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On certain spaces of vector measures of bounded variation ☆

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ABSTRACT

If (Σ, X) is a measurable space and X a Banach space we investigate the X-inheritance of copies of ℓ_∞ in certain subspaces $\Delta(\Sigma, X)$ of $bvca(\Sigma, X)$, the Banach space of all X-valued countable additive measures of bounded variation equipped with the variation norm. Among the consequences of our main theorem we get a theorem of J. Mendoza on the X-inheritance of copies of ℓ_∞ in the Bochner space $L_1(\mu, X)$ and other of the author on the X-inheritance of copies of ℓ_∞ in $bvca(\Sigma, X)$.

1. Preliminaries

In what follows (Ω, Σ) will always be a measurable space and X a Banach space over the field $\mathbb{K}$ of real or complex numbers. We shall denote by $ca(\Sigma, X)$ the Banach space of all countably additive measures $F : \Sigma \to X$ provided with the semivariation norm $\|F\|_\Sigma$, while $cca(\Sigma, X)$ will stand for the closed subspace of $ca(\Sigma, X)$ of all those measures with relatively compact range. We represent by $ca^+(\Sigma)$ the set of positive and finite measures defined on Σ and denote by $bvca(\Sigma, X)$ the Banach space of all X-valued countably additive measures $F : \Sigma \to X$ of bounded variation equipped with the variation norm $|F|_\Sigma$. If $\mu \in ca^+(\Sigma)$ then $bvca_\mu(\Sigma, X)$ stands for the subspace of $bvca(\Sigma, X)$ of all those measures $F \in bvca(\Sigma, X)$ such that $F \ll \mu$. A Banach space X is said to have the Radon–Nikodým property (RNP) with respect to a finite measure space (Ω, Σ, μ) if every $F \in bvca_\mu(\Sigma, X)$ has a Bochner μ-integrable X-valued derivative. If X has the RNP with respect to every finite measure space (Ω, Σ, μ), it is said that X has the RNP [2]. Following [9] we denote by $\mathcal{M}_1(\Sigma, X)$ the (closed) linear subspace of $bvca(\Sigma, X)$ consisting of all measures $F \in bvca(\Sigma, X)$ with the Radon–Nikodým property, that is, such that for each $\mu \in ca^+(\Sigma)$ with $F \ll \mu$ there exists a density $f \in \mathcal{L}_1(\mu, X)$ with $F(E) = (B)\int_E f\, d\mu$ for every $E \in \Sigma$. According to [9, Theorem 5.22] the space $\mathcal{M}_1(\Sigma, X)$ is linearly isometric to $ca(\Sigma) \widehat{\otimes}_\pi X$. Thus if $F \in \mathcal{M}_1(\Sigma, X)$ then $F \in cca(\Sigma, X) = ca(\Sigma) \widehat{\otimes}_\varepsilon X$. If X has the RNP with respect to each $\mu \in ca^+(\Sigma)$ then clearly $\mathcal{M}_1(\Sigma, X) = bvca(\Sigma, X)$. Particularly, if X has the RNP then $\mathcal{M}_1(\Sigma, X) = bvca(\Sigma, X)$.

If each $\mu \in ca^+(\Sigma)$ is purely atomic, then $ca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X contains, respectively, a copy of c_0 or ℓ_∞ [4]. If X has the Radon–Nikodým property with respect to each $\mu \in ca^+(\Sigma)$, then $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does [6]. As a consequence, if each $\mu \in ca^+(\Sigma)$ is purely atomic, then $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X contains, respectively, a copy of c_0 or ℓ_∞. If there exists a nonzero atomless measure $\mu \in ca^+(\Sigma)$, the latter statement is no longer true [11]. However, if the range space of the measures is a dual Banach space X^*, then $bvca(\Sigma, X^*)$ has a copy of c_0 if and only if X^* does [10]. For further information about the inheritance of copies of c_0 or ℓ_∞ in other spaces of vector-valued functions or operators we refer the reader to the excellent tract [1].

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2. Results

In what follows $\Delta(\Sigma, X)$ will stand for any closed linear subspace of $bvca(\Sigma, X)$, i.e. a Banach space of countably additive measures $F : \Sigma \to X$ of bounded variation equipped with the variation norm $|F|_\Sigma$. Given $\Delta(\Sigma, X)$, if Γ is a sub-σ-algebra of Σ and Y is a closed linear subspace of X we denote by $\Delta[\Gamma, Y]$ the linear subspace of $bvca(\Gamma, Y)$ consisting of the Y-valued restrictions to Γ of the elements of $\Delta(\Sigma, X)$, i.e.

$$\Delta[\Gamma, Y] = \{F|_\Gamma\colon\ F \in \Delta(\Sigma, X),\ F(\Gamma) \subseteq Y\},$$

equipped with the norm $|\cdot|_\Gamma$ of $bvca(\Gamma, Y)$. If $\mu \in ca^+(\Sigma)$ we represent by $\Delta_\mu[\Gamma, Y]$ the linear subspace of $\Delta[\Gamma, Y]$ consisting of those $G \in \Delta[\Gamma, Y]$ with $G \ll \mu|_\Gamma$.

Theorem 2.1. *If $\Delta(\Sigma, X)$ contains an isomorphic copy of ℓ_∞ then either X contains an isomorphic copy of ℓ_∞ or there exist a countably generated sub-σ-algebra Γ of Σ, a scalar measure $\mu \in ca^+(\Sigma)$ and a closed and separable linear subspace Y of X such that the closure of $\Delta_\mu[\Gamma, Y]$ in $bvca_{\mu|_\Gamma}(\Gamma, Y)$ contains an isomorphic copy of ℓ_∞.*

Proof. Let J be an isomorphism from ℓ_∞ into $\Delta(\Sigma, X)$ and denote by $\{e_n\colon n \in \mathbb{N}\}$ the unit vector sequence of ℓ_∞. For each pair $m, n \in \mathbb{N}$ let $\{E^m_{n,i}\colon 1 \leqslant i \leqslant k(m,n)\}$ be a finite partition of Ω by elements of Σ verifying that

$$|Je_n|_\Sigma \leqslant \sum_{i=1}^{k(m,n)} \|Je_n(E^m_{n,i})\| + \frac{1}{m}.$$

Let us denote by Λ the algebra generated by the countable family

$$\{E^m_{n,i}\colon 1 \leqslant i \leqslant k(m,n);\ m, n \in \mathbb{N}\}.$$

Observe that Λ is also a countable family [7, 1.5 Theorem C] and denote by Γ the σ-algebra generated by Λ. Since clearly $\Omega \in \Gamma$, then Γ is a sub-σ-algebra of Σ.

Define $T : \Delta(\Sigma, X) \to \Delta[\Gamma, X]$ by $TF = F|_\Gamma$. This map is well defined, linear and bounded since $|F|_\Gamma|_\Gamma \leqslant |F|_\Sigma$ for all $F \in \Delta(\Sigma, X)$. Thus $T \circ J$ is a bounded map from ℓ_∞ into $\Delta[\Gamma, X]$. Further, given $m \in \mathbb{N}$, by virtue of the definition of Γ one has

$$|Je_n|_\Sigma \leqslant \sum_{i=1}^{k(m,n)} \|Je_n(E^m_{n,i})\| + \frac{1}{m} \leqslant |Je_n|_\Gamma|_\Gamma + \frac{1}{m},$$

which implies that $|Je_n|_\Sigma = |Je_n|_\Gamma|_\Gamma = |(T \circ J)e_n|_\Gamma$ for every $n \in \mathbb{N}$.

Let Y denote the closure in X of the linear cover of the countable subset $\bigcup_{n=1}^\infty Je_n(\Lambda)$ of X formed by the union of the images of the countable set Λ by the measures Je_n. Let us suppose that $\Lambda = \{A_n\colon n \in \mathbb{N}\}$. Then assume that X does not contain a copy of ℓ_∞ and define $J_n : \ell_\infty \to X$ by $J_n\xi = (J\xi)(A_n)$ for each $n \in \mathbb{N}$. Since ℓ_∞ does not live in X and J_n is a bounded linear operator for each $n \in \mathbb{N}$, all the operators J_n are weakly compact. So, according to [5], there exists an infinite subset N of $\mathbb{N}$ such that

$$J_n\xi = \sum_{i=1}^\infty \xi_i J_n e_i$$

for each $n \in \mathbb{N}$ and $\xi \in \ell_\infty(N)$. So one has

$$J\xi(A_n) = \sum_{i=1}^\infty \xi_i Je_i(A_n)$$

in X for every $\xi \in \ell_\infty(N)$ and $n \in \mathbb{N}$. But since $Je_i(A_n) \in Y$ for every $i, n \in \mathbb{N}$ and Y is closed, we get that $J\xi(A_n) \in Y$ for every $\xi \in \ell_\infty(N)$ and $n \in \mathbb{N}$, i.e. $J\xi(A) \in Y$ for every $\xi \in \ell_\infty(N)$ and $A \in \Lambda$. By the classic theorem on monotone classes [7, 1.6 Theorem B], the family $\{E \in \Sigma\colon\ J\xi(E) \in Y\ \forall\xi \in \ell_\infty(N)\}$ contains the sub-σ-algebra Γ generated by Λ. So we conclude that $J\xi(A) \in Y$ for every $\xi \in \ell_\infty(N)$ and $A \in \Gamma$.

Hence $J\xi|_\Gamma \in \Delta[\Gamma, Y]$ for $\xi \in \ell_\infty(N)$, i.e. $(T \circ J)\xi \in \Delta[\Gamma, Y]$ for each $\xi \in \ell_\infty(N)$ or, in other words, $T(J(\ell_\infty(N))) \subseteq \Delta[\Gamma, Y]$. There is no loss of generality by identifying N with $\mathbb{N}$.

If $\mu := \sum_{n=1}^\infty 2^{-n}|Je_n|_\Sigma$ then $\mu \in ca^+(\Sigma)$, and since $Je_n|_\Gamma \in bvca_{\mu|_\Gamma}(\Gamma, Y)$ for all $n \in \mathbb{N}$, setting $\xi^m = (\xi_1, \ldots, \xi_m, 0, 0, \ldots)$ one has $J\xi^m|_\Gamma \in bvca_{\mu|_\Gamma}(\Gamma, Y)$ for every $m \in \mathbb{N}$ and $\xi \in \ell_\infty$. Thus $J\xi^m|_\Gamma \ll \mu|_\Gamma$ for every $m \in \mathbb{N}$ and $\xi \in \ell_\infty$. But we can go beyond this. Let us show the following.

Claim. *$J\xi|_\Gamma \ll \mu|_\Gamma$ for each $\xi \in \ell_\infty$.*

Proof. Since we are assuming that X does not contain a copy of ℓ_∞ the linear operators $J_E : \ell_\infty \to X$ defined by $J_E\xi = J\xi(E)$ for each $E \in \Sigma$ are weakly compact and consequently a standard argument (see for instance the proof of main theorem of [6]) shows that $E \mapsto \sum_{n=1}^{\infty} \xi_n J_E e_n$ is an X-valued countable additive measure on Σ of bounded variation and that the map $S : \ell_\infty \to bvca(\Sigma, X)$ given by $S\xi(E) = \sum_{n=1}^{\infty} \xi_n J_E e_n$ is well defined, bounded and verifies that $S\xi \ll \mu$ for all $\xi \in \ell_\infty$. Fix ξ and note that

$$S\xi(A_n) = \sum_{i=1}^{\infty} \xi_i J_{A_n} e_i = \sum_{i=1}^{\infty} \xi_i J e_i(A_n) = J\xi(A_n)$$

for every $n \in \mathbb{N}$. Consequently $S\xi$ coincides with $J\xi$ on the algebra Λ. Now assume that $A \in \Gamma$ satisfies that $\mu(A) = 0$. Then $S\xi(A) = \mathbf{0}$ since $S\xi \ll \mu$ and hence $x^* S\xi(A) = 0$ for every $x^* \in X^*$. Given that Λ is an algebra and the countably additive measure $x^* J\xi|_\Gamma$ is an extension of the bounded, scalarly valued, countably additive measure $x^* S\xi|_\Lambda$ to the σ-algebra Γ, Hahn's extension theorem [3, Corollary III.5.9] guarantees that $x^* J\xi|_\Gamma = x^* S\xi|_\Gamma$ and, consequently, that $x^* J\xi(A) = 0$. Since this is true for every $x^* \in X^*$, it follows that $J\xi(A) = \mathbf{0}$. So $J\xi|_\Gamma \ll \mu|_\Gamma$, which completes the proof of the claim. □

Summarizing: first we have seen that $(T \circ J)\xi \in \Delta[\Gamma, Y]$ for every $\xi \in \ell_\infty(N)$, where N is an infinite subset of $\mathbb{N}$, and then we have proved that $(T \circ J)\xi \ll \mu|_\Gamma$ for each $\xi \in \ell_\infty(N)$, so that $(T \circ J)\xi \in \Delta_\mu[\Gamma, Y]$ for every $\xi \in \ell_\infty(N)$. Consequently, $T \circ J$ is a bounded linear operator from $\ell_\infty(N)$ into $\Delta_\mu[\Gamma, Y]$. Since $|(T \circ J)e_n|_\Gamma = |Je_n|_\Sigma$ for every $n \in N$, then $\inf_{n \in \mathbb{N}} |(T \circ J)e_n|_\Gamma > 0$ and Rosenthal's ℓ_∞ theorem guarantees that the completion of $\Delta_\mu[\Gamma, Y]$, that is, the closure of $\Delta_\mu[\Gamma, Y]$ in $bvca_{\mu|_\Gamma}(\Gamma, Y)$, contains a copy of ℓ_∞. □

Corollary 2.2. *If $\Delta_\mu[\Gamma, Y]$ is separable for every countably generated sub-σ-algebra Γ of Σ, every $\mu \in ca^+(\Gamma)$ and every closed and separable linear subspace Y of X, then X contains a copy of ℓ_∞ if $\Delta(\Sigma, X)$ does.*

Proof. If $\Delta(\Sigma, X)$ contains a copy of ℓ_∞ but X does not then Theorem 2.1 provides a countably generated sub-σ-algebra Γ of Σ, a scalar measure $\mu \in ca^+(\Sigma)$ and a closed and separable linear subspace Y of X such that the closure of $\Delta_\mu[\Gamma, Y]$ in $bvca_{\mu|_\Gamma}(\Gamma, Y)$ contains a copy of ℓ_∞, contradicting the hypothesis. □

Corollary 2.3. *If every closed and separable linear subspace of X has the Radon–Nikodým property, then X contains a copy of ℓ_∞ if $\Delta(\Sigma, X)$ does.*

Proof. If Γ is a sub-σ-algebra of Σ, $\mu \in ca^+(\Gamma)$ and Y is a closed linear subspace of X then, by hypothesis, $\Delta_\mu[\Gamma, Y]$ is linearly isometric to a subspace of $L_1(\Gamma, \mu|_\Gamma, Y)$. Hence, if Γ is a countably generated sub-σ-algebra of Σ and Y is separable, then $\Delta_\mu[\Gamma, Y]$ is linearly isometric to a linear subspace of the separable Banach space $L_1(\Gamma, \mu|_\Gamma, Y)$. According to Corollary 2.2 this implies that X contains a copy of ℓ_∞ if $\Delta(\Sigma, X)$ does. □

Corollary 2.4. *(See Mendoza [8].) If $\Delta(\Sigma, X) = \{F \in \mathcal{M}_1(\Sigma, X)\colon\ F \ll \mu\}$ with $\mu \in ca^+(\Sigma)$, then X contains a copy of ℓ_∞ if $\Delta(\Sigma, X)$ does.*

Proof. First note that if Γ is a sub-σ-algebra of Σ then $\Delta[\Gamma, X]$ is isomorphic to a subspace of $L_1(\Gamma, \mu|_\Gamma, X)$. In fact, if $G \in \Delta[\Gamma, X]$ there exists $F \in \mathcal{M}_1(\Sigma, X)$ with $F \ll \mu$ such that $F|_\Gamma = G$. So there is $f \in L_1(\Sigma, \mu, X)$ satisfying that $F(E) = \int_E f \, d\mu$ for every $E \in \Sigma$ and, according to [2, Chapter V, Theorem 4], there exists a unique $E(f \mid \Gamma) \in L_1(\Gamma, \mu|_\Gamma, X)$, the so-called conditional expectation of f relative to Γ, such that $G(A) = \int_A E(f \mid \Gamma) \, d\mu|_\Gamma$ for every $A \in \Gamma$. Since

$$|G|_\Gamma = \int_\Omega \left\| E(f \mid \Gamma)(\omega) \right\| d\mu|_\Gamma(\omega)$$

the map $G \mapsto E(f \mid \Gamma)$ is a linear isometry from $\Delta[\Gamma, X]$ into $L_1(\Gamma, \mu|_\Gamma, X)$. Hence, if Γ is a countably generated sub-σ-algebra of Σ and Y is a closed and separable linear subspace of X then $\Delta[\Gamma, Y]$ is linearly isometric to a linear subspace of $L_1(\Gamma, \mu|_\Gamma, Y)$. Since $L_1(\Gamma, \mu|_\Gamma, Y)$ is separable, we apply Corollary 2.2 to get the conclusion. □

Corollary 2.5. *$\mathcal{M}_1(\Sigma, X)$ contains a copy of ℓ_∞ if and only if X does.*

Proof. If J is an isomorphism from ℓ_∞ into $\mathcal{M}_1(\Sigma, X)$, set $\mu = \sum_{n=1}^{\infty} 2^{-n} |Je_n|_\Sigma$. Then let $\Delta(\Sigma, X) := \{F \in \mathcal{M}_1(\Sigma, X)\colon F \ll \mu\}$ be as in Corollary 2.4 and define $\pi : \mathcal{M}_1(\Sigma, X) \to \Delta(\Sigma, X)$ so that $\pi(F)$ is the μ-continuous part of F supplied by the Lebesgue decomposition theorem of F. The fact that F has the Radon–Nikodým property assures that $\pi(F) \in \mathcal{M}_1(\Sigma, X)$ and hence $\pi(F) \in \Delta(\Sigma, X)$. Clearly π is a continuous linear projection and, consequently, $T := \pi \circ J$ is a bounded map from ℓ_∞ into $\Delta(\Sigma, X)$. Since $Te_n = Je_n$ for all $n \in \mathbb{N}$, Rosenthal's ℓ_∞ theorem ensures that $\Delta(\Sigma, X)$ contains a copy of ℓ_∞. So Corollary 2.4 applies. □

Corollary 2.6. *(See Ferrando [6].) If X has the Radon–Nikodým property with respect to each $\mu \in ca^+(\Sigma)$, then $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does.*

Proof. Just notice that if X has the RNP with respect to each $\mu \in ca^+(\Sigma)$ then clearly $bvca(\Sigma, X) = \mathcal{M}_1(\Sigma, X)$, so we can use Corollary 2.5. □

3. Remark

In [8] the containment of a copy of ℓ_∞ in $L_1(\Omega, \Sigma, \mu, X)$ is reduced by means of [3, Lemma III.8.5] to the presence of a copy of ℓ_∞ in a space $L_1(\Omega_1, \Sigma_1, \mu_1, X)$, where $(\Omega_1, \Sigma_1, \mu)$ is a separable finite measure space (see also [1, Theorem 1.6.2]) and then, assuming that X contains no copy of ℓ_∞, an application of Drewnowski's lemma [5] allows to locate a copy of ℓ_∞ in a space $L_1(\Omega_1, \Sigma_1, \mu_1, X_0)$ with separable X_0, getting a contradiction. In the proof of Theorem 2.1 we have followed as far as possible a similar (but not identical) strategy, working with the measures rather than with the functions.

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