Functional $T$-observers
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ABSTRACT

This article is a contribution to behavioral observer theory which was started by Valcher and Willems in 1999 and which was recently exposed by Fuhrmann in a comprehensive survey article. It is also a further development of the article on $T$-observers by Oberst and the author. For a given continuous or discrete time linear time-invariant behavior we assume that a linear function of a trajectory (e.g., some components) can be measured. We are interested in estimating another linear function of this trajectory.

We generalize the notions of $T$-observability and $T$-observers introduced by Oberst and the author. $T$ denotes a multiplicatively closed subset of the ring of operators. For different choices of $T$, $T$-observability coincides with observability, reconstructibility, trackability, or detectability, a $T$-observer is an exact, dead-beat, tracking, or asymptotic observer. We show the equivalence of $T$-observability and the existence of $T$-observers and give a constructive parametrization of all $T$-observers. Corresponding results for proper $T$-observers are also presented.

Partial observation of the state of a Kalman state space system (compare e.g. Fuhrmann’s work) is a special case of our setting, and so are the observers of certain unknown components of a behavior studied by Bisiacco, Valcher, and Willems. The first result on functional observers in context with Rosenbrock equations or polynomial matrix descriptions is due to Wolovich (1974).

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0. Introduction

This paper extends and simplifies our article [2], primarily by the consideration of functional or partial observers which have also been studied in special cases in the literature, in particular as reduced...
order observers already by Luenberger, compare [5, 13, Theorem 7.3.23, 3, Sections 3 and 5]. The behavioral observers discussed by Valcher et al. [11,10,1] and Fuhrmann [3, Section 4] also fit into this setting.

Our aim in detail is the following: For a given (discrete or continuous) linear behavior \( B \), we assume that we can measure some image \( P \circ w \) of a trajectory \( w \in B \). This image could for example consist of some components of \( w \). We are interested in estimating the signal \( w \) itself, or, more generally, another image \( Q \circ w \) (e.g. some other components, or a linear combination of certain components). Observers that estimate not the complete trajectory but just a function thereof are sometimes referred to as functional or partial observers. One particular case that has received considerable attention in the past is the estimation of a function \( Kx \) of the state \( x \) of a Kalman state space system, compare for example Fuhrmann’s introduction [3, pp. 44–46]. Our setting also includes the situation of such a system that may even be subject to unknown inputs. We introduce the term of Fuhrmann’s introduction [3, pp. 44–46]. Our setting also includes the situation of such a system that

\[ Q \circ B \]\n
of the ring of operators as in [2].

The equivalence of the existence of a \( T \)-observability signifies that, whenever two measured signals \( P \circ w \) and \( P \circ \tilde{w} \) are equal, the difference between the corresponding signals \( Q \circ w \) and \( Q \circ \tilde{w} \) (that we want to estimate) is \( T \)-small, i.e., negligible if \( T \) has been chosen appropriately. For different choices of the set \( T \), \( T \)-observability coincides with observability, reconstructibility, trackability, or detectability (compare e.g. [11, Definition 2.1, 3, pp. 62, 104, 105]).

We will characterize the property of \( T \)-observability and then define \( T \)-observers, i.e., input/output behaviors such that, when the measured signal \( P \circ w \) is used as input, the output is an estimation for \( Q \circ w \). Again, the use of the set \( T \) in the definition allows the simultaneous treatment of exact, dead-beat, tracking, asymptotic, and other observers (cf. [11, Definition 3.1, 3, pp. 77, 109, 110]).

The contents of the paper are the following: in Section 1 we will introduce the localized signal module \( D_T \), and the localized \( D_T \)-behaviors \( B_T \). The results established here will be the basis for the techniques applied in Section 2 where the problems described above will be treated and solved. Section 3 consists of the algorithms necessary in order to make the theory constructive and of an example.

1. Preparations

We will use the one-dimensional behavior theory summarized in the introduction of [2]. In this section some further results and their proofs taken from the book [6] will be listed.

Let \( D \) denote the polynomial ring \( F[s] \) over some arbitrary field \( F \), and let \( F \) be an injective cogenerator over \( D \). We repeat the injectivity and cogenerator properties in the following two results:

\[ \text{Result 1.1.} \text{ The following properties of a module } D_F \text{ over the principal ideal domain } D \text{ are equivalent:} \]

\[ 1. F \text{ is injective, i.e., } \text{Hom}_D(-, F) \text{ is an exact functor.} \]
\[ 2. F \text{ is divisible over } D, \text{ i.e., each equation } f \circ y = u, 0 \neq f \in D, \text{ has a solution } y \in F \text{ for given right side } u \in F. \]
\[ 3. \text{Every } D \text{-monomorphism } f : F \to M \text{ splits, i.e., there is a (non-unique) submodule } M' \text{ with } M = M' \oplus f(F). \]

\[ \text{Proof [4, Chapter 1, Section 3].} \]

\[ \text{Result 1.2.} \text{ For an injective } D \text{-module } F \text{ the following properties are equivalent:} \]

\[ 1. F \text{ is a cogenerator, i.e., } \text{Hom}_D(-, F) \text{ is a faithful functor.} \]
\[ 2. \text{If } M \text{ is nonzero, then so is } \text{Hom}_D(M, F). \]
3. \( \mathcal{F} \) contains each simple \( \mathcal{D} \)-module up to isomorphism. Recall that, up to isomorphism, a simple \( \mathcal{D} \)-module is of the form \( \mathcal{D}/\mathcal{D}q \) where \( q \) is a prime or irreducible polynomial.

4. For any two behaviors \( B_1 = \{ w \in \mathcal{F}; R_1 \circ w = 0 \}, R_i \in \mathcal{D}^{k_i \times \ell} \), the following equivalence holds:

\[
B_1 \subseteq B_2 \iff \mathcal{D}^{1 \times k_2}R_2 \subseteq \mathcal{D}^{1 \times k_1}R_1 \iff \exists X \in \mathcal{D}^{k_2 \times k_1} : R_2 = XR_1.
\]

**Proof** [4, Section 19A, 7, Corollary 2.47]. □

Furthermore, let \( T \) be a multiplicatively closed subset of \( \mathcal{D} \setminus \{0\} \). Without loss of generality we assume that \( T \) is saturated. The set \( T \) gives rise to the quotient ring \( \mathcal{D}_T \) and to the quotient module \( \mathcal{F}_T \):

\[
\mathcal{D}_T = \left\{ \frac{f}{t} \in F(s); f \in \mathcal{D}, t \in T \right\} \subseteq F(s),
\]

\[
\mathcal{F}_T = \left\{ \frac{w}{t}; w \in \mathcal{F}, t \in T \right\},
\]

where

\[
\frac{w_1}{t_1} = \frac{w_2}{t_2} \in \mathcal{F}_T : \iff \exists t \in T : t \circ (t_2 \circ w_1 - t_1 \circ w_2) = 0.
\]

Our first aim is to establish the direct sum decomposition \( \mathcal{F} = \mathcal{F}_T \oplus t_1(\mathcal{F}) \) where \( t_1(\mathcal{F}) := \{ w \in \mathcal{F}; \exists t \in T : t \circ w = 0 \} \) denotes the \( T \)-torsion submodule of \( \mathcal{F} \).

**Lemma 1.3.** The torsion module

\[
t(\mathcal{F}) = \{ w \in \mathcal{F}; \exists d \in \mathcal{D} \setminus \{0\} : d \circ w = 0 \}
\]

is injective over \( \mathcal{D} \).

**Proof.** We show that \( t(\mathcal{F}) \) is divisible over \( \mathcal{D} \), i.e., that any equation \( d \circ w = v \) with given \( d \in \mathcal{D} \setminus \{0\} \) and \( v \in t(\mathcal{F}) \) has a solution \( w \in t(\mathcal{F}) \): Since \( \mathcal{F} \) is injective over \( \mathcal{D} \) we know that such a solution \( w \) exists in \( \mathcal{F} \). Now we use the fact that \( v \in t(\mathcal{F}), \) i.e., there exists a \( d \in \mathcal{D} \setminus \{0\} \) such that \( d \circ v = 0 \). We deduce:

\[
0 = \tilde{d} \circ v = \tilde{d} \circ (d \circ w) = \tilde{d}d \circ w.
\]

Since \( \tilde{d}d \neq 0 \), it follows that \( w \) is really contained in \( t(\mathcal{F}) \). □

In the following, let \( \mathcal{P} \) be the representative system of all prime elements of \( \mathcal{D} \) consisting of the monic irreducible polynomials. We will study the primary decomposition

\[
t(\mathcal{F}) = \bigoplus_{q \in \mathcal{P}} \mathcal{F}_q \quad \text{where}
\]

\[
\mathcal{F}_q := \bigcup_{k=1}^{\infty} \text{ann}_\mathcal{F}(q^k) = \bigcup_{k=1}^{\infty} \left\{ w \in \mathcal{F}; q^k \circ w = 0 \right\} = \left\{ w \in \mathcal{F}; \exists k : q^k \circ w = 0 \right\}.
\]

We divide the prime elements in \( \mathcal{P} \) into two subsets: \( \mathcal{P}_1 := \mathcal{P} \cap T, \mathcal{P}_2 := \mathcal{P} \setminus \mathcal{P}_1 \).

**Lemma 1.4.** For \( q \in \mathcal{P} \) and \( \mathcal{F}_q \) as above, the localization \( (\mathcal{F}_q)_T \) has the following form:

\[
(\mathcal{F}_q)_T \begin{cases} 
= 0 & \text{if } q \in \mathcal{P}_1, \\
\cong \mathcal{F}_q & \text{if } q \in \mathcal{P}_2.
\end{cases}
\]
Proof

1. If \( q \in \mathcal{P}_1 \), then any element \( w \in \mathcal{F}_q \) satisfies \( q^k \circ w = 0 \), \( q^k \in T \), for some \( k \). Consequently, every element \( \frac{w}{t} \in (\mathcal{F}_q)_T \) satisfies \( \frac{w}{t} = \frac{q^k \circ w}{q^k t} = \frac{0}{q^k t} = 0 \in \mathcal{F}_T \). We deduce that \( (\mathcal{F}_q)_T = 0 \).

2. If \( q \in \mathcal{P}_2 \), \( \gcd(t, q^k) = 1 \) for all \( t \in T \) and for all \( k \). Then the canonical map \( \mathcal{F}_q \rightarrow (\mathcal{F}_q)_T \) is bijective: To prove injectivity, assume that \( w \in \mathcal{F}_q \) is mapped to zero, i.e., \( \frac{w}{t} = 0 \). On the other hand, since \( w \in \mathcal{F}_q \), we know that \( q^k \circ w = 0 \) for some \( k \). Consequently (by the Euclidean Algorithm), \( \gcd(t, q^k) \circ w \) is zero as well. Since \( \gcd(t, q^k) = 1 \), this means that \( w = 0 \).

Now let \( \frac{w}{t} \) be an element of \( (\mathcal{F}_q)_T \), i.e., \( w \in \mathcal{F}_q \) with \( q^k \circ w = 0 \) for some \( k \). Since \( \gcd(t, q^k) = 1 \), the Euclidean Algorithm yields \( a, b \in \mathcal{D} \) such that \( aq^k + bt = 1 \). Then the following implications hold:

\[
q^k \circ w = 0 \Rightarrow aq^k \circ w = (1 - bt) \circ w = 0 \Rightarrow w = bt \circ w.
\]

Consequently, the element \( b \circ w \in \mathcal{F}_q \) is mapped to \( \frac{b \circ w}{t} = \frac{bow}{t} = \frac{w}{t} \), i.e., the canonical map is really surjective. \( \square \)

Corollary 1.5. The \( T \)-torsion submodule

\[
t_T(\mathcal{F}) = \{ w \in \mathcal{F}; \ \exists t \in T : t \circ w = 0 \}
\]

has the following decomposition:

\[
t_T(\mathcal{F}) = \bigoplus_{q \in \mathcal{P}_1} \mathcal{F}_q.
\]

Consequently, we can now interpret \( t_T(\mathcal{F}) \) as a direct summand of \( t(\mathcal{F}) \):

\[
t(\mathcal{F}) = \bigoplus_{q \in \mathcal{P}_1} \mathcal{F}_q \oplus \bigoplus_{q \in \mathcal{P}_2} \mathcal{F}_q = t_T(\mathcal{F}) \oplus \bigoplus_{q \in \mathcal{P}_2} \mathcal{F}_q.
\]

Proof

\[
t_T(\mathcal{F}) = t_T(t(\mathcal{F})) = \ker( t(\mathcal{F}) \rightarrow t(\mathcal{F})_T )
\]

\[
= \ker( \bigoplus_{q \in \mathcal{P}} \mathcal{F}_q \rightarrow \bigoplus_{q \in \mathcal{P}} (\mathcal{F}_q)_T )
\]

\[
= \bigoplus_{q \in \mathcal{P}_1} \mathcal{F}_q,
\]

where the last equality holds due to Lemma 1.4. \( \square \)

Theorem 1.6. Consider the injective cogenerator \( \mathcal{D} \mathcal{F} \).

1. The \( T \)-torsion submodule \( t_T(\mathcal{F}) \) of \( \mathcal{F} \) is injective and therefore a direct summand of \( \mathcal{F} \), i.e.,

\[
\mathcal{F} = \mathcal{F}' \oplus t_T(\mathcal{F})
\]

for some \( \mathcal{D} \)-module \( \mathcal{F}' \).

2. The canonical map \( \mathcal{F} \rightarrow \mathcal{F}_T, w \mapsto \frac{w}{t} \), is surjective and induces an isomorphism \( \mathcal{F}' \cong \mathcal{F}_T \). Therefore we identify

\[
\mathcal{F}' = \mathcal{F} / t_T(\mathcal{F}), \quad w = \frac{w}{1} = \bar{w}
\]
Remark 1.7

Proof

1. As direct summand of the injective module \( t(F) \) the module \( t_T(F) \) is injective and therefore a direct summand of \( F \) (see [4, Proposition 3.4]).

2. Let \( \frac{w}{t} \in F_T \). Since \( F \) is divisible there is \( z \in F \) with \( w = t \circ z \), hence

\[
\frac{w}{t} = \frac{t \circ z}{t} = \frac{z}{1} \in \text{im}(\text{can} : F \to F_T) \quad \text{and} \quad \frac{w}{1}.
\]

The direct decomposition \( F' \oplus t_T(F) = F \) induces the isomorphism \( F' \cong F/t_T(F), w \mapsto \bar{w} \), hence \( F' \cong F/t_T(F) \cong F_T \).

3. (a) As direct summand of \( F, F_T \) is an injective, hence divisible \( D \)-module. Let \( \frac{f}{t} \neq 0 \) in \( D_T \).

Then \( f \circ : F_T \to F_T \) is surjective (divisibility), and \( t^{-1} \circ : F_T \to F_T \) is even bijective, thus \( \frac{f}{t} \circ = t^{-1} \circ f \circ : F_T \to F_T \) is surjective and \( F_T \) is \( D_T \)-divisible and thus \( D_T \)-injective.

(b) The set \( \mathcal{P}_2 \) of irreducible polynomials is a representative system of primes in \( D_T \) [2, p. 2424], hence the modules \( D_T/D_Tq = (D/Dq)_T, q \in \mathcal{P}_2 \), are all simple \( D_T \)-modules up to isomorphism. Since \( D \) \( F \) is a cogenerator there are embeddings \( D/Dq \to F \) which give rise to monomorphisms \( D_T/D_Tq = (D/Dq)_T \to F_T \) since \((-)_T \) is exact. By Result 1.2 this signifies that \( F_T \) is a \( D_T \)-cogenerator. \( \square \)

Remark 1.7

1. The module \( F' \) is not unique and can in general not be constructed, for instance for \( D = \mathbb{C}[s] \) and \( F = \mathbb{C}^\infty(\mathbb{R}, \mathbb{C}) \) or \( F = \mathbb{D}'(\mathbb{R}, \mathbb{C}) \).

2. Note that a trajectory \( w = w_T + (w - w_T) \) in \( F = F_T \oplus t_T(F) \) is in the standard examples (i.e., by appropriate choice of the set \( T \), cf. for instance [2, Example 2.16]) essentially described by the part \( w_T \) because \( (w - w_T) \) is annihilated by some \( t \in T \), i.e., it is \( T \)-small and thus negligible.

We will now repeat the definition of \( T \)-autonomy and \( T \)-stability (cf. [2, Theorem and Definition 2.15]):

Reminder 1.8

1. A behavior \( B = \{w \in F^\ell; R \circ w = 0\}, R \in \mathbb{D}^{k \times \ell}, \) is called \( T \)-autonomous if

\[
\exists t \in T : t \circ B = 0.
\]

Trajectories of a \( T \)-autonomous behavior are \( T \)-small.

2. An input/output behavior [2, p. 2419] \( B = \{y \in F^{p+m}; P \circ y = Q \circ u\} \) where \( P \in \mathbb{D}^{p \times p} \) with \( \det(P) \neq 0 \) and \( Q \in \mathbb{D}^{p \times m} \) is called \( T \)-stable if its autonomous part \( B^0 = \{y \in F^{p}; P \circ y = 0\} \) is \( T \)-autonomous. This is equivalent to \( P \in \text{Gl}_p(\mathbb{D}_T) \), i.e., \( P \) has an inverse with entries in \( \mathbb{D}_T \) or \( \det(P) \in T \). Then the transfer matrix \( H := P^{-1}Q \) belongs to \( \mathbb{D}_T^{p \times m} \) and the difference between any two possible outputs to the same input is \( T \)-small.
Theorem 1.9. Consider the decomposition
\[ \mathcal{F} = \mathcal{F}' \oplus t_\mathcal{T}(\mathcal{F}), \text{ hence also} \]
\[ \mathcal{F}^\ell = \mathcal{F}'^\ell \oplus t_\mathcal{T}(\mathcal{F})^\ell \ni w = w_\mathcal{T} + (w - w_\mathcal{T}) \]
and a behavior
\[ B = \left\{ w \in \mathcal{F}^\ell; R \circ w = 0 \right\} \overset{\text{Malgrange}}{\cong} \text{Hom}_\mathcal{T}(M, \mathcal{F}), \text{ where } M := \mathcal{D}^{1 \times \ell} / \mathcal{D}^{1 \times k} R. \]

Then \( B \) inherits the decomposition, in detail:

1. \( B = B' \oplus t_\mathcal{T}(B) \) with
   \[ B' := B \cap \mathcal{F}'^\ell = \left\{ w' \in \mathcal{F}'^\ell; R \circ w' = 0 \right\}, \quad t_\mathcal{T}(B) = B \cap t_\mathcal{T}(\mathcal{F})^\ell. \]

2. \( B' \cong B_T \cong \text{Hom}_\mathcal{T}(M_T, \mathcal{F}_T) \), hence we also identify
   \[ B' = B_T, \quad w' = \frac{w}{1} \]
   like \( \mathcal{F}' = \mathcal{F}_T \). In particular, \( B' = B_T \) is an \( \mathcal{F}_T \)-behavior.

3. \( B \) is \( T \)-autonomous
   \[ \iff \exists t \in T \text{ with } t \circ B = 0 \iff \exists t \in T \text{ with } tM = 0 \]
   \[ \iff B_T = 0 \iff M_T = 0 \]
   \[ \iff R \text{ has a left inverse in } \mathcal{D}_T^{\ell \times k}. \]

Proof

1. Clear.

2. The decomposition from (1) implies
   \[ B_T = B'_T \oplus t_\mathcal{T}(B)_T = B' T. \]
   Since \( \mathcal{F}' \cong \mathcal{F}_T \) is a \( \mathcal{D}_T \)-module, the multiplication \( t \circ: \mathcal{F}'^\ell \to \mathcal{F}'^\ell \) is bijective for all \( t \in T \).
   Since \( t^{-1} \circ R \circ w' = R \circ t^{-1} \circ w' \) for \( w' \in \mathcal{F}'^\ell \), this implies \( t^{-1} \circ B' \subseteq B' \) and thus that \( B' \) is a \( \mathcal{D}_T \)-submodule of \( \mathcal{F}'^\ell \), hence
   \[ B' \cong B'_T = B_T = \left\{ \tilde{w} \in \mathcal{F}_T^\ell; R \circ \tilde{w} = 0 \right\}. \]

   Since
   \[ M_T = (\mathcal{D}^{1 \times \ell} / \mathcal{D}^{1 \times k} R)_T = \mathcal{D}_T^{1 \times \ell} / \mathcal{D}_T^{1 \times k} R \]
   the standard Malgrange isomorphism implies
   \[ \text{Hom}_\mathcal{T}(M_T, \mathcal{F}_T) = \text{Hom}_\mathcal{T}(\mathcal{D}_T^{1 \times \ell} / \mathcal{D}_T^{1 \times k} R, \mathcal{F}_T) \]
   \[ \cong \left\{ \tilde{w} \in \mathcal{F}_T^\ell; R \circ \tilde{w} = 0 \right\} = B_T. \]

3. \( B \) is by definition \( T \)-autonomous iff there exists a \( t \in T \) such that \( t \circ B = 0 \), i.e., \( B \subseteq t_\mathcal{T}(\mathcal{F})^\ell \)
   and hence \( B \subseteq B \cap \mathcal{T}(\mathcal{F})^\ell = t_\mathcal{T}(B) \). Since the decomposition \( B = B_T \oplus t_\mathcal{T}(B) \) holds, this is equivalent to \( B_T = 0 \) or \( M_T = 0 \). The last equivalence has already been shown in \( [2, \text{Theorem and Definition 2.15}] \). \( \square \)

Corollary 1.10. Let \( B_i = \left\{ w \in \mathcal{F}'^{(i)}; R_i \circ w = 0 \right\}, R_i \in \mathcal{D}^{k(i) \times \ell(i)}, i = 1, 2, 3, \) be three \( \mathcal{F} \)-behaviors with modules \( M_i = \mathcal{D}^{1 \times \ell(i) / \mathcal{D}^{1 \times k(i)}} R_i \) and let \( P \in \mathcal{D}^{(2) \times \ell(1)}, Q \in \mathcal{D}^{(3) \times \ell(2)} \). Assume that the sequences
\[ B_1 \overset{P}{\longrightarrow} B_2 \overset{Q}{\longrightarrow} B_3 \]

or, equivalently,
\[ M_1 \overset{(op)\text{ind}}{\leftarrow} M_2 \overset{(oQ)\text{ind}}{\leftarrow} M_3 \]

are well-defined and exact, i.e.,
\[ P \circ B_1 = \{ w \in B_2; \ Q \circ w = 0 \} . \]

Then also the sequences
\[ B_1T \overset{P}{\longrightarrow} B_2T \overset{Q}{\longrightarrow} B_3T \]

and
\[ M_1T \overset{(op)\text{ind}}{\leftarrow} M_2T \overset{(oQ)\text{ind}}{\leftarrow} M_3T \]

are exact.

**Proof.** This follows directly from the exactness of the functor \((-)_T\). \(\square\)

**Remark 1.11.** Let \( R \in D_{k \times \ell} \), \( Q \in D_{q \times \ell} \), and \( B := \{ w \in \mathcal{F}^{\ell}; \ R \circ w = 0 \} \). The previous corollary implies in particular that
\[ B_T = \{ w_T \in \mathcal{F}^{T \ell}; \ R \circ w_T = 0 \} \]

and
\[ (Q \circ B)_T = Q \circ B_T. \]

**2. Theory**

In the following we consider an arbitrary behavior \( \mathcal{B} \) and two image behaviors \( P \circ \mathcal{B} \) and \( Q \circ \mathcal{B} \):
\[ \mathcal{B} := \{ w \in \mathcal{F}^{\ell}; \ R \circ w = 0 \} , \ R \in D_{k \times \ell} , \]
\[ P \circ \mathcal{B} = \{ w_1 \in \mathcal{F}^m; \ \exists w \in \mathcal{B} : w_1 = P \circ w \} , \ P \in D_{m \times \ell} , \]
\[ Q \circ \mathcal{B} = \{ Q \circ w \in \mathcal{F}^{q}; \ w \in \mathcal{B} \} , \ Q \in D_{q \times \ell} . \]

**Definition and Lemma 2.1.** \( Q \circ \mathcal{B} \) is called \( T \)-observable from \( P \circ \mathcal{B} \) \( \iff \) \( Q \circ w \) is \( T \)-observable from \( P \circ w \) for all \( w \in \mathcal{B} : \)
\[ Q \circ (\ker(P \circ : \mathcal{B} \longrightarrow \mathcal{F}^m)) = \{ Q \circ w; \ w \in \mathcal{F}^{\ell}, \ (R) \circ w = 0 \} \]

is \( T \)-autonomous. This is equivalent to the existence of a matrix \( Y \in D_{T}^{q \times (k+m)} \) such that
\[ Y \left( \begin{array}{c} R \\ P \end{array} \right) = Q. \]

**Proof.** \( Q \circ \mathcal{B} \) is \( T \)-observable from \( P \circ \mathcal{B} \) \( \iff \)
\[ \iff \quad \{ Q \circ w; \ w \in \mathcal{F}^{\ell}, \ (R) \circ w = 0 \} \text{ is } T \text{-autonomous} \]
\[ \iff \quad \{ Q \circ w; \ w \in \mathcal{F}^{\ell}, \ (R) \circ w = 0 \}_T = 0 \]
\[ \iff \quad \{ Q \circ w_T; \ w_T \in \mathcal{F}^{T \ell}, \ (R) \circ w_T = 0 \} = 0 \]
\[
\begin{align*}
&\iff \left[ \begin{array}{c} \mathbf{w}_T \in \mathcal{F}_T^\ell; \ (R_P) \circ \mathbf{w}_T = 0 \Rightarrow Q \circ \mathbf{w}_T = 0 \end{array} \right] \\
&\iff \left\{ \begin{array}{c} \mathbf{w}_T \in \mathcal{F}_T^\ell; \ (R_P) \circ \mathbf{w}_T = 0 \end{array} \right\} \subseteq \left\{ \mathbf{w}_T \in \mathcal{F}_T^\ell; \ Q \circ \mathbf{w}_T = 0 \right\} \\
&\iff \exists \mathbf{Y} \in D_T^{q \times (k+m)} : \ Y \left( \begin{array}{cc} R_P \\ P \end{array} \right) = Q,
\end{align*}
\]

where the last equivalence holds since \(\mathcal{F}_T\) is a cogenerator over \(\mathcal{D}_T\) by Theorem 1.6 and due to Result 1.2. \(\Box\)

**Remark 2.2.** The interpretation of \(T\)-observability is the following: we assume that we can measure the signal \(P \circ \mathbf{w}\) for a trajectory \(\mathbf{w} \in \mathcal{B}\) and that we are interested in an estimation for \(Q \circ \mathbf{w}\). \(T\)-observability signifies that, whenever \(P \circ \mathbf{w} = P \circ \tilde{\mathbf{w}}\) for \(\mathbf{w}, \tilde{\mathbf{w}} \in \mathcal{B}\), then the difference between \(Q \circ \mathbf{w}\) and \(Q \circ \tilde{\mathbf{w}}\) is \(T\)-small.

The above definition is a generalization of the one given in [2, Definition and Corollary 2.18]. Similarly, the following definition of \(T\)-observers generalizes Definitions 3.1 and 4.1 in [2].

**Definition 2.3.** An input/output behavior

\[
\mathcal{B}_{\text{obs}} = \left\{ \begin{array}{c} \hat{\mathbf{w}} \\ \mathbf{w}_1 \end{array} \right\} \in \mathcal{F}^{q+m}; \ P_{\text{obs}} \circ \hat{\mathbf{w}} = Q_{\text{obs}} \circ \mathbf{w}_1 \}
\]

is called a \(T\)-observer of \(Q \circ \mathcal{B}\) from \(P \circ \mathcal{B}\): \(\iff\)

\[
\mathcal{B}_{\text{err}} := \left\{ \hat{\mathbf{w}} - Q \circ \mathbf{w} \in \mathcal{F}^q; \ w \in \mathcal{B}, \left( \begin{array}{cc} \hat{\mathbf{w}} \\ P \circ \mathbf{w} \end{array} \right) \in \mathcal{B}_{\text{obs}} \right\} \text{ is } T\text{-autonomous.}
\]

Fig. 1 shows the interconnection diagram of this situation. Note that we can assume without loss of generality that \(P_{\text{obs}}\) is square since any input/output behavior can be described in this form. We use this assumption for simplicity of the notation.

**Example 2.4.** For different standard choices of the set \(T\), the terms \(T\)-observability and \(T\)-observer have the following significances (for the discrete or continuous standard signal modules), compare for example [11, Definitions 2.1 and 3.1, 3, pp. 62, 77, 104, 105, 109, 110]:

1. \(T := \{1\}\) (resp. \(T := F \setminus \{0\}\) if we want to choose a saturated \(T\)): \(T\)-observability coincides with observability, and a \(T\)-observer is an exact observer.
2. \(T := \{s^k; \ k \in \mathbb{N}\}\) (resp. \(T := \{cs^k; \ c \in F, c \neq 0, k \in \mathbb{N}\}\)): For this choice in the discrete standard case, \(T\)-observability is called reconstructibility, a \(T\)-observer is a dead-beat observer (cf. also [10,1]).
3. \(T := \mathcal{D} \setminus \{0\}\): \(T\)-observability is sometimes denoted as trackability, a \(T\)-observer as tracking observer (cf. for example Fuhrmann’s paper [3]). Valcher and Willems [11, Definition 3.1] call a \(T\)-observer with this choice of \(T\) just observer.
4. \(T := \{t \in \mathcal{D}; \mathcal{V}_C(t) \subseteq \Lambda_1\}\) where \(\mathcal{V}_C(t) := \{\lambda \in \mathbb{C}; \ t(\lambda) = 0\}\) and

![Interconnection diagram of the trajectory w ∈ B, the observed signal P ∘ w, and the observer system Bobs that produces an approximation \(\hat{\mathbf{w}}\) for the signal Q ∘ w.](image-url)
Corollary 2.8 (Parametrization). Assume that \( Q \circ B \) is \( T \)-observable from \( P \circ B \), and let \( (-X, H_{obs}) \in D_T^{q \times (k+m)} \) satisfy \( (-X, H_{obs}) (R_P) = Q \). Let \((P_{obs}, -Q_{obs})\) be the controllable realization of \( H_{obs} \). Then, for any matrix \( A \in D_T^{q \times q} \) with \( \det(A) \in T \),

\[
\Lambda_1 := \begin{cases} 
\{ \lambda \in \mathbb{C}; \ Re(\lambda) < 0 \} & \text{in the continuous case,} \\
\{ \lambda \in \mathbb{C}; |\lambda| < 1 \} & \text{in the discrete case.}
\end{cases}
\]

With this important choice, a \( T \)-observable behavior is \textit{detectable}, and a \( T \)-observer is an asymptotic observer.

Lemma 2.5. Any \( T \)-observer \( B_{obs} \) is automatically \( T \)-stable.

Proof. \( B_{obs} \) is a \( T \)-observer if and only if \( B_{err} \) is \( T \)-autonomous if and only if \( B_{err}^0 = \emptyset \), which is equivalent to \( B_{obs} \) being \( T \)-stable (cf. [2, Theorem 2.15]).

Theorem 2.6. A \( T \)-stable \( IO \) behavior \( B_{obs} \) as above is a \( T \)-observer of \( Q \circ B \) from \( P \circ B \) if and only if there is a matrix

\[
X \in D_T^{q \times k} \quad \text{such that} \quad H_{obs} P - Q = XR \quad \text{or} \quad (-X, H_{obs}) (R_P) = Q.
\]

Proof. \( B_{obs} \) is a \( T \)-observer if and only if \( B_{err} \) is \( T \)-autonomous if and only if \( B_{err}^T = 0 \) if and only if the following implication holds in \( \mathcal{F}_T \):

\[
R \circ w_T = 0, \quad P_{obs} \circ \hat{w}_T = Q_{obs} \circ (P \circ w_T) \quad \Rightarrow \quad \hat{w}_T = Q \circ w_T = 0.
\]

But since \( P_{obs}^{-1} Q_{obs} = H_{obs} \in D_T^{q \times m} \) due to the assumed \( T \)-stability of \( B_{obs} \) (compare Reminder 1.8.2) and \( \mathcal{F}_T \) is a \( D_T \)-module, we know that

\[
P_{obs} \circ \hat{w}_T = Q_{obs} \circ (P \circ w_T) \quad \Leftrightarrow \quad \hat{w}_T = H_{obs} \circ (P \circ w_T) = H_{obs} P \circ w_T \quad \text{in} \quad \mathcal{F}_T.
\]

Consequently, we have shown that \( B_{obs} \) is a \( T \)-observer if and only if

\[
R \circ w_T = 0 \quad \text{implies that} \quad H_{obs} P \circ w_T - Q \circ w_T = 0 \quad \text{in} \quad \mathcal{F}_T, \quad \text{i.e.,}
\]

\[
\left\{ w_T \in \mathcal{F}_T^f; \quad R \circ w_T = 0 \right\} \subseteq \left\{ w_T \in \mathcal{F}_T^f; \quad (H_{obs} P - Q) \circ w_T = 0 \right\}.
\]

Since \( \mathcal{F}_T \) is a cogenerator over \( D_T \), this is equivalent to the existence of a matrix \( X \) in \( D_T^{q \times k} \) such that

\[ H_{obs} P - Q = XR. \]

Theorem 2.7. There exists a \( T \)-observer of \( Q \circ B \) from \( P \circ B \) if and only if \( Q \circ B \) is \( T \)-observable from \( P \circ B \).

Proof. If an observer exists the equation \(( -X, H_{obs}) (R_P) = Q \) of Theorem 2.6 implies \( T \)-observability by Definition and Lemma 2.1. On the other hand, if \( Q \circ B \) is \( T \)-observable from \( P \circ B \) and if \(( -X, H_{obs}) \in D_T^{q \times (k+m)} \) satisfies \(( -X, H_{obs}) (R_P) = Q \) according to Definition and Lemma 2.1, then the controllable realization \( B_{obs} = \left\{ \left( \hat{w}_T \right) \in D_T^{q \times m}; \quad P_{obs} \circ \hat{w}_T = Q_{obs} \circ w_T \right\} \) of \( H_{obs} \) with \( D_T^{1 \times q} P_{obs} = \left\{ \xi \in D_T^{1 \times q}; \quad \xi H_{obs} \in D_T^{1 \times m} \right\} \) satisfies \(( -X, H_{obs}) (R_P) = Q \). Let \((P_{obs}, -Q_{obs})\) be the controllable realization of \( H_{obs} \). Then, for any matrix \( A \in D_T^{q \times q} \) with \( \det(A) \in T \),
\( \mathcal{B}_{\text{obs}}(H_{\text{obs}}, A) := \left\{ \left( \begin{array}{c} \hat{W} \\ W_1 \end{array} \right) \in \mathcal{F}^{q+m}; \; AP_{\text{obs}} \circ \hat{W} = AQ_{\text{obs}} \circ W_1 \right\} \)

is a \((T\text{-stable})\) \(T\)-observer of \(Q \circ B\) from \(P \circ B\) and all such \(T\)-observers can be obtained by this method.

In other words: the matrices \((-X, H_{\text{obs}}) \in D_T^{q \times (k+m)}\) with \((-X, H_{\text{obs}})(R_p) = Q\) parameterize all controllable \(T\)-observers of \(Q \circ B\) from \(P \circ B\), and the triples \((-X, H_{\text{obs}}, A)\) where in addition \(A \in D_T^{q \times q}\) with \(\det(A) \in T\) parameterize all not necessarily controllable ones.

Two left inverses \((-X, H_{\text{obs}})\) give rise to the same controllable \(T\)-observer \(\iff H_{\text{obs}1} = H_{\text{obs}2}\).

**Proof.** The controllable realization \(\mathcal{B}_{\text{obs}} = \left\{ \left( \begin{array}{c} \hat{W} \\ W_1 \end{array} \right), \; P_{\text{obs}} \circ \hat{W} = Q_{\text{obs}} \circ W_1 \right\} \) of \(H_{\text{obs}}\) is a \(T\)-observer by the proof of the previous theorem. Multiplying both \(P_{\text{obs}}\) and \(Q_{\text{obs}}\) by a matrix \(A\) with determinant in \(T\) preserves both \(T\)-stability and the transfer matrix of the behavior. Consequently, Theorem 2.6 yields that the resulting behavior is still a \(T\)-observer of \(Q \circ B\) from \(P \circ B\).

On the other hand, Theorem 2.5 yields that any \(T\)-observer \(\mathcal{B}_{\text{obs}}\) must be \(T\)-stable (\(\Rightarrow H_{\text{obs}} \in D_T^{q \times m}\), and Theorem 2.6 implies that the transfer matrix \(H_{\text{obs}}\) must satisfy \((-X, H_{\text{obs}})(R_p) = Q\) for some \(X \in D_T^{q \times k}\). Hence, \(\mathcal{B}_{\text{obs}}\) must have the form

\[
\mathcal{B}_{\text{obs}} = \left\{ \left( \begin{array}{c} \hat{W} \\ W_1 \end{array} \right) \in \mathcal{F}^{q+m}; \; AP_{\text{obs}} \circ \hat{W} = AQ_{\text{obs}} \circ W_1 \right\},
\]

where \((P_{\text{obs}}, -Q_{\text{obs}})\) is the controllable realization of such a matrix \(H_{\text{obs}}\), and \(A \in D_T^{q \times q}\). \(T\)-stability of \(\mathcal{B}_{\text{obs}}\) implies that \(\det(A) \in T\). \(\square\)

**Remark 2.9.** The existence of a matrix \(Y \in D_T^{q \times (k+m)}\) such that \(Y(R_p) = Q\) can be checked via the Smith form of \((R_p)\). If such a matrix does exist, the set of all matrices with these properties is available.

For the details of those computations, cf. Section 3, Algorithm 3.1.

Hence, the theory presented here is completely constructive.

**Corollary 2.10.** There exists a proper \(T\)-observer of \(Q \circ B\) from \(P \circ B\) if and only if

\[
\exists (-X, H_{\text{obs}}) \in D_T^{q \times k} \times S^{q \times m} : (-X, H_{\text{obs}})(R_p) = Q,
\]

where \(S := D_T \cap F(s)_{pr}\) denotes the ring of all proper \(T\)-stable rational functions. If this is the case, all such matrices give rise to proper \(T\)-observers by the same construction as in Corollary 2.8. The parametrization result from that corollary holds mutatis mutandis if only proper \(T\)-stable matrices \(H_{\text{obs}}\) are considered.

**Remark 2.11.** If the set \(T\) contains an element \(s - \alpha\) for some \(\alpha \in F\), then the condition of the previous corollary can be checked and all matrices satisfying this condition (if there are any) can be computed by means of Algorithms 3.1 and 3.2 in Section 3.

Note that, for the important case \(F = \mathbb{C}\), the assumption that \(T\) does not contain such an element \(s - \alpha\) would imply that \(S = F\). This is obviously too restrictive and hence not interesting.

We will now relate the present theory to the results obtained by our predecessors, in particular by Valcher and coworkers [11,10,9,1], by Fuhrmann [3], by Vidyasagar [12], and by Wolovich [13] in a series of examples.

**Example 2.12** (Comparison to Valcher and Willems [11,10], Bisiacco et al. [1], and Fuhrmann [3]). One important choice for the matrices \(P\) and \(Q\) is the following:

\[
\mathcal{B} = \left\{ \left( \begin{array}{c} W_r \\ W_m \\ W_i \end{array} \right) \in \mathcal{F}^{r+m+1}; \; R_r \circ W_r = R_m \circ W_m + R_i \circ W_i \right\},
\]

\[
R_r \in D^{p \times r}, \; R_m \in D^{p \times m}, \; R_i \in D^{p \times i},
\]
\[ P = (0, \text{id}_m, 0) \in \mathcal{D}^{m \times (r+m+i)}, \quad P \circ \begin{pmatrix} w_r \\ w_m \\ w_i \end{pmatrix} = w_m. \]

\[ Q = (\text{id}_r, 0, 0) \in \mathcal{D}^{r \times (r+m+i)}, \quad Q \circ \begin{pmatrix} w_r \\ w_m \\ w_i \end{pmatrix} = w_r. \]

Here the components of \( w \in B \) are divided into three sets: the relevant variables \( w_r \) that shall be estimated, the measured variables \( w_m \), and the irrelevant variables \( w_i \) that are not known and of no interest. This is the setting studied in [10,1]. In [11,3] the special case \( i = 0 \) is considered (of course, the case including irrelevant variables can be reduced to that case by elimination of the irrelevant variables).

\( w_r \) is \( T \)-observable from \( w_m \)

\[ \iff Q \circ B \text{ is } T \text{-observable from } P \circ B \]

\[ \iff \exists (-X, H_{\text{obs}}) \in \mathcal{D}^{r \times (p+m)} \text{ with } \]

\[ (-X, H_{\text{obs}}) \begin{pmatrix} R_r \\ -R_m \\ \text{id}_m \\ 0 \end{pmatrix} = (\text{id}_r, 0, 0) \]

\[ \iff \exists (-X, H_{\text{obs}}) \in \mathcal{D}^{r \times (p+m)} \text{ with } \begin{cases} -XR_r = \text{id}_r, \\ XR_m + H_{\text{obs}} = 0, \\ XR_i = 0 \end{cases} \]

\[ \iff \exists X \in \mathcal{D}^{r \times p} : -X(R_r, -R_i) = (\text{id}_r, 0). \]

Compare [1, Theorem 3, iv and Corollary 4, iv, 3, p. 106, Proposition 4.2] for this result.

Let \( H_i \in \mathcal{D}^{k \times p}, k := i - \text{rank}(R_i) \), be a universal left annihilator of \( R_i \) w.r.t. \(\mathcal{D} \), i.e., the sequence

\[ \mathcal{D}^{1 \times k} \xrightarrow{\circ H_i} \mathcal{D}^{1 \times p} \xrightarrow{\circ R_i} \mathcal{D}^{1 \times i} \]

is exact or \( \text{im}(\circ H_i) = \ker(\circ R_i) \). Then \( H_i \) is also a universal left annihilator of \( R_i \) w.r.t. \(\mathcal{D}_T \) since localization preserves exactness. Consequently, \( T \)-observability is equivalent to

\[ \exists X \in \mathcal{D}^{r \times p} \text{ with } -XR_r = \text{id}_r, \quad \exists Z \in \mathcal{D}^{r \times k} : X = ZH_i \]

\[ \iff \exists Z \in \mathcal{D}^{r \times k} \text{ with } -ZH_iR_r = \text{id}_r \]

\[ \iff \Gamma := H_iR_r \text{ has a left inverse } -Z \in \mathcal{D}^{r \times k}. \]

Compare [1, Theorem 3, iii, and Corollary 4, iii] for this last condition. In this case, \( H_{\text{obs}} \in \mathcal{D}^{q \times m} \) is determined by

\[ H_{\text{obs}} = -XR_m = -ZH_iR_m \]

and hence \((\text{id}_r, -H_{\text{obs}}) = (-XR_r, XR_m) = -Z(H_iR_r, -H_iR_m)\). If

\[ B_{\text{obs}} = \left\{ \begin{pmatrix} \hat{w}_2 \\ \hat{w}_1 \end{pmatrix} \in \mathcal{D}^{p+m}, \quad P_{\text{obs}} \circ \begin{pmatrix} \hat{w}_2 \\ \hat{w}_1 \end{pmatrix} = Q_{\text{obs}} \circ \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \right\} \]

is any \( T \)-observer of \( w_r \) from \( w_m \), i.e., \( P_{\text{obs}} \in \mathcal{D}^{p \times p} \), \( \det(P_{\text{obs}}) \in T \), \( Q_{\text{obs}} = P_{\text{obs}}H_{\text{obs}} \), then

\[ (P_{\text{obs}}, -Q_{\text{obs}}) = P_{\text{obs}}(\text{id}_r, -H_{\text{obs}}) = -P_{\text{obs}}Z(H_iR_r, -H_iR_m) = L(\Gamma, -\Phi), \]

where \( L := -P_{\text{obs}}Z \) and \( \Phi := H_iR_m \). It can easily be seen that an input/output behavior \( B_{\text{obs}} \) is a \( T \)-observer if and only if \((P_{\text{obs}}, -Q_{\text{obs}}) = L(\Gamma, -\Phi)\) for some \( L \in \mathcal{D}^{r \times k} \) such that
\[ L \Gamma \in \mathcal{D}^{r \times r}, \quad \det(L \Gamma) \in T, \quad L \Phi \in \mathcal{D}^{r \times m} \]

compare [1, (14)].

**Example 2.13.** Also the problems of partially observing the state of a Kalman state space system (cf. for example Fuhrmann [3, Section 3, pp. 60–104]) and of observing the pseudo state of a Rosenbrock system (polynomial matrix description = differential operator representation, cf. [9,2, Section 4, pp. 2436–2447]) are special cases of the present framework:

\[ B = \left\{ \left( \begin{array}{c} x \\ u \end{array} \right) \in \mathbb{R}^{n+m}, \ A \circ x = B \circ u \right\}, \]

\[ A \in \mathcal{D}^{n \times n}, \ B \in \mathcal{D}^{n \times m}, \ \det(A) \neq 0, \]

\[ P = \left( \begin{array}{cc} C & D \\ 0 & \text{id}_m \end{array} \right) \in \mathcal{D}^{(p+m) \times (n+m)}, \ P \circ \left( \begin{array}{c} x \\ u \end{array} \right) = \left( \begin{array}{c} C \circ x + D \circ u \\ u \end{array} \right) =: \left( \begin{array}{c} y \\ u \end{array} \right). \]

\[ Q = (K, 0) \in \mathcal{D}^{k \times (n+m)}, \ Q \circ \left( \begin{array}{c} x \\ u \end{array} \right) = K \circ x =: z. \]

Here \( z \) is \( T \)-observable from \( \left( \begin{array}{c} y \\ u \end{array} \right) \)

\[ \iff Q \circ B \text{ is } T \text{-observable from } P \circ B \]

\[ \iff \exists (-X, H_y, H_u) \in \mathcal{D}_T^{k \times (n+p+m)} : (-X, H_y, H_u) \left( \begin{array}{cc} A & -B \\ C & D \\ 0 & \text{id}_m \end{array} \right) = (K, 0) \]

\[ \iff \exists (-X, H_y, H_u) \in \mathcal{D}_T^{k \times (n+p+m)} : \begin{cases} -XA + H_yC = K, \\ XB + H_yD + H_u = 0 \end{cases} \]

\[ \iff \exists (-X, H_y) \in \mathcal{D}_T^{k \times (n+p)} : -XA + H_yC = K \]

\[ \iff \exists (-X, H_y) \in \mathcal{D}_T^{k \times (n+p)} : (-X, H_y) \left( \begin{array}{c} A \\ C \end{array} \right) = K. \]

Compare [3, p. 68, Proposition 3.3] for the case of partial state observers for Kalman state space equations (i.e., \( A = (s \text{id}_n - A') \)) and the matrices \( A', B, C, D, \) and \( K \) are constant) and [9, Proposition 2.4, 2, Theorems 4.3 and 4.4] for the case of pseudo state observers for Rosenbrock systems (with \( K := \text{id}_n \)). This equation and its proper analogue also generalize [13, Eq. (7.3.24)], Wolovich's Theorem 7.3.23 in [13] was the first result on functional observers for special observable Rosenbrock equations.

If the above observability condition is satisfied, then \( H_u \in \mathcal{D}_T^{k \times m} \) is determined by

\[ H_u = -XB - H_yD. \]

Note that \( H_u \) is automatically proper if we assume that \( H_y \in \mathcal{S}^{k \times p} \subseteq F(s)^{k \times p} \) and that \( KH_1 \) (where \( H_1 := A^{-1}B \)) and \( H_2 := CA^{-1}B + D \) are proper, i.e., under these conditions there exists a proper \( T \)-observer of \( z \) from \( \left( \begin{array}{c} y \\ u \end{array} \right) \) if and only if

\[ \exists (-X, H_y) \in \mathcal{D}_T^{k \times n} \times \mathcal{S}^{k \times p} \text{ such that } (-X, H_y) \left( \begin{array}{c} A \\ C \end{array} \right) = K \]

(compare [2, Theorems 4.3 and 4.5]):

\[ H_u = -XB - H_yD = (-XA)(A^{-1}B) - H_yD \]

\[ = (K - H_yC)(H_1) - H_yD = KH_1 - H_y(CH_1 + D) = KH_1 - H_yH_2 \in F(s)^{k \times m}. \]
Example 2.14 (Comparison to Luenberger Observers [5]). We consider observable Kalman state space equations, i.e.,

\[ s \circ x = A'x + Bu, \]
\[ y = Cx + Du, \]
where \( A' \in \mathbb{F}^{n \times n}, B \in \mathbb{F}^{n \times m}, C \in \mathbb{F}^{p \times n}, \) and \( D \in \mathbb{F}^{p \times m}. \) By the previous example, \( x \) is \( T \)-observable from \( \begin{bmatrix} y \\ u \end{bmatrix} \) if and only if there exist matrices \( X \in \mathcal{D}_T^{n \times n} \) and \( H_y \in \mathcal{D}_T^{n \times p} \) such that

\[ (-X, H_y) \left( s \frac{\mathrm{id}_n - A'}{C} \right) = \mathrm{id}_n. \]  

This is satisfied for any choice of \( T \) since observability is assumed. Moreover, with \( H_u := -XB - H_yD, \) the controllable realization of \( (H_y, H_u) \) is a \( T \)-observer, and any \( T \)-stable IO behavior with this transfer matrix is so as well.

A Luenberger Observer is defined as

\[ \mathcal{B}_{\text{Lue}} := \left\{ \begin{bmatrix} \hat{x} \\ y \\ u \end{bmatrix} \in \mathbb{F}^{n+p+m}; \ s \circ \hat{x} = (A' - MC)\hat{x} + My + (B - MD)u \right\} \]

for some matrix \( M \in \mathbb{F}^{n \times p} \) such that the spectrum of \( A' - MC \) is contained in the open left half plane, i.e., such that \( \det(s \frac{\mathrm{id}_n - (A' - MC)}{C}) \in T \) where \( T \) is chosen as the set of polynomials with roots only in the open left half plane.

\( \mathcal{B}_{\text{Lue}} \) is \( T \)-stable by construction due to the choice of the matrix \( M \), and its transfer matrix is

\[ H_{\text{Lue}} := (s \frac{\mathrm{id}_n - (A' + MC)}{C})^{-1}(M, B - MD) =: (H_y, H_u). \]

We check whether \( H_y \) satisfies \( \text{(3)} \) for some \( X \in \mathcal{D}_T^{n \times n} \):

\[ -X(s \frac{\mathrm{id}_n - A'}{C} + H_yC = \mathrm{id}_n \]
\[ \iff -X(s \frac{\mathrm{id}_n - A'}{C} + (s \frac{\mathrm{id}_n - A'}{C} + MC)^{-1}MC = \mathrm{id}_n \]
\[ \iff -(s \frac{\mathrm{id}_n - A'}{C} + MC)X(s \frac{\mathrm{id}_n - A'}{C}) + MC = s \frac{\mathrm{id}_n - A'}{C} + MC \]
\[ \iff -(s \frac{\mathrm{id}_n - A'}{C} + MC)X = \mathrm{id}_n \]
\[ \iff X = -(s \frac{\mathrm{id}_n - (A' - MC)}{C})^{-1}. \]

\( X \) is contained in \( \mathcal{D}_T^{n \times n} \) since \( \det(s \frac{\mathrm{id}_n - (A' - MC)}{C}) \in T \). Moreover, it can easily be seen that \( H_u = (s \frac{\mathrm{id}_n - (A' - MC)}{C})^{-1}(B - MD) \) is really equal to \( -XB - H_yD \). Hence, the Luenberger Observers are indeed contained in the set of all \( T \)-observers parameterized in Corollary 2.8.

3. Practical computations

Algorithm 3.1 (cf. [12, p. 152, Lemma 4]). Let \( R \) be a principal ideal domain, e.g., \( R = \mathcal{D}_T \), with quotient field \( K = \text{quot}(R) \) and let \( A \in K^{a \times c}, B \in K^{b \times c}. \) The following algorithm determines whether there exists a matrix \( Y \in R^{b \times a} \) such that \( YA = B \) and, if this is the case, gives a parametrization of all such matrices. Let

\[ \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix} = UAV, \quad E = \begin{pmatrix} e_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_r \end{pmatrix}, \quad r = \text{rank}(A) \]

be the Smith form of \( A \) with respect to \( R \). Then the following equivalences hold:

\[ \exists Y \in R^{b \times a} : YA = B \]
The following equivalences hold:

\[ \exists Y \in R^{b \times a} : YU^{-1} UAV = BV \]

\[ =: \bar{Y} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} =: \bar{B} \]

\[ \Leftrightarrow \exists \bar{Y} \in R^{b \times a} : \bar{Y} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = \bar{B} \]

\[ \Leftrightarrow \exists \bar{Y} \in R^{b \times a} : \bar{Y} e_j \begin{cases} \bar{Y} e_j & \text{for } 1 \leq j \leq r, \\ 0 & \text{for } r < j \leq c, \end{cases} \quad 1 \leq i \leq b. \]

\[ \Leftrightarrow \forall i \in \{1, \ldots, b\} \forall j \in \{1, \ldots, r\} : \bar{Y} e_j^{-1} \in R, \]

\[ \forall i \in \{1, \ldots, b\} \forall j \in \{r + 1, \ldots, c\} : \bar{Y} = 0. \]

If this is the case, define \( \bar{Y} \in R^{b \times a} \) by

\[ \bar{Y} e_j := \begin{cases} \bar{Y} e_j^{-1} & \text{for } 1 \leq j \leq r, \\ 0 & \text{for } r < j \leq c, \end{cases} \quad 1 \leq i \leq b. \]

Then \( Y := \bar{Y} U \in R^{b \times a} \) satisfies

\[ YA = B. \]

Furthermore (parametrization):

\[ \left\{ Y' \in R^{b \times a} : Y' A = B \right\} = Y + R^{b \times (a-r)} U_2, \]

where \( U := \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \in R^{(r+(a-r)) \times a} \), i.e., \( U_2 \) is a universal left annihilator of \( A \).

**Proof.** We only have to prove the result on parametrization:

Let \( Y' \) be an arbitrary matrix in \( R^{b \times a} \) and \( U_2 \in R^{(a-r) \times a} \) a universal left annihilator of \( A \). Then the following equivalences hold:

\[ Y' A = B \Leftrightarrow Y' A = YA \Leftrightarrow (Y' - Y) A = 0 \]

\[ \Leftrightarrow Y' - Y \in \ker(\sigma A) = \im(\sigma U_2) = R^{b \times (a-r)} U_2 \]

\[ \Leftrightarrow Y' \in Y + R^{b \times (a-r)} U_2. \]

Now consider in particular the ring \( R := D_T \). The following algorithm can be used to determine whether there exist proper matrices among those parameterized in the previous algorithm. Moreover, a parametrization of all proper such matrices is given if there are any.

In particular, Algorithm 3.1 can be applied in order to find \((X, H_{\text{obs}}) \in D_T^{q \times (k+m)}\) such that \((X, H_{\text{obs}})^{(R)} = Q\) and a \( T\)-stable matrix \( U_2 \) such that any pair \((X', H'_{\text{obs}})\) satisfying this condition is of the form \((X, H_{\text{obs}}) + ZU_2\) for some \( T\)-stable matrix \( Z \). Then the following algorithm can be used to find all \( T\)-stable matrices \( Z \) such that \( H'_{\text{obs}} = H_{\text{obs}} + ZU_2 \) is proper (\( U_{22} \) consists of the last \( m \) columns of \( U_2 \)).

**Algorithm 3.2.** Assume that \( T \) contains an element \( s - \alpha \) for some \( \alpha \in F \) and define

\[ \sigma := (s - \alpha)^{-1}. \]

Let \( C \in D_T^{b \times d}, D \in D_T^{n \times d}. \) The following algorithm determines whether there exists a matrix \( Z \in D_T^{b \times n} \) such that \( C + ZD \) is proper and, if this is the case, gives a parametrization of all such matrices.
1. Let 
\[
\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = UDV, \quad E = \begin{pmatrix} e_1 & & \\ & \ddots & \\ 0 & & e_r \end{pmatrix}, \quad r = \text{rank}(D)
\]
be the Smith form of $D$ with respect to $S$, i.e., with respect to $F[\sigma]$ (cf. [2, Definition and Lemma 2.14]).

For each elementary divisor $e_j \in F(s) = F(\sigma)$, $1 \leq j \leq r$, find $f_j, g_j \in F[\sigma]$ such that 
\[
e_j = \frac{f_j}{g_j} \quad \text{and} \quad \gcd(f_j, g_j) = 1.
\]

Then:
\[
\exists Z \in D_T^{b \times n}: C + ZD \in S_T^{b \times d} 
\Leftrightarrow \begin{cases} (CV)_{ij}g_j \in D_T & \text{for } 1 \leq j \leq r, 1 \leq i \leq b \\ (CV)_{ij} = 0 & \text{for } r < j \leq d, 1 \leq i \leq b 
\end{cases}
\]

Assume now that this condition is satisfied.

2. Construction of $Z \in D_T^{b \times n}$ with $C + ZD \in S_T^{b \times d}$:

- $1 \leq i \leq b, 1 \leq j \leq r$: Define $\tilde{Z}_{ij}$ by the following steps:
  
  Since by assumption $(CV)_{ij}g_j \in D_T$, there exist $h_{ij} \in D$ and $t_{ij} \in T$ such that 
  \[
  (CV)_{ij}g_j = \frac{h_{ij}}{t_{ij}}.
  \]
  
  Define 
  \[
  k_{ij} := \deg(t_{ij}) - \deg(h_{ij}).
  \]
  
  Since gcd($f_j, g_j$) = 1 by definition, the Euclidean Algorithm yields $a_j^1, a_j^2 \in F[\sigma]$ such that 
  \[
  a_j^1f_j + a_j^2g_j = 1.
  \]
  
  Case 1: $k_{ij} \geq 0$: Define 
  \[
  \tilde{Z}_{ij} := -(CV)_{ij}g_ja_j^1.
  \]
  
  Case 2: $-k_{ij} > 0$: find a representation 
  \[
  f_j = \sigma^\ell f_j^1, \quad f_j^1 \in F[\sigma], \quad \gcd(f_j^1, \sigma) = 1.
  \]
  
  Then, again by the Euclidean Algorithm and since gcd($f_j^1, \sigma^{-k_{ij}}$) = 1, there exist $a_j^3, a_j^4 \in F[\sigma]$ such that 
  \[
  a_j^3f_j^1 + a_j^4\sigma^{-k_{ij}} = 1.
  \]
  
  Now define 
  \[
  \tilde{Z}_{ij} := -(a_j^3\sigma^{-\ell_j} + a_j^4a_j^1\sigma^{-k_{ij}})(CV)_{ij}g_j.
  \]

- $1 \leq i \leq b, r < j \leq n$: Define 
  \[
  \tilde{Z}_{ij} := 0.
  \]

Then 
\[
Z := \tilde{Z}U \in D_T^{b \times n}
\]
satisfies $C + ZD \in S_T^{b \times d}$.
3. Parametrization: The following bijection is valid:

\[
\{ Z' \in D_T^{b \times \ell}; \ C + Z'D \in S^{b \times d} \} \approx S^{b \times r} \times D_T^{b \times (n-r)},
\]

where

\[ G := \text{diag}(\sigma^{-\ell_1}g_1, \ldots, \sigma^{-\ell_r}g_r, 1, \ldots, 1) \in F(s)^{n \times n}. \]

Proof. See [2, Corollaries 3.9–3.14]. □

Example 3.3. As an example for the described theory and algorithms we consider the continuous standard case, i.e., the signal module \( F = C^\infty(\mathbb{R}, \mathbb{C}) \) or, more general, \( F = D'(\mathbb{R}, \mathbb{C}) \). We assume that we know the system equations \( R \circ w = 0 \) of the behavior \( \mathcal{B} \), that we can measure the image \( P \circ w \) and want to estimate \( Q \circ w \) as in Section 2 with the following data:

\[
R := \begin{pmatrix} s^2 + s & s \\ -2 & s + 1 \end{pmatrix},
\]

\[
P := (s + 1 \quad -s),
\]

\[
Q := \begin{pmatrix} s & -1 \\ 2 & -1 \end{pmatrix}.
\]

We are interested in proper asymptotic observers, i.e., we choose the set

\[ T := \{ t \in D; \text{ all complex zeroes of } t \text{ have real part } < 0 \}. \]

By Corollary 2.10 such an observer exists if and only if there is a matrix \((-X, H_{\text{obs}}) \in D_T^{2 \times 2} \times S^{2 \times 1}\) such that \((-X, H_{\text{obs}}) \begin{pmatrix} R \\ P \end{pmatrix} = Q\). In order to check this, we study the existence of such matrices in \( D_T^{2 \times (2+1)}\) (without properness) first: Since the Smith Form of \( \begin{pmatrix} R \\ P \end{pmatrix} \) is \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), Algorithm 3.1 yields that such matrices do indeed exist, and that any of them is of the form \((-X, H_{\text{obs}}) = (-X^0, H_{\text{obs}}^0 + ZU_2\) for some \( Z \in D_T^{2 \times 1}\) where

\[
X^0 = \begin{pmatrix} 1/4 (s + 2)(s - 1)(s + 1) & -1/4 \left(s^4 + 4s^3 + 3s^2 - 6s - 4\right) \\ 1/2 \left(s^2 + s + 1\right)(s + 1) & -1/2 s \left(s^3 + 4s^2 + 6s + 5\right) \end{pmatrix},
\]

\[
H_{\text{obs}}^0 = \begin{pmatrix} 1/4 (s + 2)(s - 1) \left(s^2 + 3s + 4\right) \\ 1/2 \left(s^2 + s + 1\right) \left(s^2 + 3s + 4\right) \end{pmatrix}, \quad \text{and}
\]

\[
U_2 = \begin{pmatrix} - (s^2 + 1) & s(s + 1)^2 & s \left(s^2 + 2s + 3\right) \end{pmatrix}.
\]

The matrix \( H_{\text{obs}}^0\) is the transfer matrix of an asymptotic observer, but it is obviously not proper. We use Algorithm 3.2 in order to find the proper ones among the matrices \( H_{\text{obs}} = H_{\text{obs}}^0 + ZU_2\) where \( U_2 = (U_{21}, U_{22}) \in D^{1 \times (2+1)}\). In our case such matrices do really exist, and they are exactly those with

\[ Z = (\tilde{Z} + WG)\hat{U} \]

for some free parameter \( W \in S^{2 \times 1}\) and

\[
\tilde{Z} = \begin{pmatrix} \left(s^5 + 6s^4 + 13s^3 + 12s^2 + 8s + 2\right)(s + 2)(s - 1)(s^2 + 3s + 4) \\ \left(s^5 + 6s^4 + 13s^3 + 12s^2 + 8s + 2\right)(s^2 + s + 1) \left(s^2 + 3s + 4\right) \end{pmatrix} \left(s + 1\right)^3.
\]
\[ G = \left( \frac{2}{(s+1)^3} \right) \]  and
\[ \tilde{U} = (-1/2). \]

Choosing the parameter \( W \) as zero, we get \( Z = \tilde{Z}\tilde{U} \) and thus
\[
H_{\text{obs}} = \begin{pmatrix}
\frac{(s+2)(s-1)(s^2+1)}{4(s+1)^3} & \frac{(s^2+3s+4)(s^2-2s-1)}{2(s+1)^3} \\
\frac{(s^2+3s+4)(s^2-2s-1)}{2(s+1)^3} & \frac{(s^2+3s+4)(s^2-2s-1)}{2(s+1)^3}
\end{pmatrix}.
\]

This matrix is obviously both proper and \( T \)-stable. Checking whether \( X := X^0 + ZU_{21} \) is also \( T \)-stable and \((−X, H_{\text{obs}})(R^P) = Q \) shows that these conditions are indeed satisfied. So all that is left to do is constructing the controllable realization \( B_{\text{obs}} = \{ (\hat{w}_{w_1}) \in \mathcal{X}^{2+1}; P_{\text{obs}} \circ \hat{w} = Q_{\text{obs}} \circ w_1 \} \). The pair \((P_{\text{obs}}, −Q_{\text{obs}})\) defining this behavior can be obtained as a universal left annihilator of \((H_{\text{obs}}^R Id_1)\) or of \( d \cdot (H_{\text{obs}}^R Id_1) \) where \( d \) is a common denominator of \( H_{\text{obs}} \), cf. [2, Res. 2.10]. The universal left annihilator of a matrix with entries in \( D \) can be computed by means of the Smith form, compare for example [2, Definition and Lemma 2.7]. In our case we get that
\[
P_{\text{obs}} = \begin{pmatrix}
1/2 (s^2 + s + 1) & -1/4 (s + 2)(s - 1) \\
-4/3 (s + 1)^8 & 2/3 (s + 1)^8
\end{pmatrix}, \quad \text{and}
\]
\[
Q_{\text{obs}} = \begin{pmatrix}
0 & -1/4 (s + 2)(s - 1) \\
- (s^2 + 1)(s^2 + 3s + 4)(s^2 - 2s - 1)
\end{pmatrix}.
\]

Since \( \det(P_{\text{obs}}) = (s + 1)^8 \) is obviously contained in \( T \), \( B_{\text{obs}} \) is \( T \)-stable as predicted by Lemma 25.

We want to check whether the error behavior \( B_{\text{err}} = \{ \hat{w} - Q \circ w \in \mathcal{X}^{2}; w \in B, (\hat{w}_{w_1}) \in B_{\text{obs}} \} =: \{ v \in \mathcal{X}^{2}; R_{\text{err}} \circ v = 0 \} \) is really \( T \)-autonomous. Computing \( B_{\text{err}} \) using the Theorem on images of behaviors (cf. [7, p. 24. (34), 8, Theorem 6.2.6, 2, Res. 2.8]) yields that
\[
R_{\text{err}} = \begin{pmatrix}
4/3 (s + 1)^8 & -2/3 (s + 1)^8 \\
1/2 (s^2 + s + 1) & -1/4 (s + 2)(s - 1)
\end{pmatrix}.
\]

The determinant of \( R_{\text{err}} \) is \( (s + 1)^8 \in T \), i.e., \( B_{\text{err}} \) is really \( T \)-autonomous.

Of course we can try to choose another parameter \( W \in S^{2 \times 1} \) such that the resulting observer is of a simpler form. Some trying shows that the choice
\[
W := \begin{pmatrix}
7s^7 + 62s^6 + 227s^5 + 462s^4 + 573s^3 + 446s^2 + 225s + 46 \\
-5s^7 + 34s^6 + 112s^5 + 213s^4 + 261s^3 + 196s^2 + 90s + 17
\end{pmatrix}
\]
\[
\begin{pmatrix}
4(s^2 + 2s + 3)(s + 1)^2 \\
2(s^2 + 2s + 3)(s + 1)^2
\end{pmatrix}
\]
is beneficial: it leads to the transfer matrix
\[
H_{\text{obs}} = \begin{pmatrix}
-2 \\
2
\end{pmatrix}
\]
and thus to the asymptotic observer defined by
\[
P_{\text{obs}} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad Q_{\text{obs}} = \begin{pmatrix}
-2 \\
2
\end{pmatrix}.
In this case the error behavior is described by the matrix

$$R_{\text{err}} = \begin{pmatrix} 0 & s^2 + 2s + 3 \\ 1 & -3/2s - 1/2 \end{pmatrix}$$

with $\det(R_{\text{err}}) = -s^2 - 2s - 3 = -(s + 1 - i\sqrt{2})(s + 1 + i\sqrt{2}) \in T$.

The problem of algorithmically finding “minimal” $T$-observers among all those parameterized in the theory is not yet solved.

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References