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PHYSICS LETTERS B

Physics Letters B 624 (2005) 125–134

www.elsevier.com/locate/physletb

Heisenberg saturation of the Froissart bound from AdS-CFT

Kyungsik Kang, Horatiu Nastase

Brown University, Providence, RI 02912, USA

Received 11 April 2005; accepted 1 August 2005

Available online 8 August 2005

Editor: M. Cvetič

Abstract

In a previous paper, we have analyzed high energy QCD from AdS-CFT and proved the saturation of the Froissart bound (a purely QCD proof of which is still lacking). In this Letter we describe the calculation in more physical terms and map it to QCD language. We find a remarkable agreement with the 1952 Heisenberg description of the saturation (pre-QCD!) in terms of shockwave collisions of pion field distributions. It provides a direct map between gauge theory physics and the gravitational physics on the IR brane of the Randall–Sundrum model. Saturation occurs through black hole production on the IR brane, which is in QCD production of a nonlinear pion field soliton of a Born–Infeld action in the hadron collision, that decays into free pions.

1. Introduction

The Froissart bound [1] is a bound on the behaviour of the total cross section at high center of mass energies $s \to \infty$, with saturation of the type

$$\sigma_{\text{tot}} \sim \frac{A}{M^2} \ln^2 \frac{s}{s_0}, \tag{1.1}$$

where A is a numerical constant and M is the mass of the smallest excitation in the theory. In pure Yang–Mills, M would be the mass of the lightest glueball, $M_1 = \alpha \Lambda_{\text{QCD}}^{-1}$ (we could take by definition $\alpha = 1$, but we will keep it). If the lightest state is an almost Goldstone boson, like the pion of QCD, then $M = m_\pi$ and $A \leqslant \pi$, so that $A/M^2 \leqslant 60$ mb.

In QCD, experimentally, one first found the "soft pomeron" behaviour, $\sigma_{\text{tot}} \sim s^{0.09}$ at large energies [2] (cited in the 2001 PDG [3]) $\sqrt{s} \geqslant 9$ GeV, which was then argued to be replaced by a statistically better fit for the maximal Froissart behaviour (1.1) plus a reaction-dependent constant term in σ_{tot} [4] (cited in the 2004 PDG [5]), that fits all data above $\sqrt{s} = 5$ GeV, and with $A/M^2 = 0.32$ mb, far less than 60 mb. Theoretically, there is no good explanation for the expected saturation of the Froissart bound.[1]

E-mail address: nastase@het.brown.edu (H. Nastase).

[1] For an earlier attempt, based on l-plane analyticity, see [6].

0370-2693
doi:10.1016/j.physletb.2005.08.001

Paradoxically, Heisenberg has found in 1952 [7] a simple physical model that saturates the bound, long before the bound was proposed and even before QCD! It is a simple effective field theory model, but full of physical insight, as we shall see.

In [8] we have used AdS-CFT [9] to analyze the high energy behaviour of gauge theories in the large s, fixed t regime. We have found that the last energy regime corresponds indeed to the maximal Froissart behaviour. Our analysis was in the large N, large 't Hooft coupling $g_{\mathrm{YM}}^2 N$ regime, but we have shown that at $s \to \infty$, the corrections due to finite N and g_{YM} are negligible. Thus our proof applies to real QCD as well.

In this Letter, we will describe the calculation in physical terms, making use of the results in [8], to which we refer the reader for full details. While doing this, we will see a remarkable agreement with Heisenberg's description and learn how the bound is saturated in QCD, while also gaining insight into the dual gravity physics. We should note that our analysis only applied to the case where M_1 is lightest. If m_π is lightest, there is just a simple order-of-magnitude argument that the bound will be saturated, but is not an exact proof. This remains to be investigated in further work. For a possible modification of the Heisenberg model to take into account glueballs, see also [10].

We will first describe Heisenberg's calculation, then our AdS-CFT proof and then we will compare them.

2. Heisenberg model

Heisenberg's description starts with scattering of two hadrons of size $\sim 1/M_H$ in the center of mass frame. Lorentz contraction by the factor $1/\gamma = \sqrt{1-\beta^2}$ shrinks the size of the hadrons in the direction of motion, thus the two colliding hadrons look like pancakes, as in Fig. 1. That is not surprising, and one would say that if the impact parameter is $b > 1/M_H$, there would be no interaction.

Heisenberg says however that surrounding the hadron there is a pion field distribution (cloud of virtual pions), with radius $\sim 1/m_\pi$, also Lorentz contracted in the direction of motion, thus also looking like a pancake. And in the limit of $s \to \infty$, when the hadrons and the pion distributions will look like shockwaves (zero size in the direction of motion, thus delta function distributed), he argues that the hadron size becomes irrelevant (we could say that the hadrons "dissolve" into the pion field), and one has a collision of shockwaves of pion field distributions. Thus the details of the hadron become irrelevant, and only its ability to create pions is a relevant factor.

Then the collision of the pion field shockwaves is analyzed, and the energy radiated away in the collision is calculated. For a free massive scalar pion

$$(\Box - m_\pi^2)\phi = 0 \tag{2.1}$$

the energy radiated $\mathcal{E}$ as a function of the single meson energy E_0 (or its de Broglie frequency) is found to satisfy

$$\frac{d\mathcal{E}}{dE_0} = A = \text{const} \tag{2.2}$$

up to a maximum energy $E_{0,m} = \gamma m_\pi$, and then the number of emitted pions satisfies

$$\frac{dn}{dE_0} = \frac{A}{E_0}. \tag{2.3}$$

Since the pion energy has to be larger than m_π, we get

$$\mathcal{E} = A(E_{0,m} - m_\pi), \qquad n = A \ln\frac{E_{0,m}}{m_\pi} \tag{2.4}$$

which implies that the average energy of emitted mesons would increase as s (thus also $\gamma \simeq \sqrt{s}/M_H$) increases.

$$\langle E_0 \rangle \equiv \frac{\mathcal{E}}{n} \simeq \frac{E_{0,m}}{\ln\frac{E_{0,m}}{m_\pi}} = \gamma m_\pi \frac{1}{\ln\gamma}. \tag{2.5}$$

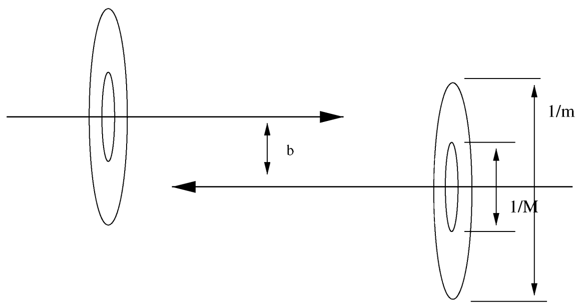

Fig. 1. Hadron scattering in the center of mass frame. M = hadron mass, m = pion mass. Also, A–S shockwave scattering on the IR brane. M = dual particle size. m = KK graviton mass (gravitational field in 5d, with given boundary conditions).

But this is clearly not satisfactory. In the limit of high energy scattering, we cannot treat the pion as free. It is clear that nonlinearities will flatten the linear growth of $\langle E_0\rangle$ with $\sqrt{s}$. Heisenberg then tried to do it with a simple $\lambda\phi^4$ interaction, but that clearly did not work, as it is now understood this is not a high energy correction. Now we know that the pion action is a simple $\lambda\phi^4$ theory, but in terms of isomultiplet states of an SU(2) matrix (the linear sigma model) $\Sigma = \sigma + \tau^a\phi^a$

$$\mathcal{L}_L = \frac{1}{4}\operatorname{tr}\partial_\mu\Sigma\partial^\mu\Sigma + \frac{\mu^2}{4}\operatorname{tr}\Sigma^+\Sigma - \frac{\lambda}{16}\big[\operatorname{tr}\big(\Sigma^+\Sigma\big)\big]^2 \tag{2.6}$$

in which

$$\Sigma = (v+s)U = (v+s)e^{i\tau^a\pi^a/v} \tag{2.7}$$

and at low energies we can integrate out the "absolute value", the field s, of Σ, and get the nonlinear sigma model

$$\mathcal{L}_{\rm NL} = \frac{v^2}{4}\operatorname{tr}\partial_\mu U\partial^\mu U^+ \tag{2.8}$$

that contains derivative interactions for the pions π^a. As this was before QCD was discovered, and before the pion was described as an isomultiplet of an SU(2) field, Heisenberg took the simplest model in terms of a single scalar field, with a remarkable intuition for the physics, as we see now!

Indeed, he took the Dirac–Born–Infeld (DBI)-like action for the scalar pion

$$S = l^{-4}\int d^4x\sqrt{1+l^4\big[(\partial_\mu\phi)^2 + m^2\phi^2\big]} \tag{2.9}$$

with l a length scale, and obtained

$$\frac{d\mathcal{E}}{dE_0} = \frac{A}{E_0} \Rightarrow \frac{dn}{dE_0} = \frac{A}{E_0^2} \tag{2.10}$$

for the same energy region, $m_\pi < E_0 < E_{0,m} = \gamma m_\pi$, and

$$\begin{aligned}
&\mathcal{E} = A\ln\frac{E_{0,m}}{m_\pi},\\
&n = \frac{A}{m_\pi}\left(1-\frac{m_\pi}{E_{0,m}}\right)\\
&\quad\Rightarrow \langle E_0\rangle \equiv \frac{\mathcal{E}}{n} = m_\pi\frac{\ln E_{0,m}/m_\pi}{1-m_\pi/E_{0,m}}\\
&\qquad\qquad = m_\pi\frac{\ln\gamma}{1-1/\gamma} \simeq m_\pi\ln\gamma
\end{aligned} \tag{2.11}$$

and now the average energy of the emitted meson is almost independent of $\gamma = \sqrt{s}/M_H$, and almost equal to m_π.

Finally, the last step in computing the cross section for the pion shockwave scattering is to postulate that the energy loss is proportional to the total energy, with a proportionality constant that is exponentially decreasing in the impact parameter

$$\mathcal{E} = \alpha\sqrt{s}, \quad \alpha = e^{-bm_\pi}. \tag{2.12}$$

This is motivated by the fact that the pion distribution has a transverse size $\sim 1/m_\pi$ and more precisely, the pion wavefunction is expected to decrease exponentially with the distance from the hadron, and it seems reasonable to assume that the "degree of inelasticity" coefficient α is proportional to the overlap of pion wavefunctions.

Then the cross section is found from $\sigma \simeq \pi b_{\rm max}^2$, where $b_{\rm max}$ is the maximum impact parameter for which we can still create pions, namely when the energy loss $\mathcal{E}$ is of the order of the average emitted pion energy $\langle E_0\rangle$. Then

$$\begin{aligned}
&e^{-b_{\rm max}m_\pi}\sqrt{s} = \langle E_0\rangle\\
&\quad\Rightarrow b_{\rm max} \simeq \frac{1}{m_\pi}\ln\frac{\sqrt{s}}{\langle E_0\rangle} \Rightarrow \sigma_{\rm tot} \simeq \frac{\pi}{m_\pi^2}\ln^2\frac{\sqrt{s}}{\langle E_0\rangle}.
\end{aligned} \tag{2.13}$$

Since $\langle E_0\rangle$ is almost independent of s, we get the maximal Froissart behaviour. But note that this behaviour was obtained only because $\langle E_0\rangle$ was independent of s, which came from the DBI-like action for the scalar pion. If we had a free pion or a $\lambda\phi^4$ theory, we would not get it (we would get a constant $\sigma_{\rm tot}$). One would think that the minimum energy emitted $\mathcal{E}$ is m_π anyway, but the correct answer is the average pion energy, which for a free pion would grow linearly with $\sqrt{s}$, and we need the higher derivative, DBI-like action to get $\langle E_0\rangle \sim m_\pi$.

We should mention here that Heisenberg treats also the case of several "pion" varieties, and finds that the lightest variety dominates the high energy behaviour. Also it is clear that the model applies equally well to the case of pure gauge theory, when the lightest variety of "pion" is actually the lightest glueball. This is in fact the case that we have studied in detail in AdS-CFT. But we will be a bit cavalier in the Heisenberg description and talk about pions and lightest glueballs interchangeably.

3. Dual saturation of the bound

Let us now describe our gravity dual (AdS-CFT) version of the saturation in [8]. Polchinski and Strassler [11] (see also [12]) have shown that at high energies in a gauge theory, one can describe the scattering of colourless states by scattering in a very simple model of gravity dual. One takes the $AdS_5 \times X_5$ gravity dual to a conformal theory

$$\begin{aligned} ds^2 &= \frac{r^2}{R^2} d\vec{x}^2 + \frac{R^2}{r^2} dr^2 + R^2\, ds_X^2 \\ &= e^{-2y/R}\, d\vec{x}^2 + dy^2 + R^2\, ds_X^2 \end{aligned} \tag{3.1}$$

and cuts the warp factor $e^{-2y/R}$ off in the IR, at a $r_{\min} \sim R^2 \Lambda_{\rm QCD}$ (equivalently, $y_{\max}$), which hides our ignorance of what happens in the IR (for the gravity dual at small r), that leads the theory to become nonconformal. This simple model is enough to obtain many features of the nonconformal gauge theory.

Scattering in the gauge theory of a mode with momentum p and wavefunction e^{ipx} corresponds to scattering in AdS of a mode with local momentum $\tilde{p}_\mu = (R/r) p_\mu$ and wavefunction $e^{i\tilde{p}x}\psi(r,\Omega)$, with Ω coordinates on X. Then the amplitudes in gauge theory are related to amplitudes in AdS by

$$\mathcal{A}_{\rm gauge}(p) = \int dr\, d^5\Omega \sqrt{g} \mathcal{A}_{\rm string}(\tilde{p}) \prod_i \psi_i. \tag{3.2}$$

High energy scattering in AdS can be defined relative to the string tension $\alpha' = R^2/(g_s N)^{1/2}$, and in the gauge theory by the gauge theory string tension $\hat{\alpha}' = \Lambda_{\rm QCD}^{-2}/(g_{\rm YM}^2 N)^{1/2}$ (with $g_s = g_{\rm YM}^2$). The two are related by $\sqrt{\alpha'}\tilde{p}_{\rm string} \leqslant \sqrt{\hat{\alpha}'} p_{\rm QCD}$. We can see that the AdS scale R corresponds in gauge theory to $\Lambda_{\rm QCD}^{-1}$.

Giddings then noticed that one will start producing black holes when one reaches the Planck scale $M_P = g_s^{-1/4}\alpha'^{-1/2}$ in AdS, and correspondingly $\hat{M}_P = N^{1/4}\Lambda_{\rm QCD}$ in the gauge theory [13]. Since the black hole horizon radius in D dimensions grows with energy as $r_H \sim E^{1/(D-3)}$, the simplest model for the cross section for black hole formation, a black disk with radius $r_H(E = \sqrt{s})$ (all the collision energy is taken as mass of the formed black hole), gives

$$\sigma \sim \pi r_H^2 \sim E^{\frac{2}{D-3}} \tag{3.3}$$

which means (for a 10-dimensional gravity dual), $\sigma \sim s^{1/7}$.

When the black hole size r_H reaches the AdS size R, we have

$$E \sim M_P^8 \sim M_P^8 R^7 \to E = E_R = M_P (R M_P)^7 \tag{3.4}$$

and the corresponding gauge theory energy scale is $\tilde{E}_{R,{\rm QCD}} = N^2 \Lambda_{\rm QCD}$. This is the maximal behaviour one can have, so it should correspond to the Froissart behaviour. How do we see that?

The cut-off AdS is just the 2-brane Randall–Sundrum model [14], if we cut-off also in the UV (unnecessary, but does not change physics). If we put a point mass of $m = \sqrt{s}$ on the IR brane, we get for the linearized metric perturbation

$$h_{00,{\rm lin}} \sim G_4 \sqrt{s} \frac{e^{-M_1 r}}{r}, \qquad G_4^{-1} = M_P^3 R, \tag{3.5}$$

where $M_1 = j_{1,1}/R$ is the mass of the lightest KK mode ($j_{1,1}$ is the first zero of the Bessel function J_1). If we consider the position of the horizon of the formed black hole to be roughly when $h_{00,{\rm lin}} \sim 1$, we get

$$\begin{aligned} r_H &\sim \frac{1}{M_1} \ln\bigl(G_4 \sqrt{s} M_1\bigr) \\ \Rightarrow \sigma &\sim \pi r_H^2 \sim \frac{\pi}{M_1^2} \ln^2\bigl(\sqrt{s}\, G_4 M_1\bigr) \end{aligned} \tag{3.6}$$

and if $\sigma_{\rm QCD} = \sigma$ we obtain the maximal Froissart behaviour, with mass scale given by the lightest KK mode, of the order of $R \leftrightarrow \Lambda_{\rm QCD}^{-1}$, and corresponds to the lightest pure gauge theory excitation = glueball. Note that now indeed $r_H \sim 1/M_1 \sim R$, as argued.

The case when there is also an almost Goldston boson (like the pion) of mass lighter than that of the lightest glueball can also be modelled by making the radion of the Randall–Sundrum model (the distance between the UV and IR branes) dynamical, and giving it a mass M_L by a Goldberger–Wise stabilization [15] or flux stabilization for gravity duals of the Polchinski–Strassler [16] type. Then, if $M_L < M_1$, the brane will bend under the mass $m = \sqrt{s}$ on the IR brane, and the linearized radion change (bending) will be

$$\left.\frac{\delta L}{L}\right|_{\rm lin} \sim G_4 \sqrt{s} (M_L R) \frac{e^{-M_1 r}}{r} \tag{3.7}$$

and the maximal Froissart behaviour in gauge theory is obtained when the bending in the gravity dual becomes of order 1,

$$\frac{\delta L}{L}\bigg|_{\text{lin}} \sim 1 \Rightarrow \sigma_{\text{QCD}} = \sigma \sim \frac{\pi}{M_L^2}\ln^2\left(\sqrt{s}\, G_4 M_L R\right). \tag{3.8}$$

But there were a number of open questions that we set out to analyze: Why taking a point mass on the IR brane works? We would like a dynamical statement, involving scattering in AdS. Why does $\sigma_{\text{BH}} \sim \pi r_H^2$ give a good estimate and why $\sigma = \sigma_{\text{QCD}}$? Why does $h_{00,\text{lin}} \sim 1$ (or $\delta L/L|_{\text{lin}} \sim 1$) give a good estimate of the horizon (maximum impact parameter) size? And do string corrections, corresponding to finite N and $g_{\text{YM}}^2 N$ corrections in gauge theory, modify the results? In answering these questions, we have found a model [8] that looks remarkably like Heisenberg's.

We want to study the scattering in AdS of two almost massless modes, at high energy. The model for this was due to 't Hooft [17] and it involves the observation that at energies close to the Planck scale (but not above it), the massless particles are described by geometry and gravity alone. One particle produces a gravitational shockwave, specifically the Aichelburg–Sexl [18] solution for flat 4d, of the type

$$ds^2 = 2\,dx^+\,dx^- + \left(dx^+\right)^2 \Phi\left(x^i\right)\delta\left(x^+\right) + d\vec{x}^2. \tag{3.9}$$

The function Φ satisfies the Poisson equation

$$\Delta_{D-2}\Phi\left(x^i\right) = -16\pi G p \delta^{D-2}\left(x^i\right). \tag{3.10}$$

The second particle is just a null geodesic scattering in this metric, and 't Hooft showed that this scattering matches Rutherford scattering due to single graviton exchange. He also suggested that at energies above M_P both particles should be represented by shockwaves. The reason why only gravitational interactions are relevant is because interactions due to massive fields are finite range, and at small $r = \sqrt{x^i x^i}$ the function Φ diverges, thus creating an infinite time delay for the massive interactions.

In [19] the question of putting A–S type shockwaves inside a general warped compactification and other manifolds was analyzed. In all cases of interest, one adds a shockwave term to the background metric of the type $(dx^+)^2\Phi(x^i)\delta(x^+)$, where Φ still satisfies the Poisson equation (3.10) in the background, and where Δ_{D-2} is the Laplacean in the background, at $\partial_{x^+} = \partial_{x^-} = 0$ (independent of light-cone coordinates). Note that therefore this is not just the Laplacean for an AdS space of lower dimension, since it contains the dimension explicitly (as a term $d\partial_y$).

By comparison, the procedure of calculating the linearized metric perturbation $h_{00,\text{lin}}$ due to a static point mass, uses the Poisson equation with Δ_{D-1}, the Laplacean at $\partial_t = 0$ (static solution), but then there is no a priori reason to expect that the full black hole solution still has the same features. By contrast, for A–S shockwaves in the spaces of interest (warped compactifications and cut-off AdS), we saw that the linearized solution is the exact solution! For an early attempt at using shockwaves in AdS (similar to ours) for AdS-CFT, see [20], and later [21] used shockwave arguments to argue for the cross section for black hole creation. The linearization phenomenon for shockwaves in AdS and on braneworlds was observed in [22].

So we have two A–S shockwaves in the gravity dual background scattering, one travelling in the x^+ direction, one in the x^- direction, at impact parameter b, but they must create a black hole, which is a highly nonlinear, uncalculable process. Fortunately, following earlier calculations in flat $d = 4$ in [23], generalized by us to curved higher-dimensional space in [24], we can find when a "trapped surface" forms at the interaction point $x^+ = x^- = 0$, and by a GR theorem we know that there will be a horizon forming outside it, thus a black hole, for which we have a lower bound on the mass.

The last piece of information needed is to turn this classical scattering process into a quantum amplitude that we can put into the Polchinski–Strassler formalism. For 't Hooft scattering, one calculated an amplitude for the scattering using an eikonal formalism, finding that $S = e^{i\delta}$ (where δ is the eikonal) is $= e^{ip^+\Phi}$, with p^+ the momentum of the second photon, interacting with the A–S solution. Now, we use also an eikonal form for the quantum amplitude, with the eikonal being the simplest thing one can have, a black disk:

$$\begin{aligned}
&\text{Re}\big(\delta(b,s)\big) = 0, \qquad \text{Im}\big(\delta(b,s)\big) = 0, \quad b > b_{\max}(s),\\
&\text{Im}\big(\delta(b,s)\big) = \infty, \quad b < b_{\max}(s),
\end{aligned} \tag{3.11}$$

where $b_{\max}(s)$ is what we find from the classical A–S scattering. Then the imaginary part of the forward amplitude reproduces the classical $\pi b_{\max}^2(s)$ result for σ_{tot}, but now we have a quantum $2 \to 2$ amplitude that we can put in the Polchinski–Strassler formula (3.2).

When we plug in several relevant forms for $b_{\max}(s)$ like as^{β} and $a\ln s$, we find that when we translate to gauge amplitudes, all we get is we multiply σ_{string} with a model-dependent constant, and we modify the subleading behaviour. But more importantly, we find that most of the extra dimension (r) integration in the gauge theory amplitude is concentrated near the IR brane, for these high energy behaviours.

We have seen that in the simple Giddings description [13], Froissart behaviour should come in when the black hole size reaches the AdS size. But if most of the gauge theory amplitude comes from near the IR brane, the scattering will look as if it happens on the IR brane itself (due to its large size, the formed black hole will not "see" that is outside the IR brane). The A–S shockwave solution living on the IR brane is found to be

$$\begin{aligned}\Phi(r,y) &= \frac{4G_{d+1}p}{2\pi}e^{-\frac{d|y|}{2R}} \\ &\quad\times\int\frac{d^{d-2}\vec{q}}{(2\pi)^{d-2}}e^{i\vec{q}\vec{x}}\frac{I_{d/2}(e^{-|y|/R}Rq)}{qI_{d/2-1}(Rq)} \\ &= \frac{4G_{d+1}p}{(2\pi)^{\frac{d-4}{2}}}\frac{e^{-\frac{d|y|}{2R}}}{r^{\frac{d-4}{2}}} \\ &\quad\times\int_0^\infty dq\,q^{\frac{d-4}{2}}J_{\frac{d-4}{2}}(qr)\frac{I_{d/2}(e^{-|y|/R}Rq)}{I_{d/2-1}(Rq)}.\end{aligned}\tag{3.12}$$

This solution is found by imposing normalizability at $y=\pm\infty$ (on the UV brane), and matching conditions (periodicity) at the IR brane for the Poisson equation solution. Therefore it can be defined as the first KK mode for the effective theory of massless modes on the IR brane. At large r it becomes

$$\begin{aligned}&\Phi(r,y=0)\simeq R_s\sqrt{\frac{2\pi R}{r}}C_1e^{-M_1r}, \\ &C_1=\frac{j_{1,1}^{-1/2}J_2(j_{1,1})}{a_{1,1}}, \\ &J_1(z)\sim a_{1,1}(z-j_{1,1}),\quad z\to j_{1,1}\end{aligned}\tag{3.13}$$

and we can see the same exponential drop as in $h_{00,\text{lin}}$, only the power or r is different, due to the fact that we have a solution of Δ_{D-2} (massless perturbation), as opposed to Δ_{D-1} (static massive perturbation).

If we take two A–S waves on the IR brane scattering at $b=0$, the condition that determines the shape and size of the trapped surface is

$$(\nabla\Psi)^2+e^{2|y|/R}(\partial_y\Psi)^2=4,\quad \Psi=\Phi+\zeta,\tag{3.14}$$

where ζ is defined perturbatively in y by the condition that the above matches also $\Psi=C=\text{const}$ (for full details see [24] and [8]).

One finds that the condition for the trapped surface size r at $y=0$ is

$$\frac{3r}{2R^2}\Phi(r,y=0)=1\tag{3.15}$$

which we see that is similar to the approximate condition for the horizon r_H that [13] had, namely $h_{00,\text{lin}}\sim 1$ (but the power of r and the constants are different).

But one can do better, one can find an approximate condition for the trapped surface at nonzero b,

$$\left(\frac{3r}{2R^2}\Phi(r,y=0)\right)^2\left(1-\frac{b^2}{2r^2}\right)=1\tag{3.16}$$

which gives a maximum b that satisfies it that is approximately

$$\begin{aligned}&b_{\max}(s)=\frac{\sqrt{2}}{M_1}\ln[R_sM_1K], \\ &K=\frac{3\sqrt{\pi}}{\sqrt{2}j_{1,1}^{3/2}}\simeq 0.501\end{aligned}\tag{3.17}$$

and $R_s=G_4\sqrt{s}$, $G_4=1/(RM_{P,5}^3)$, $M_1=j_{1,1}/R$ ($j_{1,1}\simeq 3.83$).

As we mentioned, then the gauge theory cross section is (via the Polchinski–Strassler formalism)

$$\sigma_{\text{tot}}=\bar{K}\pi b_{\max}^2(\tilde{s})\tag{3.18}$$

with $\bar{K}$ a model dependent constant, $\tilde{s}=s\hat{\alpha}'/\alpha'$, or equivalently by keeping s fixed and replacing R by $\Lambda_{\text{QCD}}^{-1}$ and $M_{P,5}$ by $N^{1/4}\Lambda_{\text{QCD}}$ (gauge theory quantities), and we get the expected Froissart behaviour.

Up to now we have discussed strictly speaking the usual AdS-CFT limit, of large N and large g_{YM}^2N, since this corresponds to small α' and g_s string corrections in the gravity dual. But in [24] we have also analyzed string corrections using a model by Amati and Klimcik [25]. They obtained a string-corrected A–S wave by matching the 't Hooft scattering of a superstring in an arbitrary shockwave profile Φ, $S=e^{ip^+\Phi}$, with a resummed eikonal superstring calculation $S=e^{i\delta}$ [26], and finding the Φ that equates the two results. Then

$$\Phi(y) = -q^v \int_0^\pi \frac{4}{s} : a_{\text{tree}}\big(s, y - X^d(\sigma_d, 0)\big) : \frac{d\sigma_d}{\pi}, \tag{3.19}$$

where $2p_v q^v = -s$ and $b = x^u - x^d$ is the variable y. Note that this corresponds both to α' corrections, given by a_{tree}, and to g_s corrections, since the eikonal form $e^{i\delta}$ resummed "ladder diagrams", which are predominant at $s \to \infty$, $e^{i\delta} = \sum_h (i\delta)^h/h! \sim \sum_h (g_s)^h (a_{\text{tree}})^h/h!$ ($h =$ loop number). We have found that by scattering two of these modified A–S waves the effect on $b_{\max}$, now called $B_{\max}$ to avoid confusion with $b \sim y$ is

$$B_{\max} = \frac{R_s}{\sqrt{2}}\big(1 + e^{-\frac{R_s^2}{8\alpha' \log(\alpha' s)}}\big) \tag{3.20}$$

in the regime where the exponent is large (in absolute value). Thus at $s \to \infty$, the string corrections to black hole production are exponentially small! Although this result was obtained for flat 4d, it is not hard to imagine that it will remain true in the warped compactification case, for scattering on the IR brane. And as string corrections are mapped to large N, large 't Hooft coupling corrections, we can say that the Froissart behaviour will also apply for the case of real QCD (small N, small 't Hooft coupling)!

Note that we are talking here about string corrections to the scattering itself, but of course there will be corrections to the gravity dual background. Since however our model was so general (no model really, just cut off AdS), we can confidently say that all that can happen is for the parameters of the theory, the AdS size R and M_P, to get renormalized. That would translate into gauge theory to a modification of the energy scales Λ_{QCD}, $\hat{M}_P$ and $\hat{E}_R$, the scale of the onset of Froissart behaviour.

4. Comparison to QCD and Heisenberg

So then we can ask the question how does our calculation translate into QCD and Heisenberg language?

We know since shortly after the Randall–Sundrum model was proposed [14,27] that we can understand it as just AdS-CFT when gravity is not decoupled from the 4d physics [28]. As usual in the AdS-CFT correspondence, KK modes of the graviton in 5d correspond to glueball excitation of the gauge theory. One can understand RG flow of the gauge theory (scale transformations) as just motion in the 5th direction in the gravity dual (as if we move a physical brane in the 5th direction). On the other hand, bulk gravity can be reduced to usual gravity + KK modes on the IR brane, coupled to the Standard Model that might live there. So on the IR brane, there is a duality between the gauge physics of glueball states and the gravity physics of KK modes, as both have their origin in bulk gravity.

With Polchinski–Strassler, we have made concrete the AdS-CFT duality part, with the gauge theory living on a 4d brane, not necessarily the IR brane. Scattering in the gauge brane corresponds to scattering in AdS, which itself can be reduced in certain cases to scattering in an effective field theory on the IR brane.

Indeed, we have found that in the Froissart regime most of the AdS scattering happens near the IR brane, and we can effectively describe it as scattering on the IR brane. The A–S shockwaves have profiles that correspond to KK gravitons (solutions to the Poisson equation with certain boundary conditions on the UV and IR branes).

From 't Hooft we know that corrections at large s due to massive modes are negligible, and only gravity is relevant. We represent the scattering of dual particles just by scattering of gravitational A–S shockwaves. Thus the same Fig. 1 can be used to describe the dual picture as well! We have shockwaves (limits of pancake-like distributions) colliding, and the particles dual to the hadrons "dissolve" into the KK graviton field. Indeed, as we said, from the 4d point of view, the wave profile Φ corresponds to the first KK graviton mode. Moreover, the A–S shockwaves can be actually found by boosting black holes to the speed of light (and keeping their energy fixed), see [18] and [19], the same way we boost hadrons.

The first KK graviton mode corresponds to the lightest glueball, which as we argued should replace the pion in the Heisenberg analysis for the case where the pion is heavier, so we have a direct correspondence. In Heisenberg's analysis, the "degree of inelasticity" was postulated to be e^{-bm_π}, based on the fact that the pion wavefunction around the hadron should go like e^{-rm_π}. Now we have a form for the KK graviton wavefunction (3.13), and a dynamical mechanism to check for the creation of the black hole, as given in (3.16).

So what happens in QCD when a black hole forms in the dual description? Pretty much the same, in a certain sense. As Heisenberg describes, if we had a free massive pion, the shockwave collision will produce in the interaction regime (after the collision) just free spherical waves (like two small plane water waves colliding in a pond). This would correspond on the dual side to having a free graviton. Clearly, nothing would happen there, and as Heisenberg described, in that case $\langle E_0\rangle \sim \sqrt{s}$ and thus a constant $\sigma_{\rm tot}$. We need to take instead a nonlinear pion as in the DBI-like action (2.9). Then as we said $\langle E_0\rangle$ is almost independent of s and we find the maximal Froissart behaviour. Heisenberg actually calculates perturbatively the pion field in the interaction region, but it cannot be calculated everywhere. One should find the equivalent of the black hole formation for the scalar pion, i.e., a highly singular nonlinear structure. For A–S collision in flat 4d, the metric after the interaction was also found perturbatively in [29], but one cannot draw any conclusions from it about the nonperturbative solution. Luckily, we had the trapped surface formalism that relied on the metric at the interaction point.

So a dual black hole is in QCD a highly nonlinear soliton being formed in the collision of two pion field distributions, and that further decays into free perturbative waves (radiated pions). Heisenberg could not calculate the soliton form, but he could calculate the average energy of the pions emitted in the decay of the soliton, finding as we saw $\langle E_0\rangle \simeq m_\pi \ln\gamma$. That, coupled with the assumption of "degree of inelasticity" mirroring the pion field distribution around the hadron, was enough to derive the maximal Froissart behaviour.

In our case, we have the wavefunction profile Φ in (3.13) that shows exponential decay, and then we use the full 5d GR rules to calculate whether a black hole (pion field soliton) forms and decays, as in (3.16), so we do not need to calculate the average energy of gravitons emitted in the decay of the black hole and postulate a "degree of inelasticity" for the energy loss, but the idea is the same! Presumably, there should be an effective purely 4d description of the KK graviton mode Φ of mass M_1 that plays the role of the DBI action (2.9) for Heisenberg. It is tempting to assume that it is exactly the same action.

In fact, we have been talking about pions until now, but as we mentioned we should really speak about lightest glueballs, as our analysis was done only for that case. But the simple analysis in [13] that we mentioned before applies also for an almost Goldston boson like the radion of Randall–Sundrum (3.7), (3.8). The remarkable thing is that the action for the massless radion (position of the IR brane) is the DBI action! Indeed, we can easily check that for a codimension one brane, in the straight gauge X^μ: $(X^a=\xi^a, X)$ (where ξ^a are worldvolume coordinates and $X=l_s^2\phi$ is the radion = 5th coordinate, and we did the rescaling to a canonical dimension 4d field ϕ), the action is

$$\mathcal{L}=l_s^{-4}\sqrt{\det\bigl(\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}\bigr)}=l_s^{-4}\sqrt{g}\sqrt{1+(\partial_a X)^2}$$
$$=l_s^{-4}\sqrt{g}\sqrt{1+l_s^4(\partial_a\phi)^2}, \tag{4.1}$$

where the contraction in the square root and the metric g on the right-hand side are done with the 4d reduced metric g_{ab}. And Heisenberg's suggestion that the pion mass be put inside the square root as in (2.9) is suggestive that maybe one should try to do the same in radion stabilization mechanisms. We are not aware whether this was considered in the literature, but since as Heisenberg points out this is relevant for getting the right nonlinear behaviour in the QCD side, one should perhaps consider it as a nonlinear extension of the Goldberger–Wise stabilization [15].

Also, we have expressed the hope that the KK graviton and black hole creation can be described by a DBI-like action, maybe exactly (2.9) for the wavefunction Φ, but it is not clear that one can. After all, we made use of the full GR at least to deduce from the appearance of a trapped surface that a horizon will form outside it, and probably the whole nonlinear and tensor nature of gravity is needed. But one could maybe look for a 4d effective action for the massive KK graviton $g^{(1)}_{\mu\nu}$. Born–Infeld-type actions for gravity have been considered before. For instance, [30] analyzes a more general action, of the type

$$\mathcal{L}=\sqrt{-\det(g_{\mu\nu}+aR_{\mu\nu}+bX_{\mu\nu})}, \tag{4.2}$$

where $X_{\mu\nu}$ is an expression that can be quadratic or higher in curvatures and can be used to put almost any action in this DBI form. More to the point, the action for $b=0$, $a=1$ (true Born–Infeld), when written in first order form (see, for instance, [31])

$$\int \sqrt{\det\bigl(R^{ab}(\omega) + l^{-2}e^a e^b\bigr)} \tag{4.3}$$

can be rewritten as a Lanczos–Lovelock action, is the equivalent of the odd-dimensional Chern–Simons action for gravity (as a gauge theory of the Poincaré group), and in fact can be obtained by dimensional reduction from it, in any dimension [31]. In 4d, it is rewritten as (the contraction of local gauge indices is done with ϵ^{abcd})

$$R \wedge R + 2l^{-2} R \wedge e \wedge e + l^{-4} e \wedge e \wedge e \wedge e, \tag{4.4}$$

where the first term is topological, the second and third are Einstein–Hilbert and cosmological constant terms. Thus this action is the usual Einstein–AdS theory with a topological term, so regular 4d gravity is already a Born–Infeld type action.

Perhaps also the hoped-for effective action for the KK graviton looks like a BI action, something like

$$\sqrt{\det\bigl(g_{\mu\nu} + R_{\mu\nu}\bigl(g^{(1)}_{\rho\sigma}\bigr) + X_{\mu\nu}\bigl(g^{(1)}_{\rho\sigma}\bigr)\bigr)}, \tag{4.5}$$

where $g_{\mu\nu}$ is the massless 4d graviton and $g^{(1)}_{\rho\sigma}$ is the massive KK graviton, with $R_{\mu\nu}$ bilinear in $g^{(1)}_{\rho\sigma}$ (and the rest of metric fields are $g_{\mu\nu}$) and $X_{\mu\nu}$ a mass term also bilinear in it.

So we used Heisenberg's description to make some conjectures about the dual gravity theory (for a radion and KK graviton effective action), and from the gravity dual calculation we have found a precise description of the mechanism for Froissart saturation. But this mechanism is in terms of the effective field theory of pions and lightest glueball states. It is clear that this is the only thing that we can learn from AdS-CFT, as AdS-CFT deals with gauge invariant quantities. However, maybe this precise description can be used to learn about a QCD proof of the saturation too. Finally, it would be nice to have a precise dual description for the case the pion is the lightest field too.

In conclusion, one can only be amazed by Heisenberg's physical insight, well ahead of his time. His effective field theory description matches exactly the gravity dual description of the saturation. His use of the DBI action for the pion describes the radion action and maybe the KK graviton. The DBI action seems to be in the same universality class as the action for the real (SU(2)) pions, generating the same physics.

Acknowledgements

We would like to thank Micha Berkooz for discussions. This research was supported in part by DOE grant DE-FE0291ER40688-Task A.

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