JOURNAL OF ALGEBRA **207**, 1–42 (1998)
ARTICLE NO. JA987431

The p-Intersection Subgroups in Quasi-Simple and Almost Simple Finite Groups

P. Fleischmann and W. Lempken

Institute for Experimental Mathematics, University of Essen, 45326 Essen, Germany

P. H. Tiep

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

Communicated by Gernot Stroth

Received May 7, 1996

DEDICATED TO G. O. MICHLER ON THE OCCASION OF HIS 60TH BIRTHDAY

Let p be a fixed prime and G a finite group. A proper subgroup $X < G$ is called a *p-intersection subgroup* if $X \cap X^g$ is a p-group for each $g \in G \setminus X$, but X is not a p-group. In this paper we classify the p-intersection subgroups in the quasi-simple and almost simple finite groups.

1. INTRODUCTION

Let G be a finite group containing a proper subgroup X such that $X \cap X^g = 1$ for each $g \in G \setminus X$, i.e., X is a self-normalizing *TI*-subgroup in G. Because of a classical theorem of Frobenius, G contains a nontrivial normal subgroup N such that G is a semidirect product of N and X. Nowadays such groups are called Frobenius groups for obvious reasons.

In this paper we generalize the concept of Frobenius groups in the following way.

DEFINITION 1.1. Let $\mathscr{E}$ be a property of groups and G a finite group.

(i) A proper subgroup $H < G$ will be called an *$\mathscr{E}$-intersection subgroup* if H does not have property $\mathscr{E}$, but $H \cap H^g$ has property $\mathscr{E}$ for every $g \in G \setminus H$. The set of $\mathscr{E}$-intersection subgroups will be denoted by $\mathscr{I}_{\mathscr{E}}$ or $\mathscr{I}_{\mathscr{E}}(G)$.

0021-8693/98 $25.00

(ii) If π is a set of primes and $\mathscr{E}$ is the property of being a π-group, then an $\mathscr{E}$-intersection subgroup will be called a *π-intersection subgroup*, or a *p-intersection subgroup* if $\pi = \{p\}$. The corresponding $\mathscr{I}_{\mathscr{E}}(G)$ will be denoted by $\mathscr{I}_{\pi}(G)$ or $\mathscr{I}_{p}(G)$, respectively.

Note that if $\mathscr{E}$ is the property that a group is trivial, then G is a Frobenius group if and only if $I_{\mathscr{E}}(G) \neq \emptyset$; in particular, in this situation G is not simple. This raises the question about structural consequences for different properties $\mathscr{E}$. As a first natural generalization, we investigate the case where $\mathscr{E}$ is the property of being a p-group for some fixed prime p. Clearly, this will include the Frobenius groups, but, as the following example shows, simple groups also occur: take $G \cong GL_3(2)$ and $\Sigma_4 \cong X \in \mathscr{I}_2(G)$.

The main result of this paper is the classification of p-intersection subgroups in quasi-simple and almost simple finite groups G. Recall that G is called quasi-simple if G is perfect and $G/Z(G)$ is nonabelian simple; furthermore, G is called almost simple if $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for some nonabelian simple group S.

The strategy of proof is as follows. First we classify the p-intersection subgroups of the nonabelian simple groups. This, in turn, will be used to obtain the corresponding results for the almost simple groups and the quasi-simple groups, using well-known data on automorphism groups and Schur multipliers of the simple groups (e.g., consult 2.7, [6], and [18]). In Section 4 we shall deal with the alternating groups, in Section 5 the analysis is carried out for finite groups of Lie type, while it is done for the sporadic groups in Section 6.

It may be worth noting that these results will be used in a forthcoming paper [14] to classify all primitive permutation groups in which two-point stabilizers are p-groups.

2. BASIC DEFINITIONS AND RESULTS

G will always denote a finite group and p a prime. The notation used is standard and can be found in [21] or [15]. In particular, $\pi(G)$ denotes the set of prime divisors of $|G|$; for any set π of primes, π' is the complement of π in the set of all primes. Moreover, $O_{\pi}(G)$ denotes the largest normal π-subgroup of G, with $O_p(G) := O_{\{p\}}(G)$ and $O(G) := O_{\{2\}'}(G)$; $Z(G)$ is the center of G, and $F(G)$ denotes the Fitting-subgroup of G. For any $g \in G$, $|g|$ is the order of g; the notation $=_H$, $\leq_H$, etc., will indicate equality or containment, etc., up to conjugacy in the subgroup $H \leq G$. $X < \cdot G$ means that X is a maximal subgroup of G.

The sets of Fermat- and Mersenne-primes will be denoted by $\mathscr{F}$ and $\mathscr{M}$, respectively.

First we record some obvious facts about the set $\mathscr{I}_{\mathscr{E}}(G)$ and its elements.

LEMMA 2.1. *Let G be a finite group.*

(i) *If property non-$\mathscr{E}$ is inherited by subgroups, then $\mathscr{I}_{\mathscr{E}}(G) = \emptyset$.*

(ii) *If $H \in \mathscr{I}_{\mathscr{E}}(G)$, then $N_G(H) = H$. If $H < G$ is minimal among the subgroups of G that do not have property $\mathscr{E}$, then $H \in \mathscr{I}_{\mathscr{E}}(G)$ if and only if $N_G(H) = H$.*

(iii) *Suppose that the property $\mathscr{E}$ is inherited by subgroups. Then we have the following*:

If $U, V \in \mathscr{I}_{\mathscr{E}}(G)$, then either $U \cap V \in \mathscr{I}_{\mathscr{E}}(G)$ or $U \cap V$ has property $\mathscr{E}$.

If $H \in \mathscr{I}_{\mathscr{E}}(G)$, then $N_G(S) \leq H$ for any $S \leq H$ not having property $\mathscr{E}$.

If $H \leq G$ and $X \in \mathscr{I}_{\mathscr{E}}(G)$ with $H \nleq X$, then either $X \cap H < H$ has property $\mathscr{E}$ or $X \cap H \in \mathscr{I}_{\mathscr{E}}(H)$.

LEMMA 2.2. *Suppose S is simple and $X \in \mathscr{I}_{\mathscr{E}}(S)$. Then*

$$X = \langle U \leq X | U \text{ has property } \mathscr{E} \rangle = \langle X \cap X^g | g \in S \setminus X \rangle.$$

Proof. Without loss we may assume that $X \neq 1$, and so S is nonabelian. Put $X^* := \langle X \cap X^g | g \in S \setminus X \rangle$ and $X_{\mathscr{E}} := \langle U \leq X | U$ has property $\mathscr{E} \rangle$. Note that $X^* \trianglelefteq X$ *and* $X^* \leq X_{\mathscr{E}} \leq X$. *By a theorem of Wielandt* ([21], *Satz V*.7.5), *there exists a normal subgroup $N \trianglelefteq S$ with $N \cap X = X^*$. Since S is not a Frobenius group, $X^* \neq 1$, and thus $N = S$ as well as $X = X^* = X_{\mathscr{E}}$.* ∎

LEMMA 2.3. *Let X be a nontrivial proper subgroup of the group G with $\pi(X) \neq \{p\}$ for some prime p. Then the following are equivalent*:

(i) *$X \in \mathscr{I}_p(G)$.*

(ii) *$N_G(Y) \leq X$ for any nontrivial $Y \leq X$ with $\pi(Y) \neq \{p\}$.*

(iii) *X is strongly q-embedded in G for $p \neq q \in \pi(X)$, i.e., $N_G(Q) \leq X$ for q-subgroups $Q \neq 1$ of X.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) being clear, we assume (iii) and let $g \in G \setminus X$. Now suppose $1 \neq Q \in Syl_q(X \cap X^g)$ for some prime $q \neq p$. If $Q \leq \tilde{Q} \in Syl_q(G)$, then $Z(\tilde{Q}) \leq N_G(Q) \leq X \cap X^g$; hence $\tilde{Q} \leq C_G(Z(\tilde{Q})) \leq X \cap X^g$ and $Q = \tilde{Q}$. So Q and ${}^gQ$ are in $Syl_q(X)$. Hence there is $x \in X$ with $xg \in N_G(Q) \setminus X$, a contradiction that finally proves the claim. ∎

LEMMA 2.4. *Let p be a prime, $X \in \mathscr{I}_p(G)$, and $H \le G$. Then the following hold*:

(i) *If $p \neq q \in \pi(X)$, then $Syl_q(X) \subseteq Syl_q(G)$ and* $\gcd(|X|, |G:X|)$ *is a power of p.*

(ii) *$Z(G) \le Core_G(X) \le O_p(G) \cap X \le O_p(X)$.*

(iii) *If $H \nleq X$, then either $X \cap H$ is a p-group or $X \cap H \in \mathscr{I}_p(H)$.*

Proof. All claims are immediate in view of Lemma 2.3. ∎

LEMMA 2.5. *Let $N \trianglelefteq G$, $X < G$, and let p be a prime.*

(i) *If $X \in \mathscr{I}_p(G)$ and if $X \cap N$ is a p-group, then either $XN = G$ or $XN/N \in \mathscr{I}_p(G/N)$.*

(ii) *If $X \in \mathscr{I}_p(G)$ and if $X \cap N$ is not a p-group, then $XN = G$ and $X \cap N \in \mathscr{I}_p(N)$.*

(iii) *Suppose that $N \le O_p(G)$. Then $XN/N \in \mathscr{I}_p(G/N)$ if and only if $XN \in \mathscr{I}_p(G)$.*

(iv) *Suppose that $\pi(G/N) = \{p\}$. If X is a minimal element of $\mathscr{I}_p(N)$ or if $X \in \mathscr{I}_p(N)$ such that $X^G = X^N$, then $N_G(X) \in \mathscr{I}_p(G)$. In particular, $\mathscr{I}_p(G) \neq \emptyset$ iff $\mathscr{I}_p(N) \neq \emptyset$.*

Proof. Assume first that $X \in \mathscr{I}_p(G)$, that $X \cap N$ is a p-group, and that $XN < G$. Clearly, XN/N is not a p-group. Let U/N be a nontrivial q-subgroup of XN/N for a prime $q \neq p$ and $g \in G$ with $gN \in N_{G/N}(U/N)$. Then we can assume that $U = QN$ for some $Q \in Syl_q(U)$ with $Q \le \tilde{Q} \in Syl_q(X) \subseteq Syl_q(G)$. Since ${}^gQ, Q \in Syl_q(U)$, there is $u = \gamma n \in U$ with $\gamma \in Q$ and $n \in N$ such that $N_G(Q) \ni ug = \gamma ng$. Therefore $ng \in N_G(Q)$, whence $gN \in XN/N$. Hence $XN/N \in \mathscr{I}_p(G/N)$ by Lemma 2.3. This proves part (i).

For part (ii) observe that N is not contained in X, and thus $X \cap N \in \mathscr{I}_p(N)$; now apply the Frattini argument to N and some $Q \in Syl_q(X \cap N) \subseteq Syl_q(N)$ with $p \neq q \in \pi(X \cap N)$.

The claims in (iii) are obvious in view of part (i).

Henceforth assume that $\pi(G/N) = \{p\}$ and $X \in \mathscr{I}_p(N)$. Let $Q \neq 1$ be a q-subgroup of $Y := N_G(X)$ for some prime $q \neq p$ and let $g \in N_G(Q)$. Clearly, $Q \le N_N(X) = X$, and so $Q \le X \cap X^g \in \mathscr{I}_p(N)$.

If X is maximal in $\mathscr{I}_p(N)$, then $X \cap X^g = X$, and so $g \in Y$; hence $Y \in \mathscr{I}_p(G)$ by Lemma 2.3. If $X^G = X^N$, then $X^g = X^n$ for some $n \in N$, and thus $Q \le X \cap X^n$; therefore $n \in N_N(X) = X$, and so $g \in Y$. Again we get $Y \in \mathscr{I}_p(G)$.

The remaining claims in part (iv) are easy consequences of part (ii). ∎

COROLLARY 2.6. *If G is minimal subject to $\mathscr{I}_p(G) \neq \emptyset$ (relative to taking subgroups and factor-groups), then either G is a nonabelian simple group, or for any proper normal subgroup $N \lhd G$ and $X \in \mathscr{I}_p(G)$ we have $G = NX$ with $N \cap X$ a p-group.*

COROLLARY 2.7. *Let G be a quasi-simple group and $L := G/Z(G)$. Then $\mathscr{I}_p(G) \neq \emptyset$ if and only if $\mathscr{I}_p(L) \neq \emptyset$ and $Z(G)$ is a p-group. Moreover, each $X \in \mathscr{I}_p(G)$ is the complete inverse image in G of some $Y \in \mathscr{I}_p(L)$, i.e., $\mathscr{I}_p(L) = \{X/Z(G) | X \in \mathscr{I}_p(G)\}$.*

Proof. Suppose $X \in \mathscr{I}_p(G)$ and let $x \neq 1$ be a p'-element of X. By Lemma 2.3, $X \geq C_G(x) \geq Z(G)$. now for any $g \in G \setminus X$, $Z(G) \leq X \cap X^g$, and so $Z(G)$ is a p-group. It remains to apply Lemma 2.5(iii). ∎

Remark. As Corollary 2.7 indicates, a check (using [6], [18] for instance) whether p divides the order of the Schur multiplier of L for the simple groups L with $\mathscr{I}_p(L) \neq \emptyset$ reduces the quasi-simple case to the simple case. This easy treatment is left to the reader.

LEMMA 2.8. *Suppose $G = G'K$ with $K \lhd G$ and $G' \cap K \leq Z(G)$; moreover, assume that either $G' \leq H \leq G$ or $H = G/Z$ with $Z \leq Z(G)$. If $\mathscr{I}_p(H) = \emptyset$, then $\mathscr{I}_p(G) = \emptyset$.*

Proof. Suppose that $X \in \mathscr{I}_p(G)$. Note that $[X, K] \leq G' \cap K \leq Z(G) \leq X$, and thus $K \leq N_G(X) = X$; in particular, K is a p-group. In the first case, G/H is a p-group, and so $\mathscr{I}_p(H) \neq \emptyset$ by Lemma 2.5(iv). In the second case, $Z \leq X \cap O_p(G)$, and so $X/Z \in \mathscr{I}_p(H)$ by Lemma 2.5(iii). ∎

LEMMA 2.9. *Let G be a finite group and $X_1 \leq X_2 \leq G$ with $X_1, X_2 \in \mathscr{I}_p(G)$, and $[X_2 : X_1]$ a power of p. If $O_{p'}(X_2) \neq 1$, then $X_1 = X_2$.*

Proof. By the assumption we have

$$|O_{p'}(X_2)X_1| = \frac{|O_{p'}(X_2)|\,|X_1|}{|O_{p'}(X_2) \cap X_1|} = |X_1| \cdot p^x$$

for some $x \in \mathbb{N}$. Hence, $O_{p'}(X_2) \leq X_1$, and thus $X_2 \leq N_G(O_{p'}(X_2)) \leq X_1$. ∎

THEOREM 2.10. *Let G be a finite group, p be an odd prime, and $X \in \mathscr{I}_p(G)$ with even order. Moreover, let $T \in Syl_2(X), T_1 := \Omega_1(T)$ and $Q := O(G)$. Then one of the following holds:*

(i) *T is cyclic or (generalized) quaternion and $C_G(T_1) \leq X$, as well as $G = QC_G(T_1) = QX$.*

(ii) *$Q := O_p(G)$ and $G/Q \cong SL_2(q)$ or ${}^2B^2(q)$, where $q = 2^a \geq 4$ and $p = q - 1$ is a Mersenne prime. Moreover, $X = QN_G(T)$ with $X/QT \cong \mathbb{Z}_p$.*

Proof. Note that X is strongly embedded in G, and thus we can apply Bender's Theorem [2]. In view of Glauberman's Z^*-theorem, the claims in (i) are evident. We are left with the situation where G has a normal series $1 \trianglelefteq Q \trianglelefteq L \trianglelefteq G$ such that $Q = O(G)$, and G/L is isomorphic to a subgroup of $Out(L/Q)$ with $|G/L|$ odd and $L/Q \cong SL_2(q)$, ${}^2B_2(q)$, or $PSU_3(q)$ for some $q = 2^a \geq 4$; moreover, $X = QN_G(T)$ for some $T \in Syl_2(G)$.

If $Q \neq O_p(G)$, we get $G = X$ by 2.3, which, of course, is absurd. Therefore, $Q = O_p(G)$. Now put $\overline{G} := G/Q$ as well as $X_0 := X \cap L$; note that $\overline{X}_0 = N_{\overline{L}}(\overline{T}) \in \mathcal{I}_p(\overline{L})$ with $2 \in \pi(\overline{X}_0) \neq \{p\}$. Assume first that $\overline{L} \cong PSU_3(q)$. Then $\overline{X}_0 = \overline{T} : \overline{K}$, where $\overline{K}$ is cyclic of order $(q^2 - 1)/d$, with $d = \gcd(3, q+1)$. Since $N_{\overline{L}}(\overline{K}) \leq N_{\overline{L}}(\overline{Y})$ for any $\overline{Y} \leq \overline{K}$, and since $N_{\overline{L}}(\overline{K}) \nleq \overline{X}_0$, $\overline{K}$ must be a p-group. So $(q^2-1)/d = p^f$, with $f \geq 1$; as $\gcd(q-1, q+1) = 1$, we get $q = 2$, a contradiction.

So we have $\overline{L} \cong SL_2(q)$ or $L \cong {}^2B_2(q)$; in any case $\overline{X}_0 = \overline{T} : \overline{K}$, where $\overline{K}$ is cyclic of order $q-1$. Since $N_{\overline{L}}(\overline{K})$ is not contained in $\overline{X}_0$, $q - 1 = p^f$ for some $f \geq 1$. Now we easily verify that $f = 1$, and thus $p = q - 1$ is a Mersenne prime; in particular, a is prime as well.

Suppose now $\overline{G} \neq \overline{L}$. As a is prime, we can assume without loss that $\overline{G} = \overline{L} : \overline{A}$ and $\overline{X} = \overline{X}_0 : \overline{A}$, where $\overline{A} = \langle \phi \rangle \cong \mathbb{Z}_a$ and ϕ acts as the standard Frobenius field automorphism on $\overline{L}$. Now we obtain a contradiction, because $C_{\overline{L}}(\phi)$ is not contained in $\overline{X}_0$. This proves that $\overline{G} = \overline{L}$; so (ii) follows. ∎

We will need the following result, which is an easy consequence of the classification of finite simple groups.

LEMMA 2.11. *Let S be a nonabelian finite simple group and $\alpha \in \mathrm{Aut}(S)$ an element whose order is coprime to $|S|$. Then $C_S(\alpha)$ is not nilpotent.*

Proof. We can assume $1 \neq \alpha$. It is well known from the "classification theorem" that the only nonabelian finite simple groups that have coprime automorphisms are among the simple Chevalley groups $G(q)$ or those of twisted type. In this case α is conjugate to some field automorphism and thus fixes elementwise a subgroup $H \cong G(p)$, defined over the prime field. But such a group is never nilpotent. ∎

COROLLARY 2.12. *Let S be a nonabelian simple group, $S \trianglelefteq G \leq \mathrm{Aut}(S)$, and $\hat{X} \in \mathcal{I}_p(G)$. Then $X := \hat{X} \cap S \in \mathcal{I}_p(S)$, $G = S\hat{X}$, and $\hat{X} = N_G(X)$.*

Proof. Suppose that X is a p-group. Then there is $x \in \hat{X}$ such that $|x| = \ell \in \pi(\hat{X}) \setminus \{p\}$. Moreover, $C_S(x) = C_X(x)$ is nilpotent, and so Lemma 2.11 implies that $\ell \in \pi(S)$. But then $\ell \in \pi(C_S(x))$, contrary to $\pi(X) =$

$\{p\}$. Now Lemma 2.5(ii) yields $X \in \mathscr{I}_p(S)$ and $G = S\hat{X}$, as well as $\hat{X} = N_G(X)$. ∎

3. SOME ARITHMETICAL LEMMAS

When dealing with groups of Lie type, we shall need the following arithmetical results.

LEMMA 3.1. *Let p be a prime, $q \in \mathbb{Z}$ an integer coprime to p, and $\nu_p(q)$ the p-adic valuation of q. Then the following holds*: $\nu_p(q^m - 1) = \nu_p(q - 1) + \nu_p(m)$ *if p is odd and divides $q - 1$ or $p = 2$ and 4 divides $q - 1$. If 4 divides $q + 1$, then*

$$\nu_2(q^m - 1) = \begin{cases} 1 & \text{if } m \text{ is odd} \\ \nu_2(q + 1) + \nu_2(m) & \text{if } 0 < m \text{ is even}. \end{cases}$$

Proof. This is well known. ∎

LEMMA 3.2. *Let $\epsilon \in \{\pm 1\}$ and $d, m, q \in \mathbb{N}$ such that $1 < q$ and d is a proper divisor of m. If*

$$\left| \frac{(q\epsilon)^m - 1}{(q\epsilon)^d - 1} \right| \leq (q\epsilon - 1)^2,$$

then

$$(\epsilon, q, m, d) \in \{(\pm 1, q, 2, 1), (-1, q, 3, 1), (-1, q, 4, 2), \\ (-1, 2, 4, 1), (-1, 2, 6, 3)\}.$$

Proof. If $\epsilon = 1$, then $q^2 - 2q + 1 \geq 1 + q^d + q^{2d} + \cdots + q^{d(m/d-1)}$ implies $d = 1$ and $m = 2$. Hence, in the following we suppose that $\epsilon = -1$ and put $X := |((-q)^m - 1)/((-q)^d - 1)|$.

In case $m \equiv d \equiv 1 \bmod 2$, we have $q^2 + 2q + 1 \geq X = (q^d - 1)[q^{m-2d} + q^{m-4d} + \cdots + q^d] + 1$. This yields $d = 1$ and, consequently, $m = 3$.

Suppose next that $m \equiv d \equiv 0 \bmod 2$. Then $X = 1 + q^d + \cdots + q^{d(m/d-1)} \leq q^2 + 2q + 1 \leq q^3 + 1$, and thus $d = 2$, as well as $m = 4$.

Finally, we consider the case $m \equiv 0 \bmod 2$ and $d \equiv 1 \bmod 2$. If $m \equiv 2 \bmod 4$, then, using the previous considerations,

$$\frac{q^m - 1}{q^{m/2} + 1} \frac{q^{m/2} + 1}{q^d + 1} \leq (q + 1)^2$$

implies $m/2 = 3$, $d = 1$, or $d = m/2$. In the first case we get the contradiction $(q^3 - 1)(q^2 - q + 1) \le q^2 + 2q + 1$. The second possibility leads to $m = 2$, $d = 1$, or to $q = 2$, $m = 6$, $d = 3$. If $m \equiv 0 \bmod 4$, then, again together with the previous results,

$$\frac{q^m - 1}{q^{m/2} - 1} \frac{q^{m/2} - 1}{q^d + 1} \le (q + 1)^2$$

implies $m = 4$, $d = 1$, and $q = 2$. ∎

COROLLARY 3.3. *Let* $q \in \mathbb{Z} \setminus \{0 \pm 1\}$, $a \in \mathbb{N}$, $a > 1$ *with* $|q^a - 1| = |q - 1| \cdot |\gcd(q - 1, a)|$. *Then* $(q, a) \in \{(-2, 2), (-2, 3), (-3, 2)\}$.

Proof. By Lemma 3.2, $a \in \{2, 3, 4\}$. Now the result follows from a direct inspection of these cases. ∎

LEMMA 3.4. *Let* $1 \neq q \in \mathbb{Z}$ *be an integer and* $m \in \mathbb{N}$. *Then* $(q^m - 1)/(q - 1)\gcd(q - 1, m) \in \mathbb{Z}$.

Proof. Put $X := (q^m - 1)/(q - 1)\gcd(q - 1, m)$ and let p be a prime dividing $q - 1$. If p does not divide m, then $\nu_p(X) \ge 0$. So assume that p divides $\gcd(q - 1, m)$. If $2 < p$ or $2 = p$ and $4 | q - 1$, then $\nu_p(X) = \nu_p(m) - \nu_p(\gcd(q - 1, m)) \ge 0$ by Lemma 3.1. If $p = 2 | \gcd(q - 1, m)$ and $4 | q + 1$, then $\nu_2(X) = \nu_2(q^m - 1) - 1 - \nu_2(\gcd(q - 1, m)) \ge 0$, and we are done. ∎

LEMMA 3.5. *Let* $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ *and* $m \in \mathbb{N}$ *odd*, $m > 1$. *Then*

$$\left| \frac{q^m - q}{q - 1} \right| = m \Leftrightarrow (q, m) = (-2, 3).$$

Proof. Suppose that $|(q^m - 1)/(q - 1)| = m$. Since $1 + |q| + |q|^2 + \cdots + |q|^{m-1} > m$, we get $q' := -q > 1$. Now $m = 1 + q'(q' - 1)(1 + q'^2 + \cdots + (q'^2)^{(m-3)/2}) \ge 1 + q'(q' - 1)^{(m-1)/2} \ge m$; this in turn implies $m = 3$ and $q' = 2$. ∎

LEMMA 3.6. *Let* $q \in \mathbb{Z} \setminus \{0, \pm 1\}$, $m \in \mathbb{N}$, *and* p *be a prime such that*

$$\left| \frac{q^m - 1}{(q - 1)\gcd(q - 1, m)} \right| = p^s > 1.$$

Then one of the following holds:

(i) *m is an odd prime dividing $p - 1$.*

(ii) *$m = p = 2$ and $q = \pm 2^{s+1} - 1$.*

(iii) $(q, m, p^s) \in \{(-2, 4, 5), (-2, 6, 7), (-3, 4, 5)\}$.

Proof. Put $X_0 := (q^m - 1)/(q-1)$ and $X := X_0/\gcd(q-1, m)$. Also note that $m > 1$.

Suppose first that $m = bc$ with $1 < b, c$. Now we get $X = ABCD$, with $A := (q^m - 1)/(q^b - 1)\gcd(q^b - 1, c)$, $B := \gcd(q^b - 1, c)/\gcd(q-1, c)$, $C := \gcd(q-1, c)\gcd(q-1, b)/\gcd(q-1, m)$, $D := (q^b - 1)/(q-1)\gcd(q-1, b)$, and $A, B, C, D \in \mathbb{Z}$. Note that $|A| = p^\alpha$ with $\alpha \geq 1$, because otherwise Corollary 3.3 implies $q^b \in \{-2, -3\}$, which, of course, is absurd. Moreover, observe that $\gcd(q^m - 1)/(q^b - 1), D) = \gcd(c, D)$.

Suppose, in addition, that $\gcd(c, p) = 1$, and thus $1 = \gcd(c, D) = \gcd(A, D)$. Since $\alpha \geq 1$, we get $|D| = 1$, and so Corollary 3.3 yields

$$(q, b) \in \mathscr{S} := \{(-2, 2), (-2, 3), (-3, 2)\}. \tag{1}$$

If $\gcd(p, m) = 1$, then $\gcd(p, b) = 1$ and thus $(q, c) \in \mathscr{S}$ as well; noting that $(q, m) = (-2, 9)$ yields the contradiction $|X| = 57$, we easily verify that the remaining possibilities lead to case (iii).

Suppose now that p divides m. The arguments leading to (1) now show that c is a prime different from $p = b \in \{2, 3\}$. Evaluating $|A| = p^\alpha$ for each of the three possibilities in (1), we easily derive a contradiction now.

Suppose next that $m = p^r$ for some $r \in \mathbb{N}$.

Assume that p is odd. Then $k := ord_{\mathbb{Z}_p^*}(q) | \gcd(p-1, m) = 1$, and so $k = 1$ and p divides $q - 1$. Now 3.1 implies $\nu_p(X_0) = \nu_p(m) = r$, and thus $|X_0| = m$. But now Lemma 3.5 implies $(q, m) = (-2, 3)$, which in turn leads to $|X| = 1$, a contradiction.

So we have $p = 2$, and q is odd. Clearly, if $r = 1$, then case (ii) holds. So we suppose that $r > 1$. If 4 divides $q^{2^{r-1}} - 1$, then $q^{2^{r-1}} + 1 = 2$, a contradiction. If 4 divides $q^{2^{r-1}} + 1$, then $q^{2^{r-1}} - 1 = |q - 1|$, which implies $q = -2$ and $r = 2$, a contradiction.

We are left with the situation where m is a prime different from p. If $p = 2$, then q and m are odd, and so we get $s = \nu_2(X) = \nu_2(m) = 0$ by Lemma 3.1, a contradiction. Therefore, p is odd.

Now let $k := ord_{\mathbb{Z}_{p^r}^*}(q)$, where $r = \nu_p(q^m - 1) \geq 1$. Note that k divides $\gcd(m, p-1) \in \{1, m\}$. If $k = 1$, we get $1 \leq s = \nu_p(X) = \nu_p(X_0) = 0$, a contradiction. Hence we have $k = m$, and so m divides $p - 1$. This is case (i). ∎

COROLLARY 3.7. *Let $q \in \mathbb{Z} \backslash \{0, \pm 1\}$, $m_i \in \mathbb{N}$, and p be a prime such that*

$$\left| \frac{q^{m_i} - 1}{(q-1)\gcd(q-1, m_i)} \right| = p^{s_i} > 1$$

for $i = 1$ and 2. Then $m_1 = m_2$.

Proof. In view of Lemma 3.6, only the case $p > 2$ has to be considered. We can assume that $m_1 < m_2$ are primes dividing $p - 1$. Then p^{m_1} divides $\gcd(q^{m_1} - 1, q^{m_2} - 1) = q - 1$, and we get the contradiction $m_1 = \nu_p((q^{m_1} - 1)/(q - 1)\gcd(q - 1, m_1)) = 0$. ∎

LEMMA 3.8. *Let r be a prime, $q = r^m$, $\epsilon = \pm 1$, and $q - \epsilon = 3 \cdot 2^x$, with $m > 1$ and $x > 0$. Then $\epsilon = 1$ and $q \in \{5^2, 7^2\}$.*

Proof. If $\epsilon = -1$, then $q + 1 = 3 \cdot 2^x$; hence the order of r modulo $q + 1$ is $2m$ and divides $\phi(q + 1) = 2^x$. Hence m is a power of 2. We get $r^m \equiv (\pm 1)^m \equiv 1 \mod (3)$, and 3 divides $q - 1$ but not $q + 1$. So we conclude $\epsilon = 1$.

The order $ord_{q-1}(r) = m$ divides $\phi(q - 1) = 2^x$. Hence $m = 2^t$ and $(r^{m/2} - 1)(r^{m/2} + 1) = 3 \cdot 2^x$. Suppose that 3 divides $r^{m/2} - 1$. Then $r^{m/2} + 1 = 2^s$; hence $m = 2$ and $r = 2^p - 1 \in \mathscr{M}$ is a Mersenne prime. Moreover, $r - 1 = 2(2^{p-1} - 1) = 3 \cdot 2^y$. We conclude $y = 1$, $p = 3$, and $r = 7$. Suppose that 3 divides $r^{m/2} + 1$. Then $m = 2$ and $r - 1 = 2^s$, i.e., $r = 2^{2^t} + 1 \in \mathscr{F}$ is a Fermat prime with $t > 0$. From $r + 1 = 2(2^{2^t - 1} + 1) = 3 \cdot 2^y$, we conclude $y = 1$, $t = 1$, and $r = 5$. ∎

4. THE ALTERNATING GROUPS

In this section we classify all p-intersection subgroups of the simple alternating groups $\mathscr{A}_n$.

THEOREM 4.1. *Let $G \cong \mathscr{A}_n$, with $n \geq 5$, $p \in \pi(G)$, and suppose that $X \in \mathscr{I}_p(G)$. Then $X = N_G(Q)$, with $Q \in Syl_q(G)$ for a suitable prime q; moreover, one of the following holds:*

(i) $n = 5$, $p = 3$, $q = 2$, *and* $X \cong \mathscr{A}_4$;

(ii) $n = 6$, $p = 2$, $q = 3$, *and* $X \cong 3^2 : \mathbb{Z}_4$;

(iii) $q = n - 2 = 2^f + 1 \in \mathscr{F}$, $p = 2$, *and* $X = Q : K$, *where* $Q \cong \mathbb{Z}_q$ *and* $K \cong \mathbb{Z}_{q-1}$.

(iv) $q = n - 1$ *is odd*, $(q - 1)/2 = p^f$, *and* $X = Q : K$, *where* $Q \cong \mathbb{Z}_q$ *and* $K \cong \mathbb{Z}_{(q-1)/2}$.

(v) $q = n$ *is odd*, $(n - 1)/2 = p^f$, *and* $X = Q : K$, *with* $Q \cong \mathbb{Z}_n$ *and* $K \cong \mathbb{Z}_{(n-1)/2}$.

Proof. The subgroup structure of the groups $\mathscr{A}_n$ with $5 \leq n \leq 8$ is well known, and it is an easy exercise to verify the claimed results for these cases. Henceforth we assume that $X \in \mathscr{I}_p(G)$, that $n \geq 9$, and that q is the minimal prime divisor of $|X|$ with $q \neq p$.

If $q = 2$, then $|X|$ is even and p is odd; but then Theorem 2.10 provides a contradiction. Therefore, q is odd and X contains a product $\alpha = \beta_1 \beta_2 \cdots \beta_k$ of k pairwise disjoined q-cycles. Now $X \geq C_G(\alpha) \ni \beta_1$, and so we can assume $\mathscr{Z} := \langle (123 \cdots q) \rangle < X$. Note that $N := N_G(\mathscr{Z}) \leq X$.

Suppose now that $q \leq n - 2$. Then 2 divides $|N|$, and the choice of q implies $p = 2$. If $q \leq n - 3$, then N contains a 3-cycle; thus $q = 3$. Now $N \cong (\mathbb{Z}_3 \times \mathscr{A}_{n-3}) : \mathbb{Z}_2$ is a maximal subgroup of G; hence $X = N$. But now we find a G-conjugate N^g of N such that $N \cap N^g \geq (\mathbb{Z}_3 \times \mathscr{A}_{n-6} \times \mathbb{Z}_3) : \mathbb{Z}_2$. This contradiction proves that $q = n - 2$; in particular, q and n are odd and $N = \mathscr{Z} : K$, where K is cyclic of order $q - 1$. The minimality of q now implies $q - 1 = 2^f$; so $q = 2^f + 1 \in \mathscr{F}$. If $\pi(X) \neq \{2, q\}$, then the choice of q ensures $\pi(X) = \{2, q, n\}$, and so X contains a subgroup $R \cong \mathbb{Z}_n$. But now $X \geq N_G(R) \cong \mathbb{Z}_n : \mathbb{Z}_{(n-1)/2}$, and so $n - 1$ is a power of 2 as well; now we get $n = 5$, a contradiction. Hence we have $\pi(X) = \{2, q\}$. In particular, X cannot be transitive on $\underline{n} := \{1, 2, \ldots, n\}$, and thus $\underline{q} := \{1, 2, \ldots, q\}$ and $\Omega := \{n - 1, n\}$ are the X-orbits on $\underline{n}$. In particular, X is isomorphic to a solvable subgroup of Σ_{n-2}. Since n is odd, $Q := O_2(X)$ has fixed points on both $\underline{q}$ and Ω; hence $Q = 1 \neq O_q(X)$, and thus $X = N$.

Suppose next that $q = n - 1$. Since $n \geq 9$, q is odd, and $N = \mathscr{Z} : K$, with K being cyclic of order $(q - 1)/2$. The choice of q implies that $(q - 1)/2 = p^f$ for some $f \geq 1$; hence $\pi(X) = \{p, q\}$; in particular, X is solvable. If X acts transitively on $\underline{n}$, then n divides $|X|$, and so $p = 2$ and $n = q + 1 = 2(2^f + 1)$ is a power of 2. We get the contradiction $f = 0$. Therefore $X \leq G_n \cong \mathscr{A}_{n-1}$. Since $q \equiv 1 \bmod (p)$, the group $O_p(X)$ has fixed points on $\underline{q}$, while X is transitive on $\underline{q}$. From this we deduce $O_p(X) = 1 \neq O_{p'}(X)$, and hence $X = N$.

We are left to consider the case where $q = n$. Then $N = \mathscr{Z} : K$, with K being cyclic of order $(n - 1)/2$. As in the previous case, we see that $(n - 1)/2 = p^f$ and $\pi(X) = \{q, p\} = \{p, n\}$, as well as $O_p(X) = 1$; thus $X = N$. ∎

COROLLARY 4.2. *Let $\mathscr{A}_n \triangleleft G \leq \mathrm{Aut}(\mathscr{A}_n)$, with $n \geq 5$, and suppose that $X \in \mathscr{I}_p(G)$. Then $p = 2$, and one of the following holds*:

(i) *$n = 6$, $X = N_G(Q)$, with $Q \in Syl_3(G)$.*

(ii) *$X = N_G(Q)$, where $Q \cong \mathbb{Z}_q$, with $q = 2^f + 1 \in \{n - 2, n - 1, n\} \cap \mathscr{F}$.*

Proof. Put $X_0 := X \cap G'$. Since X is not a 2-group, $X_0 \neq 1$. If X_0 is a p-group, then X_0 is a proper normal subgroup of X and $\pi(X) = \{2, p\}$; but then Theorem 2.10 yields a contradiction. Hence, by Lemma 2.4, $X_0 \in \mathscr{I}_p(G')$, and the claim follows from the previous theorem, noting that $p = 2$ by Theorem 2.10, and neither $\mathscr{A}_4$ nor Σ_4 are in $\mathscr{I}_3(\Sigma_5)$. ∎

TABLE I
p-Intersection Maximal Subgroups in Alternating and Symmetric Groups

G	p	X
$\mathscr{A}_5$	3	$\mathscr{A}_4$
$\mathscr{A}_5 \leq G \leq \Sigma_5$	2	$N_G(\mathbb{Z}_3)$
$\mathscr{A}_6 \leq G \leq Aut(\mathscr{A}_6)$	2	$N_G(3^2)$
$\{\mathscr{A}_6, \Sigma_6\} \not\ni G \leq Aut(\mathscr{A}_6)$	2	$N_G(\mathbb{Z}_5)$
$\mathscr{A}_n$, n prime $\notin \{7, 11, 17, 23\}$	p with $(n-1)/2 = p^f$	$N_G(\langle(12 \cdots n)\rangle) \cong \mathbb{Z}_n : \mathbb{Z}_{(n-1)/2}$
$\Sigma_n, 5 \leq n \in \mathscr{F}$	2	$N_G(\langle(12 \cdots n)\rangle) \cong \mathbb{Z}_n : \mathbb{Z}_{n-1}$

COROLLARY 4.3. *Let $\mathscr{A}_n \trianglelefteq G \leq \mathrm{Aut}(\mathscr{A}_n)$ with $n \geq 5$, and suppose that $X \in \mathscr{I}_p(G)$ is maximal in G. Then the triple (G, p, X) is as listed in Table I.*

Proof. The claims for $n \in \{5, 6\}$ can be verified directly, using the well known subgroup structure of Σ_5 and $\mathrm{Aut}(\mathscr{A}_6)$. Henceforth we assume that $n \geq 7$.

Applying Theorem 4.1, we see that cases (iii) and (iv) have to be discarded, because then X is not maximal in G: $X < St_G(\{1, 2, \ldots, q\})$. A similar argument shows that in the case of Corollary 4.2(ii) we must have $q = n \in \mathscr{F}$.

Now let X be as listed in the last two lines of Table I and suppose X is not maximal: $X < M < G$. Since M is transitive of prime degree n, well-known theorems of Galois and Burnside imply that M is a doubly transitive permutation group, $S \coloneqq \mathrm{soc}(M)$ is nonabelian simple, and $S \trianglelefteq M \leq \mathrm{Aut}(S)$. Using the list of doubly transitive permutation groups with nonabelian socle (see [4] or the main result of [19]), one arrives at one of the following exceptions: $n \in \{7, 11, 17, 23\}$, and $X < M < G \cong \mathscr{A}_n$ is one of the "exceptional" embeddings $7:3 < PSL_2(7) < \mathscr{A}_7$, $17:8 < PSL_2(16):4 < \mathscr{A}_{17}$, $11:5 < M_{11} < \mathscr{A}_{11}$, and $23:11 < M_{23} < \mathscr{A}_{23}$. ∎

5. FINITE GROUPS OF LIE TYPE

In this section $LIE(r)$ denotes the set of finite simple (twisted or untwisted) groups of Lie type defined over a finite field of characteristic r.

THEOREM 5.1. *Let $G \in LIE(r)$, and suppose that $X \in \mathscr{I}_p(G)$ with $r \in \pi(X)$.*

(i) *If $X = N_G(U)$ for some $U \in Syl_r(G)$, then $r \neq p$.*

(ii) *If* $r \neq p$, *then* $X = N_G(U) = U:T$, *where* $U \in Syl_r(G)$, $T \neq 1$ *is a cyclic* p-*group, and exactly one of the following five cases occurs*:

(1) $G \cong SL_2(q)$ *or* $G \cong {}^2B_2(q)$, *where* $q = 2^q \geq 4$ *and* $p = q - 1$ *is a Mersenne prime*.

(2) $G \cong PSL_2(r^m)$, *where* $r^m = 2 \cdot p^x + 1$ *with odd primes* r *and* p; *moreover, if* $m \neq 1$, *then* $r = 3$ *and* m *is an odd prime dividing* $p - 1$.

(3) $G \cong PSL_2(q)$ *with* $p = 2$ *and* $q = r \in \mathscr{F}$ *or* $(q, r) = (9, 3)$.

(4) $G \cong PSU_3(r)$ *with* $= p = 2$ *and* $r \in \{3, 5\}$.

(5) $G \cong {}^2G_2(3)' \cong PSL_2(2^3)$ *with* $p = 2$ *and* $r = 3$.

In particular, G *has Lie-rank* 1 *and* X *is a Borel subgroup of* G.

Proof. For (i) just observe that $X = U:T$, with $T \neq 1$ and $N_G(T) \neq T$. Henceforth we assume that $r \neq p$ and seek to prove the claims in (ii).

Clearly, as X is strongly r-embedded in G, $X \geq B := N_G(U) = U:T$ for some $U \in Syl_r(G)$ and a suitable complement T of U in B. If G has a Lie-rank of at least 2, then G is generated by the maximal parabolic subgroups containing B, and hence $G = \langle N_G(R) | 1 \neq R \leq U \rangle \leq X$, which, of course, is absurd. Therefore G has Lie-rank 1 and thus is isomorphic to one of the following groups: $PSL_2(q)$, $PSU_3(q)$, ${}^2B_2(q)$, ${}^2G_2(q)'$. Note that $q = 2^{2m+1} \geq 8$ in the third case and $q = 3^{2m+1} \geq 3$ in the last case. Moreover, T is cyclic of order $(q-1)/\gcd(q-1, 2)$, $(q^2-1)/\gcd(q+1, 3)$, $q - 1$, or $q - 1$, respectively. Since B is maximal in G and $G = \langle B, N_G(T) \rangle$, we see that $X = B$ and $N_G(T)$ is not contained in X; in particular, $T \in Syl_p(B)$.

Now suppose that $p > 2$. If $r = 2$, then $2 \in \pi(X)$, and so Theorem 2.10 shows that case (1) holds. Thus we may assume that r is odd; in particular, $G \cong PSL_2(q)$, with $q = r^m = 2 \cdot p^x + 1$. Observe that m is odd and $(r^m - 1)/(r-1)\gcd(r-1, m) = p^{x'}$ with $x' \leq x$.

Now suppose that $m > 1$ and thus $x' > 0$. Then Lemma 3.6 shows that m is a prime divisor of $p - 1$. If $p | r - 1$, then Lemma 3.1 implies $x' = \nu_p((r^m - 1)/(r-1)\gcd(r-1, m)) = \nu_p(m) = 0$, a contradiction. Thus we have $\gcd(p, r-1) = 1$. As $r^m - 1 = 2p^x$, we then get $r = 3$. This is case (2).

If $p = 2$ and $G \not\cong PSU_3(q)$, then $|T| = 2^x > 1$ implies $q \in \mathscr{F} \cup \{9\}$; in particular, ${}^2B_2(q)$ does not occur. This leads to case (3) or to case (5). If $p = 2$ and $G \cong PSU_3(q)$, then, by Lemma 3.8, $|T| = (q^2 - 1)/\gcd(q + 1, 3) = 2^x > 1$ implies $q \in \{3, 5\}$, i.e., case (4) holds. ∎

THEOREM 5.2. *Let* G *be a finite nonabelian simple group,* p *be a prime, and* $X \in \mathscr{I}_p(G)$. *If there exists* $q \in \pi(X) \backslash \{2, p\}$ *such that* $Q \in Syl_q(X)$ *is* *not cyclic, then one of the following holds*:

(i) $G \cong PSL_2(3^m)$, $q = 3$, *and* $3^m = 2p^x + 1$, *with odd primes*, m, p *such that* m *divides* $p - 1$; *moreover*, X *is a Sylow* 3-*normalizer of* G.

(ii) $G \cong PSL_2(9) \cong \mathscr{A}_6$ *and* $X \cong 3^2 : \mathbb{Z}_4$ *with* $q = 3$ *and* $p = 2$.

(iii) $G \cong PSU_3(q)$ *and* $X \cong q^{1+2} : \mathbb{Z}_8$ *with* $q \in \{3, 5\}$ *and* $p = 2$.

(iv) $G \cong PSL_3(4)$ *and* $X \cong 3^2 : Q_8$ *with* $q = 3$ *and* $p = 2$.

(v) $G \cong M_{11}$ *and* $X \cong 3^2 : SD_{16}$ *with* $q = 3$ *and* $p = 2$.

Proof. Since X is strongly q-embedded in G, we can apply [16, Theorem 24.1], and obtain a well-specified list of possibilities for the pairs (G, q). using Theorem 5.1 together with the information in [6] on the subgroup structure of the groups in the list obtained, it is now an easy exercise to reduce to the cases (i)–(v) of Theorem 5.2. ∎

Now we set up some notation concerning *maximal tori* in simple groups of Lie type.

Let S be such a simple group. Then S occurs as a section of a suitable finite group of Lie type, i.e., a group $\tilde{S} := \tilde{\mathbf{S}}_F$ of F-fixed points of a connected reductive algebraic group $\tilde{\mathbf{S}}$, defined over $\mathbf{F}_q$ with Frobenius endomorphism $F : \tilde{\mathbf{S}} \to \tilde{\mathbf{S}}$.

Each $\tilde{\mathbf{S}}$ contains a *maximally split* F-stable maximal torus, i.e., a torus $\tilde{\mathbf{T}}$ contained in an F-stable Borel subgroup. Recall that the $\tilde{\mathbf{S}}_F$-conjugacy classes of maximal F-stable tori are classified by the F-conjugacy classes of $W := N_{\tilde{\mathbf{S}}}(\tilde{\mathbf{T}})$, where w, w' are F-conjugate if and only if $w' = \tilde{w}^{-1} w F(\tilde{w})$ for suitable $\tilde{w} \in W$.

For each $g \in \tilde{\mathbf{S}}$, let $\mathscr{L}_F(g) := g^{-1}F(g)$. If ${}^g\tilde{\mathbf{T}}$ is an F-stable maximal torus of $\tilde{\mathbf{S}}$, then $\mathscr{L}_F(g)\tilde{\mathbf{T}} \in W$ and

$$({}^g\tilde{\mathbf{T}})_F = {}^g\left(\tilde{\mathbf{T}}_{\mathscr{L}_F(g)F}\right).$$

By Lang's theorem, for any element $w \in W$, there is a $g \in \tilde{\mathbf{S}}$ with $\mathscr{L}_F(g)\tilde{\mathbf{T}} = w$, hence a "finite torus" $\tilde{T}_w := {}^g\tilde{\mathbf{T}}_{w \cdot F} \leq \tilde{S}$. If $S := \tilde{S}/Z(\tilde{S})$ or $\tilde{S}'/Z(\tilde{S}')$, then T_w will always denote $\tilde{T}_w/Z(\tilde{S})$ or $(\tilde{T}_w \cap \tilde{S}')/Z(\tilde{S}')$. An element $t \in T_w$ is *regular* if the *connected centralizer* $C_{\tilde{\mathbf{S}}}(\tilde{t})^0$ is equal to the torus $\mathbf{T}_w := {}^g\tilde{\mathbf{T}}$. Here ${}^g\tilde{\mathbf{T}}$ and T_w are related as above, and $\tilde{t}$ will always denote a preimage of t in $\tilde{\mathbf{S}}$. We will call T_w a *maximal torus* of S.

LEMMA 5.3. *Suppose that* $t \in T_w$ *is regular*; *then* $N_S(T_w) = N_{\tilde{S}'}(\tilde{\mathbf{T}}_w)/Z(\tilde{S}')$.

Proof. Clearly, $N_S(T_w) \geq N_{\tilde{S}'}(\tilde{\mathbf{T}}_w)/Z(\tilde{S}')$. Suppose that $t \in T_w$ is regular and $x \in N_S(T_w)$. Then $\tilde{t} \in \tilde{\mathbf{T}}_w \cap (\tilde{T}_w \cap \tilde{S})^{\tilde{x}} \leq \tilde{\mathbf{T}}_w \cap \tilde{\mathbf{T}}_w^{\tilde{x}}$. Since $\tilde{t}$ is regular, $\tilde{x} \in N_{\tilde{S}'}(\tilde{\mathbf{T}}_{\mathrm{w}})$. ∎

A key ingredient of our further arguments is the following statement:

LEMMA 5.4. *Let $S \in LIE(r)$ and $X \in \mathscr{I}_p(S)$. If X is not conjugate to a Borel group of S, then X contains at least one normalizer $N_S(T_w)$ of a maximal torus T_w of S. If, moreover, $r \notin \pi(X)$, then any p'-element t of X is contained in such a unique maximal torus $T_w \leq N_S(T_w) \leq X$ and is regular in there.*

Proof. If $r \in \pi(X)$, we can assume that $r = p$ by 5.1. Hence, in any case X contains a $\{p, r\}'$-element $t \neq 1$. Since every semisimple element of $\tilde{S}$ is contained in a maximal torus of $\tilde{S}$ and since $C_S(t) \leq X$, we get the first assertion.

Now suppose that $r \notin \pi(X)$. According to a classical result on centralizers of semisimple elements of groups of Lie type (cf. Theorem 3.5.4 of [5], also see Theorem 4.2.2 of [17] for a more expanded version), the connected centralizer $C_{\tilde{\mathbf{S}}}(\tilde{t})^0$ is reductive, and hence $C_{\tilde{S}}(\tilde{t})$ contains a central product of finite groups of Lie type $H \in LIE(r)$ and a torus. Since $\gcd(r, |Z(\tilde{S})|) = 1 = \gcd(r, [\tilde{S} : \tilde{S}'])$, we get $C_{\tilde{S}}(\tilde{t}) \cong \tilde{T}_w$, a maximal torus in which $\tilde{t}$ is regular. Hence $\tilde{t}$ does not lie in any other maximal torus of $\tilde{S}$, so T_w is the only maximal torus of S containing t. ∎

In view of the last result, it is important to have more information on overgroups in S or $\tilde{S}$ of given maximal tori. In particular, the following result on solvable overgroups of maximal tori will be very useful:

THEOREM 5.5 (Seitz [30]). *Let $q = r^m$, $\tilde{T}$ be a maximal torus of $\tilde{S}$, and $\tilde{T} \leq \tilde{X} \leq \tilde{S}$.*

(i) *If $q > 7$ and if $\tilde{X}$ is solvable, then $\tilde{X} = O_r(\tilde{X})N_{\tilde{X}}(\tilde{T})$ and $O_r(\tilde{X})$ is a product of $\tilde{T}$-root subgroups of $\tilde{S}$.*

(ii) *If $r > 3$ and $q > 11$, then the normal closure of $\tilde{T}$ in $\tilde{X}$ is generated by $\tilde{T}$ and the $\tilde{T}$-root subgroups of $\tilde{S}$ contained in $\tilde{X}$.*

Notice that $\tilde{T}$-root subgroups of $\tilde{S}$ are either r-subgroups or products of certain finite groups of Lie type.

Now we will explain more precisely which pairs of groups $(\tilde{S}, S)$ we are looking at:

DEFINITION 5.6. Let r be a prime, $m \in \mathbb{N}$, and $q = r^m$. In the sequel $(\tilde{S}, S)$ will be one of the following pairs of groups:

(A) For Dynkin type A_{n-1}, $n > 1$: $\tilde{S} := S_n^{\epsilon}(q) = SL_n(q)$ if $\epsilon = 1$ and $SU_n(q)$ if $\epsilon = -1$ and $S = PS_n^{\epsilon}(q) := \tilde{S}/Z(\tilde{S}) = PSL_n(q)$ if $\epsilon = 1$ and $PSU_n(q)$ if $\epsilon = -1$.

(B) For Dynkin type B_n, $n > 1$, q *odd*: $\tilde{S} := SO_{2n+1}(q)$, the special orthogonal group and $S := (\tilde{S})' \cong P\Omega_{2n+1}(q)$.

(C) For Dynkin type C_n, $n > 1$: $\tilde{S} := Sp_{2n}(q)$, the symplectic group and $S := \tilde{S}/Z(\tilde{S}) = PSp_{2n}(q)$.

(D) For Dynkin type ${}^{\epsilon}D_n$, $n > 3$: $\tilde{S} := SO_{2n}^{\epsilon}(q)$, the special orthogonal group and $S := \tilde{S}/Z(\tilde{S}') = P\Omega_{2n}^{\epsilon}(q)$;

(E) For Dynkin type ${}^2B_2$, ${}^3D_4$, ${}^{(2)}G_2$, ${}^{(2)}F_4$, ${}^{(2)}E_6$, E_7, and E_8: $\tilde{S}$ denotes the finite group of Lie type (possibly twisted), coming from the simply connected algebraic group of the same Dynkin type: whereas $S := \tilde{S}/Z(\tilde{S})$.

In all cases we exclude the solvable groups $PSL_2(2)$, $PSL_2(3)$, $PSU_3(2)$, ${}^2B_2(2)$, etc.; moreover, in (B) and (C) we exclude the exceptional case $(n, q) = (2, 2)$, in which $(\tilde{S}, S) \cong (\Sigma_6, \mathscr{A}_6)$.

Now we give a more precise description of the maximal tori in $\tilde{S}$ and S for groups of classical type. Let $\mathbf{S}$ be one of the following algebraic groups: $\mathbf{S} = SL_n(\overline{\mathbf{F}}_q)$ in case (A), $\mathbf{S} = SO_{2n+1}(\overline{\mathbf{F}}_q)$ in case (B), $\mathbf{S} = SP_{2n}(\overline{\mathbf{F}}_q)$ in case (C), and $\mathbf{S} = SO_{2n}(\overline{\mathbf{F}}_q)$ in case (D).

In each of these cases a maximally split torus can be described as the set $\tilde{\mathbf{T}} = (\overline{\mathbf{F}}_q^*)^{\mathbb{N}}$ of functions $f : \mathbb{N} := \{1, 2, \dots, n\} \to \overline{\mathbf{F}}_q^*$, where in case (A) we have to restrict to those functions that satisfy $\prod_{i \in \mathbb{N}} f(i) = 1$, e.g.,

$$f = t = \begin{pmatrix} t_1 & 0\cdots & 0\cdots & 0\cdots \\ 0\cdots & t_2 & \cdots & \\ 0\cdots & \cdots & \cdots & \\ 0\cdots & \cdots & \cdots & t_n \end{pmatrix} \in \tilde{\mathbf{T}} \le SL_n\left(\overline{\mathbf{F}}_q\right)$$

or

$$f = t = \begin{pmatrix} t_1 & 0\cdots & 0\cdots & 0\cdots & & & & \\ 0\cdots & t_2 & \cdots & \cdots & & 0 & & \\ 0\cdots & \cdots & \cdots & \cdots & & & & \\ 0\cdots & \cdots & \cdots & t_n & & & & \\ & & & & t_1^{-1} & 0\cdots & 0\cdots & 0\cdots \\ & & & & 0\cdots & t_2^{-1} & \cdots & \cdots \\ & 0 & & & 0\cdots & \cdots & \cdots & \cdots \\ & & & & 0\cdots & \cdots & \cdots & t_n^{-1} \end{pmatrix}$$

$$\in \tilde{\mathbf{T}} \le Sp_{2n}\left(\overline{\mathbf{F}}_q\right).$$

All Weyl groups $W = N_G(\tilde{\mathbf{T}})/\tilde{\mathbf{T}}$ of classical groups are subgroups of the wreath product $\mathbb{Z}_2 \wr \Sigma_{\mathbb{N}}$, which is the Weyl group in cases (B) and (C). So W is acting naturally on $\tilde{\mathbf{T}}$ (viewed as a set of functions) by ${}^{(z,\sigma)}f(i) =$

$(f(\sigma^{-1}(i)))^{-1^{z(i)}}$, where $z(i) \in \{0,1\}$. In case (A) we have $W = \Sigma_{\mathbb{N}}$, whereas in (D), W is the subgroup of index 2 of $\mathbb{Z}_2 \wr \Sigma_{\mathbb{N}}$ consisting of elements with an even number of 1's in the base group $\mathbb{Z}_2^n$. The action of F on $\tilde{\mathbf{T}}$ can be described as follows: in case (A): $F(f)(i) = f(i)^{\epsilon q}$ $\forall i \in \mathbb{N}$; in cases (B), (C), and (D) (with $\epsilon = 1$); $F(f)(i) = f(i)^q$ $\forall i \in \mathbb{N}$; in case (D) with $\epsilon = -1$, $F(f)(i) = f(i)^q$ $\forall i \in \mathbb{N}\backslash\{n\}$ and $F(f)(n) = f(n)^{-q}$.

Let $w \in W$ be a representative of the F-class $[w]$; then w contains n_i^+ positive and n_i^- negative pairwise disjoint i-cycles with $\sum_{i=1}^n i \cdot (n_i^+ + n_i^-) = n$. (A negative i-cycle $(1,2,\ldots,i)^-$ maps $1 \mapsto 2 \mapsto 3 \cdots \mapsto i \mapsto -1 \mapsto -2 \cdots$ and has order $2i$).

Now the F-conjugacy classes of W and the corresponding tori are given as follows:

In the cases (A) to (C) and (D) with $\epsilon = 1$, the F-conjugacy classes coincide with conjugacy classes of W. In case (D) and $\epsilon = -1$, they can be identified with the W_{D_n}-orbits in the coset $W_{D_n} \cdot (n^-) \subseteq W_{B_n}$, where (n^-) is the negative 1-cycle $n \mapsto -n$ in W_{B_n}.

In case (A) all cycles are positive, i.e., $n_i^- = 0$ for all i and $\tilde{\mathbf{T}}_{w\cdot F}$ is isomorphic to a subgroup of index $q - \epsilon$ in the corresponding maximal torus $\check{\mathbf{T}}_{w\cdot F}$ of the general linear or unitary group with

$$\check{\mathbf{T}}_{w\cdot F} \cong \prod_{i=1}^{n} \mathbb{Z}_{q^i-\epsilon^i}^{n_i^+}.$$

In cases (B) to (D) and $\epsilon = 1$,

$$\tilde{\mathbf{T}}_{w\cdot F} \cong \prod_{i=1}^{n} \left(\mathbb{Z}_{q^i-1}\right)^{n_i^+} \times \left(\mathbb{Z}_{q^i+1}\right)^{n_i^-},$$

where in case (D) the number of negative cycles is even. In case (D) and $\epsilon = -1$, $\tilde{\mathbf{T}}_{w\cdot F}$ is given in the same way as in the case $\epsilon = 1$, if the cycle type of w is replaced by that of $w \cdot (n^-)$. In particular, the number of negative cycles has to be odd.

For Coxeter tori we will use the notation $\tilde{T}_{cox} := \tilde{T}_{w_{cox}}$ and $T_{cox} := T_{w_{cox}}$. Notice that $w_{cox} = w$ $(12 \cdots n)$ in case (A), $w_{cox} = w$ $(12 \cdots n)^-$ in cases (B) and (C), and $w_{cox} = (12 \cdots n-1)^-(n)^-$ in case (D) and $\epsilon = 1$. In case (D) and $\epsilon = -1$, we define $w_{cox} := (12 \cdots n-1)^-$. For quasi-split tori we will use the notation $\tilde{T}_1 := \tilde{T}_{id}$ and $T_1 := T_{id}$.

In all cases, except (D) and $\epsilon = -1$, we have $N_{\tilde{S}}(\tilde{\mathbf{T}}_w)/\tilde{T}_w \cong C_W(w)$, whereas in case (D) and $\epsilon = -1$, $N_{\tilde{S}}(\tilde{\mathbf{T}}_{w\cdot F})/\tilde{T}_w \cong C_W(w(n^-))$. Clearly, $N_{\tilde{S}}(T_w) \geq N_{\tilde{S}}(\tilde{\mathbf{T}}_w)$.

The following theorem classifies the elements in $\mathscr{I}_p(S)$ for $p > 2$. Because of Theorem 2.10 we can assume in this case that $X \in \mathscr{I}_p(S)$ has odd order.

THEOREM 5.7. *Let $p > 2$, $(\tilde{S}, S)$ as above, and $X \in \mathcal{I}_p(S)$ of odd order. If X is not a Borel-subgroup in S, then one of the following holds*:

In case (A)

(i) $n = p+1$, $S \cong PS^{\epsilon}_{p+1}(q)$, *and*

$$X = {}_S N_S\big(T_{(12\cdots p)}\big) \cong \mathbb{Z}_{|((\epsilon q)^p - 1)/\gcd(\epsilon q - 1, p+1)|} : \mathbb{Z}_p,$$

with $q - \epsilon = \gcd(q - \epsilon, p+1)\cdot p^x$ *for some* $x \in \mathbb{N}$; *moreover*, $(q, \epsilon, p) \neq (2, -1, 3)$.

(ii) $p = 3$, $S \cong L_4(3)$, *and* $X \cong E(3^3) : \mathbb{Z}_{13} : \mathbb{Z}_3$; *moreover, S contains two conjugacy classes of* 3-*intersection subgroups of the given type*.

(iii) $n = p$, $S \cong PS^{\epsilon}_p(q)$, *and*

$$X = {}_S NS(T_{cox}) \cong \mathbb{Z}_{|((\epsilon q)^p - 1)/(\epsilon q - 1)\gcd(\epsilon q - 1, p)|} : \mathbb{Z}_p.$$

In case (D)

(i) $n = p$, $S \cong P\Omega^{\pm}_{2p}(q)$, $S = {}_S N_S(T_{(123\cdots p)^{\pm}}) \cong \mathbb{Z}_{(q^n - \epsilon)/\gcd(q^n - \epsilon, 4)|} : \mathbb{Z}_p$, *and* $q - \epsilon = \gcd(q - \epsilon, 4)\cdot p^x$ *for some* $x \in \mathbb{N}$. *Here the* $+$ *corresponds to* $\epsilon = 1$ *and the* $-$ *to* $\epsilon = -1$.

(ii) $n = p = 5$, $S \cong P\Omega^{+}_{10}(5)$, *and* $X = Q : N$, *with* $Q = O_5(X)$ *being elementary abelian of order* 5^5 *or* 5^{10} *and* $N \cong \mathbb{Z}_{781} : \mathbb{Z}_5$.

In case (E)

$S \cong {}^{\epsilon}E_6(q)$, $p = 3$, $X = {}_S N_S(T)$, *with* $T := T_{24}$ (*see* [11]), $N_S(T)/T \cong \mathbb{Z}_9$, *and* $T \cong \mathbb{Z}_{(q^6 + \epsilon q^3 + 1)/\gcd(q - \epsilon, 3)}$.

Proof. By Lemma 5.4 we know that $X \geq N_w := N_S(T_w)$ for some maximal torus T_w of S. Since $|C_W(w)|$ divides $|N_w|$, W cannot contain the element $w_0 = -id$; therefore $\tilde{S}$ is of type ${}^{(2)}A$, ${}^{(2)}D_n$ with n odd or of type ${}^{(2)}E_6$.

(1) Suppose we are in case (A). If w consists of n_i cycles of length i, then $|N_w/T_w|$ is divisible by $\prod_i i^{n_i} n_i!$. Since this is an odd number, we conclude that $n \geq 3$; furthermore, $n_i = 0$ whenever i is even and $n_i \leq 1$ if i is odd. Moreover,

$$\left| \frac{1}{(\epsilon q - 1)\gcd(\epsilon q - 1, n)} \prod_{i \text{ odd}, n_i = 1} \big((\epsilon q)^i - 1\big)\cdot i \right| \qquad \text{divides } |N_w|.$$

If a is odd with $n_a > 0$ and $1 < a < n-1$, then there exists $t_a = {}^g(c, c^{\epsilon q}, \ldots, c^{\epsilon q^{a-1}}, 1, \ldots, 1)Z(\tilde{S}) \in X$ of order $|((\epsilon q)^q - 1)/(\epsilon q - 1)|$, which is not regular in T_w, hence $C_S(t_a)$ has even order. By Lemma 3.6 we see that $|((\epsilon q)^a - 1)/(\epsilon q - 1)|$ is a power of p and a is a prime dividing $p - 1$.

If $1 < a < b < n$ with $n_a, n_b > 0$ and thus $a + b \leq n$, then

$$\text{w.l.o.g. } 1 \neq p^x = \left|\frac{(\epsilon q)^a - 1}{\epsilon q - 1}\right| = \gcd\left(\left|\frac{(\epsilon q)^a - 1}{\epsilon q - 1}\right|, \left|\frac{(\epsilon q)^b - 1}{\epsilon q - 1}\right|\right)$$
$$= \left|\frac{(\epsilon q)^{\gcd(a,b)} - 1}{\epsilon q - 1}\right| = 1,$$

a contradiction. So either w is conjugate to $(1)(234 \cdots n)$ and n is even, or w is conjugate to $(123 \cdots n)$ and n is odd.

Now define $\tilde{n} := n$ if n is odd and $\tilde{n} := n - 1$ otherwise. Suppose that $\tilde{n} = c_1 c_2$ with $1 < c_1, c_2 < \tilde{n}$. Since $((\epsilon q)^{c_i} - 1)/(\epsilon q - 1)$ divides $((\epsilon q)^{n-1} - 1)/\gcd(\epsilon q - 1, n)$ if $\tilde{n} = n - 1$ and $((\epsilon q)^n - 1)/(\epsilon q - 1)\gcd(\epsilon q - 1, n)$ otherwise, the same argument as above shows that the $|((\epsilon q)^{c_i} - 1)/(\epsilon q - 1)|$ are p-powers and $c_1 = c_2$ is an odd prime dividing $p - 1$. So we may assume that $\tilde{n} = c^2$ with prime $c \neq p$. Then c is again the order of a suitable element of a maximal torus $T_{w'} \leq X$. By our previous arguments we must have

$$|T_{w'}| = |T_w| = \left|\frac{(\epsilon q)^{n-1} - 1}{\gcd(\epsilon q - 1, n)}\right| \text{ or } \left|\frac{(\epsilon q)^n - 1}{(\epsilon q - 1)\gcd(\epsilon q - 1, n)}\right|,$$

respectively. We conclude: $0 \equiv (\epsilon q)^{c^2} - 1 \equiv \epsilon q - 1 \bmod c$, hence $c | \epsilon q - 1$. This implies that X contains a quasi-split torus $T_{id} \not\cong T_w$. Since $n! | \, |N_{id}| \equiv 0 \bmod 2$, we derive a contradiction.

So $\tilde{n}$ is a prime. If $\tilde{n} \neq p$, we conclude, with c replaced by $\tilde{n}$, that $\tilde{n}$ divides $\epsilon q - 1$, leading to a contradiction. Hence $\tilde{n} = p$ and $n \in \{p, p + 1\}$ in case (A).

(2) Consider the case (A) with $n = p + 1$. Now we may assume that $X \geq N := N_S(T_{(12 \cdots p)}) \cong \mathbb{Z}_{|((\epsilon q)^p - 1)/\gcd(\epsilon q - 1, p + 1)|} : \mathbb{Z}_p$. Hence $(q - \epsilon)/\gcd(q - \epsilon, p + 1)$ divides $|X|$. If an odd prime $s \neq p$ divides this number, then X would contain a conjugate of $N_S(T_{id})$ with even order, a contradiction. So we have $q - \epsilon = \gcd(q - \epsilon, p + 1) \cdot p^x$.

Now let $t \in N \cap N^g$ be a p'-element and $g \in S \setminus N$. Then $t \in T \cap T^g$ with $T := T_{(12 \cdots p)}$. Hence t is singular in T. So the order $|\tilde{t}|$ of a preimage $\tilde{t} \in \tilde{S}$ must be a divisor of $\gcd((\epsilon q)^p - 1, (\epsilon q)^i - 1) = \epsilon q - 1$ with some $1 \leq i < p$, and $\tilde{t}$ is $\tilde{\mathbf{S}}$-conjugate to $\mathrm{diag}(c, c, \ldots, c, c^{-p})$. Hence $\tilde{t}^{p^x} \in Z(\tilde{S})$ and $t^{p^x} = 1$. So $t = 1$, and either $N \in \mathscr{I}_p(S)$ or N is a p-group.

Using Fermat's Theorem together with Lemma 3.1 and Lemma 3.5, we easily verify that N is a p-group iff $(q,\epsilon,p)=(2,-1,3)$. Moreover, an easy inspection shows that $\mathscr{I}_3(U_4(2))=\emptyset$. Henceforth we may assume that $(q,\epsilon,p)\neq(2,-1,3)$ and $N\in\mathscr{I}_p(S)$ as well as $N\leq X\in\mathscr{I}_p(S)$. Now we want to prove that $N=X$.

For that we assume first that r is a prime divisor of $|X|$ and thus $r=p$ as well as $q-\epsilon=\gcd(q-\epsilon,p+1)$, with $q=r^m$ for some $m\geq 1$.

Assume that $\epsilon=1$. Then we get $m=1$, $q=p=3$, and $N\cong\mathbb{Z}_{13}:\mathbb{Z}_{13}$; moreover, an inspection of the subgroup structure of $S\cong L_4(3)$ reveals that either $X=N$ or $X=O_3(X):N$ with $O_3(X)\cong E(3^3)$. In any case X is not maximal in S.

Assume now that $\epsilon=-1$. Then we get $m=1$, $q=p$, and $T\cong\mathbb{Z}_{((p)^p+1)/(p+1)}$ as well as $N/T\cong\mathbb{Z}_p$. Now observe that T is minisotropic (i.e., is not contained in a proper parabolic subgroup of S). As X is solvable and $O_p(X)=1$, we have $O_{p'}(X)\neq 1$. Since any p'-element of X is contained in a conjugate of T and since $N\in\mathscr{I}_p(S)$, $|S:N|$ is a power of p; so Lemma 2.9 implies $X=N$.

Next assume that X is a r'-group and consider preimages $\tilde{X}\geq\tilde{N}\cong\mathbb{Z}_{q^p-\epsilon}:\mathbb{Z}_p$ of X and N and their action on the natural module $V\cong\mathbb{F}_q^{p+1}$. Let s be a Zsigmondy prime of $q^{2p}-1$ if $\epsilon=-1$ and a Zsigmondy prime of q^p-1 if $\epsilon=1$ (a prime s is called a Zsigmondy prime of $q^{\ell}-1$ iff it divides $q^{\ell}-1$ but not q^m-1 for $1\leq m<\ell$). Also observe that such an s exists, because otherwise a well-known result of Zsigmondy would lead to $(q,\epsilon,p)=(2,-1,3)$, a contradiction. Now s divides $q^p-\epsilon$ but no $q^i-\epsilon$ for any $i<p$. Let $t\in\tilde{T}$ with $|t|=s$; then $V|_{\tilde{X}}\cong V_1\oplus W$ with $\dim_{\mathbb{F}_q}(V_1)\leq 1$, t acts trivially on V_1 and irreducibly on W. Let $K:=C_{\tilde{X}}(V_1)\trianglelefteq\tilde{X}$; clearly, K acts faithfully on W. Let $Y:=O_p(K)$ and consider $H_1:=Y:\langle t\rangle$. Then H_1 acts faithfully and irreducibly on W. Now we easily deduce that all characteristic abelian subgroups of Y are cyclic, so Y is of symplectic type. Since $p>2$, we have $Y=\mathscr{E}*Z$ with $\mathscr{E}$ extraspecial of type p^{2a+1}, $Z=Z(Y)$ cyclic, and $\mathscr{E}$ char Y.

Suppose that $1\neq\mathscr{E}$. Since $p=\operatorname{ord}(q)\bmod(s)|s-1$, we have $p<s$ and $[t,Z(\mathscr{E})]=1$. If $1\neq t^j\in C_{\langle t\rangle}(\mathscr{E})$, then $\mathscr{E}\leq C_{\tilde{S}}(t^j)\leq\tilde{T}$, a contradiction. Hence we can apply [1, 36.1, p. 192]; since $C_W(\langle t\rangle)=0$ we get $p=2$, a contradiction.

This shows that $\mathscr{E}=1$ and Y is a cyclic p-group with $[Y,t]=1$. Suppose $O_{p'}(X)=1$; then $O_{p'}(\tilde{X})\leq Z(\tilde{X})$. Considering the Fitting group $F(K)=Y\times O_{p'}(K)$, we get $t\in C_K(Y)=C_K(F(K))\leq F(K)$, and thus $t\in Z(\tilde{X})$, a contradiction. So $O_{p'}(X)\neq 1$ and $X=N$ by Lemma 2.9.

(3) Now we consider the case (A) with $n=p$. If $t\in N\cap N^g$ is a p'-element and $g\in S\setminus N$, then, as above, a preimage $\tilde{t}$ of t in $\tilde{T}:=\tilde{T}_{(12\cdots p)}$ satisfies $t\in\tilde{T}\cap\tilde{T}^g$ and is not regular in $\tilde{T}$. Hence the order $|\tilde{t}|$ divides

$\epsilon q - 1$ and $\tilde{t} \in Z(\tilde{S})$. Again we see that $N \in \mathcal{I}_p(S)$. The proof that $X = N$ for any $N \leq X \in \mathcal{I}_p(S)$ is similar to that given in the previous case. In particular, any noncentral element $\tilde{t} \in \tilde{T}$ acts faithfully and irreducibly on $V \cong \mathbb{F}^p$.

(4) Next suppose we are in case (D) with n odd. Let $N_w = N_S(T_w) \leq X$. Then $|C_{W_{B_n}}(w)|/2$ divides $|X|$ and thus is odd. This requires w to consist of exactly one positive (if $\epsilon = 1$) or one negative (if $\epsilon = -1$) n-cycle. In particular, $|N_w| = ((q^n - \epsilon)/\gcd(q^n - \epsilon, 4)) \cdot n$. Suppose that $n = c_1 c_2$ with $1 < c_i < n$. Then, as in case (A), we conclude that $(q^c i - \epsilon)/\gcd(q - \epsilon, 4) = p^{x_i}$, and hence $c_1 = c_2 = c$ an odd prime divisor of $p - 1$. Again as in case (A), c divides $(q^{c^2} - \epsilon)/\gcd(q - \epsilon, 4)$, hence c divides $q - \epsilon$, from which we get the contradiction $N_{id} \leq X$ or $N_{-id} \leq X$ for $\epsilon = 1$ or -1, respectively. So n is an odd prime. If $n \neq p$ we can apply the previous argument to n instead of c and get a contradiction again. So $n = p$. Notice that $\nu_2(q^p - \epsilon) = \nu_2(q - \epsilon)$. Since $(q - \epsilon)/\gcd(q - \epsilon, 4)$ divides $|X|$, the arguments above show that $(q - \epsilon)/\gcd(q - \epsilon, 4)$ must be a power of p. If this is the case, then we see, similar to the cases in (A), that all p'-elements of $N := N_S(T_{(123\cdots p)^\pm})$ are regular elements of $T_{(123\cdots p)^\pm}$ and $N \in \mathcal{I}_p(S)$.

Next we want to see that $N = X$ for any $N \leq X \in \mathcal{I}_p(S)$. Assume first that r divides $|X|$ and thus $r = p = n \geq 5$. In particular, we then have $q - \epsilon = \gcd(q - \epsilon, 4)$ and hence $q = r = p = n = 5$ as well as $\epsilon = 1$; moreover, $T_w \cong \mathbb{Z}_{781}$ and $C_S(t) = T_w$ for $1 \neq t \in T_w$. Since any p'-element of X has a conjugate in T_w and since X is solvable, we get $X = O_5(X) : N$. If $O_5(X) \neq 1$, then an inspection of the subgroup structure of $S \cong P\Omega_{10}^+(5)$ shows that $O_5(X)$ is elementary abelian of order 5^5 or 5^{10}.

Finally, we assume that X is an r'-group. Now the proof that $X = N$ for any $N \leq X \in \mathcal{I}_p(S)$ is similar to the one in (2).

(5) In the case of exceptional groups, the only maximal torus T_w whose normalizer in S has odd order occurs in case $S \cong {}^{(2)}E_6(q)$, with $T := T_w$ of order $(q^6 + \epsilon q^3 + 1)/\gcd(q - \epsilon, e)$ and $N_S(T)/T \cong \mathbb{Z}_9$. Also recall that in the notation of [11], T can be identified with T_{24}. Henceforth we can assume that $X \geq N := N_S(T)$.

Since S involves sections isomorphic to Σ_3, since $|X|$ is odd and since $3 \in \pi(X)$, we have $p = 3$. Note that $3 \neq q^6 + \epsilon q^3 + 1 \not\equiv 0 \bmod 9$; therefore T is a nontrivial $3'$-group. Now let $g \in S \setminus N$ and let $t \in N \cap N^g$ be a $3'$-element. Then $t \in T \cap T^g$ and so t is singular; hence $t = 1$. So we have $N \in \mathcal{I}_3(S)$.

Now we proceed to show that $X = N$. If $q > 7$, then this follows from Theorem 5.5, since in our case T is minisotropic, so there are no nilpotent T-root subgroups and we get $O_r(X) = 1$, whence $X = N$. So suppose that

$q \in \{2,4,3,5,7\}$. Let $\tilde{T}$ denote the complete preimage of T in $\tilde{S}$; so $\tilde{T} = Z(\tilde{S}) \times O_{3'}(\tilde{T})$ with $T \cong O_{3'}(\tilde{T})$. Now observe that $C_{\tilde{S}}(t)/Z(\tilde{S}) = C_S(tZ(\tilde{S}))$ for any $1 \neq t \in O_{3'}(\tilde{T})$, because $\gcd(|t|, |Z(\tilde{S})|) = 1$. From [11] we see that $C_{\tilde{S}}(t') = \tilde{T}_w$ for all $t' \in \tilde{T} \setminus Z(\tilde{S})$; hence $C_S(t) = T$ for any $1 \neq t \in T$. Since X is solvable, we have $1 \neq K := O_k(X)$ for some prime k. A look at orders of Sylow groups of S and at $|T|$ shows that T cannot act fixed-point freely on $Z(K)$. Hence there is $1 \neq z \in Z(K)$ and $1 \neq t \in T$ with $[t, z] = 1$. So $z \in Z(K) \cap T$ and $T = C_S(z) \geq K$. So $T = C_S(K)$char X. We conclude that $X = N$. ∎

Now we consider 2-intersection subgroups. Because of Theorem 5.1 we know that any $X \in \mathscr{I}_2(S)$ is conjugate to a Borel subgroup of S if $2 \neq r \in \pi(X)$ and $S \in LIE(r)$.

We will need the following technical lemmas:

LEMMA 5.8. *Let $H = YT$, where $Y = O_2(H) \in Syl_2(H)$ and T is cyclic of order $q^{2^k} + 1$, with $k \geq 1$, $q = r^m$, and r an odd prime such that $C_H(t) \leq T$ for all $1 \neq t \in O(T)$. If V is a faithful $\mathbb{F}_q$ H-module with* $\dim_{\mathbb{F}_q} V \in \{2^{k+1}, 2^{k+1}+1, 2^{k+1}+2\}$ *and* $\dim_{\mathbb{F}_q}(C_V(O_2(T))) \leq 1$, *then either Y is cyclic or $(k, q) = (1, 3)$.*

Proof. Put $T_0 := O(T)$ and $T_1 := O_2(T)$; note that $T_1 \cong \mathbb{Z}_2$. Suppose by way of contradiction that Y is noncyclic and that $(k, q) \neq (1, 3)$.

Observe first of all that $q^{2^k} + 1$ is not a power of 2. Now let s be an odd prime dividing $q^{2^k} + 1$. Then $\gcd(q^{2^k} + 1, q^j - 1)_{2'} = 1$ for all $1 \leq j < 2^{k+1}$, and thus $ord_{\mathbb{Z}_s^*}(q) = 2^{k+1} | x - 1$; in particular, $2^{k+1} + 1 \leq s$. Let $t \in T$ be of order s. Since t does not act trivially on V, $V|_{\langle t \rangle}$ must contain an irreducible subspace of dimension $\geq 2^{k+1}$.

(1) Put $d := \dim_{\mathbb{F}_q}(V)$ and $D := \mathrm{End}_H(V)$. Note first of all that both $\langle t \rangle$ and H act irreducibly on V whenever $d = 2^{k+1}$.

Suppose now that $d = 2^{k+1} + 1$ and that V is an irreducible $\mathbb{F}_q H$-module; so V is absolutely irreducible as DH-module, and $\dim_D(V)$ divides $\gcd(|H|, d)$. Moreover, $\dim_D(V) \geq 2$, because H is not cyclic. Since odd prime divisors of $|H|$ are greater than 2^{k+1}, $\dim_D(V) = d$ is an odd prime dividing $|H|$; in particular, $D = \mathbb{F}_q$.

Now observe that $\overline{V} := V \otimes \overline{\mathbb{F}}_q$ is irreducible and that $\overline{V}|_Y$ splits into linear $\overline{\mathbb{F}}_q Y$-modules; hence $Y' \leq \ker V = 1$ and so Y is abelian.

If $V|_Y = V_1 \oplus \cdots \oplus V_d$ with one-dimensional homogeneous components V_i, $(1 \leq i \leq d)$, then each V_i has inertia-group $I(V_i) = YT^d$; as $\gcd(|T|, q - 1) = 2$, we get $T^{2d} \leq \mathrm{Ker}(V_i)$ and thus $T^{2d} = 1$. In particular, $q^{2^k} + 1 = |T| = 2d$; this in turn leads to $(k, q) = (1, 3)$, a contradiction. Therefore,

$V|_Y$ is homogeneous. As Y is abelian, we easily conclude now that Y is cyclic, again a contradiction.

We have shown that $V = V_0 \oplus V_1$ with irreducible H-modules V_0 and V_1 such that $\dim_{\mathbb{F}_q}(V_0) = 1$ and $\dim_{\mathbb{F}_q}(V_1) = d - 1$.

As $\gcd(|T|, q-1) = 2$, $C_T(V_0) \geq T_0$ with $|T : C_T(V_0)| \leq |T : T_0| = 2$. Note that $[C_H(V_0), C_H(V_1)] \leq C_H(V) = 1$, and hence $C_H(V_1) \leq C_H(T_0) = T$. As $O(H) = 1$ and $C_H(V_1) \leq \mathrm{Core}_H(T)$, we get $C_H(V_1) \leq T_1$. Since $\dim(C_V(T_1)) \leq 1$ and $\dim(V_1) \geq 4$, we have $C_H(V_1) = 1$. Hence H acts irreducibly and faithfully on V_1.

Suppose next that $d = 2^{k+1} + 2$ and assume first that H acts irreducibly on V. Then $\dim_D V$ divides $\gcd(|H|, 2^{k+1} + 2) = 2\gcd(T, 2^k + 1) = 2$, and we conclude that $\dim_D V = 2$ and $D \cong \mathbb{F}_{q^{2^{k+1}}}$. Hence H embeds into $GL_2(q^{2^{k+1}})$ and $|T|_{2'}$ is a divisor of $\gcd(q^{2^k} + 1, q^{2^{k+1}} - 1)_{2'} = 1$ or $\gcd(q^{2^k} + 1, q^{2^{k+1}} + 1)_{2'} = 1$, a contradiction.

Assume now that $V = V_0 \oplus V_1$ with irreducible $\mathbb{F}_q H$-modules V_0 and V_1 of dimension 1 and $d-1$, respectively. Then, in the same way as above, we see that $C_H(V_1) = 1$. Now we get a contradiction to the result in the previous case.

Hence $V = V_0 \oplus V_1$ with an irreducible $\mathbb{F}_q H$-module V_1 of dimension $d-2$ and an $\mathbb{F}_q H$-module V_0 of dimension 2. As before, we see that $C_H(V_1) = 1$.

In any case, we have an irreducible and faithful H-module $W := V_1$ of dimension 2^{k+1}, such that $W_{|\langle t \rangle}$ is irreducible for any $1 \neq t \in T$ of odd prime order.

(2) Let A be a characteristic abelian subgroup of Y. Then $W = W_1 \uparrow^H$ for some irreducible $\mathbb{F}_q I$-module with $A \leq I \leq H$. Hence $[H : I]$ divides 2^{k+1}, so without loss, $T_0 \leq I$ and $W_1 = W$. We conclude that $W_{|A}$ is homogenous. In particular, all albelian characteristic subgroups of Y are cyclic. Hence Y is of symplectic type, i.e., $Y \cong \mathscr{E} * R$ with $\mathscr{E}$ extraspecial or 1 and $R = Z(Y)$ cyclic or $R \cong Q_{2^b}$, D_{2^b}, SD_{2^b} with $b \geq 4$ (SD stands for semidihedral). In the latter case, Y contains the normal cyclic subgroup $N := Z(C_Y(\Phi(Y)))$ of order 2^{b-1}, and since $N \supset C_Y(T) \leq Y \cap T \cong \mathbb{Z}_2$, we get a contradiction. So we can assume that $Y \cong \mathscr{E} * Z(Y)$ with cyclic center.

Suppose that $1 \neq \mathscr{E}$. Consider the group $H_s := Y : \langle t \rangle$ with $1 \neq t \in T$ an element of odd prime order s. Then $Z(Y) \leq Y \cap T$; hence $Z(Y) = Z(\mathscr{E}) = C_Y(t)$ and $Y = [Y, \langle t \rangle]$ is extraspecial of order 2^{2a+1}. Notice that $C_{\langle t \rangle}(Y) = 1$ (otherwise $Y \leq C_H(t^j) = T$ for some $1 \leq j < s$). So we can apply (36.1) in [1].

Since $C_W(\langle t \rangle) = 0$, we conclude that $s = 2^a + 1$ is a Fermat prime; moreover, $q^{2^k} + 1 = 2s^{\ell}$. Since $C_{Y/Z(Y)}(t') = 1$ for each $1 \neq t' \in T_0$, we get $|Y/Z(Y)| - 1 = 2^{2a} - 1 = (s-2)s = ms^{\ell}$ for some $m \in \mathbb{N}$. Hence $\ell = 1$ and $m = s - 2$. So $q^{2^k} = 2(2^a + 1) - 1 = 2^{a+1} + 1$ and $k > 0$ imply $k = 1$ and $q = 3$, a final contradiction. ∎

LEMMA 5.9. *Let $T = \langle t \rangle \times \langle c \rangle \cong \mathbb{Z}_{2^k \ell} \leq G$ with $|t| = 2^k$ and $1 < |c| = \ell$ odd, such that $C_G(\langle t \rangle) = C_G(\langle c \rangle) = T$. Let $N := N_G(T) < X \leq G$ such that X is solvable and N is maximal in X. Then $O(X) \neq 1$.*

Proof. Suppose that $O(X) = 1$. Then $1 \neq O_2(X)$ and $C_X(O_2(X)) \leq O_2(X)$. We have $[O_2(N), \langle c \rangle] \leq O_2(N) \cap \langle c \rangle = 1$; hence $O_2(N) = O_2(T)$ and $O_2(X) \cap N \leq \langle t \rangle$. In particular, $O_2(X)$ is not contained in N. Define $Y := O_2(X)\langle t \rangle$; since $Y > \langle t \rangle$ and $Y \cap N = \langle t \rangle$, there is an element $y \in N_Y(\langle t \rangle) \setminus N$. Hence $N_X(\langle t \rangle) \geq \langle N, y \rangle = X$. So $\langle t \rangle \trianglelefteq X$ and $T = C_X(\langle t \rangle) \trianglelefteq X \leq N_G(T)$, a contradiction. ∎

THEOREM 5.10. *Let $q = r^m$ and let $(\tilde{S}, S)$ be as in Definition 5.6. Suppose that $X \in \mathscr{I}_2(S)$ such that $r \notin \pi(X)$ whenever $r \neq 2$. Then one of the following holds:*

In case (A) *and $n = 2$*

(i) $S \cong PSL_2(q)$, $X =_S N_S(T_{id}) \cong D_{2(1q-1)/\gcd(2,q-1)}$ *with* $2 < q \notin \mathscr{F} \cup \{9\}$ *or* $X =_S NS(T_{cox}) \cong D_{2(q+1)/\gcd(2,q-1)}$ *with* $2 < q \notin \mathscr{M}$;

(ii) $S \cong PSL_2(7)$, $X =_S \Sigma_4^{(i)}$, $i = 1$ *or* 2. *S acts 2-transitively on the seven points and the seven lines of $P(2,2)$. The group $\Sigma_4^{(1)}$ represents a conjugacy class of point stabilizers, $\Sigma_4^{(2)}$ represents a conjugacy class of line stabilizers. In particular, $|\Sigma_4^{(i)} \cap \Sigma_4^{(i)^g}| = 4$ for $g \in S \setminus \Sigma_4^{(i)}$.*

In case (A), *$n > 2$ and $\epsilon = 1$*

(i) $n = 3$: $S \cong PSL_3(q)$, $X =_S N_S(T_{(12)}) \cong \mathbb{Z}_{(q^2-1)/\gcd(3,q-1)} : \mathbb{Z}_2$ *with $q = 3 \cdot 2^x + 1$ or $q = 2^x + 1$ and $x > 1$.*

(ii) $S \cong PSL_3(4)$, $\mathscr{I}_2(S) = \{X \leq S | X =_S N_S(T_{(12)}) \cong \mathbb{Z}_5 : \mathbb{Z}_2,\ X =_S (\mathbb{Z}_2^4 : \mathbb{Z}_5) \cdot \mathbb{Z}_2,\ X =_S \mathbb{Z}_3^2 : Q_8 < \cdot S\}$.

(iii) $n = 4$: $S \cong PSL_4(q)$, $X =_S N := N_S(T_{cox}) \cong \mathbb{Z}_{2^u(q^2+1)/2} : \mathbb{Z}_4$ *with $q = 2^u - 1 \in \mathscr{M}$*;

(iv) $n = 5$; $S \cong PSL_5(3)$, $X =_S N_S(T_{(1234)}) \cong \mathbb{Z}_{2^4 5} : \mathbb{Z}_4$.

In case (A) *and $\epsilon = -1$*

(i) $n = 3$: $S \cong PSU_3(q)$, $X =_S N_S(T_{(12)}) \cong \mathbb{Z}_{(q^2-1)/\gcd(3,q+1)} : \mathbb{Z}_2$ *with $q = r = 3 \cdot 2^x - 1$ or $q = 2^x - 1$ and $x > 1, q > 3$.*

(ii) $n = 4$: $S \cong PSU_4(q)$, $X =_S N_S(T_{cox}) \cong \mathbb{Z}_{2^{2^t}(q^2+1)/\gcd(q+1,4)} : \mathbb{Z}_4$ *with $q = 2^{2^t} + 1 \in \mathscr{F} \cup \{9\}$.*

(iii) $n = 5$: $S \cong PSU_5(q)$, $q \in \{3, 9\}$, $X =_S N_S(T_{(1234)}) \cong \mathbb{Z}_{2^4 5} : \mathbb{Z}_4$ *and $\mathbb{Z}_{2^5 41} : \mathbb{Z}_4$, respectively.*

In case (B)

$n = 2^k$: $S \cong P\Omega_{2^{k+1}+1}(q)$, $X =_S N_S(T_{cox})$ *with* $N_{\hat{S}}(\tilde{T}_{cox}) \cong \mathbb{Z}_{q^{2^k}+1} \cdot \mathbb{Z}_{2^{k+1}}$.

In case (C)

$n = 2^k$: $S \cong PSp_{2^{k+1}}(q)$, $X =_S N_S(T_{cox}) \cong \mathbb{Z}_{(q^{2^k}+1)/\gcd(q-1,2)} : \mathbb{Z}_{2^{k+1}}$.

In case (D) *and* $\epsilon = 1$

$n = 2^k + 1$: $S \cong P\Omega_{2n}(q)$, $X =_S N_S(T_{cox})$, $N_{\hat{S}}(\tilde{T}_{cox}) \cong (\mathbb{Z}_{q^{2^k}+1} \times \mathbb{Z}_{q+1}) \cdot \mathbb{Z}_{2^{k+1}}$ *with* $q \in \mathscr{M}$.

In case (D) *and* $\epsilon = -1$

$n = 2^k + 1$: $S \cong P\Omega^-_{2n}(q)$, $X =_S N_S(T_{cox})$, $N_{\hat{S}}(\tilde{T}_{cox}) \cong (\mathbb{Z}_{q^{2^k}+1} \times \mathbb{Z}_{q-1}) \cdot \mathbb{Z}_{2^{k+1}}$, *with* $q \in \mathscr{F} \cup \{9\}$.

$n = 2^k$: $S \cong P\Omega^-_{2n}(q)$, $X =_S N_S(T_{(12\cdots n)^-})$, $N_{\hat{S}}(\tilde{T}_{(12\cdots n)}) \cong \mathbb{Z}_{q^{2^k}+1} \cdot \mathbb{Z}_{2^k}$.

In case (E) *and S of type* ${}^2B_2(q)$ *with* $q = 2^{2n+1}$ *and* $n \geq 1$

$X =_S N_S(T)$ *with* $T \cong \mathbb{Z}_{q-1}, \mathbb{Z}_{q+\sqrt{2q}+1}$ *or* $\mathbb{Z}_{q-\sqrt{2q}+1}$ *and* $X/T \cong \mathbb{Z}_2, \mathbb{Z}_4$ *or* $\mathbb{Z}_4$, *respectively*.

In case (E) *and S of type* ${}^2G_2(q)$ *with* $q = 3^{2n+1}$ *and* $n \geq 1$

$X =_S N_S(T_1) \cong \mathbb{Z}_{q-1} : \mathbb{Z}_2$.

In case (E) *and S of type* ${}^3D_4(q)$

$X := N_S(T_5) \cong \mathbb{Z}_{q^4-q^3+1} \cdot \mathbb{Z}_4$ (*see* [23] *for notation*).

In case (E) *and Dynkin type* F_4

$S \cong F_4(q)$, $X =_S N_S(T_{(1234)^-}) \cong \mathbb{Z}_{q^4+1} : \mathbb{Z}_8$. ($T_{(1234)^-}$ *is a Coxeter torus of* $B_4(q) \leq F_4(q)$).

In case (E) *and Dynkin type* E_6

$S \cong E_6(q)$, $X =_S N_S(T_{19}) \cong \mathbb{Z}_{(q^2-1)(q^4+1)/\gcd(q-1,3)} : \mathbb{Z}_8$, $q \in \{3, 7\}$ (*see* [11] *for notation*).

In case (E) *and Dynkin type* E_6

$S \cong {}^2E_6(q)$, $X =_S N_S(T_{19}) \cong \mathbb{Z}_{(q^2-1)(q^4+1)/\gcd(q+1,3)} : \mathbb{Z}_8$, $q \in \{3, 5\}$ (*see* [11] *for notation*).

Proof. Let $\mathscr{I}_{2,r'} := \{Y \in \mathscr{I}_2(S) | r \notin \pi(Y)\}$ and suppose that $X \in \mathscr{I}_2(S)$. In view of Theorem 5.2, we may assume that X and hence S has cyclic Sylow s-subgroups for any odd prime $s \in \pi(X)$. Moreover, by Theorem 5.1, we can assume that $r \notin \pi(X)$ whenever $r \neq 2$.

I. Classical Groups

(1) First we consider the case (A) with $n = 2$ and thus $S \cong PSL_2(q)$. It is easy to see that each nontrivial element of odd order in T_{id} and T_{cox} is regular. Hence $N_S(T_{id})$ and $N_S(T_{cox})$ are contained in $\mathcal{I}_2(S)$. Moreover, using Dickson's complete list of subgroups given in [21, p. 213], we easily verify that $N_S(T_{id})$ and $N_S(T_{cox})$ are maximal in S, unless possibly $q \in \{4, 5, 7, 9, 11\}$; but for these remaining cases the claims follow by a trivial check.

In what follows we may assume that $n > 2$ in case (A).

(2) Note that each element of X of odd prime order lies in a finite maximal torus of S. We take $i \in \{1, \dots, n\}$ to be minimal such that there is an odd prime s dividing $q^{2i} - 1$ and an element $x \in X$ of order s. Let $\tilde{X} \le \tilde{S}$ be such that $X = (\tilde{X} \cap \tilde{S}')/Z(\tilde{S})$ and let $\tilde{x}$ be a preimage of x in $\tilde{X}$.

(3) In this section we handle the case where $i = 1$; so s divides either $q - 1$ or $q + 1$.

Recall that the Sylow s-subgroups of X and hence of S are cyclic; therefore S cannot involve sections isomorphic to $L_2(q) \times L_2(q) \cong U_2(q) \times U_2(q)$. Hence S is isomorphic to $L_3(q)$ or to $U_3(q)$.

Assume now that $S \cong L_3(q)$. As S contains subgroups isomorphic to $\mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$, s divides $q + 1$. Without loss we may assume that X contains the group $N_{(12)} := N_S(T_{(12)})$ of order $2_{(q^2-1)/\gcd(3,q-1)}$; moreover, as $\pi(X)$ does not contain any odd prime divisor of $q - 1$, we have $q - 1 = \gcd(3, q - 1)2^x$ with $x \ge 0$.

If $q \le 9$, then all of the claims can be verified by straightforward checks using the subgroup structure of S (e.g., see [6]). Henceforth we may assume that $q \ge 13$ and thus $x \ge 2$ as well as $r \ge 3$. Note that Lemma 3.8 now even implies $r \ge 5$; hence by Theorem 5.5 we get $X = N_{(12)}$. Finally, it is easy to see that all elements $t \in (T_{(12)})_{2'}$ are regular; moreover, $|N_{(12)}/T_{(12)}| = 2$, and thus $X = N_{(12)}$ is indeed contained in $\mathcal{I}_2(S)$.

Suppose next that $S \cong U_3(q)$. Suppose in addition that s divides $q + 1$. Then we may assume that X contains the image T_1 of order $(q + 1)^3/\gcd(3, q + 1)$ of a "diagonal" torus of $\tilde{S}$. Since Sylow s-subgroups of X are cyclic, we get $s = 3$ and $q + 1 = 3 \cdot 2^x$ for some $x \ge 0$. But now observe that $X \ge N_S(T_1) \ge T_1 : \langle d \rangle$ with d being the image of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

in S and $o(d) = 3$, which in turn shows that Sylow 3-subgroups of X cannot be cyclic. This contradiction shows that s divides $q - 1$.

Now we can assume that X contains a maximal torus T_{w_0} of order $(q^2 - 1)/\gcd(q + 1, 3)$. As $q > 2$ and as $\pi(X)$ cannot contain any odd prime divisor of $q + 1$, we get $q + 1 = \gcd(3, q + 1) \cdot 2^x$ with $x \geq 1$. Using Lemma 3.8 and Fermat's Theorem, we find that $q = r$.

If $q \in \{3, 5, 7, 11\}$, we easily verify the claims using the subgroup structure of S as can be found in [6]. Henceforth we may assume that $q = r \geq 13$.

As before, we can now apply Theorem 5.5 to conclude that $X = N := N_S(T_{w_0})$. Finally, notice that $N/T_{w_0} \cong \mathbb{Z}_2$. Let $t \in X \cap X^g$ for some $g \in S$ be an element of odd order. Then $t \in T_{w_0} \cap T_{w_0}^g$. If $g \notin X$, then the preimage of t in $\overline{S}$ is singular and conjugate (in $\tilde{\mathbf{S}}$) to

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a^{q-1} & 0 \\ 0 & 0 & q^{-q} \end{pmatrix}.$$

If $a = a^{q-1}$, then $a^q = a^2$; hence $a^4 = a^{q^2} = a$ and $a^{-q} = a$, i.e., $t = 1$. The same follows if $a^{-q} = a^{q-1}$. If $a^{-q} = a$, then $a^{q+1} = 1$; hence $|t|$ divides $(q + 1)/\gcd(3, q + 1)$, which is a power of 2. This shows that N is in fact a 2-intersection subgroup.

(4) In the remainder we can suppose that $1 < i$. In particular, $\gcd(3, |X|) = 1$, because otherwise we get $r = 3 \in \pi(X)$, a contradiction.

(4a) First we deal with the case (A). Let ℓ be the order of $q\epsilon$ modulo s and observe that $2 < \ell \leq n$; moreover, either $\ell = i$ is odd or $\ell = 2 \cdot i$. Also note that s divides $((q\epsilon)^\ell - 1)(q\epsilon - 1)^{n-\ell}/(q\epsilon - 1)\gcd(q\epsilon - 1, n)$, which is the order of a maximal torus T containing the element $x \in X$ of order s. Hence

$$\frac{\big((q\epsilon)^\ell - 1\big)(q\epsilon - 1)^{n-\ell}}{(q\epsilon - 1)\gcd(q\epsilon - 1, n)} \cdot \ell \cdot (n - \ell)! \,\big|\, |N_S(T)| \,\big|\, |X|.$$

Now suppose that ℓ is a prime and let m denote the order of $q\epsilon$ modulo ℓ; clearly, $m \leq \ell - 1 = i - 1$. On the other hand, ℓ divides $|X|$ and $(q\epsilon)^{2m} - 1 = q^{2m} - 1$; so the choice of i implies $i \leq m$, a contradiction. Hence ℓ is not a prime.

Suppose next that there exists a divisor v of i with $1 < v < i$. Then $|((q\epsilon)^v - 1)/(q\epsilon - 1)|$ divides $|X|$, and so the minimal choice of i implies $|((q\epsilon)^v - 1)/(q\epsilon - 1)| = 2^m$ with $m \geq 1$; moreover, since $v \,|\, |X|$ and thus $v \neq 3$, application of Lemma 3.6 and Corollary 3.3 now yields $v = 2$. We have shown that either i is a prime or $i = 4$; in particular, $\ell = 2i$.

If s divides $(q\epsilon)^i - 1$, then without loss X contains a torus of order $((q\epsilon)^i - 1)^2(q\epsilon - 1)^{n-\ell}/(q\epsilon - 1)\gcd(q\epsilon - 1, n)$ with noncyclic Sylow s-subgroups; but this is impossible. Therefore s divides $(q\epsilon)^i + 1$. In particular, as $((q\epsilon)^i - 1)/(q\epsilon - 1) \mid |X|$, we now see that $((q\epsilon)^i - 1)/(q\epsilon - 1) = 2^m$ with $m \geq 1$. Using the same arguments as above, we find $i = 2$ and $l = 4$. Furthermore, if $\epsilon = 1$, then $q = 2^m - 1 \in \mathscr{M}$; if $\epsilon = -1$, then $q = 2^m + 1 \in \mathscr{F} \cup \{9\}$.

As $(n - \ell)! \mid |X|$ and $3 \notin \pi(X)$, $\ell \leq n \leq \ell + 2$. If $n = \ell + 2 = 6$, then s divides the order of a section isomorphic to $L_4(q)$ or $U_4(q)$ according to $\epsilon = 1$ or $\epsilon = -1$, respectively. In any case, we easily see that then X contains a section isomorphic to $L_2(q) \cong U_2(q)$, contrary to $3 \notin \pi(X)$. Therefore $n \in \{4, 5\}$.

Assume now that $n = \ell + 1 = 5$. Then $|(q\epsilon - 1)/\gcd(q\epsilon - 1, 5)|$ divides $|X|$, and thus is also a power of 2. From this we readily deduce that $(q, \epsilon, s) \in \{(3, 1, 5), (3, -1, 5), (9, -1, 41)\}$. A straightforward check using the relevant subgroup structure now yields the claims as stated for the three cases just given.

Henceforth we have $n = \ell = 4$. Moreover, $N := N_S(T_{(1234)}) \leq X$ with $N \cong \mathbb{Z}_{2^m(q^2+1)/2} : \mathbb{Z}_4 \in \mathscr{I}_2(S)$ if $\epsilon = 1$ and with $N \cong \mathbb{Z}_{2^m(q^2+1)/\gcd(q+1,4)} : \mathbb{Z}_4 \in \mathscr{I}_2(S)$ if $\epsilon = -1$. Whenever $q > 11$, we can use Theorem 5.5 to prove $X = N$. If $q \leq 11$, again an easy check using the subgroup structure of the groups $L_4(3)$, $L_4(7)$, $U_4(2)$, $U_4(3)$, $U_4(5)$, and $U_4(9)$ shows that $X = N$.

(4b) We are left to deal with the cases (B) to (D). Recall that s divides $q + \delta$ for some $\delta \in \{-1, 1\}$. Now suppose that $i = a \cdot b$ with $1 < b < i$ and b odd. Then $q^a + \delta$ divides $q^i + \delta$, and so minimality of i implies that $q^a + \delta$ is a power of 2. As $a \geq 2$, we now deduce that $(q, a, \delta) = (3, 2, -1)$; but then s divides $q^{2b} - 1$, contrary to the choice of i. We have shown that either $i = 2^k \geq 2$ and $s \mid q^i + 1$ or i is an odd prime.

(4c) Now we first consider the cases (B) and (C). We can assume that $x \in T_{(12\cdots i)^{\pm}}$. Since x is regular, we must have $i = n$. Suppose that i is odd; then it is a prime and we have $x = {}^g(t, t^q, \ldots, t^{q^{i-1}})Z \in T_{(1,2,\ldots,n)^{\pm}}$. Since $N_S(T_{(1,2,\ldots,n)^{\pm}}) \leq X$, there is an element $\dot{w}$ of order n in X. Suppose $x \in T_{(1,2,\ldots,n)}$; then $q - 1$ is a power of 2. By minimality of $n = i$, the element $\dot{w}$ is contained in a maximal torus of order $q^n - 1$ or of order $q^n + 1$. The first case implies $0 \equiv q^n - 1 \equiv q - 1 \bmod n$; hence $n = 2$. The second case implies that a torus $T_{(1,2,\ldots,n)^-}$ is contained in X and hence $q + 1$ is a power of 2. We get $0 \equiv q^n + 1 \equiv q + 1 \bmod n$, and again $n = 2$. Similarly, the assumption $\bar{t} = (t, t^q, \ldots, t^{q^{i-1}}) \in T_{(1,2,\ldots,n)^-}$ yields $n = 2$. But this is a contradiction to $i > 2$.

Hence $1 < i = 2^k = n$, $x \in T_{cox} \cong T_{(1,2,\ldots,2^k)^-}$, and each $y \in T_{cox}$ of odd order is regular in T_{cox}. Moreover, all odd prime divisors of $|\tilde{X}|$ are

"Zsigmondy primes" of $q^{2^k} + 1$, i.e., they do not divide any $q^{2i} - 1$ with $i < 2^k$. In particular, we see that $[X : T_{cox}]$ is a power of 2; hence there is $Q \in Syl_2(\tilde{X})$ such that $\tilde{X} = \tilde{T}_{cox}Q = Q\tilde{T}_{cox}$. Notice that $\tilde{N} := N_{\tilde{S}}(\tilde{T}_{cox}) \leq \tilde{X}$, $\tilde{N}/T_{cox} \cong \mathbb{Z}_{2^k}$, and $N := N_S(T_{cox}) \in \mathcal{I}_2(S)$. Since $\tilde{X}$ is the product of the nilpotent groups $\tilde{T}_{cox}$ and Q, it is solvable by the theorem of Kegel and Wielandt [21, 4.3, p. 674]. We claim that $T_{cox} \trianglelefteq X$. By Lemma 2.9 it suffices to show that $|O(X)| > 1$.

We consider the natural module V of $\tilde{S}$ of dimension $2^{k+1} + 1$ in case B_n or 2^{k+1} in case C_n. Suppose that $r = 2$. Then $S = \tilde{S} \cong Sp_{2^{k+1}}(q)$, and we see, as in Lemma 5.8, that V_X is irreducible and faithful. Hence $O_2(X) = 1$ and $O(X) \neq 1$. So we can assume that $r > 2$.

Suppose that $Y := O_2(\tilde{X}) \neq 1$. Now we can apply Lemma 5.8 with $H := Y.\tilde{T}_{cox}$. If Y is not cyclic, then we conclude $S \cong PSp_4(3) \cong P\Omega_5(3)$. This simple group is also isomorphic to $SU_4(2)$, and we are reduced to the groups of case (A). Now our previous results give a contradiction. So $Y = Z(Y) \cong \mathbb{Z}_2 \leq Z(\tilde{X})$. Now we consider the Fitting group $F(\tilde{X}) = O_2(\tilde{X}) \times O(F(\tilde{X}))$. Since $C_{\tilde{X}}(F(\tilde{X})) \leq F(\tilde{X})$, we get $1 \neq O(F(\tilde{X}))$ and thus $1 \neq O(X)$.

(4d) Next we deal with the case (D). As in (4a), we see that $n \leq i + 1$. Suppose that $n = i + 1$ and i is an odd prime. Then s divides $q^i \pm 1$. In case $\epsilon = 1$, we get

$$s \big| \big((q^i \pm 1)(q \pm 1)i\big)_{2'} = \big(\big|N_S\big(\mathbf{T}_{(12\,\cdots\,i)^{\mp}(n)^{\mp}}\big)\big|\big)_{2'} \big|\, |X|,$$

whereas in case $\epsilon = -1$ we get

$$s \big| \big((q^i \pm 1)(q \mp 1)i\big)_{2'} = |N_S(\quad {}_{2\ldots i)^{\pm}(n)^{\mp}}|_{2'} \big|\, |X|.1$$

Let $j \leq n$ be such that i divides $q^{2j} - 1$. Then $i \leq j$ (by the choice of s); hence $j = i$ or $i + 1$. If $j = i + 1$, we get $0 \equiv q^{2(i+1)} - 1 \equiv q^4 - 1 \mod (i)$. Since i is an odd divisor of $|X|$, we get the contradiction $i \leq 2$. If $j = i$, then we conclude in a similar way that i divides $q^2 - 1$, yielding the contradiction $i = 2$.

Now suppose that $n = i$ is an odd prime. Then s divides $q^n \pm 1$. In case $\epsilon = 1$, we get $s | ((q^n - 1)n)_{2'} = (|N_S(\mathbf{T}_{(12\,\cdots\,n)^+})|)_{2'} \,|\, |X|$, whereas in case $\epsilon = -1$ we get $s | ((q^n + 1)n)_{2'} = (|N_S(\mathbf{T}_{(12\,\cdots\,i)^-})|)_{2'} \,|\, |X|$.

Let $j \leq n$ be such that i divides $q^{2j} - 1$. Then $j = i = n$, and we get $0 \equiv q^{2n} - 1 \equiv q^2 - 1 \mod (i)$, yielding the contraction $i = 2$. Therefore $i = 2^k$ and $n = i$ or $n = i + 1$.

Suppose that $n = i$. Then, by the order formulae for Chevalley groups, s divides $q^n + 1$; since $\gcd(q^{2j} - 1, q^{2^k} + 1) | 2$ for all $j < n$, we conclude that without loss of generality, $s \in T := T_{(12\,\cdots\,n)^-}$; in particular, we have

$\epsilon = -1$. It is easy to see that all elements of odd order in T are regular, and hence $N := N_S(T) \in \mathcal{I}_2(S)$. The proof that $N = X$ for any $N \leq X \in \mathcal{I}_2(S)$ is similar to that given in (4c): We consider the natural module $V \cong \mathbb{F}_q^{2^{k+1}}$ for $\tilde{S}$. As above, we can assume that $[\tilde{X} : \tilde{T}]$ is a power of 2 and $\tilde{X}$ is solvable. If $r = 2$, we see, as above, that $O_2(\tilde{X}) = 1$, so we can assume $r > 2$, and apply Lemma 5.8 with $H := Y.\tilde{T}$ and $Y := O_2(\tilde{X})$. If Y is noncyclic, we get the contradiction $n = 2$, so Y is cyclic and we can finish as in (4c).

Finally, we suppose that $n = i + 1$. Then s divides $q^{n-1} + 1$, and the only maximal tori of order divisible by s are the Coxeter tori $T_{cox} = T_{(12\cdots 2^k)^-(n)^-}$ in case $\epsilon = 1$ and $T_{cox} = T_{(12\cdots 2^k)^-(n)^+}$ in case $\epsilon = -1$. In particular, $q + 1 = 2^m$, so $q \in \mathcal{M}$ is a Mersenne prime if $\epsilon = 1$ and $q - 1 = 2^m$, so $q \in \mathcal{F} \cup \{9\}$ is a Fermat number if $\epsilon = -1$. So $r > 2$, and it is easy to see that $N_S(T_{cox})$ is a 2-intersection subgroup. To prove "$X = N$," we can use Lemma 5.8 once more and proceed in a way similar to (4c).

II. Exceptional Groups

Let s be an odd prime divisor of $|X|$ and recall that $s \neq r$.

(5) Suppose first that $S \cong {}^2B_2(q)$ with $q = 2^{2a+1} \geq 8$. Then all of the claims can be verified easily by using the information given in [3, Chapt. XI.3].

(6) Suppose that $S \cong {}^2G_2(q)$ with $q = 3^{2a+1}$ and $a \geq 1$. (Recall that ${}^2G_2(3) \cong L_2(8) : \mathbb{Z}_3$.) As $r = 3 \notin \pi(X)$, we readily deduce from Theorem C in [24] that $X \geq N := N_S(T)$ only for tori $T \cong \mathbb{Z}_{q-1}$ with $|N : T| = 2$; moreover, $|X : N|$ is a power of 2 dividing $|S : N|_2 = 2$, and thus $X = N$.

From [3, 13.2, p. 292] we see that S is doubly transitive of degree $q^3 + 1$ with two-point stabilizer $S_{\alpha,\beta} = T$ such that any three-point stabilizer has order 2. This shows that $N \in \mathcal{I}_2(S)$.

(7) Now let S be the simple group $G_2(q)$ of order $q^6(q^6 - 1)(q^2 - 1)$. Since $(G_2(2))' \cong U_3(3)$, we can assume that $q > 2$.

Observe next that S involves a section isomorphic to $L_2(q) \times L_2(q)$ (e.g., see [7] and [24]). Since Sylow s-subgroups of S are cyclic, we conclude that s does not divide $q^2 - 1$; in particular, $s \neq 3$. Consequently, s divides $q^2 \pm q + 1$. Using the information given in [20], we now get $N := N_S(T) \leq X$ with a maximal torus T of order $q^2 \pm q + 1$; as $|N/T| = 6$, we reach a contradiction proving that $\mathcal{I}_2(S) = \emptyset$.

(8) Next let S be isomorphic to the simple group ${}^3D_4(q)$. The maximal subgroups of S have been determined by Kleidman [23]; informa-

tion on maximal tori can be found in [10], from which we take the notation here.

Since the Sylow s-subgroups of X are cyclic, we easily verify that $X \geq N := N_S(T)$ only for maximal tori T of type $T_5 \cong \mathbb{Z}_{q^4-q^2+1}$ with $N/T \cong \mathbb{Z}_4$. Moreover, we see from [10] that all nontrivial elements of T are regular; hence $N \in \mathscr{I}_2(S)$. Since N is a maximal subgroup of S (see [23]), we conclude $X = N$.

(9) Next let S be isomorphic to the simple group ${}^2F_4(q)$ with $q = 2^{2m+1}$. The maximal subgroups of S have been determined by Malle [29]; using Propositions 1.2 and 1.3 of [29] together with the fact that S has cyclic Sylow s-subgroups, we ready deduce that $\mathscr{I}_2(S) = \varnothing$.

(10) Here let S be the simple group $F_4(q)$ of order $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$ and let H be the subgroup of type $B_4(q)$. Then $H = Spin_9(q)$ and is a central 2-extension of $SO_9(q)$. Let $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a base of the root system Φ_{F_4} with α_1 and α_2 long roots. If $X \in \mathscr{I}_2(S)$, then either $H \leq X$, or $X \cap H$ is a 2-group, or $X \cap H \in \mathscr{I}_2(H)$.

Suppose that $H \leq X$. Then $r = 2$ and $X_{\pm\alpha_i} \in X$ for $i = 1, 2, 3$. Hence $X_{\pm\alpha_4} \leq C_S(\langle X_{\pm\alpha_1}\rangle) \leq X$ and $S = X$, a contradiction.

Suppose that $X \cap H$ is a 2-group. Since H has order $q^{16}(q^2-1)(q^4-1)(q^6-1)(q^8-1)$, we see that s must be a divisor of $q^4 - q^2 + 1$ and coprime to $|S|/(q^4 - q^2 + 1)$. By [32] there is only one class of maximal tori T such that $\gcd(|H|, |T|)_{2'} = 1$. A representative T_w of this class has order $q^4 - q^2 + 1$ and $|N_S(T_w)/T_w| = 12$. So $3 \mid |X|$ as well as $|H|$, a contradiction. We conclude that $X \cap H \in \mathscr{I}_2(H)$; hence $X \cap H$ is conjugate in H to $N_H(T_{(1234)^-})$. From [31] and [32] we see that $T_{(1234)^-}$ is also a maximal torus of S such that all elements whose order is not a power of 2 are regular. In particular, $N_S(T_{(1234)^-}) \in \mathscr{I}_2(S)$.

(11) Let S be the simple group ${}^\epsilon E_6(q)$ where $\epsilon = -1$ in the twisted cases and $\epsilon = 1$ otherwise. Note that $|S| = q^{36}(q^{12}-1)(q^9-\epsilon)(q^8-1)(q^6-1)(q^5-\epsilon)(q^2-1)(1/\gcd(q-\epsilon, 3))$, and let $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be a base of the root system, such that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ form a Dynkin diagram of type A_5. (See [11] for further notation.) Let $X \in \mathscr{I}_2(S)$ and $H = {}^\epsilon A_5(q)$. Since $\mathscr{I}_2(H) = \varnothing$, $X \cap H$ is a 2-group or $H \leq X$. Suppose that $H \leq X$. Then $X_{\pm[a_i]} \leq X$ for $i \neq 4$. But $X_{\pm[a_4]} = X_{+\alpha_4} \leq C_S(\langle X_{\pm[a_1]}\rangle) \leq X$, so we get the contradiction $S = X$. (Notice that in the case of ${}^2E_6(q)$, $[\alpha_1] = \{\alpha_1, \alpha_6\}$, $[\alpha_2] = \{\alpha_2, \alpha_5\}$, and $[\alpha_i] = \alpha_i$ for $i = 3, 4$.) So $\gcd(|X|_{2'}, |{}^\varepsilon A_5(q)|_{2'}) = 1$ with $|{}^\varepsilon A_5(q)| = q^{15}(q^6-1)(q^5-\epsilon)(q^4-1)(q^3-\epsilon)(q^2-1)/\gcd(q-\epsilon, 6)$. Hence s divides $q^4 + 1$, $(q^9-\epsilon)/(q^3-\epsilon) = q^6 - \epsilon q^3 + 1$, or $(q^6+1)/(q^2+1) = q^4 - q^2 + 1$.

If s divides $q^6 - \epsilon q^3 + 1$, then we can assume that $N := N_S(T_{24}) \leq X$ with $|T_{24}| = (q^6 - \epsilon q^3 + 1)/\gcd(q - \epsilon, 3)$. But $|N/T_{24}| = 3^2$, and we get the contradiction $3 \mid \gcd(|X|_{2'}, |{}^{\epsilon}A_5(q)|_{2'})$.

Suppose that s divides $q^4 - q^2 + 1$, then w.l.o.g. $N := N_S(T_{23}) \leq X$ with $|T_{23}| = (q^2 + q + 1)(q^4 - q^2 + 1)/\gcd(q - 1, 3)$ and $|N/T_{23}| = 2^2 3$, giving the same contradiction.

Hence we conclude that s divides $q^4 + 1$. In this case we can assume that $N := N_S(T_{19}) \leq X$ with $T_{19} \cong \mathbb{Z}_{(q^2-1)(q^4+1)/\gcd(q-\epsilon,3)}$ and $|N/T_{19}| = 2^3$. Moreover, we see that $(q^2 - 1)/\gcd(q - \epsilon, 3)$ is power of 2. If 3 divides $q - \epsilon$, we get $q = 7$ if $\epsilon = 1$ and $q = 5$ if $\epsilon = -1$. It is easy to see that $T_{19} = \mathbb{Z}_{(q-1)/\gcd(q-1,3)} * \tilde{T}_{(1)^-(2345)^-} \leq \tilde{H} := \mathbb{Z}_{(q-1)/\gcd(q-1,3)} * D_5(q)$ if $\epsilon = 1$ and $T_{19} = \mathbb{Z}_{(q+1)/\gcd(q+1,3)} * \tilde{T}_{(1)(2345)^-}, \leq \tilde{H} := \mathbb{Z}_{(q+1)/\gcd(q+1,3)} * {}^2D_5(q)$ if $\epsilon = -1$. Since all $(q^4 + 1)/2$ elements of odd order in T_{19} are regular, we see that $N \in \mathcal{I}_2(S)$.

Now we have to prove that $X = N = N_S(T_{19})$ if $q \in \{3, 7\}$ or $\{3, 5\}$, respectively. Our arguments show that $[X : N] = 2^y$ and $|N| = 2^8 \cdot 1201$ for $(q, \epsilon) = (7, 1)$, with $N \geq Q \in Syl_{1201}(S)$, $|N| = 2^7 \cdot 41$ for $(q, \epsilon) = (3, \pm 1)$, with $N \geq Q \in Syl_{41}(S)$, $|N| = 2^7 \cdot 313$ for $(q, \epsilon) = (5, -1)$, with $N \geq Q \in Syl_{313}(S)$. In particular, $N = N_S(Q)$ and $[S : \tilde{H}]_2 = 2$ for $(q, \epsilon) = (3, -1)$ and $[S : \tilde{H}]_2 = 1$ for $(q, \epsilon) \in \{(3, 1), (7, 1), (5, -1)\}$.

So we can assume that $X = QR$ with $R \in Syl_2(X)$ and $Q \in Syl_2(\tilde{H}) \subseteq Syl_s(S)$. Let $R \leq \tilde{R} \in Syl_2(S)$ and $R_1 \leq \tilde{R}$ with $R_1 \in Syl_2(\tilde{H})$. Then $|R|/|R \cap R_1| = |RR_1|/|R_1| \leq 2$; hence $[X : X \cap \tilde{H}] \leq 2$. From our result in case (D), we know that $Q = O_s(X \cap \tilde{H}) \operatorname{char} X \cap \tilde{H} \trianglelefteq X$; hence $X \leq N_S(Q) = N$.

(12) Now let S be the simple group of type E_7 with

$$|S| = q^{63}\big(q^{18} - 1\big)\big(q^{14} - 1\big)\big(q^{12} - 1\big)\big(q^{10} - 1\big) \\ \times\big(q^8 - 1\big)\big(q^6 - 1\big)\big(q^2 - 1\big)\frac{1}{\gcd(q - 1, 2)}.$$

Let $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ be a base of the root system and δ_0 be the highest positive root, such that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, -\delta_0\}$ form a Dynkin diagram of type A_7. Let $X \in \mathcal{I}_2(S)$ and $H := {}^{\epsilon}A_7(q) \leq S$. Since $\mathcal{I}_2(H) = \emptyset$, $X \in H$ is a 2-group or $H \leq X$. Suppose that $H \cong A_7(q) \leq X$. Then $X_{\pm\alpha_i} \leq X$ for $i \neq 5$. But $X_{\pm\alpha_5} \leq C_S(\langle X_{\pm\alpha_1}\rangle) \leq X$, so we get the contradiction $S = X$. Suppose that $H = {}^2A_7(q) \leq X$. Then $X_{\pm[\alpha_i]} \leq X$ for $i = 1, 2, 3$ and 4 with $[\alpha_1] = \{\alpha_1, -\delta_0\}$, $[\alpha_2] = \{\alpha_2, \alpha_7\}$, $[\alpha_3] = \{\alpha_3, \alpha_6\}$, and $[\alpha_4] = \{\alpha_4\}$. Hence $X_{\pm\alpha_5} \leq C_S(\langle X_{\pm[\alpha_1]}\rangle) \leq X$, and $X_{\pm\alpha_i} \leq C_S(\langle X_{\pm\alpha_5}\rangle) \leq X$ for $i \in \{1, 2, 3, 6, 7\}$. Again we get the contradiction

$S = X$. So we have $\gcd(|X|_{2'}, |H|_{2'}) = 1$ with

$$|H| = q^{28}(q^8 - 1)(q^7 - \epsilon)(q^6 - 1)(q^5 - \epsilon) \times (q^4 - 1)(q^3 - \epsilon)(q^2 - 1)\frac{1}{\gcd(q - \epsilon, 8)}.$$

Hence s divides $(q^9 - \epsilon)/(q^3 - \epsilon) = q^6 - \epsilon q^3 + 1$ or $(q^6 + 1)/(q^2 - 1) = q^4 - q^2 + 1$.

But then we can assume that $N_{\epsilon_{E_6(q)}}(T) \le X$ with $T = T_{24}$ or $T = T_{23}$, maximal tori of ${}^{\epsilon}E_6(q) \le E_7(q)$ as above. This gives the contradiction $3 | \gcd(|X|_{2'}, |H|_{2'})$.

(13) Finally, let S be the simple group of type E_8 with

$$|S| = q^{120}(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1) \times (q^{14} - 1)(q^{12} - 1)(q^8 - 1)(q^2 - 1).$$

Let $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ be a base of the root system, such that $\Pi \backslash \{\alpha_1\}$ forms a Dynkin diagram of type E_7. Let $X \in \mathscr{I}_2(S)$ and $H := E_7(q) * \mathbb{Z}_{q-1}$. Since $\mathscr{I}_2(H) = \emptyset$, $X \cap H$ is a 2-group or $H \le X$. Suppose that $H \le X$. Then $X_{\pm\alpha_i} \le X$ for $i > 1$. But $X_{\pm\alpha_1} \le C_S(\langle X_{\pm\alpha_8}\rangle) \le X$, so we get the contradiction $S = X$. Thus we have $\gcd(|X|_{2'}, |H|_{2'}) = 1$ with $|H| = (q-1)|E_7(q)|$. Hence s divides $(q^{10} + 1)/(q^2 + 1) = q^8 - q^6 + q^4 - q^2 + 1$ or $(q^{12} + 1)/(q^4 + 1) = q^8 - q^4 + 1$ or $(q^{30} - 1)/(q^{10} - 1)(q^2 + q + 1)(q^2 - q + 1) = (q^8 + q^7 - q^5 - q^4 - q^3 + q + 1)(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1)$.

Now consider the following maximal tori T_i and normalizers N (here the information and notation is taken from [12] (see also [9]):

$|T_{106}| = q^8 - q^6 + q^4 - q^2 + 1$ and $|N/T_{106}| = 2^2 5 | \, |A_5(q)| \, |E_7(q)|$;

$|T_{105}| = q^8 - q^4 + 1$ and $|N/T_{105}| = 2^3 3 | \, |E_7(q)|$;

$|T_{104}| = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$ and $|N/T_{104}| = 2 \cdot 3 \cdot 5 | \, |A_5(q)| \, |E_7(q)|$;

$|T_{109}| = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$ and $|N/T_{109}| = 2 \cdot 3 \cdot 5 | \, |A_5(q)| \, |E_7(q)|$.

Repeating the arguments above, we see that $\mathscr{I}_2(S) = \emptyset$. Theorem 5.10 has been proved. ∎

Now we proceed to consider the automorphic extensions of simple groups of Lie type, i.e., groups G satisfying $S \lhd G \le \mathrm{Aut}(S)$. Because of Corollary 2.12, $\mathscr{I}_p(G) \ne \emptyset$ implies that $\mathscr{I}_p(S) \ne \emptyset$. Notice that $\mathrm{Aut}(S) =$

$(D:F).\Delta$, where D consists of inner and diagonal automorphisms and $|D|/|S|$ divides $|S|$; $F \cong \mathbb{Z}_m$ is the cyclic group of field automorphisms $(q = r^m)$, and Δ is the group of diagram automorphisms.

We need the following lemma:

LEMMA 5.11. *Let $A := D:F$ be a semidirect product, $\hat{X} \leq G \leq A$ and $S \leq G \cap D$ be subgroups such that $S \trianglelefteq A$. Moreover, let $\ell \in \pi(\hat{X})$ be such that* $\gcd(\ell, |D/S|) = 1 = \gcd(\ell, |G|/|\hat{X}|)$. *Then for any $Q \in Syl_\ell(\hat{X})$ there is $\tilde{Q} \in Syl_\ell(A)$ and $a \in A$ such that*

$$Q^a = \left(\tilde{Q} \cap S\right) : \left(\tilde{Q} \cap F \cap \hat{X}^a\right).$$

In particular, if ℓ divides $|\hat{X}/\hat{X} \cap S|$, then $Q^a \cap F \neq 1$.

Proof. Choose $\tilde{Q} \in Syl_\ell(A)$ such that $\tilde{Q} \cap F \in Syl_\ell(F)$. Since $S \trianglelefteq A$, we have $\tilde{Q} \cap S \in Syl_\ell(S)$; moreover, $\tilde{Q} \cap S = \tilde{Q} \cap D \in Syl_\ell(D)$. Since $|A|/(|\tilde{Q} \cap S| \cdot |\tilde{Q} \cap F|) = |D| \cdot |F|/(|\tilde{Q} \cap S| \cdot |\tilde{Q} \cap F|)$ is coprime to ℓ, we conclude that $\tilde{Q} = (\tilde{Q} \cap S):(\tilde{Q} \cap F)$. Let $Q \in Syl_\ell(\hat{X})$. Then Q is also in $Syl_\ell(G)$, and there is $a \in A$ with $Q^a \leq \tilde{Q}$. Since $S \trianglelefteq G^a$, $Q^a \cap S = \tilde{Q} \cap S \in Syl_\ell(S)$ and $Q^a = (\tilde{Q} \cap S):(\tilde{Q} \cap F \cap \hat{X}^a)$. If ℓ divides $|\hat{X}/\hat{X} \cap S|$, then $Q^a > Q^a \cap S$; hence, $1 \neq \tilde{Q} \cap F \cap \hat{X}^a \leq Q^a \cap F$. ∎

LEMMA 5.12. *Let S be a nonabelian simple group of Lie type, $S \trianglelefteq G \leq$* $\mathrm{Aut}(S) = (D:F).\Delta$, *and $\hat{X} \in \mathcal{I}_p(G)$ such that $C_S(f) \nleq X := \hat{X} \cap S$ for any $f \in F$. Suppose that* $\gcd(|X|, |D/S|)_{p'} = 1$ *and all p'-elements of G/S lie in $D/S:F$. Then $\pi(\hat{X}/X) = \pi(G/S) \subseteq \{p\}$.*

Proof. Notice that $G = S\hat{X}$ by 2.12. Let $H := D:F \trianglelefteq \mathrm{Aut}(S)$. Then $G/G \cap H \cong (G/S)/(G \cap H/S)$ is a p-group by hypothesis. Moreover, $\hat{X}/\hat{X} \cap H$ is isomorphic to a subgroup of $G/G \cap H$ and hence it is a p-group, too. Since $|\hat{X}/X| = |\hat{X}/\hat{X} \cap H| \cdot |\hat{X} \cap H/X|$, we can assume that $G \leq H$. Let $1 \neq \bar{x} \in \hat{X}/X$ with $|\bar{x}| = \ell$, a prime different from p; then there is a preimage $x \in \hat{X}$ of $\bar{x}$ with $|x| = \ell^i$. Suppose that ℓ divides $|D/S|$. Since $|D|/|S|$ divides $|S|$, ℓ divides $|S|$ also, and there is $Q \in Syl_\ell(S)$ with $1 \neq Q \leq X$; hence ℓ divides $\gcd(|X|, |D/S|)$, a contradiction. We conclude that $\gcd(\ell, |D/S|) = 1 = \gcd(\ell, |G|/|\hat{X}|)$. By Lemma 5.11, we can assume that there is $1 \neq f \in \tilde{Q} \cap F$ for some $\tilde{Q} \in Syl_\ell(\hat{X})$. Hence $C_S(f) \leq X$, a contradiction. ∎

THEOREM 5.13. *Let S be a simple group of Lie type and $S \triangleleft G \leq$* $\mathrm{Aut}(S)$. *Then $\mathcal{I}_p(G) \neq \emptyset$ if and only if $\mathcal{I}_p(S) \neq \emptyset$ and $\pi(G/S) = \{p\}$ or if $p = 2$, $S \cong PSL_2(2^\ell)$ or $S \cong {}^2B_2(2^\ell)$ and $G = S:F$ with F cyclic of odd prime order ℓ.*

If $\mathcal{I}_p(G) \neq \emptyset$, then $G = S\hat{X}$ for any $\hat{X} \in \mathcal{I}_p(G)$ and one of the following holds:

(i) $\mathcal{I}_p(G) = \{N_G(X) | X \in \mathcal{I}_p(S)$ *minimal*$\}$.

(ii) $p = 2$, $G = S:F$ *with* $S = PSL_2(2^\ell)$ *and* F *cyclic of odd prime order* ℓ *as well as* $\mathcal{I}_2(G) = \{N_G(T_{cox}) | T_{cox} \leq S\}$.

(iii) $p = 2$, $G = S:F$ *with* $S = {}^2B_2(q)$, $q = 2^\ell$ *and* F *cyclic of odd prime order* ℓ; *moreover*, $\mathcal{I}_2(G) = N_G(T)^G$ *where* T *is cyclic of order* $q + \varepsilon\sqrt{2q} + 1$ *with* $\epsilon \in \{-1, 1\}$ *such that* $5 \in \pi(T)$.

(iv) $p = 2$, $S = PSL_3(4)$, $G = S:F \cong S:2$ *and* $\mathcal{I}_2(G) = \{N_G(X) | X \in \mathcal{I}_2(S)\}$.

Proof. Assume without loss that $S \in LIE(r)$ for some prime r. Clearly, if $\mathcal{I}_p(S) \neq \emptyset$ and $\pi(G/S) = \{p\}$, then $\mathcal{I}_p(F) \neq \emptyset$ by Lemma 2.5. So assume that $p = 2$ and $S \cong PSL_2(2^\ell)$ or $S \cong {}^2B_2(2^\ell)$ as well as $G = S:F$ with F cyclic of odd prime order ℓ; moreover, let $X \in \mathcal{I}_2(S)$. By Theorem 5.10 we know that $X = N_S(Q)$, where $Q = O(X)$ is a cyclic Hall-subgroup of S of known order, with X/Q being cyclic of order 2 or 4. Put $\hat{X} = N_G(Q)$ and observe that $\hat{X} = X{:}\langle f \rangle$ with f of order ℓ inducing a field automorphism on S. Now we easily verify that $\hat{X} \in \mathcal{I}_2(G)$ iff X contains the "prime subgroup" $X_0 := C_S(f)$.

Suppose that $S \cong PSL_2(2^\ell)$ and thus $X_0 \cong SL_2(2)$. Then $\hat{X} \in \mathcal{I}_2(G)$ iff $|Q| = 2^\ell + 1$, i.e., iff Q is a Coxeter torus in S. In particular, $\mathcal{I}_2(G) = \{N_G(T_{cox}) | T_{cox} \leq S\}$.

Assume now that $S \cong {}^2B_2(q)$ with $q = 2^\ell$; in particular, $X_0 \cong {}^2B_2(2) \cong F_{20}$. Since 5 does not divide $q - 1$, we find that $\hat{X} \in \mathcal{I}_2(G)$ iff $|Q| = q + \epsilon\sqrt{2q} + 1$, where $\epsilon = \{-1, 1\}$ such that $5 \mid |Q|$. So in this case $\mathcal{I}_2(G) = N_G(Q)^G$.

Throughout the remainder we assume that $\hat{X} \in \mathcal{I}_p(G)$. By Corollary 2.12, we know that $X := \hat{X} \cap S \in \mathcal{I}_p(S)$ and $G = S\hat{X}$ as well as $\hat{X} = N_G(X)$.

First we consider the case that p is odd. As $G \neq S$, Theorem 2.10 shows that $|\hat{X}|$ is odd. If X is a Borel subgroup of S, then Theorem 5.1 together with the structure of $\mathrm{Out}(PSL_2(r^m))$ implies $S \cong PSL_2(3^m)$ and $\hat{X} = X{:}\langle f \rangle$, where f induces a field automorphism on S of odd prime order m dividing $p - 1$; now we obtain $X \geq C_S(f)$, a contradiction because $|C_S(f)|$ is even. Henceforth we may assume that X is not a Borel subgroup of S, and so (S, X) is one of the pairs in Theorem 5.7; in particular, $\mathcal{I}_p(S) = X^S$, and conclusion (i) holds.

Notice that $\hat{X}/X = G/S \leq D/S:F$, since $|\hat{X}|$ is odd and Δ is a 2-group. In case (A)(i) we have $|D/S| = |\gcd(q\epsilon - 1, p + 1)|$; suppose $\ell \neq p$ is a prime dividing $|D/S|$. Then $\nu_\ell(|X|) = \nu_\ell(q\epsilon - 1) - \nu_\ell(\gcd(q\epsilon$

$- 1, p + 1)) = 0$. Hence $\gcd(|D/S|, |X|)_{p'} = 1$. In case (A)(ii), $|D/S| = |\gcd(q\epsilon - 1, p)|$; in case (D) $|D/S|$ divides 4; and in case (E), $p = 3$ and $|D/S| = \gcd(3, q - \epsilon)$. Moreover, in all of these cases, the odd order group X does not contain any "prime subgroup" $S(r)$ of S, so $C_S(f) \not\leq X$ for all $f \in F$. Now Lemma 5.12 shows that $\pi(G/S) = \{p\}$.

Now we investigate the case where $p = 2$. If $2 \neq r \in \pi(X)$, we use Theorem 5.1 again and see that $\mathrm{Out}(S)$ is a 2-group, except when $S \cong PSL_2(8)$ with $\mathrm{Out}(S) \cong \mathbb{Z}_3$ or $S \cong PSU_3(5)$ with $\mathrm{Out}(S) \cong \Sigma_3$. In the former case conclusion (i) holds, since $G = PSL_2(8) \cdot 3$ acts 2-transitively on $G/\hat{X}$, where $X = D_{18}$. If $S \cong PSU_3(5)$ we use the information in [6] to verify that $|G : S| = 2$. So in any of the cases emerging from Theorem 5.1 we have $\mathscr{I}_2(S) = X^S$ and $\mathscr{I}_2(G) = \{N_G(Y) | y \in \mathscr{I}_2(S)$ *minimal*$\}$. Henceforth we may assume that X is not a Borel subgroup in S, and so the pair (S, X) occurs in Theorem 5.10.

First notice that the only case where an odd diagram automorphism occurs is $S \cong {}^3D_4(q)$ with $q = r^m$ and $\mathrm{Out}(S) \cong \mathbb{Z}_{3m}$. Since in this case $X = N_S(T_5)$ does not contain a Sylow 3-subgroup of $S, 3 \notin \pi(\hat{X}/X)$. Therefore, in all of the possible cases, the $2'$-elements of G/S lie in $D/S : F$.

Now suppose that that S is isomorphic neither to $PSL_2(2^m)$ for some $m \geq 2$, nor to ${}^2B_2(2^m)$ for some $m \geq 3$, nor to $PSL_3(4)$ (these cases will be dealt with later). Then the "prime subgroup" $S(r)$ of S is not solvable and thus cannot be involved in X; in particular $C_S(f) \not\leq X$ for any $f \in F$. Now we are going to verify that $\gcd(|X|, |D/S|)_{2'} = 1$.

In case (A) with $\epsilon = \pm 1$ and $n \in \{2, 4\}$, $|D/S|$ is a power of 2. This is also true if $n = 3$ and $q = 2^x + \epsilon$ with $x > 1$. If $n = 3$ and $q = 3 \cdot 2^x + \epsilon$, then $|D/S| = \gcd(q - \epsilon, 3) = 3$, which does not divide $|X|$, unless $q = 4$, $\epsilon = 1$ and $X =_S \mathbb{Z}_3^2 : Q_8$. If $n = 5$, then $|D/S| = 1$ if $q = 3$ and $\epsilon = \pm 1$. If $q = 9$ and $\epsilon = -1$, then $|D/S| = 5$, which does not divide $|X| = 2^7 \cdot 41$ in this case. In the cases (B), (C), and (D), $|D/S| \in \{1, 2, 4\}$.

In case (E) $|D/S| = 1$, except for $S = {}^{\epsilon}E_6(q)$, where $|D/S| = \gcd(3, q - \epsilon)$. If $|D/S| = 3$, then $(\epsilon, q) \in \{(1, 7), (-1, 5)\}$, and 3 does not divide $|X|$ in these cases.

Thus the hypotheses of Lemma 5.12 have been verified and we get $\pi(G/S) = \{2\}$, as well as $\mathscr{I}_2(S) = X^S$ and $\mathscr{I}_2(G) = \{N_G(Y) | Y \in \mathscr{I}_2(S)$ *minimal*$\}$.

Suppose next that $S \cong PSL_3(4)$. Then $\mathrm{Out}(S) \cong \mathbb{Z}_2 \times \Sigma_3$, and $X \in \mathscr{I}_2(S)$ is S-conjugate to a group isomorphic to $D_{10}, \mathbb{Z}_2^4 : D_{10}$ or $3^2 : Q_8$. Since $|C_S(\alpha)| \in \{21, 60\}$ for $\alpha \in \mathrm{Aut}(S) \setminus S$ with $o(\alpha) = 3$, we easily verify now that $\mathscr{I}_2(G) \neq \emptyset$ if and only if $\mathscr{I}_2(S) \neq \emptyset$ and $\pi(G/S) = \{2\}$, in which case either conclusion (i) or conclusion (iv) holds.

We are left to consider the case where $S \cong PSL_2(2^m)$ for some $m \geq 2$ or $S \cong {}^2B_2(2^m)$ for some $m \geq 3$. Recall that $\mathrm{Aut}(S) = S : F$ with $F \cong \mathbb{Z}_m$, and hence $G = S : F_0$ with $1 \neq F_0 \leq F$.

Clearly, if $\pi(G/S) = \{2\}$, then conclusion (i) holds. Thus we may assume that $2 \neq \ell \in \pi(G/S)$. Now $\hat{X}$ contains a Sylow ℓ-subgroup of G, and so we may assume without loss that there exists an element $f \in F_0 \cap \hat{X}$ of order ℓ. Note that X is solvable (see Theorem 5.10) and that $X \geq C_S(f) \cong SL_2(2^{m/\ell})$ or ${}^2B_2(2^{m/\ell})$, respectively. So we get $m = \ell$ with $3 \in \pi(X)$ in case $S \cong PSL_2(2^m)$ and $5 \in \pi(X)$ in case $S \cong {}^2B_2(2^m)$. In view of the first part of this proof, we reach conclusion (ii) or (iii), respectively. ∎

Now we classify those maximal subgroups of automorphic extensions G of S that lie in $\mathscr{I}_p(G)$. We need some lemmas:

LEMMA 5.14. *Let $S = PSL_2(q)$ with $q > 3$ and $S \leq G \leq \mathrm{Aut}(S)$; then the following holds*:

(i) *$N_G(T_1)$ is not a maximal subgroup if and only if either $G = S$ and $q \in \{5, 7, 3^2, 11\}$, or $G = PGL_2(5)$, or $G = PSL_2(3^2) \cdot 2_1 \cong \Sigma_6$.*

(ii) *$N_G(T_{cox})$ is not a maximal subgroup if and only if either $G = S$ with $q \in \{7, 3^2\}$, or $G \cong PSL_2(3^2) \cdot 2_1 \cong \Sigma_6$.*

Proof. This is well known and can easily be checked by using Dickson's list of subgroups of $PSL_2(q)$.

LEMMA 5.15. (i) *Let $G_1 \trianglelefteq G_2 = G_1 M$ with $M \leq G_2$. Then M is maximal in G_2 if and only if there is no M-stable subgroup H with $G_1 \cap H < H < G_1$.*

(ii) *Let $G_1 \trianglelefteq G_2$ and $X \in \mathscr{I}_p(G_1)$. Let $X < H < G_1$ be such that $\{H\}^{G_2} = \{H\}^{G_1}$. Then $\hat{X} := N_{G_2}(X)$ is not maximal in G_2.*

Proof. (i) For $A, B, C \leq G_2$ let $[A, B]_C$ denote the inclusion ordered interval of C-stable subgroups between A and B. Then the maps $\alpha : [G_1 \cap M, G_1]_M \to [M, G_2], H \mapsto MH$ and $\beta : [M, G_2] \to [G_1 \cap M, G_1]_M, U \mapsto U \cap G_1$ are inverse isotone poset isomorphisms.

(ii) Suppose that $\hat{X}$ is maximal in G_2. Since X is not normal in $G_1, G_1 \nleq \hat{X}$ and $G_2 = G_1 \hat{X}$. Let $1 \neq Q \in \mathrm{Syl}_a(X) \cap \mathrm{Syl}_a(H)$ with prime $a \neq p$; then for any $g \in \hat{X}$, there is $x_g \in G_1$ with $H^g = H^{x_g}$ and $Q, Q^{x_g^{-1}} \in Syl_a(H)$. Hence there is $h \in H$ with $x_g^{-1} h^{-1} \in N_{G_1}(Q) \leq X$, and we get $H^g = H^{x_g} = H^{h x_g} = H$. So H ix $\hat{X}$-invariant, a contradiction by (i). ∎

THEOREM 5.16. *Let S be a simple group of Lie type and $S \trianglelefteq G \leq \mathrm{Aut}(S) = (D : F) : \Delta$, where D is generated by the inner and diagonal, F by the field, and Δ by the graph automorphisms. Then Table II displays all cases where G*

TABLE II

p-Intersection Maximal Subgroups in Finite Groups of Lie Type

S	G	p	$\hat{X}$
$L_2(4) \cong L_2(5)$	$G = S$	2	Σ_3, D_{10}
	$PGL_2(5)$	2	$\mathscr{M}(X)$
	$G = S$	3	$\mathscr{A}_4$
$L_2(7) \cong L_3(2)$	$G = S$	2	$\Sigma_4^{(i)}, i = 1, 2$
	$G = PGL_2(7)$	2	$\mathscr{M}(X)$, $N(T_1) \cong D_{12}$
	$G = S$	3	$B \cong 7:3$
$L_2(9)$	$G = S$	2	$B \cong 3^2:4$
	$G = S \cdot 2^1 \cong \Sigma_6$	2	$3^2 : D_8$
	$G = S \cdot 2_2 \cong PGL_2(9)$	2	$3^2:8, D_{20}$
	$G = S \cdot 2_3 \cong M_{10}$	2	$3^2 : Q_8, 5:4$
	$G = S \cdot 2^2 \cong P\Gamma L_2(9)$	2	$3^2 : [2^4], 10:4$
$L_2(11)$	$G = S$	2	$N(T_{cox}) \cong D_{12}$
	$PGL_2(11)$	2	D_{24}, D_{20}
	$G = S$	5	$B \cong 11:5$
$L_2(2^a), 2^a \geq 4$	$G = S$	$p = 2^a - 1 \in \mathscr{M}$	B
$L_2(q), q \notin \mathscr{F} \cup \{4, 7, 9, 11\}$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_1)$
$L_2(q), q \notin \mathscr{M} \cup \{4, 5, 9, 11\}$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_{cox})$
$L_2(2^\ell)$, ℓ an odd prime	$\lvert G/S \rvert = \ell$	2	$N(T_{cox})$
$L_2(r)$, $r = 2p^x + 1$ a prime	$G = S$	$p > 2$	B
$L_2(3^m), \begin{cases} 3m = 2p^x + 1 \\ m \mid (p-1) \\ m \text{ an odd prime} \end{cases}$	$G = S$	$p > 2$	B
$L_2(r), r \in \mathscr{F}$	$G/S \leq \mathbb{Z}_2$	2	$N(B)$
$PSU_3(q), q \in \{3, 5, 9\}$	$G/S \leq \mathbb{Z}_2$	2	$N(B)$
${}^2G_2(3)' \cong L_2(8)$	$G/S \leq \mathbb{Z}_3$	2	$N(B)$
${}^2B_2(q), q = 2^\ell > 2$, ℓ odd	$G = S$	$p = 2^a - 1 \in \mathscr{M}$	B
	$G = S$	2	$D_{2(q-1)}$ $X_\pm \cong \mathbb{Z}_{q \pm \sqrt{2q}+1} : \mathbb{Z}_4$
	$\lvert G/S \rvert = \ell$ prime	2	$X_\pm : \mathbb{Z}_\ell$ with $5 \mid \lvert X_\pm \rvert$
$L_3(4)$	$G/S \leq F \cong \mathbb{Z}_2$	2	$N(3^2 : Q_8)$
	$\Delta \leq G/S \leq \mathbb{Z}_2 \times \mathbb{Z}_2$	2	$N(3^2 : Q_8)$
${}^3D_4(q)$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_5)$
$PS_p^\epsilon(q), \epsilon = \pm 1$	$\pi(G/S) \subseteq \{p\}$	p	$N(T_{cox})$

B denotes a Borel subgroup: $L_2(q) := PSL_2(q)$, $N(H) := N_G(H)$; if $G > S$, $\mathscr{M}(X) := \{N_G(X) \mid X \in \mathscr{I}_p(S)$, maximal in $S\}$.

has a maximal subgroup $\hat{X}$ that lies in $\mathscr{I}_p(G)$. The groups $\hat{X}$ are listed up to conjugacy in G.

Proof. We know from 5.15(i) that $X < \cdot S$ implies $\hat{X} < \cdot G$. In this proof "Y is G-stable" means that $\{Y\}^G = \{Y\}^S$.

(1) First we consider the rank 1 groups occuring in Theorem 5.1. Since the Borel group B is a maximal parabolic, it is always maximal by a theorem of Borel and Tits, hence $\hat{X}$ is maximal in G. Also, $\hat{X} \in \mathcal{I}_p(G)$ by Theorem 5.13.

(2) Next we consider the cases in Theorem 5.7. Here $G \leq D : F$. In (A)(i), $X = N_S(T_{(12\cdots p)})$ is contained (up to conjugacy) in the proper G-stable subgroup H of S consisting of matrices

$$\begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix}$$

with $A \in S_p^{\epsilon}(q)$. Hence $\hat{X}$ is not maximal in G by 5.15(ii).

In (D), $X = N_S(T_{(12\cdots p)^{\pm}})$ is contained in the normalizer H of a maximal rank subgroup of type ${}^{\epsilon}A_{p-1}$. The $\tilde{S}$-conjugacy class of $\tilde{H}$ is unique, so there are at most two S-classes of H. Since G/S has odd order, these classes are G-stable and $\hat{X}$ is not maximal in G because of Lemma 5.15(ii).

In case (E), $\hat{X}$ is not maximal, as can be seen in [28].

In case (A)(ii), X is maximal in S (e.g., [25] and [26]); hence $\hat{X}$ is maximal in G.

(3) Finally, we consider the cases in Theorem 5.10.

(3a) Case (A) with $n = 2$: by Lemma 5.14 we can assume that $q \leq 11$. Here the information on maximal subgroups of G can be found in [6].

Case (A), (i) and (iii) with $n = 3$ or 5 and $\epsilon = \pm 1$: Here $S \cong PS_n^{\epsilon}(q)$, and $X =_S N_S(T_{(12\cdots n-1)})$ can be chosen to be contained in a proper subgroup

$$H = \begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix}$$

with $A \in S_{n-1}^{\epsilon}(q)$. We can assume that $G/S \leq \langle d, f \rangle$ with $d := (\cdot)^{-tr}$ and $f = (\cdot)^{(r)}$; hence H is G-stable and $\hat{X}$ is not maximal in G by Lemma 5.15(ii).

Case (i) with $S \cong PSL_3(4)$, $G/S \leq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $X \neq N_S(T_{(12)})$: Here $\mathcal{I}_2(S)$ contains the maximal subgroup $X_m =_S \mathbb{Z}_3^2 : Q_8$, so $N_G(X_m)$ is maximal and in $\mathcal{I}_2(G)$ for all G. If $G/S \leq F$, there is $N_G(X) \in \mathcal{I}_2(G)$ with $X \cong (\mathbb{Z}_2^4 : \mathbb{Z}_5)$. $\mathbb{Z}_2 \in \mathcal{I}_2(S)$, which can be chosen in the maximal parabolic $\mathscr{P} := 2^4 : SL_2(4)$. Since the conjugacy class $\{\mathscr{P}\}^S$ is G-stable, $N_G(X)$ is not maximal in G by Lemma 5.15(ii).

Case (ii) $n = 4$, $\epsilon = \pm 1$: $S \cong PS_4^{\epsilon}(q)$, $X =_S N := N_S(T_{cox})$ and $G = S \cdot N_G(T_{cox})$. There exists an element $t \in X$ (of order a power of 2) such that $X < C := C_S(\langle t \rangle)$ is a maximal rank subgroup of type $A_2(q^2)$ and $H := N_S(C) < S$. For each $n \in N_G(T_{cox})$ we have $\langle t \rangle^n = \langle t \rangle$; hence $N_G(T_{cox}) \leq N_G(H)$. We conclude that $\{H\}^G = \{H\}^S$ and $N_G(T_{cox})$ is not maximal in G.

(3b) Cases (B), (C), (D): We first consider the cases $n = 2^k$, $S \cong P\Omega_{2^{k+1}+1}(q)$, and $PSp_{2^{k+1}}(q)$ or $n = 2^k + 1$ and $S \cong P\Omega_{2n}^{\pm}(q)$. Here $X =_S N_S(T_{cox})$. Let Φ be the root system of type B_n or C_n. Up to W-conjugacy there is a unique subsystem $\tilde{\Phi} = \tilde{\Phi}_1 \uplus \tilde{\Phi}_1$ with $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ of type $B_{n/2}$ and $C_{n/2}$, respectively. This gives rise to unique $\tilde{S}$-conjugacy classes of maximal rank subgroups of type $B_{n/2}(q^2)$, respectively, $C_{n/2}(q^2)$ in $\tilde{S}$, which contain conjugates of $\tilde{T}_{cox}$. Similarly, if $\tilde{S} = SO_{2n}^{\epsilon}(q)$, there is a unique $\tilde{S}$-conjugacy class of maximal rank subgroups of type ${}^2D_{n-1}(q)$ that contain conjugates of $\tilde{T}_{cox}$.

In each case we choose such a maximal rank subgroup $\tilde{M}$ and define $\tilde{H} := N_{\tilde{S}}(\tilde{M})$; then $[\tilde{X} : N_{\tilde{M}}(\tilde{T}_{cox})] \leq 2$ and $\tilde{H} = \tilde{M} N_{\tilde{H}}(\tilde{T}_{cox}) > \tilde{M}$. We conclude $\tilde{X} = N_{\tilde{H}}(\tilde{T}_{cox}) < \tilde{H} < \tilde{S}$.

Let $\bar{X} := \tilde{X}/Z(\tilde{S})$ and $\bar{S} := \tilde{S}/X(\tilde{S})$. Since any element of $\mathrm{Aut}(S)$ can be extended to an element of $\mathrm{Aut}(\tilde{S})$, we have $S \leq \bar{S} \trianglelefteq \mathrm{Aut}(S)$, $\bar{X} \in \mathscr{I}_2(\bar{S})$, and $\bar{X} < \bar{H} < \bar{S}$ (notice $\bar{S} = \tilde{S}$ in case B_n, $\bar{S} = S$ in case C_n, and $X = \bar{X} \cap S$ in any case). Now Lemma 5.15 implies that for any G with $\bar{S} \leq G \leq \mathrm{Aut}(S)$, the group $\hat{X} = N_G(X) = N_G(\bar{X})$ is not maximal in G. Now suppose that $S \leq G_2 \leq \mathrm{Aut}(S)$ is such that $\hat{X}_2 := N_{G_2}(X) \in \mathscr{I}_2(G_2)$ is a maximal subgroup. We consider the group $G_3 = G_2 \cdot \bar{S} \leq \mathrm{Aut}(S)$. Now Lemma 5.15(ii) implies that $N_{G_3}(X) = N_{G_3}(\bar{X})$ is not maximal in G_3, but by Lemma 5.15(i) $N_{G_3}(X) = N_{G_3}(\hat{X}_2)$ is maximal—a contradiction.

Next let $n = 2^k$, $S \cong P\Omega_{2n}^{-}(q)$, and $X =_S N_S(T_{(12\cdots n)^-})$. Then $\tilde{S}$ has a unique conjugacy class of maximal rank subgroups of type ${}^2D_{n/2}(q^2)$ that contains conjugates of $T_{(12\cdots n)^-}$. Again we choose a maximal rank subgroup $\tilde{M}$ with $\tilde{H} := N_{\tilde{S}}(\tilde{M})$ and get $[\tilde{X} : N_{\tilde{M}}(\tilde{T}_{cox})] \leq 2$, as well as $\tilde{X} = N_{\tilde{H}}(\tilde{T}_{cox}) < \tilde{H} < \tilde{S}$. Now the same argument as above shows that for any G with $\pi(G/S) \subseteq \{2\}$, $\hat{X}$ is not maximal in G.

(3c) In case (E) with S of type ${}^2B_2(q)$, the claims are immediate, because $\hat{X} \cap S$ is maximal in S. If S is of type ${}^2G_2(q)$ or $F_4(q)$, there are only inner and field automorphisms and X is contained in the normalizers of proper G-stable subgroups of maximal rank ($2 \times PSL_2(q)$ and $B_4(q)$, respectively). In case ${}^3D_4(q)$, Kleidman's paper [23] contains the information that $\hat{X}$ is maximal in G. In the E_6-cases the groups $\hat{X}$ are not maximal, as can be seen in [28]. ∎

TABLE III
p-Intersection Subgroups in Sporadic Groups

G	p	X	$X < \cdot G$	G	p	X	$X < \cdot G$
M_{11}	2	$3^2 : SD_{16}$	+	$He:2$	2	$17:16$	−
″	2	$5:4$	−	$Co1$	11	$23:11$	−
″	5	$11:5$	−	$Co2$	11	$23:11$	−
J_1	2	$\Sigma_3 \times D_{10}$	+	$Co3$	11	$23:11$	−
M_{22}	2	$5:4$	−	HN	3	$19:9$	−
″	3	$7:3$	−	Fi_{23}	2	$17:16$	−
″	5	$11:5$	−	″	11	$23:11$	−
$M_{22}:2$	2	$(5 \times 2):4$	−	Fi_{24}	2	$(17 \times 2):16$	−
M_{23}	5	$11:5$	−	Fi'_{24}	2	$17:16$	−
″	11	$23:11$	+	″	11	$23:11$	−
M_{24}	11	$23:11$	−	BM	2	$(17:8 \times 2^2)'2$	−
HiS	5	$11:5$	−	″	23	$47:23$	+
McL	5	$11:5$	−	M	29	$59:29$	??
He	2	$17:8$	−	—	—	—	—

Here ?? means either $59:29 < PSL_2(59) < \cdot M$ or $59:29 < \cdot M$. The existence of $PSL_2(59)$ in M is not settled yet.

6. THE SPORADIC SIMPLE GROUPS

In this section we classify $\mathscr{I}_p(S)$ for all sporadic simple groups S and their automorphism groups. The result can be derived from the data in [6], together with elementary results from Section 2; so we omit a formal proof.

THEOREM 6.1. *Let $S \trianglelefteq G \leq \mathrm{Aut}(S)$, where S is a sporadic simple group and suppose that $X \in \mathscr{I}_p(S)$. Then the triple (G, p, X) is exactly one of those listed in Table III. A $+$ indicates that X is maximal in G.*

ACKNOWLEDGMENTS

The authors are grateful to the referee whose suggestions greatly improved the exposition of the paper.

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