphism. This is not true if we do not restrict ourselves to connected graphs.

We shall now look at particular graphs and their colorings. If $P(n)$ is the circle with $n$ vertices, then we shall show

THEOREM 82. *$\varphi$: $\widehat{P(n)} \to B^2(P(n))$ is an isomorphism.*

*Proof.* We first show that $\varphi$ is 1–1. Suppose that $\varphi(x) = \varphi(y)$, where $x$ and $y$ are vertices of $P(n)$. This means that $x$ and $y$ are always colored the same. Since it is easy to construct colorings of $P(n)$ such that $x$ and $y$ are colored differently, $\varphi$ is 1–1.

We show that $\varphi$ is onto by induction. We computed $B(P(4))$ in the first section. We leave it to the reader to verify that $B^2(P(4)) = \widehat{P(4)}$, and that $B(P(5)) = \widehat{P(5)}$. These are very short computations.

Let $p$ be a vertex of $P(n)$. Define $P_p$ to be the complex obtained by removed $p$ and the two edges containing it, and then replacing them by an edge. $P_p$ is a circle with $n-1$ vertices. We can identify $B(P_p)$ with the subcomplex of $B(P(n))$ consisting of all colorings which are nonsingular at $p$.

Let $f$ be a coloring of $B(P(n))$. $f$ restricts to a coloring of $B(P_p)$, and so by induction is given by a triangle of $P_p$, $D^p$. Now each of the triangles of $P(n)$ corresponds to a triangle of $P_p$. Except for the two triangles at, $p$, this is a 1–1 correspondence. If $E$ is a triangle of $P(n)$, let $E_p$ be the triangle corresponding to it in $P_n$. If $p$ is not in $E$ then we identify $E$ with $E_p$. Since $\varphi$ is 1–1, a simple argument shows that there is a triangle $E$ of $P(n)$ such that for every $p$, we have $E_p = D^p$.

This $E$ has the property that $E$ induces $f$ on the subcomplex of $B(P(n))$ consisting of the union of all the $B(P_p)$. If $n$ is odd this all of $B(P(n))$, while if $n$ is even the complement of the union in $B(P(n))$ is the triangle of the 2-coloring. Consequently, there are at most $n$ colorings of $B(P(n))$. Since $\varphi$ is 1–1, $\varphi$ is an isomorphism. ∎

If $X$ is the interval $I(n)$ ($P(n)$ with an edge removed) then a similar argument shows that $B^2(I(n)) = \widehat{I(n)}$.

## VIII.4. 3-*Coloring the Integers*

In this section we shall make a brief excursion into the realms of infinite sets. Let $Z$ denote the 1-complex determined by the integers. The vertices of $Z$ are the integers, and two vertices are adjacent iff they differ by $\pm 1$. Naive intuition would lead one to hope that since $Z$ is in some sense the limit of intervals $I(n)$, and since $B^2(I(n)) = \widehat{I(n)}$, perhaps $B^2(Z) = \hat{Z}$. The answer is that this is "almost" true. There is a complex $K$, a noncompact part of $B(Z)$, so that $B^2(Z) = \hat{Z} \cup B(K)$.

Let $f$ and $g$ be 2-colorings of $Z$. We shall say that $f = g$ a.e. (almost everywhere) if $f$ and $g$ agree (perhaps after a permutation of the three colors) in the complement of a finite set of vertices of $Z$. We write $((f))$ for the set of all colorings $g$ equal to $f$ a.e. Two vertices of $B(Z)$, $p$ and $q$, are equal a.e. iff the number of integers which are in $p$ and $q$, or in $q$ and not $p$ is finite. The set of all vertices of $B(Z)$ determined by $p$ is denoted $((p))$. Each equivalence class of colorings $((f))$ determines three equivalence classes of vertices. The complex formed by the equivalence class of vertices and colorings is denoted $B_\infty(Z)$. This is the complex $K$ mentioned above.

We shall approach the projection map $P$: $B(Z) \to B_\infty(Z)$ as though $P$ were a coloring fibration. (See Problem 18). Our first lemma says that although the fiber over a triangle of $B_\infty(Z)$ may not be constant, it is after we apply $B$.

LEMMA 83. *$B(P^{-1}((f)) = \hat{Z} \cup f_\infty$, where $f$ is a coloring of $Z$ and $f_\infty$ is described below.*

*Proof.* The elements of $P^{-1}((f))$ are all the colorings of $Z$ which eventually agree with $f$. Choose two integer sequences $a_n$ and $b_n$ such that the interval $A_n = (a_n, b_n)$ has the properties (1) $A_n$ is properly contained in $A_{n+1}$, (2) $f(a_n) \neq f(b_n)$. Now define $G_n$ as the subcomplex of $P^{-1}((f))$ corresponding to all colorings of $Z$ which agree with $f$ outside the open interval $A_n$. The $G_n$'s satisfy $G_n \subseteq G_{n+1}$ and the union of all the $G_n$ is $P^{-1}((f))$. $G_n$ is important because we know exactly what the subcomplex $G_n$ is. $G_n = B(W(n))$, where $W(n)$ is the circle obtained by joining the two endpoints of the interval $A_n$ by an edge.

If $g$ is a coloring of $P^{-1}((f))$, then $g$ induces a coloring of $G_n$. On $G_n$, $g$ is induced by an edge of $W(n)$, which we shall call $e_n$. How do $e_n$ and $e_m$ compare, for $n$ less than $m$? First suppose that $e_m$ is not contained in the interval $A_n$. $e_n$ in this case must be the edge of $W(n)$ joining the endpoints of $A_n$. Next suppose that $e_m$ is in the interval $A_n$. Since $\varphi$ is 1–1, we must have $e_n = e_m$. Hence, if some $e_n$ is contained in the interval $A_n$, then all higher $e_m$ are equal to $e_n$. These $e_n$'s give rise to the factor $Z$ in the lemma.

If no $e_n$ is contained in $A_n$, then $g$ is the unique coloring $f_\infty$ determined as follows: let the coloring $f$ determine the three vertices $((f_i))$ of $B_\infty(Z)$. If $q$ is a vertex of $P^{-1}((f))$, then $f_\infty(q) = i$, where $((q)) = ((f_i))$. ∎

The next lemma describes how these colorings fit together. The situation is much easier than the case of coloring fibrations.

LEMMA 84. *Suppose that $f$ and $g$ are colorings of $Z$ which are adjacent as triangles of $B(Z)$. Then $B(P^{-1}((f)) \cup P^{-1}((g))) = \hat{Z} \cup h_\infty$, where $h_\infty$ is given below.*

*Proof.* By the last lemma, a coloring $k$ of $P^{-1}((f)) \cup P^{-1}((g))$ is given by either an edge $e$ of $Z$ or $f_\infty$ on $P^{-1}((f))$, and an edge $e'$ of $Z$ or $g_\infty$ on $P^{-1}((g))$. We first see that $h_\infty = f_\infty \cup g_\infty$ is well defined, since $f_\infty = g_\infty$ on the intersection $U$ of $P^{-1}((f))$ and $P^{-1}((g))$. The coloring induced by $e$ and $g_\infty$ does not give a coloring of the union. Under $g_\infty$ any vertex of $B(Z)$ which is eventually in $U$ is given the same color, but this is certainly not the case for $e$.

Finally, we claim that we must have $e = e'$. This is easy to see, so we are through. ∎

We can now prove our theorem. Suppose that $g$ is a coloring of $B(Z)$. On each fiber, $g$ is either given by an edge $e$ or by $f_\infty$. By the last lemma, all these edges must be the same, so this gives us the term $\hat{Z}$ in $B^2(Z)$. We use the fact that any two triangles of $B(Z)$ are joined by a path of triangles, but this follows from Theorem 81, in which no assumption of finiteness was made.

Assigning an $f_\infty$ to each $P^{-1}((f))$ is the same as giving a map $B_\infty(Z) \to \Delta^2$. We have therefore proved

THEOREM 85. $B^2(Z) = \hat{Z} \cup B(B_\infty(Z))$.

At this point, it might appear that our intuition was correct, for there are no obvious colorings of the second factor. We can only show nonconstructively that it is nonempty. We first show that every finite subcomplex of $B_\infty(Z)$ has a coloring. Let $((f_1)),\ldots,((f_n))$ be a finite set of triangles of $B_\infty(Z)$. If we look at a large enough interval, we find that there are colorings $g_i$ such that $((g_i)) = ((f_i))$, and outside this interval two $g_i$'s are related iff the corresponding $f$'s are. Chosing appropriate colorings of the interval gives us a set $K$ of triangles of $B(Z)$ which is isomorphic to the set in $B_\infty(Z)$ that we began with. Since $B(Z)$ has colorings, so does this finite subcomplex. That $B_\infty(Z)$ has colorings now follows from

RADO SELECTION PRINCIPLE. *If $K$ is a graph such that all finite subgraphs are n-colorable, then $K$ is n-colorable.*

## VIII.5. *Injectives and Automorphisms*

In this section we discuss the two questions: (1) When is the map $\varphi$ injective? (2) What is the relationship between the automorphisms of $X$ and $B(X)$?

The map $\varphi(X)$ is injective if $X$ has enough colorings to *separate points.* This means that for any two vertices $p$ and $q$ of $X$, there is a coloring $f$ of $X$ such that $f(p) \neq f(q)$. If this were not the case, then $p$ and $q$ would always be colored alike and so would determine the same set of vertices in $B(X)$.

PROPOSITION 86. *The following triangulations separate points*;

(1) *any n-manifold with a global even coloring,*

(2) *any subcomplex of $B(X)$, for any $X$,*

(3) *the image of $X$ under the map $\varphi(X)$,*

(4) *all circles $P(n)$.*

*Proof.* Let $f$ be a coloring of $M^n$ with $n+1$ colors. Given two vertices $p$ and $q$ of $M$, define a new coloring $g$ of $M$ by setting $g(x) = f(x)$ for $x \neq p$ and $g(p) = n+2$. This coloring separates $p$ and $q$. Next, let $K$ be a subcomplex of $B(X)$ and let $p$ and $q$ be two vertices of $K$. There is a vertex $x$

of $X$ which lies in $p$ and not in $q$—otherwise $p$ and $q$ would be equal. Let $x$ lie in a maximal simplex $T$ of $X$. The coloring $\varphi(X)(T)$ separates the vertices $p$ and $q$.

Clearly (2) implies (3). By Theorem 82, we see (3) implies (4). ∎

There are many complexes $X$ such that $\varphi(X)$ is not injective. For instance, if $S$ is a triangulation of the sphere with exactly two odd vertices, then we know (Section I.2) that the two odd vertices are colored alike by every coloring of $S$. It is not known if this is the entire kernel, but in any case $\varphi(S)$ is not injective. Therefore, $S$ can not be contained in any $B(X)$, for any 3-complex $X$. More generally, if $M^n$ is an $n$-manifold with odd part a submanifold, then $M$ can not be a subcomplex of $B(X)$, for any $n+1$ complex $X$ (see Section VI.2).

We now discuss automorphisms of $X$ and $B(X)$. If $X$ is a complex, let $\mathrm{Aut}(X)$ denote the group of all simplicial automorphisms of $X$. For instance, if $X$ is the $n$-simplex, then $\mathrm{Aut}(X) = S_{n+1}$, the symmetric group on $n+1$ letters. If $\alpha$ is an automorphism of $X$, then $\alpha$ induces an automorphism of the colorings of $X$ by the rule $f \mapsto f\alpha$, where $f$ is a coloring. We shall denote this map by

$$\beta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(B(X))$$

$\beta$ is a homomorphism, but is not necessarily onto nor 1–1. If we take $G$ in Fig. 39A below, then $G$ has a unique coloring, so $B(G)$ is the 3-simplex. $\mathrm{Aut}(G)$ has 12 elements, while $\mathrm{Aut}(B(G))$ has 24 elements. The two vertices of $G$ of degree 3 may be intechanged without changing the coloring, so the induced automorphism is the identity.

**Proposition** 87. *If $\varphi(X)$ is injective, then $\beta$: $\mathrm{Aut}(X) \to \mathrm{Aut}(B(X))$ is also injective.*

*Proof.* Let $\beta(\alpha) =$ identity, where $\alpha$ is an automorphism of $X$. This means that if $p$ is a color, then for any coloring $f$ we have $\alpha f^{-1}(p) = f^{-1}(p)$. Given vertices $x$ and $y$ of $X$, let $f$ be so that $f(x) \neq f(y)$. Since $y$ is not in $f^{-1}(p)$, neither is $\alpha y$. In particular, $\alpha y \neq x$. Thus $\alpha$ is the identity, for we see that if $y$ is not $x$ then neither is $\alpha y$. ∎

The converse to this is false. Taking the triangulation $S$ in Fig. 39B, we have $\mathrm{Aut}(X) =$ identity, $B(X) = \Delta^3$, but $\varphi(X)$ is not injective.

**Corollary** 88. *For any integer $n$, $\mathrm{Aut}(P(n)) = \mathrm{Aut}(B(P(n)))$ under the map $\beta$. More generally, if $B^2(X) = X$, then $\beta$ is an isomorphism.*

Since the even triangulations of the sphere usually have the nicest properties, one might guess that $\beta$ is an isomorphism in this case. The

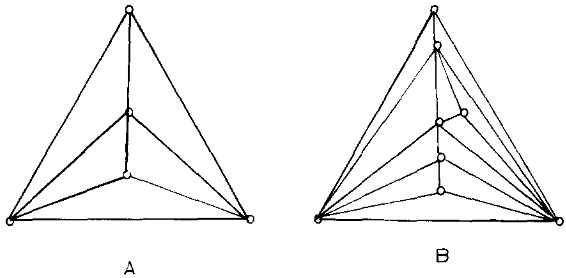



FIG. 39. Triangulations of the sphere.

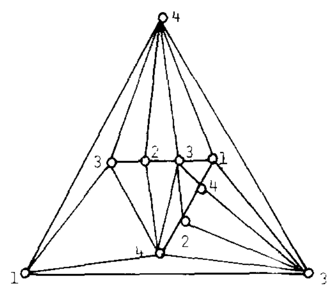



FIG. 40. Even triangulation of the sphere with 10 vertices.

example in Fig. 40 shows that this is false. Consider the coloring $f$ of the triangulation given in Fig. 40. The Kempe cycles all use colors 3 and 4, so if $T$ is the tetrahedron corresponding to $f$ in $B(X)$, then $T$ meets the rest of $B(X)$ is exactly one edge. There is an automorphism of $B(X)$ which fixes all vertices in the complement of $T$ and interchanges the two vertices corresponding to 1 and 2 of $T$. Since $X$ is an even triangulation of the sphere there are vertices in $B(X)$ of the form $(x)$, where $x$ is a vertex of $X$. Any automorphism $\alpha$ of $X$ fixing the complement of $T$ fixes all these vertices, so $\alpha$ must be the identity. Consequently, $\mathrm{Aut}(B(X))$ is larger than $\mathrm{Aut}(X)$.

VIII.6. *The Space of Colorings of Even Spheres*

In this section we shall investigate some of the effects that the topology of a complex $X$ can have on $B(X)$. We shall only prove the results for even triangulations of the 2-sphere, but analogous results hold for even triangulations of simply connected manifolds. We collect all the results in the following

THEOREM 89. *Let $X$ be an even triangulation of the 2-sphere.*

(1) *$B(X)$ is 2-connected.*

(2) *$\varphi(X)$ is injective.*

(3) *If $p$ is a vertex of $B(X)$, then* $\operatorname{lnk}(p)$ *is connected.*

(4) *There is a 2-complex $Y$ with with a 3-coloring such that* $B^2(X) = \hat{Y}$.

*Remark.* All of these results are false if $X$ is not an even triangulation of the sphere. Let $X$ be the triangulation of Fig. 2 in Section I.1. $B(X)$ consists of two disjoint components. One component $U$ is a tetrahedron, and the other $V$ is two tetrahedra joined along an edge. $X$ is a counterexample to (1). Since $B(U) = U$ and $B(V) = V$, we have $B^2(X) = U \times V$. Consequently $X$ is also a counterexample to (4). $X$ does satisfy (2) and (3) though.

We saw many examples of noninjective $X$ in the last section. To show that (3) is not always true, we describe $B(I)$, where $I$ is the icosahedron (see Problem 8, Section I.7). $B(I)$ consists of 10 tetrahedra. Consider a regular dodecahedron. It has 20 vertices, and there are 10 ways of embedding a regular tetrahedron in it. The union of these 10 regularity embedded tetrahedra is $B(I)$. Each vertex is contained in two tetrahedra, and the link is two disjoint triangles.

*Proof of the theorem.* For (1) see Theorem 55. From the last section, we know that $\varphi(X)$ is injective. We now prove (3). Think of $p$ as a set of vertices of $X$. Consider the $\mathbb{Z}_2$ sum $V = \sum \operatorname{lnk}(q)$, where $q$ is a vertex of $p$. If $e$ is an edge of $V$, then $e$ is nonsingular in any coloring which contains $p$.

Let $f$ be a coloring which contains $p$. Let us think of the vertices of $p$ as being colored 1. Let us change $f$ by Kempe cycles of type $(1, 2)$, $(1, 3)$ or $(1, 4)$ to a coloring $g$. $p$ is still a vertex of $g$. If the tetrahedra in $B(X)$ corresponding to $f$ and $g$ are $F$ and $G$, then $F - p$ and $G - p$ have a vertex in common. Hence $f$ is Kempe equivalent to a coloring $f'$ such that if $F$ and $F'$ are the corresponding tetrahedra in $B(X)$, then $F - p$ and $F' - p$ are connected by a path of edges in $\operatorname{lnk}(p)$. Moreover, $f'$ has no nonsingular $(1, 2)$, $(1, 3)$, nor $(1, 4)$ edges. Therefore we must have $ns(f') = V$, but there is at most one coloring $h$ such that $ns(h) = V$. Consequently, $\operatorname{lnk}(p)$ is connected.

We now prove (4). Let $P$ be the vertex of $B(X)$ which is in the tetrahedra corresponding to the 3-coloring, and contains no vertices of $X$ (just those of $\hat{X} - X$). We first show that if $p \in B^2(X)$ and $P \in p$, then $p$ lies in exactly one tetrahedra of $B^2(X)$. Let $p$ lie in a tetrahedron $T$ of $B^2(X)$. We think of $T$ as a coherent choice of labelings for colorings of $X$. That is, for each coloring of $X$ we have a specific choice of map $f: X \to \Delta^3$ so that $f$ and $g$ are Kempe equivalent along a $(1, 2)$ cycle for instance, then $f^{-1}((1, 2)) = g^{-1}((1, 2))$. Let the colors be 1, 2, 3, 4 and let $p$ correspond to the color 4.

We shall show that $B(X) - p$ is connected in the following sense: any triangle of $B(X) - p$ map be reached from any other by a path of triangles (any two consecutive triangles having an edge in common) not using any vertex which lies in $p$. It would then follow that there is a unique 3-coloring of $B(X) - p$, so $p$ lies in exactly one tetrahedron.

In other words, we wish to go from any coloring of $X$ to any other coloring of $X$ without ever making a change using the color 4. We begin by changing along a nonsingular $(1, 2)$ or $(1, 3)$ or $(2, 3)$ cycle. After changing along all possible such cycles, there are two possibilities: either there is a triangle colored with $1, 2, 3$ or there is not. If there is, then it follows that all triangles are colored with 1, 2, 3 so we have reached the 3-coloring's triangle. Suppose that there were a vertex colored 4. Since no $(1, 2)$, $(1, 3)$ nor $(2, 3)$ edge is nonsingular, 4 must be a global color. This is a contradiction, for this means that $p$ contains two vertices of the tetrahedron corresponding to the 3-coloring. All of $B(X) - p$ is therefore joined to the triangle of the 3-coloring, and is therefore connected.

The complex $Y$ is now determined as $B^2(X)$ minus all vertices which contain the vertex $P$ of $B(X)$, The 4-coloring of the complex $B^2(X)$ induced by the 3-coloring's tetrahedron of $B(X)$ induces a 3-coloring on the complex $Y$. ∎

## VIII.7. *Edge Coloring*

In this section we are going to study properties of edge colorings. If $G$ is a graph, then an *edge coloring in $r$ colors* is a labeling of each of the edges of $G$ with one of $r$ colors so that if two edges share a vertex, then they have different colors assigned to them. It is possible to reduce this to a problem in vertex coloring. Define the *line graph* $L(G)$ of a graph $G$ as follows: the vertices of $L(G)$ are the edges of $G$. Two vertices of $L(G)$ are joined by an edge iff the edges that they correspond to interest in a point. A coloring of $L(G)$ in $r$ colors is a coloring of the edges of $G$ in $r$ colors.

One case that is of particular interest is when all vertices of the graph have the same degree $r$. Such a graph is called a *regular graph* (*of degree $r$*). The case that has received the most attention is the case where $r$ is 3. These regular graphs of degree 3 are called *cubic graphs*. The dual of every triangulation of a surface determines a cubic graph. Precisely, if $S$ is a triangulation of a 2-manifold, let the vertices of $D(S)$, the *dual* of $S$, be the triangles of $S$. Two vertices of $D(G)$ are joined by an edge iff the corresponding triangles share an edge in $S$.

A 4-coloring of the vertices of $S$ determines what we have been calling an edge coloring (see Chapter IV), namely a labeling of the edges of $S$ with one of three colors so that every triangle has three colors on it. This is the same as an edge coloring of the dual $D(S)$.

There is one major result in this area, due to Tutte (1946). If $S$ is a triangulation of a 2-manifold, a *hamilton circuit* of $D(S)$ is a simple closed curve of $D(S)$ which passes through each vertex exactly once. This circuit determines a three coloring of the edges of $D(S)$ as follows. The edges not in the circuit are one color. Since there are an even number of triangles in $S$, the length of the circuit is even. The set of edges on the circuit split into two sets, no edge in one set meeting another in that set. This division is accomplished by taking every other edge of the circuit. Tutte proved

THEOREM 90. *If a triangular of the sphere has one hamilton circuit, then it has three.*

*Proof.* (Tutte, 1946). We can form the space of colorings of the edges of the triangulation $S$ by computing $B(L(D(S)))$. We shall abbreviate this as $BLD(S)$. $BLD(S)$ is composed of triangles. The complement of a hamilton circuit determines a vertex of this space. A vertex of $BLD(S)$ corresponds to a hamilton circuit iff the vertex lies in exactly one triangle.

If $p$ is a vertex of $BLD(S)$, let $Q(p)$ denote the set of edges of $S$ that do not lie in $p$. Since every triangle has two edges not a given color, $Q(p)$ is a $\mathbb{Z}_2$ cochain. Let $p$, $q$, $r$ be the vertices of a triangle of $BLD(S)$. Clearly, $Q(p)+Q(r)+Q(q)=0$. If $\gamma(p)$ denotes the number of triangles containing $p$ in $BLD(S)$, then we have the string of equalities:

$$0=\sum Q(p)=\sum \rho(p)\,Q(p)=\sum_{p\in H} Q(p),$$

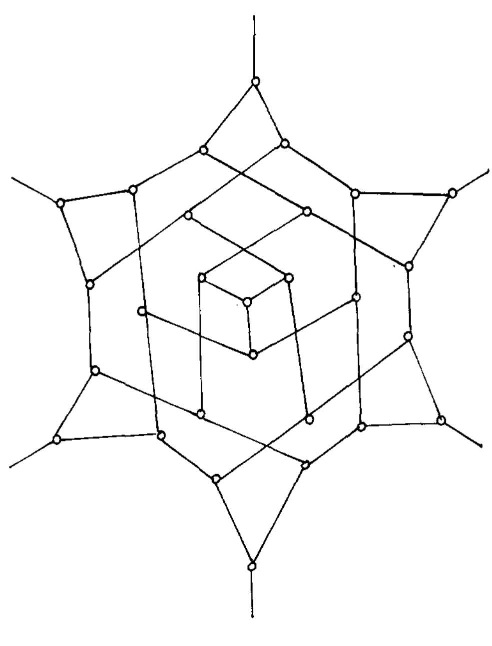


FIG. 41. The Coxeter graph.

where the first summation sign is over all triangles of $BLD(S)$, and $H$ is the set of hamilton circuits of $S$. Since the sum of all $Q(p)$, $p$ in $H$, is zero, there must be at least three members of $H$. ∎

There is undoubtably much to discover in the way of edge colorings. I would like to close this section with the description of a most remarkable graph (see Biggs, 1972, Coxeter, 1946). We begin with what is called the *Coxeter graph*, given in Fig. 41. Let $K$ be the line graph of the Coxeter graph.

AMAZING PROPERTY OF $K$. $B(K) = K \cup K$.

Biggs more or less does this by hand, although he gets the wrong result at the end. Only computer calculations of this are known. There may be some connection with the theory of projective planes (see Parsons, 1976).

## VIII.8. *Problems*

*Problem* 1. Show that the triangulation of the disk $X$ given in Fig. 42b satisfies $B(X) = \hat{X}$.

*Problem* 2. Consider the graph $K$ given in Fig. 42a. It can be thought of as the 1-skeleton of the cube with the corners cut off. If we compute the 4-colorings of $K$, show that $B_4(K) = \Delta^2 \times \partial\Delta^3$. Considering the 3-colorings of $K$, show that $B_3(K) = B(P(4))$.

*Problem* 3. How many vertices does $B(P(n))$ have? Show that this number is periodic modulo 4 with period 6.

*Problem* 4. If $I$ is the icosahedron, show that $B(I) = B^3(I)$. Show that $\mathrm{Aut}(I) = \mathrm{Aut}(B(I)) = \mathrm{Aut}(B^2(I))$.

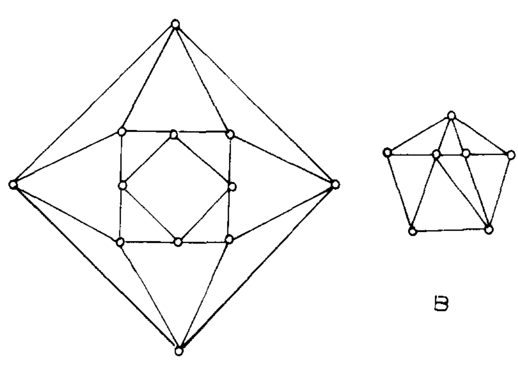



FIG. 42. Two triangulations of the disk.

*Problem* 5. Show that $B(E(P(4)))$ has 24 triangles and hence is not $B(P(4)) \cup \widehat{P(4)}$.

*Problem* 6. Find a 2-complex $D$ such that for all proper subcomplexes $C$, we have $B^2(C) = C$, but $B^2(D) \neq D$.

*Problem* 7. Find a counterexample to the conjecture: if $X$, $Y$, $Z$ are 2-complexes with $Z = X \cup Y$, $X \cap Y = A^2$, $B^2(X) = X$, $B^2(Y) = Y$, then $B^2(Z) = Z$.

*Problem* 8. Construct infinitely many 2-complexes $Z$ such that $B(Z) = B(P(4))$.

*Problem* 9. Show that there are two maps $f$ and $g$ from $P(5)$ to $B(P(5))$, such that $x$ is a member of $f(x)$ and $g(x)$ for all vertices $x$ of $P(5)$. Show that the dual of $f$ is $f$.

*Problem* 10. Give an example of an even 2-sphere with three nonadjacent vertices $p$, $q$, $r$ such that there is no 4-coloring $f$ with $f(p) \neq f(q)$, $f(p) \neq f(r)$ and $f(r) \neq f(q)$. In other words, you can not separate more than two vertices by colorings.

*Problem* 11. In $B(P(2n))$, call a vertex $p$ even if it contains an even number of vertices of $P(2n)$. Let $E$ be the set of even vertices of $B(P(2n))$ and let 0 be the set of odd vertices. How many components does the supgraph spanned by $E$ (resp. 0) have?

*Problem* 12. Show that for all $n$ and $k$, $n \geqslant 2$, $k \geqslant 0$ there is an $n$-complex $W$ with exactly $k$ colorings.

*Problem* 13. Show that the only maps from $P(2n+1) \times P(2n+1)$ to $P(2n+1)$ are given by $\pi P$, where $P$ is a projection to one of the two factors and $\pi$ is an automorphism of $P(2n+1)$. Show this is false for $P(2n)$.

*Problem* 14. If $X(n, m)$ is the triangulation of the disk given in Fig. 43, show that $B^2(X(n, m)) = \widehat{X(n, m)}$.

*Problem* 15. Let $Z$ and $P$ be as in Section 4. Let $T$ be the triangle of $B(Z)$ corresponding to the 2-coloring, and let $S$ be another triangle of $B(Z)$

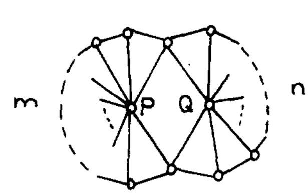



FIG. 43. Triangulations $(n, k)$ of the disk.

so that $P(T) \neq P(S)$. Show that every triangle of $P^{-1}(P(T))$ has a vertex of finite degree, but no vertex of $P^{-1}(P(S))$ has finite degree. Not all fibers of the map $P$ are isomorphic: $P$ is not a fibration.

*Problem* 16. Find a locally-6 triangulation of the torus with exactly 1 4-coloring.

*Problem* 17. Construct a theory of coloring using cubes. The objects to be colored are built out of cubes: cubulations.

*Problem* 18°. Let $H$ be a bibartite graph with $B^2(H) = \hat{H}$. Show that $H$ is either an interval or a circle $P(2n)$.

*Problem* 19. If $f: X \to Y$ is a map, we say that a map $g: Y \to X$ is a retraction if the composition $gf$ is the identity. Let $Y$ be $B^2(X)$ and let $f$ be $\varphi(X)$. In case $X = B(W)$, then $X$ has a retraction. Find an example of an even triangulation of the sphere that has no such retraction.

*Problem* 20. Let $B_c(Z)$ be all the 3-colorings of $Z$ (see Section 4) which are 2-colorings outside a finite set. Show that $B(B_c(Z)) = Z$.

The rest of these problems are unsolved.

*Problem* 21°. Define the functor $B^2$ on the category of bipartite graphs by $\tilde{B}^2(X) = H$, where $B^2(X) = \hat{H}$. Describe this functor directly. On finite graphs, does $\tilde{B}^2$ take connected graphs to connected graphs?

*Problem* 22°. Find the set of all graphs $X$ such that $B^2(X) = \hat{X}$. Does the set of 2-complexes $Y$ with the property $B^2(Y) = Y$ have any extra structure?

*Problem* 23°. What is a good reason for $B^2(P(n)) = \widehat{P(n)}$?

*Problem* 24°. Find all triangulations $D$ of the disk such that $B^2(D) = \hat{D}$. See Problems 14 and 1.

*Problem* 25°. Consider the generalized Peterson graphs $T(n, k)$. There are vertices $A_1 \cdots A_n$ and $B_1 \cdots B_n$. $A_i$ is joined to $A_{i+1}$, $A_{i-1}$, and $B_i$. $B_i$ is joined in addition to $B_{i+k}$ and $B_{i-k}$. All indices are modulo $n$. It is known that all $T(n, k)$, except for the Peterson graph $T(5, 2) = T(5, 3)$, have edge colorings in 3 colors. See Castagna (1972). Is the map $\varphi$ 1–1?

*Problem* 26°. If $S$ is an even triangulation of the 2-sphere, let $H$ satisfy $B^2(S) = \hat{H}$. Is every simple closed path of triangles of $H$ of even length?

*Problem* 27. If $S$ is a triangulation of the 2-sphere with a 3-coloring, is the number of tetrahedra of $B^2(S)$ even. Modulo 4, is it the same as the number of triangles of $S$?

*Problem* 28°. If $S$ is a triangulation of a 2-manifold and $B^2(S) = \hat{S}$, is $B(BLD(S)) = LD(S)$? (see Section 7).

*Problem* 29°. If $S^2$ is an even triangulation of the sphere, are all $B^i(S)$ connected?

*Problem* 30°. If $S$ and $H$ are even triangulations of the 2-sphere, and $B(S) = B(H)$, show that $S$ and $H$ are isomorphic. There are simple examples where this is not true if we drop the condition "even."

*Problem* 31°. If $M$ is a closed manifold with the property that $B^2(M) = \hat{M}$, then $M$ is a sphere and is a join of circles $P(2n)$ and $S^0$.

*Problem* 32°. Find all complexes $X$ such that $B(X) = r$ copies of $X$. The Coxeter graph is an example for $r = 2$. The pentagon and the triangulation of Problem 1 are examples for $r = 1$.

*Problem* 33°. If $B(X) = U \cup V$, is there a complex $Y$ such that $B(Y) = U$? If there is, let $X_u = Y$. Is $X$ a fibered product of $X_u$ and $X_v$?

*Problem* 34°. If $X$ and $Y$ separate points, then $\mathrm{Aut}(X \times Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$.

*Problem* 35°. When is $B(E(X)) = B(X) \cup B^2(X)$? More generally, suppose that we have a fibration

$$\begin{array}{ccc} E & \longrightarrow & E(Y) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B(Y) \end{array}$$

what are conditions of $f$ to insure that $B(E) = B(X) \cup B(Y)$?