The capacitated transshipment location problem under uncertainty: A computational study

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Abstract

We consider the stochastic capacitated transshipment problem for freight transportation where an optimal location of the transshipment facilities, which minimizes the total cost, must be found. The total cost is given by the sum of the total fixed cost plus the expected minimum total flow cost, when the throughput costs of the facilities are random variables with unknown probability distribution. By applying the asymptotic approximation method derived from the Extreme Value Theory, a deterministic non-linear model, which belongs to a wide class of Entropy maximizing models, is then obtained. In this paper, we present a deep analysis of the impact of different probability distributions of the random costs on the problem optimum, as well as on the number of facilities located. The computational results show a very good performance of the deterministic model when compared to the stochastic one.

1. Introduction

In this paper we focus on City Logistics systems where the freight delivery is organized in a two-tiered structure. In this family of transportation systems, called Two-Echelon Distribution Systems, the freight is not directly shipped to customers, but it is first consolidated in facilities strategically located near or inside the city center, called satellites [4, 15]. These systems are increasingly proposed in City Logistics

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projects, and this implies the development and the usage of new methods and tools for supporting the planning of the logistics system [1].

One of the main issues associated to Two-Echelon Distribution Systems is the location of intermediate depots, which highly affects the total transportation cost. While the transportation cost can be easily calculated for the links outside the urban area, for urban links this task becomes quite difficult. In fact, the transportation cost between intermediate depots and customers, which are usually located in the urban area, highly changes day-by-day and it is affected by several parameters, including the hour of the day.

Thus, whenever medium term decisions must be taken (e.g. the location of intermediate depots, which is a decision that must be valid for several years), the transportation cost between intermediate depots and customers cannot be considered deterministic, but intrinsically stochastic. Then the transportation cost becomes a stochastic variable, whose probability distribution is unknown. In fact, any assumption on the shape of this probability distribution would not be realistic, being affected by many and incommensurable parameters.

In this paper we consider the Capacitated Transshipment Location Problem under Uncertainty (CTLPU), a variant of the classical transshipment problem where the costs between intermediate depots and customers are stochastic with unknown probability distribution. More in details, given a set of origins with given supply, a set of destinations with given demand, a set of potential depot locations with deterministic fixed costs of location, upper capacity constraints for the depots, and random generalized transportation costs from origins to destinations through the depots, the CTLPU consists in finding a depot location which minimizes the total cost, given by the sum of the deterministic total fixed cost plus the expected minimum total flow cost, subject to supply, demand, and upper depot capacity constraints. Each generalized transportation cost is a random variable given by the sum of a deterministic transportation cost from an origin to a destination through an intermediate depot plus a random term, with unknown probability distribution, which represents the throughput cost of the intermediate depot.

In the huge literature on the Transshipment Facility Location Problem there are just a few papers where stochasticity is considered, but this stochasticity mainly concerns the arc capacity or the customer demand, while random costs are generally ignored. [9] considers a dynamic network flow problem where arc capacities are random variables and derive a multistage stochastic linear program. In [25] a capacitated location-allocation problem with stochastic demands is originally formulated as expected value model, chance-constrained programming and dependent-chance programming according to different criteria. [12] presents a review of some contributions to the current state-of-the-art on facility location problems. Also probabilistic models are presented where some of the input data of the location models are subject to uncertainty. A more recent review covering stochastic and some non-linear facility location problems can be found in [18]. [24] formulates the multi-location transshipment as a two-stage stochastic program with recourse, where the demand is stochastic. [17] analyzes a stochastic fractional transshipment problem with uncertain demands and prohibited routes, which is solved by reformulating the stochastic transshipment problem into an equivalent deterministic transportation problem. In [23] a two-stage linear program with recourse formulation is developed to determine the optimal storage capacity to be installed on transshipment nodes by shippers in a dynamic shipper carrier network under stochastic demand. In the first stage, the shipper decides the optimal capacity to be installed on transshipment nodes, while in the second stage, the shipper chooses a routing strategy based on the realized demand.

Just a few papers concerning location problems with stochastic costs are available in literature. Among them, [16] develops a heuristic for solving a p-median problem where the throughput costs are random variables with a given probability distribution. [19] considers a scenario-based stochastic version of a joint location-inventory model that minimizes the expected cost of locating depots, allowing costs, lead times, demand, and some other parameters to be stochastic. [22] introduces a stochastic p-median
problem where the cost for using a depot is a random variable, with unknown probability distribution. In [20] the CTLPU, where only upper capacity constraints for the depots are considered, is introduced. The problem is modeled as a stochastic program, and a deterministic approximation of it, named the Deterministic Capacitated Transshipment Location Problem (CTLPD), is given. Although the authors showed that the CTLPD is quite satisfactory when the random costs in the CTLPU follow a Gumbel distribution, no tests have been made with other probability distributions.

In this paper, we present a deep analysis of the impact of different probability distributions on the optimum of the CTLPU, as well as on the number of depots located. In particular, we analyze the performance of the deterministic approximation, the CTLPD, by comparing its results with those obtained by the CTLPU under three different probability distributions: Gumbel, Laplace, and Uniform.

2. The stochastic problem and its deterministic approximation

Let be:

\begin{align*}
I & : \text{Set of origins} \\
J & : \text{Set of destinations} \\
K & : \text{Set of potential transshipment locations} \\
L_k & : \text{Set of throughput operation scenarios at transshipment facility } k \in K \\
P_i & : \text{Supply at origin } i \in I \\
Q_j & : \text{Demand at destination } j \in J \\
U_k & : \text{Throughput capacity of transshipment facility } k \in K \\
f_k & : \text{Fixed cost of locating a transshipment facility } k \in K \\
y_k & : \text{Binary variable which takes value 1 if transshipment facility } k \in K \text{ is located, 0 otherwise} \\
c_{ij}^k & : \text{Unit transportation cost from origin } i \in I \text{ to destination } j \in J \text{ through transshipment facility } k \in K \\
\theta_{kl} & : \text{Unit throughput cost of transshipment facility } k \in K \text{ in throughput operation scenario } l \in L_k \\
s_{ij}^k & : \text{Flow from origin } i \in I \text{ to destination } j \in J \text{ through transshipment facility.}
\end{align*}

Let us assume

1. the system is balanced, i.e. \( \sum_{i \in I} P_i = \sum_{j \in J} Q_j = T \)

2. the unit throughput costs \( \theta_{kl} \) are independent and identically distributed (i.i.d.) random variables with a common and unknown probability distribution

\[
\Pr[\theta_{kl} \geq x] = F(x) \tag{1}
\]

Assumption 1 is a standard one and it is straightforward to balance the system if necessary. Assumption 2 is justified by the fact that the unit throughput costs usually vary among transshipment facilities and inside each of them in a random way and are quite difficult to be measured. Thus they become random variables with unknown probability distribution. Moreover, these random variables are
independent each other and there is no reason to consider different shapes for their unknown probability distributions [13, 14].

Let \( r_{ij}^k(\theta) \) be the stochastic generalized unit transportation cost from origin \( i \) to destination \( j \) through transshipment facility \( k \) in throughput operation scenario \( l \) given by

\[
r_{ij}^k(\theta) = c_{ij}^k + \theta_{kl}, \quad i \in I, \ j \in J, \ k \in K, \ l \in L_k
\]

with unknown probability distribution

\[
\Pr\{r_{ij}^k(\theta) \geq x\} = \Pr\{c_{ij}^k + \theta_{kl} \geq x\} = \Pr\{\theta_{kl} \geq x - c_{ij}^k\} = F(x - c_{ij}^k)
\]

Let us define

\[
\bar{\theta}_k = \min_{l \in L_k} \theta_{kl}, \quad k \in K
\]

with unknown probability distribution

\[
H(x) = \Pr\{\bar{\theta}_k \geq x\}
\]

As \( \bar{\theta}_k \geq x \Leftrightarrow \theta_{kl} \geq x, l \in L_k \) and \( \theta_{kl} \) are independent, using Equation 1 one gets

\[
H(x) = \Pr\{\bar{\theta}_k \geq x\} = \prod_{l \in L_k} \Pr\{\theta_{kl} \geq x\} = \prod_{l \in L_k} F(x) = [F(x)]^{n_k}
\]

where \( n_k = |L_k| \) is the number of the different throughput operation scenarios at the transshipment facility \( k \).

The stochastic generalized unit transportation cost from origin \( i \) to destination \( j \) through transshipment facility \( k \) is the minimum among the costs for the different throughput operation scenarios at facility \( k \) and becomes

\[
r_{ij}^k(\theta) = \min_{l \in L_k} r_{ij}^k(\theta) = c_{ij}^k + \min_{l \in L_k} \theta_{kl} = c_{ij}^k + \bar{\theta}_k, \quad i \in I, \ j \in J, \ k \in K
\]

The CTLPU may be formulated as follows

\[
\min_y \sum_{k \in K} f_k y_k + E_\theta \left[ \min_{l \in L_k} \sum_{j \in J} \sum_{k \in K} r_{ij}^k(\theta) s_{ij}^k \right]
\]
subject to

\[
\sum_{j \in J} \sum_{k \in K} s_{ij}^k = P_i, \quad i \in I \quad (9)
\]

\[
\sum_{i \in I} \sum_{k \in K} s_{ij}^k = Q_j, \quad j \in J \quad (10)
\]

\[
\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s_{ij}^k \leq U_k y_k, \quad k \in K \quad (11)
\]

\[
s_{ij}^k \geq 0, i \in I, j \in J, k \in K \quad (12)
\]

\[
y_k \in \{0, 1\}, k \in K \quad (13)
\]

where \( E_\theta \) denotes the expected value with respect to \( \theta \); the objective function (8) expresses the minimization of the total cost given by the sum of the minimum total fixed cost plus the expected minimum total flow cost; constraints (9) and (10) ensure that supply at each origin \( i \) and demand at each destination \( j \) are satisfied; constraints (11) ensure the capacity restriction at each transshipment facility \( k \); (12) are the non-negativity constraints, and (13) are the integrality constraints.

In [20] it is shown that, by applying the asymptotic approximation method derived from the Extreme Value Theory [8], the deterministic approximation of the CTLPU, i.e. the CTLPD, becomes

\[
\min_{y} \sum_{k \in K} f_k y_k + \max_{y} \left[ -\frac{1}{\beta} \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s_{ij}^k \ln s_{ij}^k - \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s_{ij}^k \left( c_{ij} - \frac{1}{\beta} \right) \right] \quad (14)
\]

subject to

\[
\sum_{j \in J} \sum_{k \in K} s_{ij}^k = P_i, \quad i \in I \quad (15)
\]

\[
\sum_{i \in I} \sum_{k \in K} s_{ij}^k = Q_j, \quad j \in J \quad (16)
\]

\[
\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s_{ij}^k \leq U_k y_k, \quad k \in K \quad (17)
\]

\[
s_{ij}^k \geq 0, i \in I, j \in J, k \in K \quad (18)
\]

\[
y_k \in \{0, 1\}, k \in K \quad (19)
\]

which is a mixed-integer deterministic non-linear model in the unknowns \( y_k \) and \( s_{ij}^k \).

We observe that the non-linearity affects only the objective function through the Entropy term

\[-\frac{1}{\beta} \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s_{ij}^k \ln s_{ij}^k\]

while all the constraints are linear.
It is interesting to note that when \( \beta \rightarrow +\infty \) problem (14)-(19) turns into the classical Capacitated Transshipment Location Problem (CTLP). In fact, the Entropy term in the objective function disappears and only the linear classical total transportation cost does remain. This is also coherent with the well-known property of the multinomial Logit model which states that this model collapses into a classical minimum transportation cost choice model as \( \beta \rightarrow +\infty \) [6].

We also observe that, provided

\[
\max_i -\frac{1}{\beta} \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s^i_j \ln s^i_j \geq 0
\]

the optimum of the classical CTLP is a Lower Bound to the CTLPU.

3. Computational results

In this section we compare the CTLPU, given by (8)-(13), with its deterministic approximation, the CTLPD, given by (14)-(19).

This section is organized as follows. As no instance for the CTLPU is available in literature, new instances are generated and introduced next (see subsection 3.1). The setting of a commercial stochastic solver to solve the CTLPU as well as the identification of an appropriate non-linear solver to solve the CTLPD are given in subsection 3.2. A detailed comparison between the CTLPU under different probability distributions and its deterministic approximation, the CTLPD, is discussed in subsection 3.3. Finally, in order to show the effect of the Entropy term in the CTLPD, in subsection 3.4 we compare the CTLPD with the classical CTLP, and we evaluate the speed of convergence of the CTLPD to the classical CTLP, when the value of parameter \( \beta \) does increase.

3.1. Instance generation

We consider a subset of the test classes given in [11], where the authors generate 4 classes of 20 instances each. Here, due to the much higher computational effort required to solve the stochastic and the non-linear problems, we consider only the first class of the above classes and generate 10 instances instead of 20, using uniform distribution with corresponding ranges according to the following criteria:

- number of depots \( |I| \) is drawn from \( U[2, 3] \)
- number of customers \( |J| \) is drawn from \( U[30, 40] \)
- number of possible locations for the transshipments \( |K| \) is drawn from \( U[10, 20] \)
- supply \( P_i \) is drawn from \( U[900, 1000] \)
- demand \( Q_j \) is drawn from \( U\left[1, \sum_{i \in I} P_i / |I|\right] \). If necessary, the demand of the last customer is adjusted so that the total demand is equal to the total supply
- capacity \( U_k \) is drawn from \( U\left[0.5avU, 3avU\right] \), where \( avU = \sum_{i \in I} P_i / |K| \)
- unit transportation cost \( c^i_j \) is drawn from \( U[1, 10] \)
- fixed cost \( f_k = TC U_k / (|I|/2) \), where \( TC \) is the total unit transportation cost over all the possible arcs
- random cost \( \theta \) is generated using three different probability distributions, Gumbel, Laplace, and Uniform, as follows (the cumulative distribution functions are considered):
Gumbel: \( \exp(-e^{\beta y}) \) (with mode equal to 0). The parameter \( \beta \) is set to 0.68 (see Subsection 3.5).

- Laplace:

\[
\begin{align*}
0.5 \exp\left(\frac{x - \mu}{b}\right) & \text{ if } x < \mu \\
1 - 0.5 \exp\left(-\frac{x - \mu}{b}\right) & \text{ if } x \geq \mu
\end{align*}
\]

with mean equal to \( \mu \). The parameters of the distribution are set such that the mean of the Laplace distribution is the same of the Gumbel one.

- Uniform:

\[
\begin{align*}
0 & \text{ if } x < a \\
\frac{x - a}{b - a} & \text{ if } a \leq x < b \\
1 & \text{ if } x \geq b
\end{align*}
\]

- The costs are generated in the range \([a, b] = [1, 10]\), such that the mean of the Uniform distribution is the same of the Gumbel one.

The random unit generalized transportation costs \( \bar{r}_{ij} \) in (8) are computed by (7). If some of them become negative, they are set to 1.

### 3.2. Stochastic solver setting and non-linear solver identification

As stated above, we compare the CTLPU with its deterministic approximation, the CTLPD.

The solution of the CTLPU is generated by implementing the stochastic model in XPress-SP, i.e. the stochastic programming module provided by XPress [7].

The tests are performed by generating an appropriate number of scenarios for each instance. In order to tune this number, we start with 50 scenarios and increase them by step 50. Then we solve each instance 10 times, reinitializing every time the pseudo-random generator of the stochastic components with a different seed, and compute the standard deviation and the mean of the optima over the 10 runs. The appropriate number of scenarios is then fixed to the smallest value ensuring for each instance a maximum ratio between the standard deviation and the mean less than 0.5% [10]. According to our tests, this value is fixed to 100 scenarios, which show a maximum ratio between standard deviation and mean equal to 0.17%.

In order to solve the deterministic approximation, i.e. the CTLPD, we consider the most efficient and effective state-of-the-art non-linear solvers: BonMIN, MinLP, KNITRO, LINGO and FilMINT. In order to have uniformity in input, output and computational results, we use the NEOS infrastructure [5] to make the tests, giving to the solvers a time limit of 1000 seconds per instance.

According to our results, BonMIN and KNITRO outperform the other solvers, obtaining the best solutions on the overall set of instances. By comparing each other these two solvers, BonMIN is 10 times faster than KNITRO, which also shows some memory problems when running large instances (with more than 50000 arcs). For these reasons, we select BonMIN (release 1.1) [2, 3] to solve the CTLPD within a
time limit of 1000 seconds. The parameters are set to their default values, which show a satisfactory behavior both in accuracy and computational effort.

3.3. Comparison between the CTLPU under different probability distributions and the CTLPD

In the following we analyze the performance of the deterministic approximation, the CTLPD, by comparing its results with those obtained from the CTLPU under the three different probability distributions. All the tests are done on a Pentium Q6600 2.4GHz Machine with 2 Gb of RAM. The parameter in the CTLPD objective function (14) and in the Gumbel distribution for the CTLPU is set to 0.68 (see Subsection 3.5).

A comparison between the optimum of the deterministic approximation, the CTLPD, and that of the stochastic model, the CTLPU, under the three different probability distributions is presented in Table 1.

The table columns have the following meaning:
- Column 1: instance number
- Column 2: optimum of the CTLPD
- Columns 3-5: optimum of the CTLPU using Gumbel, Laplace, and Uniform distributions, respectively
- Columns 6-8: percentage gap between the stochastic optimum using Gumbel, Laplace, and Uniform distributions, respectively, and the deterministic approximation one. The gap is computed as $100 \left( \frac{S - D}{S} \right)$, where $S$ is the optimum of the CTLPU and $D$ the optimum of the CTLPD.

Table 1 reports the results for each instance, as well as their mean in the last row.

The optima of the stochastic problem and its deterministic approximation are quite close together, with a mean gap around 2%. The gap is lower for the Gumbel and Laplace distributions than the Uniform one. About the negative value for some gap values present in Table 1, we remind that when the random cost $k_{ij}$ is generated and added to the deterministic unit cost in (7), if the resulting $k_{ijr}$ is negative it is set to 1. This implies a slight change in the distribution functions, producing such negative gap values.

The results of Table 1, even if satisfactory, are not sufficient to qualify the performance of the CTLPD. In fact, besides the optimum, another important comparison concerns the optimal solution of the two models. This is considered in Tables 2 and 3.

Table 1. Comparison between the optimum of the deterministic approximation, the CTLPD, and that of the stochastic model, the CTLPU, under the three different probability distributions

<table>
<thead>
<tr>
<th>Instances</th>
<th>Det</th>
<th>Objective function</th>
<th>Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Stoch</td>
<td>Gumbel</td>
</tr>
<tr>
<td>1</td>
<td>142713</td>
<td>137490</td>
<td>139664</td>
</tr>
<tr>
<td>2</td>
<td>209429</td>
<td>207013</td>
<td>209239</td>
</tr>
<tr>
<td>3</td>
<td>150860</td>
<td>144510</td>
<td>145031</td>
</tr>
<tr>
<td>4</td>
<td>167359</td>
<td>164393</td>
<td>165939</td>
</tr>
<tr>
<td>5</td>
<td>157160</td>
<td>151061</td>
<td>152683</td>
</tr>
<tr>
<td>6</td>
<td>211108</td>
<td>210291</td>
<td>210567</td>
</tr>
<tr>
<td>7</td>
<td>241405</td>
<td>243214</td>
<td>245251</td>
</tr>
<tr>
<td>8</td>
<td>248086</td>
<td>243645</td>
<td>245213</td>
</tr>
<tr>
<td>9</td>
<td>247005</td>
<td>243887</td>
<td>246621</td>
</tr>
<tr>
<td>10</td>
<td>188291</td>
<td>181987</td>
<td>184353</td>
</tr>
<tr>
<td>Mean</td>
<td>196612</td>
<td>192146</td>
<td>194456</td>
</tr>
</tbody>
</table>
Table 2 compares the optimal solutions of the CTLPD with those of the CTLPU, in terms of open facilities. The table columns have the following meaning:
- Column 1: instance number
- Columns 2-5: number of open facilities in the CTLPD and in the CTLPU under Gumbel, Laplace, and Uniform distributions, respectively
- Columns 6, 8, 10: number of open facilities which are equal in the optimal solution of the CTLPD and of the CTLPU under the three different distributions
- Columns 7, 9, 11: percentage of open facilities which are equal in the optimal solution of the CTLPD and of the CTLPU under the three different distributions.

Table 2. Comparison of the open facilities in the optimal solutions of the deterministic approximation, the CTLPD, and the stochastic model, the CTLPU, under different probability distributions

<table>
<thead>
<tr>
<th>Instances</th>
<th>Number of open facilities</th>
<th>Common open facilities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Det</td>
<td>Gumbel</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>Mean</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 3 reports the results for each instance, as well as their mean in the last row. The main conclusion we can draw from these results is that the optimal solution of the stochastic model under the three different distributions is quite similar to that of its deterministic approximation, since on average 75% of the open facilities are in common. Let us consider instance 1 with Laplace and Uniform distributions, for which the open facilities in the optimal solution are exactly the same as those of the deterministic approximation. Nevertheless, the gap between the two optima is 2.18% and 6.05% (see Table 1), respectively. We should then conclude that this gap is due to a different flow distribution in the two optimal solutions. In order to verify this conclusion we consider the optimum of the CTLPU when the open facilities are those of the CTLPD optimal solution and we compare this optimum with the original CTLPU optimum in Table 3. The table columns have the following meaning:
- Column 1: instance number
- Column 2-4: percentage gap between the stochastic optimum when the facilities are compulsory opened as in the CTLPD optimal solution and the stochastic optimum when the CTLPU can decide which facilities must be opened.

According to these results, the gap between the optimum of the CTLPU obtained with given open facilities and the original one is on average less than 0.5 for all the three distributions. This implies that the optimal decisions taken by the CTLPD and by the CTLPU in terms of open facilities are equivalent (i.e. they generate almost the same optimum) and that the gap between the CTLPD optimum and the CTLPU one when the open facilities are different is mainly due to a different flow distribution in the two optimal solutions.
Table 3. Performance of the optimal solution of the deterministic approximation, the CTLPD, when used as optimal solution of the stochastic model, the CTLPU

<table>
<thead>
<tr>
<th>Instances</th>
<th>Gumbel</th>
<th>Laplace</th>
<th>Uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.30%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>2</td>
<td>0.22%</td>
<td>0.16%</td>
<td>0.36%</td>
</tr>
<tr>
<td>3</td>
<td>0.19%</td>
<td>0.47%</td>
<td>0.34%</td>
</tr>
<tr>
<td>4</td>
<td>0.73%</td>
<td>0.08%</td>
<td>0.29%</td>
</tr>
<tr>
<td>5</td>
<td>1.31%</td>
<td>0.40%</td>
<td>0.77%</td>
</tr>
<tr>
<td>6</td>
<td>0.57%</td>
<td>0.81%</td>
<td>0.93%</td>
</tr>
<tr>
<td>7</td>
<td>0.36%</td>
<td>0.45%</td>
<td>0.96%</td>
</tr>
<tr>
<td>8</td>
<td>0.27%</td>
<td>0.49%</td>
<td>0.79%</td>
</tr>
<tr>
<td>9</td>
<td>0.53%</td>
<td>0.26%</td>
<td>0.15%</td>
</tr>
<tr>
<td>10</td>
<td>0.38%</td>
<td>0.45%</td>
<td>0.84%</td>
</tr>
</tbody>
</table>

Mean 0.49% 0.35% 0.38%

3.4. Comparison between the CTLPD and the classical CTLP

The discussion of the computational results ends by showing the behavior of the Entropy term of the CTLPD in (14). In Table 4 the contribution given by the Entropy term to the optimum of the CTLPD is presented. The table compares the optimum of the CTLPD with that of the classical CTLP, which differs from the former by the Entropy term.

The table columns have the following meaning

- Column 1: instance number
- Column 2: percentage gap between the optimum of the CTLPD and that of the classical CTLP
- Column 3: number of open facilities which are equal in the two optimal solutions
- Column 4: percentage of open facilities which are equal in the two optimal solutions.

Table 4. Contribution of the entropy term in the CTLPD

<table>
<thead>
<tr>
<th>Instances</th>
<th>Gap</th>
<th>Common open facilities</th>
<th>Common open facilities (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14.68%</td>
<td>5</td>
<td>71%</td>
</tr>
<tr>
<td>2</td>
<td>11.89%</td>
<td>6</td>
<td>75%</td>
</tr>
<tr>
<td>3</td>
<td>6.48%</td>
<td>8</td>
<td>73%</td>
</tr>
<tr>
<td>4</td>
<td>10.50%</td>
<td>5</td>
<td>71%</td>
</tr>
<tr>
<td>5</td>
<td>13.66%</td>
<td>6</td>
<td>75%</td>
</tr>
<tr>
<td>6</td>
<td>3.85%</td>
<td>10</td>
<td>77%</td>
</tr>
<tr>
<td>7</td>
<td>8.93%</td>
<td>9</td>
<td>100%</td>
</tr>
<tr>
<td>8</td>
<td>8.87%</td>
<td>6</td>
<td>75%</td>
</tr>
<tr>
<td>9</td>
<td>10.24%</td>
<td>5</td>
<td>71%</td>
</tr>
<tr>
<td>10</td>
<td>3.88%</td>
<td>11</td>
<td>92%</td>
</tr>
</tbody>
</table>

Mean 9.31% 7 78%

Table 4 reports the results for each instance, as well as their mean in the last row.

According to the results, even if a large part of the open facilities are common to the two optimal solutions, the gap between the two optima is relevant, showing an important role played by the Entropy term in the CTLPD.

We remind that when $\beta \to +\infty$ the coefficient of the Entropy term tends to 0 and the CTLPD turns into the classical CTLP.
The last test we perform is devoted to show the speed of convergence of the CTLPD to the classical CTLP, while the value of parameter $\beta$ increases. Fig. 1 reports the mean gap between the optimum of the CTLPD and that of the classical CTLP while $\beta$ varies. According to Fig. 1, the gap is almost zero for $\beta$ equal to 5, so a very fast convergence is guaranteed.

![Fig. 1. Convergence of the deterministic approximation, the CTLPD, to the classical CTLP as $\beta \to +\infty$.](image)

### 3.5. Tuning of the model in real situations

As shown by the computational results, the CTLPD is able to give an excellent approximation of the CTLP under a wide range of probability distributions. On the other hand, in order to use the model with actual data, it requires to tune the value of the parameter $\beta$ and the mean costs $c_{ij}^e$ of the CTLPD. This can be done by considering historical data. While $c_{ij}^e$ can be directly derived from the database by simple statistical computations, $\beta$ requires to consider the full distribution of the empirical costs.

Let us assume that the costs are distributed in the interval $[m, M]$ and consider the standard Gumbel distribution $G(x)=\exp(-e^x)$, then $\beta$ is calibrated as follows. If one accepts an approximation error of 0.01, then $G(x)=1 \Leftrightarrow x = -4.60$ and $G(x)=0 \Leftrightarrow x = 1.52$, and the following equations hold

\[
\beta(m - \zeta) = -4.60 \quad (20)
\]
\[
\beta(M - \zeta) = 1.52 \quad (21)
\]

where $\zeta$ is the mode of the Gumbel distribution $G(x)=\exp(-e^{\beta(x-\zeta)})$. From (20) and (21) we get for $\beta$ the following value:
\[ \beta = \frac{6.12}{M - m} \]

In our case, as \( M - m = 10 - 1 = 9 \), we get \( \beta = 0.68 \).

4. Conclusion

In this paper the Capacitated Transshipment Location Problem under Uncertainty (CTLPU), which is a stochastic program, has been approximated by an equivalent non-linear deterministic Capacitated Transshipment Location Problem (CTLPD), which belongs to a wide class of Entropy maximizing models. The performance of the CTLPD is pretty good. In fact, the mean gap between the two optima is around 2%.

The gap is smaller for the Gumbel and Laplace distributions (2.25% and 1.43%, respectively) than the Uniform one (2.93%).

Also when we consider the number of located depots which are in common between the solutions of the stochastic model and the deterministic one, the results are satisfactory. In fact the behavior of the three probability distributions is quite similar to that of the deterministic approximation, since on average 75% of open depots are in common. Moreover, additional tests prove that, even when the depots opened by the CTLPU and the CTLPD are exactly the same, there is still a small gap between the costs due to some different freight flows in the solutions.

The role of the Entropy term in the CTLPD, weighted by the non-negative parameter \( \beta \), is particularly relevant. We observe that it is highly affected by the value of the parameter \( \beta \). In fact, when this parameter increases, the contribution of the Entropy term to the optimum rapidly decreases and for \( \beta = 5 \) the CTLPD collapses into the classical linear Capacitated Transshipment Location Problem. Both the CTLPU and the CTLPD have been exactly solved for small size instances (up to 3 origins, 20 potential transshipment locations and 40 destinations) in 1000 seconds by means of existing solvers. Larger instances will likely deserve some heuristic approaches [21].

References


