A new lower bound on upper irredundance in the queens’ graph

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Abstract

The queens’ graph $Q_n$ has the squares of the $n \times n$ chessboard as its vertices, with two squares adjacent if they are in the same row, column, or diagonal. An irredundant set of queens has the property that each queen in the set attacks at least one square which is attacked by no other queen. IR($Q_n$) is the cardinality of the largest irredundant set of vertices in $Q_n$. Currently the best lower bound for IR($Q_n$) is $IR(Q_n) \geq 2.5n - O(1)$, while the best upper bound is $IR(Q_n) \leq \lfloor 6n + 6 - 8\sqrt{n + \sqrt{n} + 1} \rfloor$ for $n \geq 6$. Here the lower bound is improved to $IR(Q_n) \geq 6n - O(n^{2/3})$. In particular, it is shown for even $k \geq 6$ that $IR(Q_k) \geq 6k^3 - 29k^2 - O(k)$.

1. Introduction

The queens’ graph $Q_n$ has the squares of the $n \times n$ chessboard as its vertices, with two squares adjacent if they are in the same row, column, or diagonal. A square placed on any square $x$ is said to attack any queen placed on a square adjacent to $x$. When referring to a set of queens we usually assume a placement of these queens on squares on the chessboard.

There has been much study of various problems related to the queens’ graph (cf. [5,6]). These include the well known $n$-queens problem, which involves trying to find a way to place $n$ queens on an $n \times n$ chessboard such that no two queens attack each other. The queens domination and independent queens domination problems have also been
frequently studied. These involve finding the minimum number of queens necessary to attack all squares of, or dominate, an \(n \times n\) chessboard, with the independent domination problem requiring the additional constraint that no two queens attack each other. Here the upper irredundance problem in the queens’ graph is investigated. An irredundant set of queens has the property that every queen attacks at least one square which is attacked by no other queen. The objective in the upper irredundance problem is to determine the size of the largest irredundant set of queens (or upper irredundance number \(\text{IR}(Q_n)\)).\(^1\)

Burger et al. [2] have quoted that Weakley had shown that \(\text{IR}(Q_n) \geq 2n - 5\), and Cockayne [4] has shown that \(\text{IR}(Q_n) \leq [6n + 6 - 8\sqrt{n + 3}]\) for \(n \geq 6\). Burger et al. [2] improved these to \(\text{IR}(Q_n) \geq 2.5n - O(1)\) and \(\text{IR}(Q_n) \leq [6n + 6 - 8\sqrt{n + \sqrt{n + 1}}]\) for \(n \geq 6\). Hedetniemi et al. [6] have stated that it seems very likely that \(\text{IR}(Q_n) \leq 5n\) or possibly even \(\text{IR}(Q_n) \leq 4n\). This is disproved in this paper, by presenting a new lower bound of \(\text{IR}(Q_n) \geq 6n - O(n^{2/3})\). In fact, it is shown by computer that for \(n = 17576 = 26^3\), \(\text{IR}(Q_n) > 5n\).

We begin by stating some additional definitions that are required in the paper, and then move on to establish the new lower bound for \(\text{IR}(Q_n)\). This involves the definition of a collection of individually irredundant sets which can be combined to produce an irredundant set whose size is at least \(6k^3 - 29k^2 - O(k)\) for \(n = k^3\) where \(k\) is even and \(\geq 6\). Finally, the construction algorithm is implemented on a computer to provide exact sizes of the irredundant sets for even values of \(k\) in the range \(6 \leq k \leq 16\).

2. Definitions

Given a set \(S\) of vertices in a graph \(G\), the set of vertices with the property that each is adjacent to at least one vertex in \(S\) is called the neighbourhood \(N(S)\) of \(S\). A vertex in \(N(S)\) which is adjacent to exactly one vertex \(v\) in \(S\) is said to be a private neighbour of \(v\). Any vertex in \(S\) with at least one private neighbour is said to be irredundant. The set \(S\) is irredundant if all vertices in the set are irredundant. The upper irredundance number of a graph \(G\) is the cardinality of the largest irredundant set of vertices in \(G\), and is denoted by \(\text{IR}(G)\).

Given a chessboard with \(n\) rows and \(n\) columns (i.e. of size \(n\)), we shall number the squares starting at the top-left from 0 to \(n - 1\) across and down. Denote the square \((x, y)\) as the square that is \(x\) across and \(y\) down the board. A column (respectively row) is labeled by its \(x\) (respectively \(y\) coordinate). There are two types of diagonals. An up diagonal (or \(U\)-diagonal) runs upwards from left to right, and is numbered according to the sum of \(x\) and \(y\) coordinates of any square on the diagonal. A down diagonal (or \(D\)-diagonal) runs downwards from left to right, and is

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\(^1\) A related problem is the lower irredundance problem which involves determining the size of the smallest maximal irredundant set of queens (or lower irredundance number \(\text{ir}(Q_n)\)). The current best lower bound for \(\text{ir}(Q_n)\) is \((n + 1)/4\). This is obtained from the bound \(\text{ir}(G) \geq (\gamma(G) + 1)/2\) for any graph \(G\) (Bollobás and Cockayne [1]), and then \(\gamma(Q_n) \geq (n - 1)/2\) (due to P. Spencer, and communicated by Cockayne [3]).
numbered according to the difference $x - y$ between the $x$ and $y$ coordinates of any square on the diagonal. During this paper we shall use the symbols $C$, $R$, $U$ and $D$ to refer to the squares on a set of columns, rows, up diagonals and down diagonals, respectively.

We associate a set of queens with the set of squares occupied by the queens. We refer to queen squares as squares that are occupied by queens, and private neighbours as squares which are private neighbours of queens. If $Q$ is an irredundant set of queens, we use $Q_C$ to denote the irredundant set of queens whose private neighbours lie in the same column as their corresponding queen. If $Z$ is a set of private neighbours, then we use $Z_C$ to denote the set of all private neighbours whose corresponding queens lie in the same column. We define the sets $Q_R$, $Z_R$, $Q_D$, $Z_D$, $Q_U$, and $Z_U$ in a similar way. Note that, while the sets $Q_C \cup Z_C$, $Q_R \cup Z_R$, $Q_U \cup Z_U$ and $Q_P \cup Z_P$ are individually irredundant, they will all need to shed some elements when combined to form the irredundant set which establishes the lower bound.

If $S$ is a set of squares, we define $R(S)$, $C(S)$, $U(S)$, and $D(S)$ to be the set of rows, columns, up diagonals and down diagonals, respectively, which are occupied by squares in $S$. The set of rows, columns, up and down diagonals will be referred to collectively as lines. Throughout this paper, unless otherwise stated, it is assumed that the chessboard used is of size $n = k^3$, where $k$ is even and $\geq 6$.

### 3. Upper irredundance bound—overview

The lower bound of $\text{IR}(Q_n) \geq 6n - O(n^{2/3})$ is proved by presenting a configuration of $6k^3 - O(k^2)$ queens on a $k^3 \times k^3$ chessboard such that every queen is irredundant. We first note that an $n \times n$ chessboard has $n$ rows, $n$ columns, $2n - 1$ up diagonals and $2n - 1$ down diagonals, for a total of $6n - 2$ lines. Every irredundant queen must be in a line which is occupied by no other queen. Therefore $6n - 2$ is a trivial upper bound for $\text{IR}(Q_n)$. A line that contains more than one queen is wasted in the sense that it cannot contain a square which is a private neighbour of another square. In order to maximise the number of irredundant queens, it is necessary to minimise the number of wasted lines.

First, it is shown how to place one queen in each of $k^3$ columns so that $O(k^2)$ rows and diagonals are wasted. Similar configurations can be used for the placement of queens in rows and diagonals. Then a method is presented to combine these configurations in such a way that each configuration wastes $O(k^2)$ lines of the other configurations, so that a total of $6k^3 - O(k^2)$ lines contain a single queen. Obviously, a line containing a single queen is not sufficient for that queen to be irredundant—it must also have a private neighbour. This is achieved by placing configurations of private neighbour squares in a similar manner to the placing of the queens.

Throughout this discussion the reader might find it helpful to refer to Fig. 5 which shows the construction for $k = 6$. 
4. Row and column constructions

First, we present an irredundant set of \( k^3 - O(k^2) \) queens each of which has a private neighbour in the same column, and an irredundant set of \( k^3 - O(k^2) \) queens each of which has a private neighbour in the same row. Define

\[
B_1(x, y) = \{(x + i, y + i): 0 \leq i < k, \ 0 \leq x + i < n, \ 0 \leq y + i < n\},
\]

\[
B_2(x, y) = \bigcup_{0 \leq i < k} B_1(x + ik, y + (k - 1)k - ik),
\]

\[
B_C(x, y) = \bigcup_{0 \leq i < k} B_2(x + ik^2, y),
\]

\[
B_R(x, y) = \bigcup_{0 \leq i < k} B_2(x, y + ik^2).
\]

\[
Q_C = B_C(0, k^3/2),
\]

\[
Z_C = B_C(3k - k^2, k^3/2 - k^2),
\]

\[
Q_R = B_R(k^3/2, 0),
\]

\[
Z_R = B_R(k^3/2 - k^2, 3k - k^2).
\]

See Fig. 1 for an example with \( k = 4 \). Here a modified \( Z_C = B_C(1, k^3/2 - k^2) \) is used instead of the \( Z_C \) defined above. This is necessary since it is shown later in the paper that if \( k < 6 \), then \( U(Q_C) \cap U(Z_C) \neq \emptyset \).

From the example, it can be seen that every column contains one queen and one private neighbour. Additionally, there are \( k^2 \) rows, \( k^2 + O(k) \) \( U \)-diagonals, and \( \frac{1}{2}k^2 + O(k) \) \( D \)-diagonals containing queens. A similar number contain private neighbours. These rows and diagonals are effectively wasted since none of these lines can contain both a queen and a private neighbour.

4.1. Properties of \( Q_C, Z_C, Q_R, \) and \( Z_R \)

Lemma 1 (Down diagonals used by \( Q_C, Z_C, Q_R, \) and \( Z_R \)).

(a) There are at most \( k^2/2 + O(k) \) \( D \)-diagonals used by \( Q_C \) and \( Q_R \).

(b) There are at most \( k^2/2 + O(k) \) \( D \)-diagonals used by \( Z_C \) and \( Z_R \).

(c) \( Q_C \) and \( Q_R \) share no \( D \)-diagonals with \( Z_C \) and \( Z_R \).

Proof. Recall that \( D(S) \) denotes the set of down diagonals used by the set \( S \).

\[
D(B_1(x, y)) \subseteq \{(x + i - (y + i)): 0 \leq i < k\} = \{(x - y)\},
\]
Fig. 1. $Q_C$ and a modified $Z_C$ for $k = 4$. Filled circles represent queens ($Q_C$) and hollow circles represent private neighbours ($Z_C$).

\[
D(B_2(x, y)) \subseteq \bigcup_{0 \leq i < k} D(B_1(x + ik, y + (k - 1)k - ik)) \\
\subseteq \bigcup_{0 \leq i < k} \{(x - y - (k - 1)k + 2ik)\},
\]

\[
D(B_C(x, y)) \subseteq \bigcup_{0 \leq j \leq k} D(B_2(x + jk^2, y)) \\
\subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(x + jk^2 - y - (k - 1)k + 2ik)\},
\]

\[
D(B_R(x, y)) \subseteq \bigcup_{0 \leq j \leq k} D(B_2(x, y + jk^2)) \\
\subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(x - y - jk^2 - (k - 1)k + 2ik)\},
\]

\[
D(Q_C) = D(B_C(0, k^3/2)) \\
\subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(jk^2 - k^3/2 - (k - 1)k + 2ik)\},
\]
\[ D(Z_C) = D(B_C(3k - k^2, k^3/2 - k^2)) \]
\[ \subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(3k + jk^2 - k^3/2 - (k - 1)k + 2ik)\}, \]
\[ D(Q_R) = D(B_R(k^3/2, 0)) \]
\[ \subseteq \bigcup_{0 \leq j < k} \bigcup_{0 \leq i < k} \{(k^3/2 - jk^2 - (k - 1)k + 2ik)\}, \]
\[ D(Z_R) = D(B_R(k^3/2 - k^2, 3k - k^2)) \]
\[ \subseteq \bigcup_{0 \leq j < k} \bigcup_{0 \leq i < k} \{(k^3/2 - 3k - jk^2 - (k - 1)k + 2ik)\}. \]

From these, since \( k \) is even, it can be seen that
\[
\forall x \in D(Q_C), \quad x = k \pmod{2k},
\]
\[
\forall x \in D(Z_C), \quad x = 0 \pmod{2k},
\]
\[
\forall x \in D(Q_R), \quad x = k \pmod{2k},
\]
\[
\forall x \in D(Z_R), \quad x = 0 \pmod{2k}.
\]

Since \( k > 0 \), \( D(Q_C \cup Q_R) \cap D(Z_C \cup Z_R) = \emptyset \). Also, since the difference between any two elements of \( D(Q_C \cup Q_R) \) is \( \leq k^3 + 2k(k - 1) < k^3 + 2k^2 \), and only 1 in every 2k \( D \)-diagonals is in \( Q_C \cup Q_R \), then \( |D(Q_C \cup Q_R)| \leq k^2/2 + O(k) \). Likewise, the difference between any two elements of \( D(Z_C \cup Z_R) \) is \( \leq k^3 + 2k^2 + 6k \), so \( |D(Z_C \cup Z_R)| \leq k^2/2 + O(k) \). \( \square \)

**Lemma 2** (Up diagonals used by \( Q_C, Z_C, Q_R, \) and \( Z_R \)).

(a) There are at most \( k^2 + O(1) \) \( U \)-diagonals used by \( Q_C \) and \( Q_R \).

(b) There are at most \( k^2 + O(1) \) \( U \)-diagonals used by \( Z_C \) and \( Z_R \).

(c) \( Q_C \) and \( Q_R \) share no \( U \)-diagonals with \( Z_C \) and \( Z_R \).

**Proof.** Recall that \( U(S) \) denotes the set of negative diagonals used by \( S \).

\[ U(B_1(x, y)) \subseteq \{(x + i) + (y + i)j: \ 0 \leq i < k\} \]
\[ = \bigcup_{0 \leq i < k} \{(x + y + 2i)\}, \]
\[ U(B_2(x, y)) \subseteq \bigcup_{0 \leq i < k} U(B_1(x + jk, y + (k - 1)k - jk)) \]
\[ \subseteq \bigcup_{0 \leq i < k} \{(x + y + (k - 1)k + 2i)\}, \]
\[ U(B_C(x, y)) \subseteq \bigcup_{0 \leq j \leq k} U(B_2(x + jk^2, y)) \]
\[ \subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(x + jk^2 + y + (k - 1)k + 2i)\}, \]
\[ U(B_R(x, y)) \subseteq \bigcup_{0 \leq j \leq k} U(B_2(x, y + jk^2)) \]
\[ \subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(x + jk^2 + y + (k - 1)k + 2i)\}, \]

\[ U(Q_C) = U(B_C(0, k^3/2)) \]
\[ \subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(k^3/2 + jk^2 + (k - 1)k + 2i)\}, \]
\[ U(Z_C) = U(B_C(3k - k^2, k^3/2 - k^2)) \]
\[ \subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(k^3/2 + jk^2 - k^2 + 2k + 2i)\}, \]
\[ U(Q_R) = U(B_R(k^3/2, 0)) \]
\[ \subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(k^3/2 + jk^2 + (k - 1)k + 2i)\}, \]
\[ U(Z_R) = U(B_R(k^3/2 - k^2, 3k - k^2)) \]
\[ \subseteq \bigcup_{0 \leq j \leq k} \bigcup_{0 \leq i < k} \{(k^3/2 + jk^2 - k^2 + 2k + 2i)\}. \]

From these, since \( k \) is even, it can be seen that

\[ \forall x \in U(Q_C), \quad x = -k + 2i \pmod{k^2} \quad (0 \leq i < k), \]
\[ \forall x \in U(Z_C), \quad x = 2k + 2i \pmod{k^2} \quad (0 \leq i < k), \]
\[ \forall x \in U(Q_R), \quad x = -k + 2i \pmod{k^2} \quad (0 \leq i < k), \]
\[ \forall x \in U(Z_R), \quad x = 2k + 2i \pmod{k^2} \quad (0 \leq i < k). \]

Therefore, if \( k \geq 6 \), \( U(Q_C \cup Q_R) \cap U(Z_C \cup Z_R) = \emptyset \). Also, since the difference between the minimum and maximum elements of \( U(Q_C \cup Q_R) \) is less than \( k^3 + 2k \), and only \( k \) of every \( k^2 \) \( U \)-diagonals is in \( Q_C \cup Q_R \), then \( |U(Q_C \cup Q_R)| \leq k^2 + O(1) \). Likewise, the difference between the minimum and maximum elements of \( U(Z_C \cup Z_R) \) is less than \( k^3 + 2k \), so \( |U(Z_C \cup Z_R)| \leq k^2 + O(1) \). \( \Box \)
5. Diagonal constructions

Just as was done with the rows and columns, a similar approach is used to construct a configuration that has $2k^3 - O(k^2)$ irredundant queens whose private neighbours are on the same D-diagonal. Likewise, a configuration of $2k^3 - O(k^2)$ irredundant queens with their private neighbours along the same U-diagonals is presented. Define

$$B_3(x, y) = \{(x + i, y): 0 \leq i < k, \ 0 \leq x + i < n, \ 0 \leq y < n\},$$

$$B_4(x, y) = \bigcup_{0 \leq i < k} B_3(x, y + ik),$$

$$B_D(x, y) = \bigcup_{0 \leq i < 2k} B_4 \left( x + \frac{k^2}{2}i, y - \frac{k^2}{2}i \right),$$

$$B_U(x, y) = \bigcup_{0 \leq i < 2k} B_4 \left( x + \frac{k^2}{2}i, y + \frac{k^2}{2}i \right),$$

$$Q_D = B_D(0, k^3 - k^2),$$

$$Z_D = B_D(3k/2, k^3 - 3k/2),$$

$$Q_U = B_U(0, k),$$

$$Z_U = B_U(3k/2, 5k/2 - k^2).$$

See Fig. 2 for an example with $k = 4$, where $Z_U = B_U(k, k + 1 - k^2)$ is used instead since, for $k < 6$, the general construction gives $C(Q_U) \cap C(Z_U) \neq \emptyset$ (see Section 5.1 below). Also, $Q_U$ and $Z_U$ have been further altered by removing queens from the corners in order to avoid the situation where a queen has a private neighbour which is both in the same $U$-diagonal and off the board, and vice versa. This is further explained in Section 5.2 below.

5.1. Properties of $Q_D$, $Z_D$, $Q_U$, and $Z_U$

**Lemma 3** (Rows used by $Q_D$, $Z_D$, $Q_U$, and $Z_U$).

(a) There are at most $k^2 + O(k)$ rows used by $Q_D$ and $Q_U$.
(b) There are at most $k^2 + O(k)$ rows used by $Z_D$ and $Z_U$.
(c) $Q_D$ and $Q_U$ share no rows with $Z_D$ and $Z_U$.

**Proof.** Recall that $R(S)$ denotes the set of rows used by the set $S$. In the same way as was shown for the row and column constructions, it can be proved that

$$\forall x \in R(Q_D \cup Q_U), \quad x = 0 \ (\text{mod} \ k).$$

$$\forall x \in R(Z_D \cup Z_U), \quad x = k/2 \ (\text{mod} \ k).$$
Therefore, since \( k > 0 \), \( R(QD \cup QU) \cap R(ZD \cup ZU) = \emptyset \). Also, since \( QD \cup QU \) occupies only 1 in every \( k \) rows, and there are only \( k^3 \) rows, then \( |R(QD \cup QU)| \leq k^2 \). Likewise, \( |R(ZD \cup ZU)| \leq k^2 \). \( \square \)

Lemma 4 (Columns used by \( QD, ZD, QU, \) and \( ZU \)).

(a) There are at most \( 2k^2 + O(k) \) columns used by \( QD \) and \( QU \).
(b) There are at most \( 2k^2 + O(k) \) columns used by \( ZD \) and \( ZU \).
(c) \( QD \) and \( QU \) share no columns with \( ZD \) and \( ZU \).

Proof. Recall that \( C(S) \) denotes the set of columns used by the set \( S \). In the same way as was shown for the row and column constructions, it can be proved that

\[
\forall x \in C(QD \cup QU), \quad x = i \pmod{k^2/2} \ (0 \leq i < k),
\]

\[
\forall x \in C(ZD \cup ZU), \quad x = 3k/2 + i \pmod{k^2/2} \ (0 \leq i < k).
\]

And therefore, since \( k \geq 6 \), \( C(QD \cup QU) \cap C(ZD \cup ZU) = \emptyset \). Also, since \( QD \cup QU \) occupies only \( k \) in every \( k^2/2 \) columns, and there are only \( k^3 \) columns, then \( |C(QD \cup QU)| \leq 2k^2 \). Likewise, \( |C(ZD \cup ZU)| \leq 2k^2 \). \( \square \)
5.2. Corners of $Q_D$, $Z_D$, $Q_U$, and $Z_U$

Unlike the row and column constructions, the diagonal constructions cannot extend totally into the corner of the board. This is because using the constructions presented, some queens will be positioned off the board, while other queens will have corresponding private neighbours which are off the board. Because of this, the configurations $Q_D$ and $Q_U$ do not have a full set of $2k^2$ queens.

Consider the corner regions indicated in Fig. 3. In the top left corner, the area $A$ contains only queens from the corresponding $B_4$ configuration with index 0. The area $B$ contains only private neighbours of queens in $A$. All queens from this 0th $B_4$ configuration lie on the board. If we completely remove this configuration, along with the corresponding private neighbours, then we will remove $k^2$ queens, and will completely clear regions $A$ and $B$ in the top left corner.

Now consider the bottom right corner. Here the area $C$ contains only queens from the corresponding $B_4$ configuration with index $2k-1$. The area $D$ contains only private neighbours of queens in $C$. All queens from this $(2k-1)$th $B_4$ configuration lie on the board. If we completely remove this configuration, along with their private neighbours, then we shall remove $k^2$ queens, and will completely clear regions $C$ and $D$ in the bottom right corner. However, there is also one $B_3$ configuration in the $(2k-2)$th $B_4$ configuration of queens which lies off the board. Therefore, if we remove these $k^2 + k$ queens from the $Q_U$ configuration, we completely clear the bottom right corner, and make sure that all remaining queens are on the board, along with their private neighbours.

In total we have removed $2k^2 + k$ queens from $Q_U$. In a similar manner we remove $2k^2$ queens from $Q_D$, thereby clearing the bottom left and top right corners, and ensuring that all remaining queens are on the board, along with their private neighbours.

We point out that, by carrying out a more careful analysis, we need only remove $3k^2/2 - O(k)$ queens in each of $Q_U$ and $Q_D$. Details of this analysis is contained in Appendix A. However, for simplicity, we completely clear the corners and assume the $2k^2 + O(k)$ bound for the remainder of the paper.

![Fig. 3. The top left and bottom right corners of the board. The area between each double line and its closest corner has all queens and private neighbours removed.](image-url)
6. Combining the constructions

In order to combine the row, column and diagonal constructions, some queens and private neighbours will need to be removed. If a queen \( a \) belonging to configuration \( A \) (where \( A \in S = \{ Q_C \cup Z_C, Q_R \cup Z_R, Q_D \cup Z_D, Q_U \cup Z_U \} \) attacks the private neighbour of a queen \( b \) in configuration \( B \in S, \) where \( B \neq A \), then either \( a \) and its private neighbour in \( A \) or \( b \) and its private neighbour in \( B \) must be removed.

We now describe informally the removals in two stages. A formal definition of the resulting irredundant set is given in Section 7.

6.1. Central removals

Here we remove some queens and private neighbours in the centre of the board.

The goal here is to ensure the following:

1. No queen in \( Q_U \) attacks along a \( D \)-diagonal the private neighbour of a queen from another configuration.
2. No queen in \( Q_D \) attacks along a \( U \)-diagonal the private neighbour of a queen from another configuration.
3. No queen in \( Q_R \) attacks along a column the private neighbour of a queen from another configuration.
4. No queen in \( Q_C \) attacks along a row the private neighbour of a queen from another configuration.

This can be achieved carrying the following sets of removals in the central region shown in Fig. 4:

1. Remove all elements of \( Q_C \cup Z_C \) between the double lines \( a \) and \( b \).
2. Remove all elements of \( Q_R \cup Z_R \) between the double lines \( c \) and \( d \).
3. Remove all elements of \( Q_U \cup Z_U \) between the double lines \( e \) and \( f \).
4. Remove all elements of \( Q_D \cup Z_D \) between the double lines \( g \) and \( h \).

Let us check, for example, that these removals will ensure that condition (1) above is satisfied. As all queens and private neighbours in the region \((Q_C \cup Z_C)\) between the vertical double lines \( a \) and \( b \) have been removed there is no remaining square in \( Z_C \) that is attacked in this way. Also, there is no remaining square in \( Z_D \) which attacked in this way as we have removed all queens and private neighbours in the region \((Q_D \cup Z_D)\) between the \( D \)-diagonal double lines \( g \) and \( h \). Finally, there is no remaining square in \( Z_R \) which is attacked in this way as we have removed all queens and private neighbours in the region \((Q_R \cup Z_R)\) between the horizontal double lines \( c \) and \( d \). In a similar way we can check that the remaining conditions are also satisfied.

The number of queens that need to be removed in order to clear the centre of the board is as follows:

1. From \( Q_C \): \( 4k^2 + O(k) \).
2. From \( Q_R \): \( 2k^2 + O(k) \).
3. From \( Q_U \): \( 6k^2 + O(k) \).
4. From \( Q_D \): \( 4k^2 + O(k) \).
Thus, a total of $16k^2 + O(k)$ queens and their private neighbours are removed from the centre of the board.

6.2. Other removals

We now need to deal with other types of interactions between queens and private neighbours of other configurations. Since the centre has been removed, we know that there are no interactions of the following types:

1. A queen from $QR$ with any private neighbour in the same column.
2. A queen from $QC$ with any private neighbour in the same row.
3. A queen from $QU$ with any private neighbour in the same $D$-diagonal.
4. A queen from $QD$ with any private neighbour in the same $U$-diagonal.

From Lemmas 1–4 we know that there are no interactions of the following types:

1. A queen from $QR$ or $QC$ with any private neighbour from $ZC$ or $ZR$ in the same $U$- or $D$-diagonal.
2. A queen from $QU$ or $QD$ with any private neighbour from $ZD$ or $ZU$ in the same row or column.

We now attend to the remaining interactions:

1. Any queen from $QR$ with a private neighbour from $ZU$ or $ZD$ in the same row is removed. By Lemma 3 at most $k^2 + O(k)$ queens need to be removed to avoid these interactions.
2. If a queen from $QR$ or $QC$ attacks a private neighbour from $ZU$ in the same $U$-diagonal, then remove the queen in $QU$ corresponding to the attacked private...
neighbour in $Z_U$. By Lemma 2 at most $k^2 + O(k)$ queens need to be removed to avoid these interactions.

(3) If a queen from $Q_R$ or $Q_C$ attacks a private neighbour from $Z_D$ in the same $D$-diagonal, then remove the queen in $Q_D$ corresponding to the attacked private neighbour in $Z_D$. By Lemma 1 at most $k^2/2 + O(k)$ queens from $Q_D$ need to be removed to avoid these interactions.

(4) Remove any queen from $Q_C$ with a private neighbour from $Z_U$ or $Z_D$ in the same column. By Lemma 4 at most $2k^2 + O(k)$ queens need to be removed to avoid these interactions.

(5) Remove any queen from $Q_D$ with a private neighbour from $Z_C$ or $Z_R$ in the same $D$-diagonal. By Lemma 1 at most $k^2/2 + O(k)$ queens need to be removed to avoid these interactions.

(6) If a queen from $Q_U$ or $Q_D$ attacks a private neighbour from $Z_R$ in the same row, then remove the queen in $Q_R$ corresponding to the attacked private neighbour in $Z_R$. By Lemma 4 at most $k^2 + O(k)$ queens need to be removed to avoid these interactions.

(7) If a queen from $Q_U$ or $Q_D$ attacks a private neighbour from $Z_C$ in the same column, then remove the queen in $Q_C$ corresponding to the attacked private neighbour in $Z_C$. By Lemma 4 at most $2k^2 + O(k)$ queens need to be removed from $Q_C$ to avoid these interactions.

(8) Remove any queen from $Q_U$ with a private neighbour from $Z_R$ or $Z_C$ in the same $U$-diagonal. By Lemma 2 at most $k^2 + O(k)$ queens need to be removed to avoid these interactions.

The above actions result in the removal of a total of at most $9k^2 + O(k)$ queens.

6.3. Combining all removals

Combining the central removals ($16k^2 + O(k)$), the corner removals ($4k^2 + O(k)$), and the other removals ($9k^2 + O(k)$), gives a total of at most $29k^2 + O(k)$ queens and private neighbours which are lost. Thus the total number of queens and private neighbours remaining is at least $6k^3 - 29k^2 - O(k)$. Due to part (c) of Lemmas 1–4 and the removals considered, the remaining portions of $Q_C$, $Q_R$, $Q_D$, and $Q_U$ form an irreduntant set, with private neighbours in $Z_C$, $Z_R$, $Z_D$, and $Z_U$. Hence

\[ \text{IR}(Q_{k^3}) \geq 6k^3 - 29k^2 - O(k) \]  

(1)

**Theorem 5.** $\text{IR}(Q_n) \geq 6n - O(n^{2/3})$.

**Proof.** Follows from (1) by taking $k = 2\lfloor \sqrt[3]{n}/2 \rfloor$ and forming an irreduntant set on part of the board. □

7. Formal definition of the irreduntant set

So far we have presented an informal definition of what parts are to be removed from the partial configurations. We now define formally the final irreduntant set $X$ of
(queen, private neighbour) pairs. $X$ has size $6k^3 - 29k^2 - O(k)$, and is built up as follows:

$$Q'_C = \{(a, b) \in B_C(0, k^3/2) : (a < k^3/2 - 2k^2 \text{ or } a \geq k^3/2 + 2k^2), a \neq i, 3k/2 + i (\mod k/2), (0 \leq i < k)\},$$

$$Z'_C = \{(a, b) \in B_C(3k - k^2, k^3/2 - k^2) : (a < k^3/2 - 2k^2 \text{ or } a \geq k^3/2 + 2k^2), a \neq i, 3k/2 + i (\mod k/2), (0 \leq i < k)\},$$

$$X_C = \{((a, b), (c, d)) : (a, b) \in Q'_C, (c, d) \in Z'_C, a = c\},$$

$$Q'_R = \{(a, b) \in B_R(k^3/2, 0) : (b < k^3/2 - k^2 \text{ or } b \geq k^3/2 + k^2), b \neq 0, k/2 (\mod k)\},$$

$$Z'_R = \{(a, b) \in B_R(k^3/2 - k^2, 3k - k^2) : (b < k^3/2 - k^2 \text{ or } b \geq k^3/2 + k^2), b \neq 0, k/2 (\mod k)\},$$

$$X_R = \{((a, b), (c, d)) : (a, b) \in Q'_R, (c, d) \in Z'_R, b = d\},$$

$$Q'_D = \{(a, b) \in B_D(0, k^3 - k^2) : a - b \geq -k^3 + k^2, a - b < k^3 - k^2, \quad (a - b < -2k^2 \text{ or } a - b \geq 2k^2), a - b \neq 0, k (\mod 2k)\},$$

$$Z'_D = \{(a, b) \in B_D(3k/2, k^3 - 3k/2) : a - b \geq -k^3 + k^2, a - b < k^3 - k^2, \quad (a - b < -2k^2 \text{ or } a - b \geq 2k^2), a - b \neq 0, k (\mod 2k)\},$$

$$X_D = \{((a, b), (c, d)) : (a, b) \in Q'_D, (c, d) \in Z'_D, a - b = c - d\},$$

$$Q'_U = \{(a, b) \in B_U(0, k) : a + b \geq k^2, a + b < 2k^3 - k^2, \quad (a + b < k^3 - 3k^2 \text{ or } a + b \geq k^3 + 3k^2), a + b \neq 2k + 2i, -k + 2i (\mod k^2) \quad (0 \leq i < k)\},$$

$$Z'_U = \{(a, b) \in B_U(3k/2, 5k/2 - k^2) : a + b \geq k^2, a + b < 2k^3 - k^2, \quad (a + b < k^3 - 3k^2 \text{ or } a + b \geq k^3 + 3k^2), a + b \neq 2k + 2i, -k + 2i (\mod k^2) \quad (0 \leq i < k)\},$$
\[ X_U = \{((a, b), (c, d)) : (a, b) \in Q'_U, (c, d) \in Z'_U, a + b = c + d\}, \]

\[ X = X_R \cup X_C \cup X_D \cup X_U. \]

8. Computer constructions

A computer program was written to implement the construction algorithm. However, rather than completely clearing the corners and centre of the board, the program only removes queens necessary to avoid the conflicts described in Sections 5.2 and 6.1. Fig. 5 shows a result from this program for \( k = 6 \). Table 1 gives the exact sizes of the constructed irredundant sets for values of \( k \) in the range \( 6 \leq k \leq 26 \).

![Figure 5](image)

Fig. 5. A configuration of 563 irredundant queens on a 216×216 board (\( k = 6 \)). Filled circles represent queens and hollow circles represent private neighbours.
Table 1
Lower bounds for $\text{IR}(Q_{k^3})$ for $6 \leq k \leq 26$

<table>
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<tr>
<th>$k$</th>
<th>$k^3$</th>
<th>$\text{IR}(Q_{k^3}) \geq$</th>
<th>$\text{IR}(Q_{k^3})/k^3 \geq$</th>
</tr>
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<td>17576</td>
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<td>5.06</td>
</tr>
</tbody>
</table>

9. Conclusions

The proof presented that $\text{IR}(Q_n) \geq 6n - O(n^{2/3})$ involved constructing a configuration of $6n - O(n^{2/3})$ irredundant queens on an $n \times n$ board. A property of this configuration is that the board is not dominated. It may be possible that this configuration could be altered or augmented in some way so that the board is indeed dominated. Doing so could lead to a bound for $\Gamma(Q_n)$ which is better than the current bound of $\Gamma(Q_n) > 2.5n$. The new lower bound for $\text{IR}(Q_n)$ is a significant improvement over the previous lower bound, in that it comes a lot closer to the theoretical upper bound. However, the bound $\text{IR}(Q_{k^3}) \geq 6k^3 - 29k^2 - O(k)$ can be improved in the $k^2$ term, possibly to $25k^2$, by a more conservative removal of queens in the central and corner regions. For example, only at most $3k^2$ rather than $4k^2$ queens need to be removed from the four corners (see Appendix A for a detailed analysis). Also, the construction could be done more generally for any board size, rather than just for boards of size $k^3$. Finally, it seems likely, although not proven, that $6n - O(n^{2/3})$ is also an upper bound for $\text{IR}(Q_n)$.

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Appendix A. Corner removals

Here we show that by using a more careful analysis we avoid “off-board” problems by removing fewer than $2k^2$ queens and private neighbours from each of the configurations $Q_U$ and $Q_D$.

Let $Q'_U$, $Z'_U$, $Q'_D$, and $Z'_D$, $0 \leq j < 2k - 1$, be the $j$th $B_4$ configuration for $Q_U$, $Z_U$, $Q_D$, and $Z_D$, respectively.
We first calculate the number of queens that need to be removed from the configuration $Q_U$. In the top left corner all the queens are on the board, but some of their private neighbours are off the board. Consider the queens in configuration $Q^0_U$. We determine which $B_3$ configurations have private neighbours in the configuration $Z^0_U$. Specifically, the $j$th $B_3$ queen configuration will have the $l$th $B_3$ as its private neighbour configuration if and only if $j = l - k + 3$, that is iff $j = 0, 1$ or 2. For these configurations, the corresponding private neighbour will be off the top of the board if and only if $5k/2 - k^2 + (j + k - 3)k < 0$, that is iff $j = 0$. The $k$ queens in this $B_3$ configuration are therefore removed.

The remaining queens in configuration $Q^0_U$, namely those with index $j = 3, 4, \ldots, k - 1$, have private neighbours in the configuration $Z^l_U$. Specifically, the $j$th queen $B_3$ configuration in $Q^0_U$ has private neighbours in the $(j - 3)$th private neighbour $B_3$ configuration in $Z^l_U$, which will be off the top of the board if and only $5k/2 - k^2 + k^2/2 + (j - 3)k < 0$, that is iff $j \leq k/2$. We must remove the $k(k/2 - 2)$ corresponding queens from $Q^0_U$. Thus a total of $k + k(k/2 - 2) = k(k/2 - 1)$ queens are removed from the top left corner.

In the bottom right corner, some queens in $Q^{2k-2}_U$ and $Q^{2k-1}_U$ are positioned off the board. Specifically, in $Q^{2k-2}_U$ the $B_3$ configuration with index $k - 1$ is off the board. This accounts for $k$ queens. In $Q^{2k-1}_U$ the $B_3$ configurations with indices $j \geq k/2 - 1$ are positioned off the board. These account for $k(k/2 + 1)$ queens.

However, for $k \geq 8$ there are still some queens on the board in $Q^{2k-1}_U$ without any private neighbours. The queens in $Q^{2k-1}_U$ with private neighbours in $Z^{2k-1}_U$ have $B_3$ indices $0, 1$ and $2$. The remaining on-board queens with $B_3$ indices $3, 4, \ldots, k/2 - 2$ have no private neighbours, and must therefore be removed. This accounts for $k(k/2 - 4)$ queens. Note that for $k = 6$ no queens are accounted for here as all the on-board queens have private neighbours in $Q^{2k-1}_U$. We shall include this $k(k/2 - 4)$ term in the remainder of the analysis, and adjust the formula for the case $k = 6$ at the end.

Thus a total of $k + k(k/2 + 1) + k(k/2 - 4) = k(k - 2)$ queens are removed from the bottom right corner, making a total of $k(k/2 - 1) + k(k - 2) = k(3k/2 - 3)$ queens to be removed from the configuration $Q_U$.

We now calculate the number of queens that need to be removed from the configuration $Q_D$. In the bottom left corner all the queens are on the board, but some of their private neighbours are off the board. These private neighbours occur in $Z^0_D$ and $Z^l_D$. The $j$th $B_3$ configuration in $Q^0_D$ has private neighbours in the $l$th $B_3$ configuration in $Z^l_D$ iff $-k^3 + k^2 - jk = 3k/2 - k^3 + 3k/2 - l$, that is, iff $j = k + l - 3$, which happens only for $j = k - 1, k - 2$ and $k - 3$. Amongst these values of $j$, the corresponding $B_3$ configuration in $Z^l_D$ lies off the bottom of the board iff $j = k - 1$. The $j$th $B_3$ configuration in $Q^0_D$ has private neighbours in the $l$th $B_3$ configuration in $Z^l_D$ iff $j = l - 3$, that is iff $j = 0, 1, 2, \ldots, k - 4$. Amongst these values of $j$ the corresponding $B_3$ configuration in $Z^l_D$ lies off the board iff $k^3 - 3k^2 + k^2/2 + (j + 3)k > k^3$, that is, iff $j \geq (k - 3)/2$. Thus the $B_3$ configurations in $Q^0_D$ with indices $j = k/2 - 1, k/2, \ldots, k - 4$ must be removed. This accounts for $k(k/2 - 2)$ queens. Thus a total of $k(k/2 - 1)$ queens are removed from the bottom left corner of $Q_D$. 


In the top right corner, no queens in configuration \( Q_D^{(2k-2)} \) will be positioned off the board. However, the queens in configuration \( Q_D^{(2k-1)} \) will be positioned off the board if they belong to a \( B_3 \) configuration with index \( j < k/2 \). This accounts for \( k^2/2 \) queens.

Now, all private neighbours in \( Z_D^{(2k-1)} \) will be positioned on the board. The queens in \( Q_D^{(2k-1)} \) have private neighbours in \( Z_D^{(2k-1)} \) iff their \( B_3 \) index is \( j = k - 3, k - 2 \) or \( k - 1 \). The on-board queens with \( B_3 \) configuration index \( j = k/2, k/2 + 1, \ldots, k - 4 \) must be removed. This accounts for \( k(k/2 - 3) \) queens. Thus a total of \( k(k - 3) \) queens are removed from the top right corner of the configuration \( Q_D \). This makes a total of \( k(3k/2 - 4) \) queens which are removed from configuration \( Q_D \).

Summing up, we can say that a total of \( k(3k/2 - 3) + k(3k/2 - 4) = 3k^2 - 7k \) (or \( 3k^2 - 6k \) for \( k = 6 \)) queens are removed from the corners of the the diagonal configurations.

References