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# Tame equipped posets

Alexander Zavadskij*

*Department of Mathematics, University of Kiev, pr. Wozduhoflotskij, 31, Kiev 03680, Ukraine*

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**Abstract**

Tame equipped posets and equipped posets with involution are described.
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## 0. Introduction

The paper is devoted to the investigation of some tame matrix problems of mixed type over the pair $(\mathbb{R}, \mathbb{C})$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers. These problems are naturally determined by posets (i.e. partially ordered sets) with a simple additional structure, called equipped posets.

Remind that representations of ordinary posets over an arbitrary field were introduced by Nazarova and Roiter in [14], where they constructed the algorithm of differentiation with respect to a maximal element. Using some modification of that algorithm and the results from [16], Nazarova obtained in [12] a tameness criterion.

Later in [13] representations of posets with involution were considered and a tameness criterion was proved under the assumption that all objects of the corresponding vectroid (i.e. of the category determining the problem) have trivial endomorphism rings.

* Present address: Departamento de Mathematicas, Universidad Nacional de Colombia, Bogota, Colombia.

*E-mail address:* zavad@matematicas.unal.edu.co (A. Zavadskij).

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Further, it was worked out in [23] the group of differentiation algorithms I−V for arbitrary posets with involution. They were successfully applied in [2], where among other things a tameness criterion for that class was proved.

It can be observed that in the situations listed above, there occurred only vectroids such that all their schurian [1] objects have trivial endomorphism rings. As for the vectroids, not satisfying this restriction, until recent time their representation theory remained relatively low-investigated. One can mention among the most known facts the characterization of the representation-finite $K$-structures and the classification of representations of some critical vectroids, given by Dlab and Ringel [4–7] in connection with their work on valued graphs, and also the description of the representation-finite schurian vectroids, obtained by Klemp and Simson [10].

In the present paper we are concerned with the situation of vectroids containing schurian objects with nontrivial endomorphism rings. Namely, we deal mainly with the matrix problems corresponding to those vectroids over the field $\mathbb{R}$, which have at most two-dimensional schurian objects with the endomorphism rings $\mathbb{R}$ or $\mathbb{C}$ and satisfy some additional conditions. Such problems are well interpreted as the problems on representations of equipped posets or equipped posets with involution (see precise definitions in Section 1).

Our main goal is to describe equipped posets, equipped posets with primitive involution and equipped posets with arbitrary involution of tame type (these are Theorems A–D in Section 1). For this, in Sections 3−14 the new group of differentiation algorithms VII−XVII for equipped posets (with and without involution) is built. Along with the mentioned algorithms I−V from [23] they allow to differentiate practically any tame matrix problems from the considered class, including the problems of Gel'fandian type (being as a rule of infinite growth).

During the work we were forced to direct the main efforts towards the construction of the algorithms. Despite of their (partially illusory) complexity in substantiation, their combinatorial action is, in fact, wonderfully simple and convenient for applications in tame situation. The reader can see that the proof of the tameness condition of the main Theorem C, presented in Section 17, is very short modulo algorithms. It is based only on some standard properties of Differentiations (Section 15) and on rather transparent combinatorics (Section 16).

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## 1. Formulation of the main results

A poset $\mathscr{P}$ (with a partial order relation $\leqslant$) will be called *equipped* if the following conditions are satisfied:

[1] An object of a vectroid is called *schurian* if its endomorphism ring is a division ring. The vectroid itself is called *schurian* if all its objects are schurian.

(i) the points of $\mathscr{P}$ are divided into *single* and *double*;
(ii) the order relations between points are divided into *weak* and *strong*;
(iii) if $x < y$ is a weak relation, then both points $x, y$ are double and, moreover, in the case $x < t < y$ the relations $x < t$ and $t < y$ are weak too (and automatically the point $t$ is double).

By $x \prec y$ ($x \lhd y$) we denote a weak (strong) relation between points and by $x < y$ an arbitrary relation.

The equipment is regarded to be *trivial* if there are no double points, that is the poset $\mathscr{P}$ is *ordinary*.

Each equipped poset $\mathscr{P}$ determines some matrix problem of mixed type over the pair $(\mathbb{R}, \mathbb{C})$ in the following way. Consider a finite rectangular matrix $M$ partitioned into vertical stripes $M_x$, $x \in \mathscr{P}$ (which may be empty), where the elements of a stripe $M_x$ belong to the field $\mathbb{R}(\mathbb{C})$ if the point $x$ is single (double). One can apply the following *admissible transformations* to the matrix $M$:

(a) $\mathbb{R}$-elementary transformations of rows of the whole matrix $M$;
(b) $\mathbb{R}$-elementary ($\mathbb{C}$-elementary) transformations of columns of a stripe $M_x$ if the point $x$ is single (double);
(c) in the case of a weak relation $x \prec y$ additions of columns of the stripe $M_x$ to the columns of the stripe $M_y$ with coefficients in $\mathbb{C}$;
(d) in the case of a strong relation $x \lhd y$ independent additions both real and imaginary parts of columns of the stripe $M_x$ to the real and imaginary parts (in any combinations) of columns of the stripe $M_y$ with coefficients in $\mathbb{R}$ (certainly, it is assumed that for $y$ single there are no additions to the zero imaginary part of $M_y$).

The partitioned matrices $M$ of such kind (called also *matrix representations* of $\mathscr{P}$) are assumed to be *equivalent* (or *isomorphic*) if they can be turned into each other with help of the admissible transformations. And the *matrix problem*, determined by an equipped poset $\mathscr{P}$, is a problem on classification of indecomposable (in the natural sense) matrices up to equivalence.

The *dimension* of a partitioned matrix $M$ is a vector $d = \underline{\dim}\, M = (d_0; d_x \mid x \in \mathscr{P})$, where $d_0$ is the number of rows in $M$ and $d_x$ is the number of columns in the stripe $M_x$.

Analogously to the case of the pair $(\mathbb{R}, \mathbb{C})$, one can consider a matrix representation of $\mathscr{P}$ over the pair of polynomial rings $(\mathbb{R}[t], \mathbb{C}[t])$ as well as over the pair of free algebras of two generators $(\mathbb{R}\langle X, Y\rangle, \mathbb{C}\langle X, Y\rangle)$ (at the moment we do not introduce any admissible transformations for such representations).

Each $(\mathbb{R}[t], \mathbb{C}[t])$-representation $L$ of an equipped poset $\mathscr{P}$ naturally generates a *real series* (as a rule, infinite) of $(\mathbb{R}, \mathbb{C})$-representations by substituting square matrices $A$ over $\mathbb{R}$ (reduced to some canonical form under the real similarity transformations) for the variable $t$, and scalar matrices $\lambda E$ of the same size for the numbers $\lambda \in \mathbb{C}$.

It appears, in tame situation it is not enough to deal with the real series only. If the mentioned representation $L$ is such that all its single point stripes are scalar (i.e. do not contain the variable $t$), then $L$ generates also a *complex series* of $(\mathbb{R}, \mathbb{C})$-representation by substituting square matrices $A$ over $\mathbb{C}$ (reduced to the Jordan normal form) for the variable $t$, and the scalar matrices $\lambda E$ for the scalars $\lambda \in \mathbb{C}$ (examples of complex series are given in (10.10) and, for a more general situation, in (14.4)).

We will say also (in accordance with the just described two situations) that an $(\mathbb{R}[t], \mathbb{C}[t])$-representation $L$ generates a series *over* $\mathbb{R}$ (*over* $\mathbb{C}$).

Denote by $\mu(d)$ the least number of series (of any kind, both real and complex) containing almost all indecomposables of a given dimension $d$ (considered up to isomorphism). An equipped poset is called *tame* if $\mu(d) < \infty$ for all $d$.

Let now $W$ be a representation of $\mathscr{P}$ over the pair $(\mathbb{R}\langle X, Y\rangle, \mathbb{C}\langle X, Y\rangle)$ of free algebras of two variables. Then $W$ naturally generates an $(\mathbb{R}, \mathbb{C})$-representation $W_{A,B}$ by substituting a pair of square real matrices $A, B$ of equal size for the variables $X, Y$ and the scalar matrices $\lambda E$ of the same size for the scalars $\lambda$.

An equipped poset $\mathscr{P}$ is called *wild* if for some fixed $(\mathbb{R}\langle X, Y\rangle, \mathbb{C}\langle X, Y\rangle)$-representation $W$ and for a complete set of indecomposable and pairwise nonequivalent (under the common real similarity transformations) pairs $A, B$ all the generated representations $W_{A,B}$ are also indecomposable and pairwise nonequivalent. Such representation $W$ we call a *wild generator*.

Let $\mathscr{P}^0$ be the subset of all single points of an equipped poset $\mathscr{P}$.

The *evolvent* of the set $\mathscr{P}$ (with respect to $\mathscr{P}^0$) is an ordinary poset $\overset{\vee}{\mathscr{P}}$ of the form

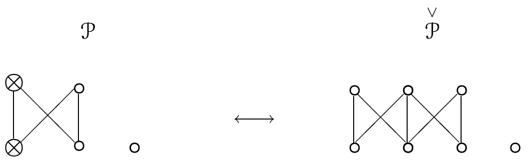
$$\overset{\vee}{\mathscr{P}} = \mathscr{P}^0 + \bigcup_{x \in \mathscr{P} \setminus \mathscr{P}^0} \{x', x''\},$$

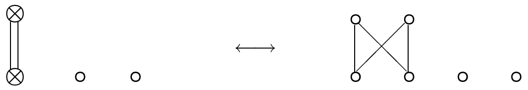
obtained from $\mathscr{P}$ by replacing each point $x \in \mathscr{P} \setminus \mathscr{P}^0$ by a pair of new incomparable points $x', x''$ with the order relation $\leqslant$ defined as follows:

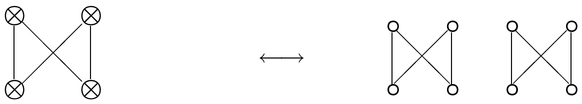
(1) the order relations inside $\mathscr{P}^0$ remain without changes;
(2) each of two points $x', x''$ inherits all previous order relations of the point $x$ with the points of the subset $\mathscr{P}^0$;
(3) if $x, y \in \mathscr{P} \setminus \mathscr{P}^0$, then $\begin{cases} x \prec y \;\Leftrightarrow\; x' < y' \text{ and } x'' < y'', \\ x \lhd y \;\Leftrightarrow\; \{x', x''\} < \{y', y''\}, \end{cases}$
where $\{x', x''\} < \{y', y''\}$ means $x' < y',\ x' < y'',\ x'' < y'$ and $x'' < y''$.

**Typical examples.** Below the symbol $\circ(\otimes)$ denotes a single (double) point and a single (double) line between double points denotes a weak (strong) relation.

(a)



(b) 

(c) 

One of the main goals of the present paper is to prove the following theorem.

**Theorem A.** *An equipped poset $\mathscr{P}$ is tame* (*wild*) *if its evolvent $\overset{\vee}{\mathscr{P}}$ is tame* (*wild*).

As usually, one can reformulate the tameness condition of the theorem in terms of quadratic forms or minimal wild subsets.

The associated with $\mathscr{P}$ *Tits quadratic form* $f = f_{\mathscr{P}}$ is given by the formula

$$f(d) = d_0^2 + \sum_{x \in \mathscr{P}} f_x d_x^2 + \sum_{x<y} p_{xy} f_x f_y d_x d_y - d_0 \sum_{x \in \mathscr{P}} f_x d_x,$$

where $f_x = 1$ ($f_x = 2$) for a single (double) point $x \in \mathscr{P}$ and $p_{xy} = 1/2 (p_{xy} = 1)$ for $x \prec y$ ($x \lhd y$). Naturally, it coincides with the usual unit form for ordinary posets [8, 5.7] if $\mathscr{P}$ contains no double points.

Remind that the form $f$ is called *weakly nonnegative* if $f(d) \geqslant 0$ for each vector $d \geqslant 0$.

In this paper a disjoint union of subsets $X, Y \subset \mathscr{P}$ is called a *sum* and denoted by $X + Y$. The sum $X + Y$ is *cardinal* (*ordinal*) [1] if $X$ and $Y$ are incomparable (if each point of $X$ is less than each point of $Y$ or conversely).

A chain (i.e. a linearly ordered set) of the form $x_1 \prec x_2 \prec \cdots \prec x_n$ is called *weak*. If, moreover, $x_1 \prec x_n$, it is called *completely weak*. The *length* of a chain is the number of its points.

Denote by $(\widetilde{p}_1, \ldots, \widetilde{p}_k, q_1, \ldots, q_l)$ a cardinal sum of $k + l$ chains among which $k$ chains are completely weak with the lengths $p_1, \ldots, p_k$ and $l$ chains are ordinary with the lengths $q_1, \ldots, q_l$ (possibly, $k = 0$ or $l = 0$).

Set $N_1 = (1, 1, 1, 1, 1)$, $N_2 = (1, 1, 1, 2)$, $N_3 = (2, 2, 3)$, $N_4 = (1, 3, 4)$, $N_5 = (N, 5)$, $N_6 = (1, 2, 6)$, where $(N, m)$ denotes an ordinary cardinal sum of $m$-point chain and four-point subset $N = \{a < b > c < d\}$.

Also denote $W_1 = (\widetilde{1}, 1, 1, 1)$, $W_2 = (\widetilde{1}, \widetilde{1}, 1)$, $W_3 = (\widetilde{1}, \widetilde{1}, \widetilde{1})$, $W_4 = (\widetilde{1}, 1, 2)$, $W_5 = (\widetilde{2}, 1, 1)$, $W_6 = (\widetilde{1}, \widetilde{2})$, $W_7 = (\widetilde{2}, 3)$, $W_8 = (\widetilde{3}, 2)$, $W_9 = (\widetilde{4}, 1)$.

We recall that according to Nazarova's theorem [12], an ordinary poset is tame if and only if it contains none of the subsets $N_1, \ldots, N_6$ (and this is equivalent to the weak nonnegativity of its quadratic form). In view of our remarks at the end of Section 17, it follows that the combinatorial tameness condition of Theorem A may be presented in several equivalent forms.

**Lemma 1.1.** *For an equipped poset $\mathscr{P}$ the next assertions are equivalent.*

(a) *The evolvent $\overset{\vee}{\mathscr{P}}$ is tame, i.e. contains none of the subsets $N_1, \ldots, N_6$.*
(b) *$\mathscr{P}$ contains none of the subsets $N_1, \ldots, N_6$ and $W_1, \ldots, W_9$.*
(c) *The Tits quadratic form $f_{\mathscr{P}}$ is weakly nonnegative.*

For example, all depicted above equipped posets are tame. [2]

Since Theorem A admits a very simple formulation, the first idea, which comes naturally to the mind, is to try to reduce the result to the case of ordinary posets. [3] Despite of this, we do not have at our disposal any evident construction establishing direct connection between indecomposables of $\mathscr{P}$ and $\overset{\vee}{\mathscr{P}}$. [4] That is why we give the complete independent proof of the result for the equipped situation (embracing, in particular, the ordinary case).

The proof is based on elaboration and application the corresponding differentiation technique which in some cases leads out of the class of equipped posets and forces to consider more general problems. In particular, we consider the class of equipped posets with involution and prove for it the analogous results.

An *equipped poset with involution* is an equipped poset satisfying (except of the listed above conditions (i)–(iii)) two additional conditions:

(iv) on the set of all points of $\mathscr{P}$ an involution $*$ is given which maps single points into single and double into double (and which is in no connection with the order relation); so, single points are divided into *small* ($x = x^*$) and *big* ($x \neq x^*$) and double points are divided into *simple double* ($x = x^*$) and *bidouble* ($x \neq x^*$);
(v) each bidouble point $x$ is assigned the number $g(x) = g(x^*) \in \{\pm 1\}$ called its *genus* (or *genus of the pair $x, x^*$*).

In the case $x \neq x^*$ we call the points $x$ and $x^*$ *equivalent* and write $x \sim x^*$.

The involution * will be called *primitive* if it leaves fixed all double points (i.e. there are no bidouble points).

Naturally, a matrix representation of an equipped poset with involution is such a representation $M$ of an equipped poset that the vertical stripes $M_x$ and $M_{x^*}$ (related to the equivalent points $x \sim x^*$) have the same numbers of columns. And the corresponding *matrix problem* consists of classification of indecomposable matrices up to equivalence determined by the listed above transformations of type (a),

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[2] One can show that the first one is two-parameter and the rest are of infinite growth.

[3] There are some more arguments in favour of this idea. The criteria for an equipped poset to be representation-finite or one-parameter can be formulated in the form, analogous to Theorem A, as it follows immediately from [10,26] respectively.

[4] An idea to try to use the induced representations of $\overset{\vee}{\mathscr{P}}$ over $\mathbb{C}$ was expressed recently by C.M. Ringel (and also by S. Kasjan). It may occur, subsequently it will help to simplify the proof.

(c), (d) and also by the following ones (b$'$)–(b$'''$) replacing the transformations of type (b):

(b$'$) $\mathbb{R}$-elementary ($\mathbb{C}$-elementary) transformations of columns of a stripe $M_x$ if the point $x$ is small (simple double);
(b$''$) the same $\mathbb{R}$-elementary transformations of columns of the stripes $M_x$ and $M_{x^*}$ if $x \sim x^*$ are big points;
(b$'''$) the same (conjugate) $\mathbb{C}$-elementary transformations of columns of the stripes $M_x$ and $M_{x^*}$ if $x \sim x^*$ are bidouble points of genus 1 (of genus $-1$).

**Remarks**

(1) The *conjugate* $\mathbb{C}$-elementary transformations of columns of the stripes $M_x$ and $M_{x^*}$ are generated by transformations of two types:
  (a) multiplications of two corresponding to each other columns of the stripes $M_x$ and $M_{x^*}$ by mutually conjugate complex numbers $\lambda, \overline{\lambda} \neq 0$;
  (b) addition of the $i$th column of the stripe $M_x$ to its $j$th column with a coefficient $\lambda \in \mathbb{C}$ and simultaneous addition of the $i$th column of the stripe $M_{x^*}$ to its $j$th column with the conjugate coefficient $\overline{\lambda}$.
(2) Obviously, in the case of a primitive involution, the transformations (b$'''$) disappear but the others remain.
(3) In a more partial case, under absence of all double points, we obtain the problem on representations of ordinary posets with involution over $\mathbb{R}$, which was considered (over an arbitrary field) in [2,13,23].

As for the tame (wild) equipped posets with involution, they are defined entirely analogously to the purely equipped situation.

Later on a subset $X \subset \mathscr{P}$ will be called *small* (*big*, *double* , ...) if all its points are small (big, double, ...). A subset, consisting of two (three, four) mutually incomparable points, is called in this paper a *dyad* (*triad*, *tetrad*).

Assume that an equipped poset with involution $\mathscr{P}$ contains such a big point $a$ that the set of all points, incomparable with $a$, is a small chain of length $n \geqslant 0$ with the elements $c_i$. Then, using some simple reduction of the stripe $M_a$ and partially the stripes $M_{c_i}$ one can transit from representations of $\mathscr{P}$ to representations of some new equipped poset with involution $\mathscr{P}'_a$ obtained from $\mathscr{P}$ by deleting the point $a$ and replacing the equivalent point $a^*$ by a small chain of length $n + 2$ (see Lemma 15.6 and its proof for details). Making such reduction as many times as possible, in the case of a finite poset $\mathscr{P}$ one can always get rid of big points with the mentioned property.

Therefore, it is quite natural (and very useful for the formulation of the result) to consider equipped posets with involution being *reduced*, i.e. satisfying the condition:

(R) each big point is incomparable with some big point or some double point or with some small dyad.

Now let $\mathscr{P}^0$ be the set of all small points [5] of an equipped poset with primitive involution $\mathscr{P}$. Keeping without any change the presented above definition of the *evolvent* $\overset{\vee}{\mathscr{P}}$ of the set $\mathscr{P}$ with respect to the subset $\mathscr{P}^0$, we receive an opportunity to formulate laconically the tameness and wildness conditions in the considered situation (generalizing Theorem A).

**Theorem B.** *A reduced equipped poset with primitive involution $\mathscr{P}$ is tame* (*wild*) *if its evolvent $\overset{\vee}{\mathscr{P}}$* (*with respect to the subset of all small points $\mathscr{P}^0$*) *is tame* (*wild*).

In the case of an equipped poset with arbitrary involution $\mathscr{P}$, denote by $\mathscr{P}_{\mathrm{prim}}$ the set with primitive involution obtained from $\mathscr{P}$ by deleting all bidouble points. This subset plays important role in the formulation of the corresponding statement (generalizing Theorem B).

**Theorem C.** *A reduced equipped poset with involution $\mathscr{P}$ is tame if it satisfies two conditions*:

(a) *each bidouble point is comparable with all other points*;
(b) *the evolvent $\overset{\vee}{\mathscr{P}}_{\mathrm{prim}}$ of the subset $\mathscr{P}_{\mathrm{prim}}$ is tame*.

*Otherwise $\mathscr{P}$ is wild*.

Taking into account the described above reducing procedure (which leads to validity of the condition (R)), we receive a natural consequence of Theorem C.

**Theorem D.** *An equipped poset with involution $\mathscr{P}$ is tame if each of its bidouble points is comparable with all other points and the subset $\mathscr{P}_{\mathrm{prim}}$ is tame*. *Otherwise $\mathscr{P}$ is wild*.

Certainly, it is expected (but is not proved in the present paper) that none of the equipped posets with involution can be both tame and wild. In other words, it is expected that the presented above sufficient tameness conditions are also necessary, i.e. Theorems A–D are in fact criteria. In this connection we would like to pay attention to the following two natural questions concerning matrix problems over an arbitrary infinite base field $k$.

Denote by $\mu_n(d)$ the least number of at most $n$-parameter (over $k$) series containing almost all indecomposables of dimension $d$.

[5] In the case of a trivial involution small and single points coincide, and the current utilization of the symbol $\mathscr{P}^0$ agrees with the previous one.

**Question 1.** If the problem is wild, whether for each $n$ there exists $d$ such that $\mu_n(d) = \infty$?

**Question 2.** If the problem is not wild, whether there exists $n$ such that $\mu_n(d) < \infty$ for all $d$?

Intuitively the first question looks like a simple one and may be considered as the easiest part of the corresponding tame–wild dichotomy statement, analogous to the result proved by Drozd [3] for boxes over algebraically (separably) closed fields. The positive answer to it (in the case $k = \mathbb{R}$) converts the Theorems A–D into criteria. Since at the moment we do not have a formal proof, it is actual to obtain a suitable one using only "elementary algebraic geometry" reasonings. [6] As for the second question, seems, it is a less transparent and more complicated one.

## 2. Further preliminaries and notations

We hope that our exposition of the material, concerning the topic, is rather detailed and self-contained. At the same time, the interested reader may also use the books [8,17,18], where some introductional and additional material on representations of posets and vector space categories (vectroids) can be found.

Recall that a *vectroid* $\mathscr{L}$ over a field $k$ (considered in fact already in [15]) is any $k$-linear subcategory (usually not full) of the category of finite-dimensional $k$-spaces with indecomposable and pairwise nonisomorphic objects.

Let $\mathscr{L}(X, Y)$ be the space of morphisms from the object $X$ into the object $Y$. Modules over the ring $\bigoplus_{X,Y} \mathscr{L}(X, Y)$ are called *$\mathscr{L}$-modules* (or modules over the vectroid $\mathscr{L}$). A natural right $\mathscr{L}$-module $\mathscr{L}_0 = \bigoplus_X X$ is the *basic module* of the vectroid $\mathscr{L}$.

A *representation* of the vectroid $\mathscr{L}$ over $k$ is any right $\mathscr{L}$-submodule $U$ of the tensor product

$$U \subset U_0 \bigotimes_k \mathscr{L}_0,$$

where $U_0$ is some $k$-space. In this work all representations are supposed to be *finite-dimensional*, i.e. satisfying the conditions $\dim_k U_0 < \infty$ and $\dim_k(U/\operatorname{rad} U) < \infty$, where $\operatorname{rad} U = U \cdot \operatorname{rad} \mathscr{L}$ is the *radical of a representation* $U$ and $\operatorname{rad} \mathscr{L} = \bigoplus_{X,Y} \operatorname{rad} \mathscr{L}(X, Y)$ is the *radical of the vectroid* $\mathscr{L}$ (the ideal generated by all noninvertible morphisms).

A *morphism* from a representation $U \subset U_0 \bigotimes_k \mathscr{L}_0$ into a representation $V \subset V_0 \bigotimes_k \mathscr{L}_0$ is any $k$-linear map $\varphi : U_0 \longrightarrow V_0$ for which $(\varphi \otimes 1)(U) \subset V$. The *category of representations* of $\mathscr{L}$ is denoted by *$\mathscr{L}$-sp*.

[6] Some general sketch of the positive solution of Question 1 (based, as in [3], on comparing the linear and quadratic growths of variety dimensions) was outlined during ICRA-9 by Crawley-Boevey.

Obviously, $U = \bigoplus_X U_X$, where $U_X \subset U_0 \bigotimes_k X$, hence, the representation $U$ can be identified with the collection

$$U = \big(U_0; U_X \mid X \in \mathrm{Ob}\,\mathscr{L}\big),$$

where $U_X\, \mathscr{L}(X, Y) \subset U_Y$ for all $X, Y$. Then a morphism $U \longrightarrow V$ of the category $\mathscr{L}$-*sp* has to be considered as any $k$-linear map $\varphi : U_0 \longrightarrow V_0$ with the condition $(\varphi \otimes 1_X)(U_X) \subset V_X$ for all $X$.

The *dimension* of a representation $U$ is a vector $d = \underline{\dim}\, U = (d_0; d_X \mid X \in \mathrm{Ob}\,\mathscr{L})$, where $d_0 = \dim_k U_0$ and $d_X = \dim_{\mathscr{L}(X,X)} U_X/(\mathrm{rad}\, U)_X$ (it is clear that $(\mathrm{rad}\, U)_X = \sum_Y U_Y\ \mathrm{rad}\,\mathscr{L}(Y, X)$). The representation $U$ is called *sincere* if its dimension $\underline{\dim}\, U$ has no zero coordinates. Vectroids, having at least one sincere indecomposable representation, are called *sincere*.

We often use below the abridge notation $(\mathrm{rad}\, U)_X = \underline{U}_X$. By $\dot{U}_X$ we denote any set of generators of the $\mathscr{L}(X, X)$-module $U_X$ taken modulo its radical $\underline{U}_X$, i.e. satisfying the condition $(\underline{U}_X + \dot{U}_X)\mathscr{L}(X, X) = U_X$.

Let $X^* = \mathrm{Hom}_k(X, k)$ be the dual space for a $k$-space $X$ and $f^* : Y^* \longrightarrow X^*$ be the dual homomorphism for a given homomorphism of $k$-spaces $f : X \longrightarrow Y$.

For each vectroid $\mathscr{L}$ over $k$ the *dual vectroid* $\mathscr{L}^*$ with the objects $X^*$ and morphisms $f^*$ is naturally defined. So, $\mathscr{L}^*(Y^*, X^*) = (\mathscr{L}(X, Y))^* = \{f^* \mid f \in \mathscr{L}(X, Y)\}$, in particular, the endomorphism rings $\mathrm{End}\, X$ and $\mathrm{End}\, X^*$ are antiisomorphic.

If $U \subset U_0 \bigotimes_k \mathscr{L}_0$ is a representation of $\mathscr{L}$, then the corresponding *dual representation* $U^\perp \subset U_0^* \bigotimes_k \mathscr{L}_0^*$ has the form $U^\perp = \big\{\varphi \in U_0^* \bigotimes_k \mathscr{L}_0^* \mid \varphi(U) = 0\big\}$. Obviously, $U^{\perp\perp} \simeq U$.

When writing $U$ in the form $U = (U_0; U_x \mid X \in \mathrm{Ob}\,\mathscr{L})$, it holds $U^\perp = (U_0^*; U_{X^*}^* \mid X^* \in \mathrm{Ob}\,\mathscr{L}^*)$, where $U_{X^*}^* = \{\varphi \in U_0^* \bigotimes_k X^* \mid \varphi(U_X) = 0\}$.

Later on a vector space over $k$ with the base $e_1, \dots, e_n$ is denoted by $k\{e_1, \dots, e_n\}$ or simply by $\{e_1, \dots, e_n\}$, if there is no doubt about the field. A direct sum of $m$ copies of a space or representation $U$ (of a linear map $f$) is denoted by $U^m (f^m)$.

By $|A|$ we denote the number of elements of a set $A$. Often a one-point set is identified with a point: $\{a\} = a$.

For a subset $A \subset \mathscr{P}$ denote by $U_0^A$ a direct sum of $|A|$ copies of a space $U_0$ numbered by the points of $A$.

In this paper we consider vectroids over the field $\mathbb{R}$.

• Each equipped poset (without involution) $\mathscr{P}$ determines a vectroid $\mathscr{L} = \mathscr{L}_{\mathscr{P}}$ over $\mathbb{R}$ with the objects $X$ corresponding to the points $x \in \mathscr{P}$. Namely, the object $X$ coincides with a copy $\mathbb{R}_x(\mathbb{C}_x)$ of the field $\mathbb{R}(\mathbb{C})$ and has the endomorphism ring $\mathbb{R}(\mathbb{C})$ if the point $x$ is single (double ). Moreover, if $X \neq Y$, it holds

$$\mathscr{L}(X, Y) = \begin{cases} \mathbb{C} & \text{for } x \prec y, \\ \mathrm{Hom}_{\mathbb{R}}(X, Y) & \text{for } x \lhd y, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to comprehend that the classification of representations of such vectroid $\mathscr{L}$ corresponds exactly to the matrix problem (a)–(d) from Section 1. Setting $\mathscr{P}$-*sp* $=$ $\mathscr{L}$-*sp*, we may write representations in the form $U = (U_0; U_x \mid x \in \mathscr{P})$.

• Let $\mathscr{P}$ be an equipped poset with primitive involution and $\Theta$ be the set of the equivalence classes of its points with respect to the involution $*$. Here the objects of the vectroid $\mathscr{L} = \mathscr{L}_{\mathscr{P}}$ are in one-to-one correspondence with the classes from $\Theta$. Certainly, the equipped subposet, obtained from $\mathscr{P}$ by deleting all big points, determines the same full subvectroid in $\mathscr{L}$ as in the previous case. In addition, each pair $x \sim x^*$ of big points is conformed to a two-dimensional object $\mathbb{R}_x \oplus \mathbb{R}_{x^*}$ of $\mathscr{L}$ with the trivial endomorphism ring in the case of incomparable $x, x^*$ (*pencil*) and with the endomorphism ring

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

in the case $x < x^*$.

The morphism sets between different objects in $\mathscr{L}$ are generated by the following $\mathbb{R}$-spaces (but, certainly, do not necessarily coincide with them):

(1) $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}_x, \mathbb{R}_y)$, where $x \lhd y$ are any single points;
(2) $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}_x, \mathbb{C}_y)$ ($\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}_y, \mathbb{R}_x)$), where $x$ is single, y is double and $x \lhd y$ ($y \lhd x$);
(3) $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}_x, \mathbb{C}_y)$ ($\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_x, \mathbb{C}_y)$), where $x \lhd y$ ($x \prec y$) are double points.

The classification of representations of $\mathscr{L}$ corresponds to the matrix problem (a), (b′), (b″), (c) and (d) from Section 1.

• Let $\mathscr{P}$ be an equipped poset with arbitrary involution and $\Theta$ be the same set of classes as before. Again the objects from $\mathscr{L}$ are in one-to-one correspondence with the classes from $\Theta$. The equipped subposet with primitive involution $\mathscr{P}_{\mathrm{prim}}$, obtained from $\mathscr{P}$ by deleting all bidouble points, determines exactly the same full subvectroid in $\mathscr{L}$ as in the previous case. Moreover, each pair $x \sim x^*$ of bidouble points of genus 1 or $-1$ is conformed to a four-dimensional object $\mathbb{C}_x \oplus \mathbb{C}_{x^*}$ with the endomorphism ring of the form:

($\alpha$) $\Lambda_1 = \left\{ \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \,\middle|\, u, v \in \mathbb{C} \right\}$ or $\Lambda_2 = \left\{ \begin{pmatrix} u & v \\ 0 & \overline{u} \end{pmatrix} \,\middle|\, u, v \in \mathbb{C} \right\}$
for $x \prec x^*$(note that $\Lambda_1 \not\simeq \Lambda_2$),

($\beta$) $\overline{\Lambda}_1 = \left\{ \begin{pmatrix} u & h \\ 0 & u \end{pmatrix} \,\middle|\, \begin{matrix} u \in \mathbb{C} \\ h \in H \end{matrix} \right\}$ or $\overline{\Lambda}_2 = \left\{ \begin{pmatrix} u & h \\ 0 & \overline{u} \end{pmatrix} \,\middle|\, \begin{matrix} u \in \mathbb{C} \\ h \in H \end{matrix} \right\}$,
where $H = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$, for $x \lhd x^*$,

($\gamma$) $\Gamma_1 = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \,\middle|\, u \in \mathbb{C} \right\}$ or $\Gamma_2 = \left\{ \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix} \,\middle|\, u \in \mathbb{C} \right\}$
for incomparable $x$ and $x^*$(this is possible in wild situation only).

The morphism sets between different objects in $\mathscr{L}$ are generated by the same $\mathbb{R}$-spaces (1)–(3) as in the previous case. The classification of representations of the vectroid $\mathscr{L}$ corresponds to the problem (a), (b′), (b″), (b‴), (c) and (d) from Section 1.

As usually, an equipped poset with involution is called *m-parameter*, if the least number of series, containing (up to isomorphism) almost all indecomposables of each given dimension, is equal to $m$.

**Lemma 2.1.** *The set* $\Psi = \{a \lhd a^*\}$, *where* $a \sim a^*$ *are bidouble points, is one-parameter.*

**Proof.** If you take a matrix representation $M$ of $\Psi$ and reduce completely the stripe $M_a$, it appears in the stripe $M_{a^*}$ the following matrix problem (A) with a pencil and two simple double points which leads out of the class of equipped posets with involution:

$$\text{(A)}\qquad \begin{array}{cccc} & \mathbb{C} \longrightarrow & \mathbb{R} \longrightarrow & \mathbb{C} \\ \mathbb{R} & \multicolumn{1}{|c|}{\mathbb{C}} & \multicolumn{1}{c|}{\mathbb{C}} & \multicolumn{1}{c|}{\mathbb{C}} \\ \cline{2-4} \end{array}$$

(here all the matrices are over $\mathbb{C}$, the transformations of rows are over $\mathbb{R}$ and of columns over $\mathbb{R}$ or $\mathbb{C}$ dependently of a stripe, moreover, there are additions of columns over $\mathbb{C}$ from the left to the right).

But, from the other hand, if you consider a matrix representation $N$ of the well-known critical one-parameter equipped poset $L_1 = \{a, r, s\}$, where $a$ is double and $r, s$ are small (see Appendix A–C), and you reduce completely the stripes $N_a$ and $N_r$, assuming additionally that

$$N_a = \boxed{\begin{matrix} E \\ iE \end{matrix}}$$

(i.e. considering even a partial case), then it appears in the third stripe $N_s$ exactly the problem (A). □

For a real vector space $U_0$ we will consider often the complex space $\widetilde{U}_0 = U_0 \bigotimes_{\mathbb{R}} \mathbb{C}$ which is usually called the *complexification* of $U_0$. Since $\widetilde{U}_0$ is canonically identified with the direct sum $U_0 \oplus \mathrm{i}U_0$, for each $\mathbb{R}$-subspace $W \subset \widetilde{U}_0$ its *real* and *imaginary* parts $\operatorname{Re} W, \operatorname{Im} W \subset U_0$ are defined (they coincide if $W$ is a $\mathbb{C}$-subspace).

**Remark.** Later on, depending on situation, the elements of the $\mathbb{C}$-space $\widetilde{U}_0$ are written both in the form of sums $u + \mathrm{i}v$ and ordered pairs $(u, v)$, where $u, v \in U_0$. In the same manner, for two subspaces $X, Y \subset U_0$ the direct sum $X \oplus \mathrm{i}Y \subset \widetilde{U}_0$ is denoted sometimes by $(X, Y)$.

For an element $z = u + \mathrm{i}v \in \widetilde{U}_0$ set $\widehat{z} = v + \mathrm{i}u$. Obviously, $\widehat{\widehat{z}} = z$.

By $|A|$ we denote the number of elements of a set $A$. Often a one-point set is identified with a point: $\{a\} = a$.

For a subset $A \subset \mathscr{P}$ and a vector space (over some field) $X$ denote by $X^A$ a direct sum of $|A|$ copies of the space $X$ numbered by the points of $A$.

If $X$ is a real space and the subset $A$ consists either of single points only or of double points only, then denote

$$\{X^A\} = \begin{cases} X^A & \text{if } A \text{ consist of single points,} \\ \widetilde{X}^A & \text{if } A \text{ consist of double points.} \end{cases}$$

Sometimes the classes $K \in \Theta$, consisting of two equivalent points $x \sim x^*$, will be written in the form of ordered pairs $K = (x, x^*)$.

Later on it will be natural to consider a representation $U$ of an equipped poset with involution $\mathscr{P}$ in the form $U = (U_0; U_K \mid K \in \Theta)$, where $U_x \subset U_0$ ($U_x \subset \widetilde{U}_0$) for a small (simple double) point $x$ and $U_{(x,x^*)} \subset U_0^2$ ($U_{(x,x^*)} \subset \widetilde{U}_0^2$) for a pair of big (bidouble) points $x \sim x^*$.

Let $\mathscr{P}$ be an equipped poset (without involution).

As $U_x \subset \widetilde{U}_0$ for a double point $x \in \mathscr{P}$, one can set $U_x^- = U_x \cap U_0$ and $U_x^+ = \operatorname{Re} U_x = \operatorname{Im} U_x$ (obviously, $U_x^- \subset U_x^+ \subset U_0$). For a single point $x$ suppose $U_x^- = U_x^+ = U_x$. Note that $U_x^+ \subset U_y^-$ for any points $x, y$ with the condition $x \lhd y$.

The following simple, but useful, relations are used frequently in our considerations. For single points $\xi, \eta$: (a) $\widetilde{U_\xi + U_\eta} = \widetilde{U}_\xi + \widetilde{U}_\eta$; (b) $\widetilde{U_\xi \cap U_\eta} = \widetilde{U}_\xi \cap \widetilde{U}_\eta$; (c) $(\widetilde{U}_\xi)^+ = U_\xi$.

For double points $x, y$: (a) $(U_x + U_y)^+ = U_x^+ + U_y^+$; (b) $(U_x \cap U_y)^- = U_x^- \cap U_y^-$; (c) $U_x^+ \subset U_y^- \Leftrightarrow U_x \subset \widetilde{U_y^-} \Leftrightarrow \widetilde{U_x^+} \subset U_y$.

For a single point $\xi$ and double points $x, y$: (a) $U_x^+ \subset U_\xi \Leftrightarrow U_x \subset \widetilde{U}_\xi$; (b) $U_\xi \subset U_y^- \Leftrightarrow \widetilde{U}_\xi \subset U_y$.

It will be convenient to write similar lattice relations in the abridge notations, omitting the symbols of a space $U$ and of intersection $\cap$ (the last one is supposed to be replaced by the point). For instance, the mentioned relations for double points $x, y$ will appear in the form: (a) $(x + y)^+ = x^+ + y^+$; (b) $(xy)^- = x^- y^-$; (c) $x^+ \subset y^- \Leftrightarrow x \subset \widetilde{y^-} \Leftrightarrow \widetilde{x^+} \subset y$.

For a subset $A \subset \mathscr{P}$ denote

$$U_A^- = \bigcap_{\alpha \in A} U_a^-, \quad U_A^+ = \sum_{a \in A} U_a^+ \quad (\text{by definition, } U_\varnothing^- = U_0 \text{ and } U_\varnothing^+ = 0).$$

For $a \in \mathscr{P}$ we use notations $a^\vee = \{x \in \mathscr{P} \mid a \leqslant x\}$, $a_\wedge = \{x \in \mathscr{P} \mid x \leqslant a\}$, $a^\triangledown = \{x \in \mathscr{P} \mid a \unlhd x\}$, $a_\vartriangle = \{x \in \mathscr{P} \mid x \unlhd a\}$. For $A \subset \mathscr{P}$ set $A^\triangledown = A + \{x \in \mathscr{P} \mid A \lhd x\}$, $A_\vartriangle = A + \{x \in \mathscr{P} \mid x \lhd A\}$,

$$A^{\mathrm{up}} = \bigcup_{a \in A} a^\vee, \quad A_{\mathrm{down}} = \bigcup_{a \in A} a_\wedge.$$

By $\min A$ ($\max A$) we denote the set of all minimal (maximal) points of a subset $A \subset \mathscr{P}$.

We write $A < B$ if $a < b$ for all $a \in A$, $b \in B$ (the notations $A \prec B$ and $A \lhd B$ have the analogous sense).

A *convex envelope* of a subset $A \subset \mathscr{P}$ is a subset of the form $[A] = \{x \in \mathscr{P} \mid a' \leqslant x \leqslant a''$ for some $a', a'' \in A\}$. A (closed) *segment* $[a, b] = \{x \in \mathscr{P} \mid a \leqslant x \leqslant b\}$ is its partial case for $A = \{a, b\}$ and $a \leqslant b$.

The set $[A]\backslash A$ is the *interior* of the convex envelope $[A]$.

Denote by $N(A)$ the set of all points incomparable with each point of a subset $A \subset \mathscr{P}$. Set $N(a_1, \ldots, a_n) = N(\{a_1, \ldots, a_n\})$.

A *vector*, corresponding to a set $\mathscr{P}$, is any vector $d = (d_0; d_x \mid x \in \mathscr{P})$. Let $\delta_a$, where $a \in \mathscr{P} \cup \{0\}$, be a trivial vector of the form $\delta_a = (d_0; d_x \mid x \in \mathscr{P})$, where $d_a = 1$ and $d_x = 0$ for $x \neq a$ (especially, $\delta_0 = (1; 0, \ldots, 0)$).

Denote by $[d]$ the indecomposable representation of dimension $d$ (in the case such a representation exists and is unique up to isomorphism). For example, $[\delta_0] = (\mathbb{R}; 0, \ldots, 0)$.

For $A \subset \mathscr{P}$ set $P(A) = P(\min A) = (\mathbb{R}; U_x \mid x \in \mathscr{P})$, where $U_x = \mathbb{R}$ ($U_x = \mathbb{C}$) for a single (double) point $x \in A^{\mathrm{up}}$ and $U_x = 0$ otherwise. Obviously, $P(\varnothing) = [\delta_0]$ and $P(a) = [\delta_a]$ for $a \in \mathscr{P}$. Set $P(a_1, \ldots, a_n) = P(\{a_1, \ldots, a_n\})$.

Denote by $\operatorname{Ind} \mathscr{P}$ ($\operatorname{Ind} \mathscr{R}$) a complete set of pairwise nonisomorphic indecomposables (indecomposable objects) of the set $\mathscr{P}$ (of the category $\mathscr{R}$).

Recall briefly the construction of Differentiation I (see [19,20,22]) which is used essentially below (we give a modified quintessence of the corresponding material, adapted to the considered situation).

A pair of incomparable points $(a, b)$ of an ordinary poset $\mathscr{P}$ is called I-*suitable* (i.e. suitable for Differentiation I) if $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + C$, where $C = \{c_1 < \cdots < c_n\}$ is a chain (possibly empty) incomparable with the points $a, b$.

The *derivative poset* of the set $\mathscr{P}$ with respect to the pair $(a, b)$ is a poset $\mathscr{P}'_{(a,b)} = (\mathscr{P}\backslash C) + C^- + C^+$, where $C^- = \{c_1^- < \cdots < c_n^-\}$ and $C^+ = \{c_1^+ < \cdots < c_n^+\}$ are two new chains (replacing the chain $C$) with the relations $c_i^- < c_i^+$; $a < c_i^+$ and $c_i^- < b$. It is assumed that each of two points $c_i^-, c_i^+$ inherits all order relations of the "paternal" point $c_i$ with the points of the subset $\mathscr{P}\backslash C$ (and, of course, all the induced relations are added).

If $U$ is a representation of $\mathscr{P}$, then the *derivative representation* $U'$ of the set $\mathscr{P}' = \mathscr{P}'_{(a,b)}$ is defined as follows:

$$\begin{aligned} &U'_0 = U_0, \\ &U'_{c_i^-} = U_{c_i} \cap U_b, \quad U'_{c_i^+} = U_{c_i} + U_a, \\ &U'_x = U_x \quad \text{for the remaining points } x \in \mathscr{P}'. \end{aligned} \tag{2.1}$$

Setting also $\varphi' = \varphi$ for any morphism $U \xrightarrow{\varphi} V$ of the category $\mathscr{P}$-*sp* (considered as a linear map $\varphi : U_0 \longrightarrow V_0$), we obtain the correctly defined *differentiation functor* $'$: $\mathscr{P}$-*sp* $\longrightarrow \mathscr{P}'$-*sp*.

It turns out that even for an indecomposable $U$ the derivative representation $U'$ is as a rule decomposable and contains trivial direct summands of the form $P(a)$. Taking $U' = U^{\downarrow} \oplus P^m(a)$, where $m \geqslant 0$ and $U^{\downarrow}$ does not contain the summands

$P(a)$, we obtain the *reduced* derivative representation $U^{\downarrow}$ being always indecomposable for an indecomposable $U$.

One can directly define the reduced representation $U^{\downarrow}$ as follows. In the lattice of all subspaces of the $\mathbb{R}$-space $U_0$ consider a subposet of the form

$$\begin{tikzcd}
 & & & U_0 \arrow[dl, dash] \arrow[dr, dash] & \\
 & & U_a+U_b \arrow[dl, dash] \arrow[dr, dash] & & W_0 \arrow[dl, dash] \\
 & U_a \arrow[dl, dash] \arrow[dr, dash] & & U_b \arrow[dl, dash] & \\
E_0 \arrow[dr, dash] & & U_a\cap U_b \arrow[dl, dash] & & \\
 & 0 & & &
\end{tikzcd} \tag{2.2}$$

where $E_0$ is any subspace complementing the meet $U_a \cap U_b$ in the interval $[0, U_a]$, and $W_0$ is any subspace complementing the sum $U_a + U_b$ in the interval $[U_b, U_0]$.[7] In other words, $E_0$ and $W_0$ must satisfy the following relations:

$$\begin{cases} E_0 \cap U_b = 0 \\ E_0 + (U_a \cap U_b) = U_a \end{cases} \quad \text{and} \quad \begin{cases} U_a + W_0 = U_0, \\ (U_a + U_b) \cap W_0 = U_b. \end{cases} \tag{2.3}$$

Then the poset (2.2) is a lattice (with the obvious meets and sums) and $U_0 = E_0 \oplus W_0$.

Any pair of subspaces $(E_0, W_0)$ of the space $U_0$, satisfying the condition (2.3), will be called a *complementing* pair of subspaces with respect to the pair $(U_a, U_b)$.

Now set $U^{\downarrow} = W = (W_0; W_x \mid x \in \mathscr{P}')$ with $W_x = U'_x \cap W_0$ for all $x \in \mathscr{P}'$. It is easy to verify that if $\pi : U_0/E_0 \overset{\sim}{\longrightarrow} W_0$ is the canonical isomorphism, then it holds $\pi((U'_x + E_0)/E_0) = W_x$, i.e. $U^{\downarrow}$ does not depend on the choice of $W_0$ and $E_0$.

Note that $W_a \subset W_b$, i.e. $U^{\downarrow}$ is in fact a representation of the *completed* derivative poset $\overline{\mathscr{P}}'_{(a,b)}$, obtained from $\mathscr{P}'_{(a,b)}$ by addition the only one relation $a < b$. The category $\overline{\mathscr{P}}'$-*sp* is a full subcategory of the category $\mathscr{P}'$-*sp* formed by the objects without direct summands $P(a)$, i.e. these categories differ nonessentially from each other. Clearly, $\operatorname{Ind} \overline{\mathscr{P}}' = \operatorname{Ind} \mathscr{P}' \backslash P(a)$.

**Remark.** For an arbitrary poset $Q$ the indecomposables in Ind $Q$ are chosen up to isomorphism. So, when subtracting $\operatorname{Ind} Q \backslash \{U, V, \ldots\}$, it is assumed that $\{U, V, \ldots\} \subset$ Ind $Q$.

[7] We recall from [1] that two elements $x, y$ of a lattice-interval $[p, q]$ are said to be mutually *complementing* in this interval, if $x \cap y = p$ and $x + y = q$.

In this paper the transition from $\mathscr{P}$ to $\mathscr{P}'_{(a,b)}$ (from $\mathscr{P}$ to $\overline{\mathscr{P}}'_{(a,b)}$) will be called the differentiation *without completion* (*with completion*).

Along with the differentiation, there exists also an evident "inverse" construction, so called *integration* which assigns to any representation $W$ of the completed derivative poset $\overline{\mathscr{P}}'$ the *primitive* representation $W^{\uparrow}$ of the initial poset $\mathscr{P}$ such that $(W^{\uparrow})^{\downarrow} \simeq W$. The main properties of Differentiation I are presented in the following two theorems proved in [19,22] respectively (see also [18, Chapter 9]).

**Theorem 2.2.** *The operations $\downarrow$ and $\uparrow$ induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{P}\backslash\{P(a), P(a, c_1), \dots, P(a, c_n)\} \rightleftarrows \operatorname{Ind}\overline{\mathscr{P}}'_{(a,b)} = \operatorname{Ind}\mathscr{P}'_{(a,b)}\backslash P(a).$$

**Theorem 2.3.** *The functor $'$ : $\mathscr{P}$-sp $\longrightarrow \mathscr{P}'_{(a,b)}$-sp induces an equivalence of the factor categories*

$$\mathscr{P} - sp/\langle P(a), P(a, c_1), \dots, P(a, c_n)\rangle \xrightarrow{\sim} \mathscr{P}'_{(a,b)} - sp/\langle P(a)\rangle,$$

*where by $\langle U, V, \dots\rangle$ is denoted the ideal of a category generated by all morphisms having a factorization through the objects $U, V, \dots$*

*Graphic conventions*. In the following sections we depict diagrams of some equipped posets with involution, in which small, big, simple double, bidouble points are denoted by the symbols $\circ$, $\bullet$, $\otimes$, $\odot$ respectively.

All order relations with a participation of at least one single point, as well as all weak relations between double points, are pictured by a single line. But all strong relations between double points, which are not consequences of some other relations, are pictured by a double line (or by an additional line).

If some group of points is encircled by a contour connected by some (single or double) line with some other points, it means that all points, located inside the contour, have the same order relations with the mentioned other points (determined by the type of the line).

## 3. Differentiation VII

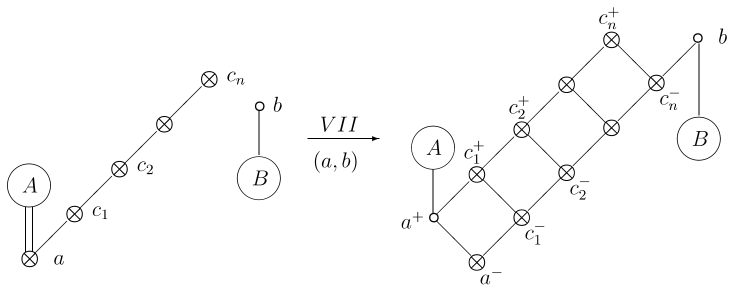
Let $\mathscr{P}$ be an equipped poset. A pair of incomparable points $(a, b)$ of the set $\mathscr{P}$, where $a$ is double and $b$ is single, is called VII-*suitable*, if $\mathscr{P} = a^{\vartriangle} + b_{\vartriangle} + C$, where $C = \{c_1 \prec \cdots \prec c_n\}$ is a completely weak chain (possibly empty) incomparable with the point $b$, and $a \prec c_1$ (note that automatically $a \prec c_n$).

The *derivative poset* of the set $\mathscr{P}$ with respect to such a pair $(a, b)$ is an equipped poset

$$\mathscr{P}'_{(a,b)} = (\mathscr{P}\backslash(a + C)) + \{a^- < a^+\} + C^- + C^+,$$

where $a^-$ is double , $a^+$ is single, $C^- = \{c_1^- \prec \cdots \prec c_n^-\}$ and $C^+ = \{c_1^+ \prec \cdots \prec c_n^+\}$ are completely weak chains, $c_i^- \prec c_i^+$ for all $i$; $a^- \prec c_1^-$; $a^+ < c_1^+$; $c_n^- < b$, and the following conditions are satisfied:

(1) each of the points $a^-, a^+$ ($c_i^-, c_i^+$) inherits all the previous order relations of the point $a(c_i)$ with the points of the subset $\mathscr{P}\backslash(a+C)$;
(2) the order relations in $\mathscr{P}'_{(a,b)}$ are induced by the relations in its subset $\mathscr{P}\backslash(a+C)$ and by the listed above relations (note that, in particular, $a^- \prec c_n^-$).




**Remark 3.1.** Differentiating the evolvent of the set $\mathscr{P}$ first with completion with respect to the pair $(a', b)$ and then without completion with respect to the pair $((a'')^+, b)$, one obtains exactly the evolvent of the derivative poset $\mathscr{P}'_{(a,b)}$ (differentiating the second time with completion too, one obtains the evolvent of the completed derivative poset $\overline{\mathscr{P}}'_{(a,b)}$).

Let $U = (U_0; U_x \mid x \in \mathscr{P})$ be a representation of the set $\mathscr{P}$, where $U_x \subset U_0 (U_x \subset \widetilde{U}_0)$ for a single (double) point $x$. The *derivative representation* $U'$ of the set $\mathscr{P}' = \mathscr{P}'_{(a,b)}$ is defined as follows:

$$
\begin{aligned}
&U_0' = U_0,\\
&U_{a^-}' = U_a \cap \widetilde{U}_b, \quad U_{c_i^-}' = U_{c_i} \cap \widetilde{U}_b,\\
&U_{a^+}' = U_a^+, \quad U_{c_i^+}' = U_{c_i} + \widetilde{U}_a^+,\\
&U_x' = U_x \quad \text{for the remaining points } x \in \mathscr{P}'.
\end{aligned} \tag{3.1}
$$

If $U \xrightarrow{\varphi} V$ is a morphism of the category $\mathscr{P}$-*sp* (considered as a linear map $\varphi : U_0 \longrightarrow V_0$), we set $\varphi' = \varphi$ and obtain the *derivative morphism* of the category $\mathscr{P}'$-*sp* . So, the *differentiation functor* $'$ : $\mathscr{P}$-*sp* $\longrightarrow \mathscr{P}'$-*sp* is correctly defined.

We shall use below indecomposables of the form

$$
D(x) = \begin{array}{c} x \\ \boxed{\begin{array}{c} 1 \\ i \end{array}} \end{array},
$$

where $x$ is double, and

$$H(x,y)=\begin{array}{cc} x & y \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{i} \\ \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \simeq \begin{array}{cc} x & y \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0} \\ \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{1} \\ \hline \end{array},$$

where $x \prec y$. Obviously, $P'(a) = P(a^+)$ and $D'(a) = H'(a, c_i) = P^2(a^+)$. A representation of $\mathscr{P}$, containing no direct summands of the form $P(a)$, $D(a)$ and $H(a, c_i)$, will be called *reduced*.

Let $(E_0, W_0)$ be the complementing pair of subspaces of the space $U_0$ with respect to the pair $(U_a^+, U_b)$ (see in Section 2 the construction of Differentiation I). Then the reduced derivative representation $U^{\downarrow}$, for which $U' = U^{\downarrow} \oplus P^m(a^+)$ with $m = \dim(U_a^+ + U_b)/U_b$, has the form $U^{\downarrow} = W = (W_0; W_x \mid x \in \mathscr{P}')$, where $W_x = U_x^{'} \cap W_0$ ($W_x = U_x^{'} \cap \widetilde{W}_0$) for a single (double) point $x$. Again $U^{\downarrow}$ does not depend,[8] up to isomorphism, on the choice of $E_0$ and $W_0$ and (due to the inclusion $W_{a^+} \subset W_b$) is in fact a representation of the completed (by the relation $a^+ < b$) poset $\overline{\mathscr{P}}'$.

Clearly, $\dim U_0^{\downarrow} \leqslant \dim U_0$, where the equality holds iff $U_a^+ \subset U_b$. Moreover, $(U_1 \oplus U_2)^{\downarrow} \simeq U_1^{\downarrow} \oplus U_2^{\downarrow}$.

Now describe the integration construction. Let $W$ be a representation of the completed derivative poset $\overline{\mathscr{P}}'$. Write each $\mathbb{C}$-space $W_{c_i^+}$ in the form $W_{c_i^+} = \underline{W}_{c_i^+} \oplus F_i \oplus H_i$, where $F_i \subset \widetilde{W}_b$ and $H_i \cap \widetilde{W}_b = 0$, and fix any $\mathbb{C}$-base $f_{i1}, \ldots, f_{im_i}$ of each subspace $F_i$ (recall that the subspace $\underline{U}_X = (\operatorname{rad} U)_X$ was already defined at the beginning of Section 2). Set $W_{a^+} = \underline{W}_{a^+} \oplus G$, where $G = \{g_1, \ldots, g_s\}$ is some complement of $\underline{W}_{a^+}$ in the whole $\mathbb{R}$-space $W_{a^+}$. Choose some new $\mathbb{R}$-space $E_0$ with the base $\{t_1, \ldots, t_s;\ e_{i1}, e_{i1}^{'}, \ldots, e_{im_i}, e_{im_i}^{'} \mid i = 1, \ldots, n\}$ and set $W^{\uparrow} = U = (U_0; U_x \mid x \in \mathscr{P}')$, where

$$\begin{aligned}
U_0 &= W_0 \oplus E_0, \\
\dot{U}_x &= W_x \quad \text{for } x \neq a, c_i, \\
\dot{U}_a &= W_{a^-} + \left\{(g_1, t_1), \ldots, (g_s, t_s)\right\} \\
&\quad + \sum_{i=1}^{n} \left\{(e_{i1}, e_{i1}^{'}), \ldots, (e_{im_i}, e_{im_i}^{'})\right\}, \qquad (3.2) \\
\dot{U}_{c_i} &= U_{c_{i-1}} + W_{c_i^-} + H_i \\
&\quad + \sum_{i=1}^{n} \left\{e_{i1} + f_{i1}, \ldots, e_{im_i} + f_{im_i}\right\} \quad (U_{c_0} = U_a).
\end{aligned}$$

[8] The argumentation for this independence is analogous to the considered above case of ordinary posets (see (2.1)–(2.3) and the text after (2.3)).

It is not difficult to see (using, especially, the placed below matrix interpretation (3.3) of the algorithm) that the representation $W^{\uparrow}$ does not depend, up to isomorphism, on the choise of the complements $F_i$, $H_i$, $G$ and of the corresponding bases. Certainly, $(W_1 \oplus W_2)^{\uparrow} \simeq W_1^{\uparrow} \oplus W_2^{\uparrow}$.

**Remark 3.2.** Obviously, it holds $\dim_{\mathbb{R}} G = \dim_{\mathbb{R}} W_{a^+}/W_{a^-}^{+}$ and $\dim_{\mathbb{C}} F_i = \dim_{\mathbb{C}}(W_{c_i^+} \cap \widetilde{W}_b)/(W_{c_i^-} + W_{c_{i-1}^+} \cap \widetilde{W}_b)$ (where $W_{c_0^+} = \widetilde{W}_{a^+}$).

**Theorem 3.3.** *In the case of Differentiation* VII *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{P} \setminus \{P(a), D(a), H(a, c_i) \mid i \in \overline{1, n}\} \rightleftarrows \operatorname{Ind}\overline{\mathscr{P}}' = \operatorname{Ind}\mathscr{P}' \setminus P(a^+).$$

**Proof.** It is enough to verify the isomorphisms $(U^{\downarrow})^{\uparrow} \simeq U$ and $(W^{\uparrow})^{\downarrow} \simeq W$ for each reduced representation $U$ of $\mathscr{P}$ and each representation $W$ of $\overline{\mathscr{P}}'$. The second isomorphism is rather obvious (take in account the defining formulas (3.1) and (3.2)). But the first one needs a substantiation, which one can give using either the more formal language (subspaces, bases) or the more visual one (matrices). In this article we prefer to use matrices as they allow to clarify better the sense and the logic of the described algorithms.

Consider the matrix $M$ of a reduced representation $U$ of $\mathscr{P}$ and, making some suitable $\mathbb{R}$-elementary transformations of its rows, place at its bottom the horizontal stripe corresponding to the subspace $U_b \subset U_0$ (obviously, everywhere above this stripe in the matrix $M_{B+b}$ we have zeroes).

$$
M = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|l}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{a}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{A}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{c_1}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{c_n}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{B}} & \multicolumn{1}{c}{\boldsymbol{b}} & \\
\hline
 & iE & & & & & & & & & & & & \\
 & & E & & & & & & & & & & & \\
 & & iE & & & & E & & & & & & & E_0 \\
 & & & E & & & & & & & & & & \\
 & & & iE & & & & & & E & & & & \\
\hline
 & & & & * & & & X_1 & & & X_n & & & Q \\
\hline
* & G & & & * & * & F_1 & Y_1 & * & F_n & Y_n & * & * & U_b \\
\hline
\multicolumn{1}{c}{a^-} & \multicolumn{1}{c}{a^+} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{c_1^-} & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{c_1^+}} & \multicolumn{1}{c}{c_n^-} & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{c_n^+}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} &
\end{array}
\quad W_0 = Q + U_b
$$

**Remark.** Here and throughout this paper all empty (all marked by $*$) blocks of matrices are supposed to be zero (to be arbitrary).

Next, we select in the upper part of the matrix $M$ the rows corresponding to the subspace $E_0$ and, using strong relations (additions) $a \lhd A$, turn into zero the block $M_A \cap E_0$ [9] (also, in accordance with the definition of the complementing pair $(E_0, W_0)$, we obtain zero at the crossing of the middle horizontal stripe $Q$ and the vertical stripe $M_a$). Then select in the matrix $Q \cap M_C$, where $C = \{c_1, \ldots, c_n\}$, a maximal number of linearly independent (all together in columns) blocks $X_1, \ldots, X_n$ of the stripe $Q$.

As a result, each matrix $M_{c_i} \cap E_0$ is divided in two vertical stripes: $Z_i$ (situated above $X_i$) and $T_i$ (the rest). Set $T_a = M_a \cap E_0$. Now the collection of matrices $T_a, T_1, \ldots, T_n$, lying in the stripe $E_0$, corresponds to a completely weak chain $a \prec c_1 \prec \cdots \prec c_n$ and can be reduced by some admissible transformations to the form (3.3) with help of the following simple fact (which can be proved easily immediately or obtained as a direct consequence of the results from [10]).

**Lemma 3.4.** *The indecomposables of the form* $[\delta_0]$, $P(x)$, $D(x)$ *and* $H(x, y)$, *where* $x \prec y$, *are all possible indecomposables of an arbitrary weak chain.*

Sketch of the proof of the lemma: taking a matrix representation $N$ of a weak chain $x_1 \prec \cdots \prec x_n$, first reduce the stripe $N_{x_1}$ to the standard form shown below

$$N = \begin{array}{c} \begin{array}{cccc} \quad x_1 & & x_2 & x_n \end{array} \\ \left[\begin{array}{cc|c|c} E & & & \\ & E & & \\ & iE & * & * \\ & & * & * \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} L \\ T \\ S \end{array},$$

then turn into zero (using additions of columns) the block $N_{\{x_2,\ldots,x_n\}} \cap L$, then reduce (using induction step) the block $N_{\{x_2,\ldots,x_n\}} \cap S$ and, at last, reduce the part $N_{\{x_2,\ldots,x_n\}} \cap T$ (using suitable additions of rows $S \xrightarrow{\mathbb{R}} T$ and columns $M_{x_1} \xrightarrow{\mathbb{C}} M_{x_i}$, $i \geqslant 2$) and obtain the desired indecomposables.

Note that the collection of matrices $X_1, \ldots, X_n$ in (3.3) can be reduced to the analogous form (this reduction is not shown in (3.3)). One can verify easily (using additions of columns $M_a \xrightarrow{\mathbb{C}} M_{c_i}$ and $M_{c_i} \xrightarrow{\mathbb{C}} M_{c_j}$ for $i \leqslant j$, and also additions of rows $Q \xrightarrow{\mathbb{R}} E_0$) that all the blocks $Z_i$ may be turned into zero. Hence, the matrix $M$ takes the form (3.3).

It will be observed that the matrices $X_i$ are totally nonsingular in columns due to the construction, and each of the matrices $G$ and $F_i$ is nonsingular in columns

[9] Let us denote by the same letter the subspace $E_0$ and the corresponding horizontal stripe of $M$.

because of the reducibility of $U$. Note also that the subspaces $H_i$ correspond to the block unions $X_i \cup Y_i$.

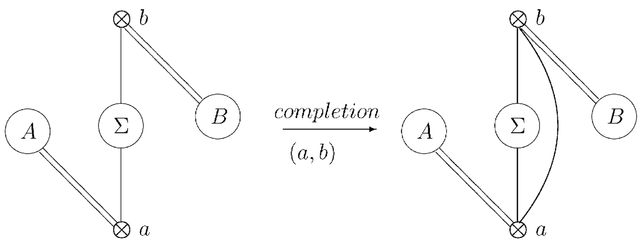
Analyzing the obtained "canonical" form (3.3) of a partially reduced matrix $M$ and comparing it with the formulas (3.1) and (3.2), we see that really the matrix realization of the representation $U^{\downarrow}$ is situated in the stripe $W_0$, and the isomorphism $(U^{\downarrow})^{\uparrow} \simeq U$ takes place. This finishes the proof of the theorem. $\square$

**Remark 3.5.** Seems (analogously to Differentiation I) the equivalence of the factor-categories $\mathscr{P}\text{-}sp\ /\langle P(a), D(a), H(a, c_i) \mid i \in \overline{1, n}\rangle \simeq \mathscr{P}'\text{-}sp\ /\langle P(a^+)\rangle$ takes place. It would be interesting to prove it, like in [22], in the categorical language.

## 4. Completion

A pair of double weakly comparable points $a \prec b$ of an equipped poset $\mathscr{P}$ will be called *special* if $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma$, where $\Sigma$ is the interior of the interval $[a, b]$.

The *completion* of $\mathscr{P}$ with respect to such special pair $(a, b)$ is a transition from $\mathscr{P}$ to a slightly different equipped poset $\overline{\mathscr{P}} = \overline{\mathscr{P}}_{(a,b)}$ obtained from $\mathscr{P}$ by *strengthening* the relation $a \prec b$, i.e. by converting it into a strong one $a \lhd b$.




**Remark.** Obviously, the evolvent of $\overline{\mathscr{P}}$ is obtained from the evolvent of $\mathscr{P}$ by two completions with respect to the ordinary special pairs $(a', b'')$ and $(a'', b')$.

It is clear that $\overline{\mathscr{P}}\text{-}sp$ is a full subcategory of the category $\mathscr{P}\text{-}sp$. Moreover, the following statement takes place.

**Lemma 4.1.** *The category $\overline{\mathscr{P}}\text{-}sp$ coincides with the full subcategory of the category $\mathscr{P}\text{-}sp$ formed by the objects without direct summands of type $D(a)$, hence,* $\operatorname{Ind}\overline{\mathscr{P}} = \operatorname{Ind}\mathscr{P} \backslash D(a)$.

**Proof.** Let $U$ be a representation of $\mathscr{P}$ not belonging to $\overline{\mathscr{P}}\text{-}sp$, i.e. satisfying the condition $U_a^+ \not\subset U_b^-$. Consider the matrix $M$ of the representation $U$:

$$M = \begin{array}{c}
\begin{array}{ccccccc} & \boldsymbol{a} & \boldsymbol{A} & \boldsymbol{\Sigma} & \boldsymbol{B} & & \boldsymbol{b} \end{array} \\
\left[\begin{array}{c|c|c|c|c|c|c}
 & E & & & & & \\
\hline
 & iE & & & & & \\
\hline
 & & * & * & & & E \\
\hline
 & & * & * & & & iE \\
\hline
 & & * & & & & \\
\hline
* & & * & * & * & E &
\end{array}\right]
\end{array}
\begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} D \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} L \\ \\ \left.\begin{array}{c} \\ \end{array}\right\} U_b^- \end{array} \tag{4.1}$$

and select in its lower part the horizontal stripe corresponding to the subspace $U_b^- \subset U_0$. Then reduce those parts of the matrices $M_a$ and $M_b$, which are situated above this stripe (considering them as a representation of a two-point chain $a \prec b$). Taking in account the previous selection of the subspace $U_b^-$, we have in the upper parts of the matrices $M_a$, $M_b$ the summands of type $D(a)$ and $D(b)$ only (see Lemma 3.4) corresponding to the horizontal stripes $D$ and $L$ of the whole matrix $M$. Note that $M_B \cap D = 0$ because $U_B^+ \subset U_b^-$.

Since $U_\Sigma \subset U_b$ and $U_a \subset U_b$, we may assume the columns of $M_\Sigma$ to be linear combinations (over $\mathbb{C}$) of the columns of $M_a$ and $M_b$. Hence, because of the relation $a \prec \Sigma$, the block $M_\Sigma \cap D$ can be turned into zero. Moreover (due to the relation $a \lhd A$), the block $M_A \cap D$ can be turned into zero as well. As a result, the matrix $M$ takes the form (4.1), where the stripe $D$ is a direct summand of $M$ (recall that all empty blocks are supposed to be zero). So, $U = V \oplus D^m(a)$ for some $V$ and some $m \geqslant 1$. $\square$

## 5. Differentiation VIII

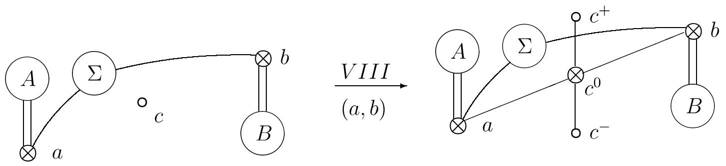
A pair of weakly comparable double points $a \prec b$ of an equipped poset $\mathscr{P}$ will be called VIII-*suitable* if $\mathscr{P} = a^\triangledown + b_\vartriangle + \Sigma + c$, where $\Sigma$ is the interior of the interval $[a, b]$ (possibly empty) and $c$ is a single point incomparable with $[a, b]$.

The *derivative poset* of the set $\mathscr{P}$ with respect to such a pair $(a, b)$ is the equipped poset

$$\mathscr{P}'_{(a,b)} = (\mathscr{P} \backslash c) + \left\{c^- < c^0 < c^+\right\},$$

obtained from $\mathscr{P}$ by replacing the point $c$ by a three-point chain $c^- < c^0 < c^+$, where $c^-$, $c^+$ are single, $c^0$ is double, $a \prec c^0 \prec b$, and the following natural conditions are satisfied:

(1) each of three points $c^-$, $c^0$ and $c^+$ inherits all the previous order relations of the point $c$ with the points of the subset $\mathscr{P} \backslash c$;
(2) the order relations in the whole set $\mathscr{P}'_{(a,b)}$ are induced by the initial relations in the subset $\mathscr{P} \backslash c$ and by the mentioned above relations.




**Remark 5.1.** Differentiating the evolvent of $\mathscr{P}$ two times (without completions) with respect to I-suitable pairs $(a', b'')$ and $(a'', b')$, we obtain exactly the evolvent of the set $\mathscr{P}'_{(a,b)}$.

For any representation $U$ of $\mathscr{P}$ define the *derivative representation* $U'$ of $\mathscr{P}'_{(a,b)}$ by the formulas:

$$
\begin{aligned}
&U'_0 = U_0,\\
&U'_{c^-} = U_c \cap U_b^-, \quad U'_{c^+} = U_c + U_a^+,\\
&U'_{c^0} = U_a + \widetilde{U}_c \cap U_b,\\
&U'_x = U_x \quad \text{for the remaining points } x \in \mathscr{P}'.
\end{aligned}
\tag{5.1}
$$

Setting $\varphi' = \varphi$ for any morphism $\varphi$ of the category $\mathscr{P}$-*sp* (considered as a linear map $\varphi : U_0 \longrightarrow V_0$), we obtain the correctly defined *differentiation functor* $' : \mathscr{P}\text{-}sp \longrightarrow \mathscr{P}'\text{-}sp$.

Consider indecomposables of the form

$$
G_1(a,c) = \begin{array}{c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{c} \\ \cline{2-3}
 & 1 & 0 \\
 & i & 1 \\ \cline{2-3}
\end{array}
\quad \text{and} \quad
G_2(a,c) = \begin{array}{c|c|cc|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{a} & \multicolumn{2}{c}{c} \\ \cline{2-4}
 & 1 & 1 & 0 \\
 & i & 0 & 1 \\ \cline{2-4}
\end{array}\,.
$$

It is easy to see that $D'(a) = G'_1(a,c) = G'_2(a,c) = D(a)$. The objects of the category $\mathscr{P}$-*sp* without direct summands of type $D(a)$, $G_1(a,c)$, $G_2(a,c)$ are called *reduced*.

Denote by $U^{\downarrow}$ such a reduced representation of the set $\mathscr{P}'$, for which $U' = U^{\downarrow} \oplus D^m(a)$, where $2m = \dim(U_a^+ + U_b^-)/U_b^-$, and define it evidently by the same scheme as for Differentiations I and VII. Namely, choosing some complementing pair $(E_0, W_0)$ of subspaces in $U_0$ with respect to the pair $(U_a^+, U_b^-)$, set $U^{\downarrow} = W$, where $W_x = U'_x \cap W_0$ $(W_x = U'_x \cap \widetilde{W}_0)$ for a single (double) point $x$. It is clear that $\dim E_0 = 2m$ and that $U^{\downarrow}$ is a representation of the completed (by the relation $a \lhd b$) derivative poset $\overline{\mathscr{P}}'_{(a,b)}$. Note that $\dim U_0^{\downarrow} = \dim U_0$ if $U_a^+ \subset U_b^-$).

Conversely, let $W$ be a representation of $\overline{\mathscr{P}}'$. In order to find the *primitive representation* $W^{\uparrow}$, satisfying the condition $(W^{\uparrow})' \simeq W \oplus D^m(a)$, present a $\mathbb{C}$-space $W_{c^0}$ in the form $W_{c^0} = (W_a + \widetilde{W}_{c^-}) \oplus F \oplus G$, where $F \subset \widetilde{W}_b^-$ and $G \cap \widetilde{W}_b^- = 0$, and choose some base $(f_1, f'_1), \dots, (f_k, f'_k)$ of the space $F$ (remark that $(W_a + \widetilde{W}_{c^-}) \oplus F = W_{c^0} \cap \widetilde{W}_b^-$).

Further, present a $\mathbb{C}$-space $\widetilde{W}_{c^+}$ in the form $\widetilde{W}_{c^+} = (W_{c^0} + \widetilde{W_a^+}) \oplus T \oplus H$ with $T \subset W_b$ and $H \cap W_b = 0$, and fix some base $(t_1, t_1'), \dots, (t_l, t_l')$ of $T$ (obviously, $(W_{c^0} + \widetilde{W_a^+}) \oplus T = W_b \cap \widetilde{W_c^+}$).

Consider now a new $\mathbb{R}$-space $E_0$ of dimension $2k + 2l$ with the base $e_1, e_1', \dots, e_k, e_k';\ q_1, q_1', \dots, q_l, q_l'$ and set $W^\uparrow = U = (U_0;\ U_K | K \in \Theta)$, where

$$\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_x = W_x \quad \text{for } x \neq a, c,\\
&\dot{U}_a = W_a \oplus \left\{(e_i, e_i')\right\}_{i=1,\dots,k} \oplus \left\{(q_j, q_j')\right\}_{j=1,\dots,l},\\
&\dot{U}_c = W_{c^-} \oplus G^+ \oplus H^+ \oplus \left\{t_j, t_j' + q_j'\right\}_{j=1,\dots,l} \oplus \left\{e_i + f_i, e_i' + f_i'\right\}_{i=1,\dots,k}.
\end{aligned} \tag{5.2}$$

**Remark 5.2**

(a) Possibly, some $t_j = 0$.

(b) Obviously, due to the construction $k = \dim_{\mathbb{C}}(W_{c^0} \cap \widetilde{W}_b^-)/(W_a + \widetilde{W}_{c^-})$, $l = \dim_{\mathbb{C}}(\widetilde{W}_{c^+} \cap W_b)/(W_{c^0} + \widetilde{W}_a^+)$ and $\dim E_0 = 2k + 2l$.

The discussion above yields:

**Theorem 5.3.** *In the case of Differentiation* VIII *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\operatorname{Ind} \mathscr{P} \backslash \{D(a), G_1(a, c), G_2(a, c)\} \rightleftarrows \operatorname{Ind} \overline{\mathscr{P}}' = \operatorname{Ind} \mathscr{P}' \backslash D(a).$$

**Proof.** As in the previous situations, we use the matrix approach in the substantiation of the algorithm. Consider the matrix $M$ of a representation $U$ of $\mathscr{P}$ and select in its lower part the horizontal stripe corresponding to the subspace $U_b^- \subset U_0$. Then reduce those parts of the matrices $M_a$, $M_b$ and $M_c$, which are located just above this stripe, considering them as a realization of some representation $V$ of a representation finite equipped poset $H_5 = \{a \prec b; c\}$.

The representations of $H_5$ are well known (see, for instance, [10,26]). Taking in account the selection of $U_b^-$, note that in accordance with [26] $V$ is decomposed into the direct sum $V = \bigoplus_i V_i$, where each $V_i$ is either a nonsincere representation of $H_5$ of type $[\delta_0]$, $D(a)$, $D(b)$, $P(c)$, $G_1(a, c)$, $G_1(b, c)$, $G_2(a, c)$, $G_2(b, c)$, or the sincere one of the form

$$\Omega = \begin{array}{c|c|c|c} \boldsymbol{a} & \boldsymbol{b} & & \boldsymbol{c} \\ \hline 1 & & & \\ i & & & 1 \\ & 1 & 1 & \\ & i & & 1 \end{array}.$$

Inserting the corresponding blocks (each with its multiplicity), present $M$ in the form (let us throw off the trivial direct summands $D(a)$):

$$M = \begin{array}{r|cccc|c|c|cccccccccc|c|ccccc|l}
 & & & \mathbf{a} & & \mathbf{A} & \mathbf{\Sigma} & & & & & & \mathbf{c} & & & & & \mathbf{B} & & & \mathbf{b} & & & \\
\hline
E_0 & & E & & & & & & E & & & & & & & & & & & & & & & \}\, G_2(a,c) \\
 & & iE & & & & & & & E \\
 & & & E & & & & & & & & & & & & & & & & & & & & \}\, G_1(a,c) \\
 & & & iE & & & & & & & & & E \\
 & & & & E & & & & & & & & & & & & & & & & & & & \}\, \Omega \\
 & & & & iE & & & & & & & & & & E \\
\hline
 & & & & & * & * & & & & E & & & & & & & & & E & & & & \}\, G_2(b,c) \\
 & & & & & * & * & & & & & E & & & & & & & & iE \\
 & & & & & * & * & & & & & & & E & & & & & & & E & & & \}\, \Omega \\
 & & & & & * & * & & & & & & & & E & & & & & & iE \\
 & & & & & * & * & & & & & & & & & & & & & & & E & & \}\, D(b) \\
 & & & & & * & * & & & & & & & & & & & & & & & iE \\
 & & & & & * & * & & & & & & & & & & & & & & & & E & \}\, G_1(b,c) \\
 & & & & & * & * & & & & & & & & & E & & & & & & & iE \\
 & & & & & * & & & & & & & & & & & E & & & & & & & \}\, P(c) \\
 & & & & & * & & & & & & & & & & & & & & & & & & \}\, [\delta_0] \\
\hline
 & * & & & & * & * & * & F & F & G & G & T & T & T & H & H & * & E & & & & & \}\, U_b^- \\
\hline
 & a & & & & & & c^- & & & c^0 & & & & c^+
\end{array} \qquad (5.3)$$

where the letters $F$, $G$, $T$, $H$ denote, in general, some nonzero blocks belonging to the corresponding vertical stripes (as for zeroes in the upper part of the stripe $M_\Sigma$, see the previous considerations concerning the operation of strengthening of a weak relation ). Now, analogously to Differentiations I and VII, using the standard (although rather bulky) matrix procedure, we select all those admissible transformations which do not change the "removed part" corresponding to the stripe $E_0$. This way leads to the justification of the definition of the derivative poset $\mathscr{P}'_{(a,b)}$ and of the relations (5.1) and (5.2). Moreover, it allows to establish immediately the isomorphisms $(U^{\downarrow})^{\uparrow} \simeq U$ and $(W^{\uparrow})^{\downarrow} \simeq W$. $\square$

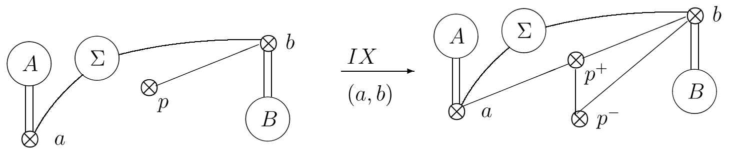
**Remark 5.4.** One can reduce the matrix $M$ to the form (5.3) in a different (more "general") way, not considering the representations of the set $H_5$. Namely, first one can choose in $M$ the stripe $U_b^-$, then the stripe $E_0$, then reduce the parts of the matrices $M_c$ and $M_b$, lying in the middle horizontal stripe (as a representation of the dyad $\{c, b\}$), and finally to obtain exactly the form (5.3) using some additions of rows of the middle stripe to the rows of $E_0$. But such way needs a more detailed description of the realized steps.

## 6. Differentiation IX

A pair of weakly comparable double points $a \prec b$ of an equipped poset $\mathscr{P}$ is called IX-*suitable* if $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma + p$, where $\Sigma$ is the interior of the interval $[a, b]$ and $p$ is a double point, incomparable with $a$, with the relation $p \prec b$.

The *derivative poset* of the set $\mathscr{P}$ with respect to the pair $(a, b)$ is an equipped poset $\mathscr{P}' = \mathscr{P}'_{(a,b)}$ obtained from $\mathscr{P}$ by replacing the point $p$ by a weak two-point chain $p^- \prec p^+$ with the additional relations $a \prec p^+ \prec b$ and $p^- \lhd b$ (plus all the induced relations).

**Remark.** Naturally, it is supposed that each of two points $p^-$, $p^+$ inherits all the previous order relations of the point $p$ with the points in $\mathscr{P} \setminus p$.




(note that in tame situation necessarily $p \prec \Sigma$ and $p^+ \prec \Sigma$).

Obviously, in the derivative poset $\mathscr{P}'_{(a,b)}$ the pair $(a, b)$ is special. So, one can always complete $\mathscr{P}'_{(a,b)}$ obtaining the *completed derivative* poset $\overline{\mathscr{P}}'_{(a,b)}$.

**Remark 6.1.** Differentiating the evolvent of $\mathscr{P}$ two times (without completions) with respect to I-suitable pairs $(a', b'')$ and $(a'', b')$, we obtain the evolvent of $\mathscr{P}'$.

The *differentiation functor* $'$ : $\mathscr{P}$*-sp* $\longrightarrow$ $\mathscr{P}'$*-sp* assigns to a representation $U$ of $\mathscr{P}$ the *derivative representation* $U'$ as follows:

$$\begin{aligned} &U'_0 = U_0, \\ &U'_{p^+} = U_a + U_p, \\ &U'_{p^-} = U_p \cap \widetilde{U}_b^-, \\ &U'_x = U_x \quad \text{for the remaining } x \in \mathscr{P}'. \end{aligned} \tag{6.1}$$

The action of the functor on a morphism $U \xrightarrow{\varphi} V$(considered as a linear map $\varphi : U_0 \longrightarrow V_0$) is defined in a standard way: $\varphi' = \varphi$.

Denote

$$D(x, y) = \begin{array}{cc} x & y \\ \hline \multicolumn{1}{|c|}{\begin{matrix}1\\ i\end{matrix}} & \multicolumn{1}{c|}{\begin{matrix}1\\ i\end{matrix}} \\ \hline \end{array}$$

for a double dyad $\{x, y\}$.

One may verify easily that $D'(a) = D'(a, p) = D(a)$. Representations of $\mathscr{P}$ without direct summands $D(a)$ and $D(a, p)$ will be called *reduced*.

A reduced representation $U^\downarrow$, for which $U' = U^\downarrow \oplus D^m(a)$ is defined evidently, analogously to the previous cases. Take some complementing pair of subspaces $(E_0, W_0)$ in $U_0$ with respect to the pair $(U_a^+, U_b^-)$ and set $U^\downarrow = W$, where $W_x = U'_x \cap W_0$ ($W_x = U'_x \cap \widetilde{W}_0$) for a single (double) point $x \in \mathscr{P}'$.

The representation $U^{\downarrow}$ does not depend, up to isomorphism, on the choice of $E_0$ and $W_0$ and, due to the inclusion $W_a^+ \subset W_b^-$, is a representation of the set $\overline{\mathscr{P}}'_{(a,b)}$ completed by the relation $a \lhd b$.

Clearly, $2m = \dim E_0 = \dim(U_a^+ + U_b^-)/U_b^-$, hence, $m > 0$ if $U_a^+ \not\subset U_b^-$.

For a description of the "inverse" construction, *integration*, consider a representation $W$ of the completed derivative poset $\overline{\mathscr{P}}'$ and present a **C**-space $W_{p^+}$ in the form $W_{p^+} = (W_a + W_{p^-}) \oplus F \oplus H$, where $F \subset \widetilde{W}_b^-$ and $H \cap \widetilde{W}_b^- = 0$. Let $f_1, \ldots, f_m$ be a base of $F$. Taking a new $\mathbb{R}$-space $E_0$ of dimension $2m$ with the base $e_1, e_1', \ldots, e_m, e_m'$, set $W^{\uparrow} = U$, where

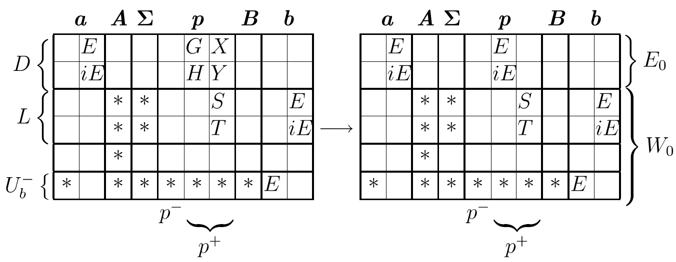
$$\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_x = W_x \quad \text{for } x \neq a, p,\\
&\dot{U}_a = W_a + \big\{(e_1, e_1'), \ldots, (e_m, e_m')\big\},\\
&\dot{U}_p = W_a + W_{p^-} + H + \big\{(f_1 + e_1, e_1'), \ldots, (f_m + e_m, e_m')\big\}.
\end{aligned} \tag{6.2}$$

The *primitive* representation $W^{\uparrow}$ does not depend (up to isomorphism) on the choise of the complements $H$, $F$ and of the base in $F$. Obviously, $(W_1 \oplus W_2)^{\uparrow} \simeq W_1^{\uparrow} \oplus W_2^{\uparrow}$.

**Theorem 6.2.** *In the case of Differentiation* IX *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\operatorname{Ind} \mathscr{P} \backslash \{D(a), D(a,p)\} \rightleftarrows \operatorname{Ind} \overline{\mathscr{P}}' = \operatorname{Ind} \mathscr{P}' \backslash D(a).$$

**Proof.** The isomorphisms $(W^{\uparrow})^{\downarrow} \simeq W$ and $(U^{\downarrow})^{\uparrow} \simeq U$ are verified with help of the matrix considerations, which are very close to the given in the proof of Lemma 4.1. Namely, consider the matrix $M$ of a representation $U$ of $\mathscr{P}$ and select, in the same way as in (4.1), the horizontal stripes $U_b^-$, $D$, $L$. Obviously, $U$ can be assumed not containing direct summands of type $\underset{\smile}{D(a)}$. Selecting now the columns of the matrix $M_p$ corresponding to the meet $U_p \cap U_b^-$ (and attached to the space $W_{p^-}$), and also some maximal collection of $\mathbb{C}$ -linearly independent columns of the matrix block $M_p \cap L$, we may reduce $M$ to the form (a) shown as follows:



(a) (b)

(6.3)

where the block

$$\boxed{\begin{matrix} S \\ T \end{matrix}}$$

is nonsingular in columns due to the construction. Since $U_a + U_p + U_\Sigma \subset U_b$, we may suppose all the columns of $M_\Sigma$ and $M_p$ to be $\mathbb{C}$-linear combinations of the columns from $M_a$ and $M_b$. Consequently, the columns of the blocks

$$\boxed{\begin{matrix} X \\ Y \end{matrix}} \quad \text{and} \quad \boxed{\begin{matrix} S \\ T \end{matrix}}$$

are linear combinations of the columns of the block

$$\boxed{\begin{matrix} E \\ iE \end{matrix}} \quad \text{(each of its own).}$$

Therefore, using additions of rows of the stripe $L$ to the rows of the stripe $D$, we may turn into zero the matrices $X$, $Y$ and to reduce the block

$$\boxed{\begin{matrix} G \\ H \end{matrix}}$$

to the form

$$\boxed{\begin{matrix} E \\ iE \end{matrix}}.$$

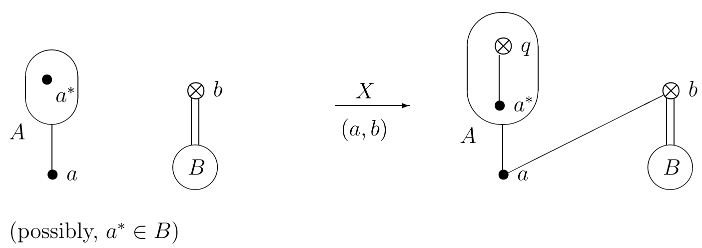
As a result, the matrix $M$ takes the form (b) which clarifies completely the isomorphisms $(U^{\downarrow})^{\uparrow} \simeq U$ and $(W^{\uparrow})^{\downarrow} \simeq W$. $\square$

## 7. Differentiation X

Let $\mathscr{P}$ be an equipped poset with a primitive involution $*$ and $\Theta$ be the set of all equivalence classes of its points (with respect to the involution).

A pair of incomparable points $(a, b)$ in $\mathscr{P}$, where $a$ is big and $b$ is double, is called X-*suitable* if $\mathscr{P} = a^{\triangledown} + b_{\vartriangle}$. The *derivative poset* of the set $\mathscr{P}$ with respect to the pair $(a, b)$ is an equipped poset with primitive involution $\mathscr{P}' = \mathscr{P}'_{(a,b)}$ obtained from $\mathscr{P}$ in the following way:

(a) the point $a^*$ is replaced by a chain $a^* < q$, where $q$ is a double point (inheriting all the order relations of the point $a^*$);
(b) the relation $a < b$ is added.



We may restrict ourselves by considering only those representations $U$ of $\mathscr{P}$, which satisfy the restriction $U_a^+ \subset U_b^+$ (in accordance with [23] for a big point $a$ set $U_a^+ = \{x \mid (x, y) \in U_{(a,a^*)}\}$). Otherwise one may add a new small point $\xi > b$ with the space $U_\xi = U_b^+$, then apply Differentiation $\overline{\alpha}$ with respect to the pair $(a, \xi)$ [23] (decreasing $\dim U_0$) and then delete $\xi$ (note that if not to introduce the relation $a^+ \subset b^+$, it would appear in $\mathscr{P}'$ one more new small point $t > q$).

The action of the *differentiation functor* $' : \mathscr{P}$*-sp* $\longrightarrow \mathscr{P}'$*-sp* is defined by the formulas

$$\begin{aligned}
&U'_0 = U_0,\\
&U'_{(a,a^*)} = U_{(a,a^*)} \cap (U_b^-, U_0),\\
&U'_q = \left\{(x', y') \mid (x, x'), (y, y') \in U_{(a,a^*)} \text{ for some } (x, y) \in U_b\right\},\\
&U'_K = U_K \quad \text{for the remaining classes } K \in \Theta'.
\end{aligned} \tag{7.1}$$

for a representation $U$ of $\mathscr{P}$ with the restriction $U_a^+ \subset U_b^+$, and also by the equality $\varphi' = \varphi$ for a morphism-map $\varphi : U_0 \longrightarrow V_0$.

Let $(E_0, W_0)$ be the complementing pair in $U_0$ with respect to the pair $(U_a^+, U_b^-)$. The *reduced* derivative representation $U^\downarrow$ is defined (uniquely up to isomorphism) by the equality $U' = U^\downarrow \oplus P^{2m}(a)$, where $2m = \dim E_0 = \dim(U_a^+ + U_b^-)/U_b^-$. Its evident form is $U^\downarrow = W$ with $W_0$ taken from the complementing pair and $W_K = U'_K \cap \{W_0^K\}$.

Denote

$$G_2(b, a) = \begin{array}{c|cc|cc|}
b & a & & a^* & \\
\hline
1 & 1 & 0 & 0 & 0\\
i & 0 & 1 & 0 & 0\\
\hline
\end{array}\,.$$

Obviously, $P'(a, b) = P^\downarrow(a, b) = P(a)$, but $G'_2(b, a) = P^2(a)$ and $G_2^\downarrow(b, a) = 0$.

If $W$ is a representation of $\mathscr{P}'$, then writing the $\mathbb{C}$-space $W_q$ in the form $W_q = \underline{W}_q \oplus F$, where $F$ is some complement with a base $(f_1, f'_1), \ldots, (f_m, f'_m)$, and considering a new $\mathbb{R}$-space $E_0$ of dimension $2m$ with a base $e_1, e'_1, \ldots, e_m, e'_m$, set $W^\uparrow = U$, where

$$\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_K = W_K \quad \text{for } K \neq \{a, a^*\}, \{b\},\\
&\dot{U}_{(a,a^*)} = W_{(a,a^*)} + \left\{(e_1, f_1), (e'_1, f'_1), \ldots, (e_m, f_m), (e'_m, f'_m)\right\},\\
&\dot{U}_b = W_b + \left\{(e_1, e'_1), \ldots, (e_m, e'_m)\right\}.
\end{aligned} \tag{7.2}$$

Here the verification of the isomorphisms $(U^\downarrow)^\uparrow \simeq U$ and $(W^\uparrow)^\downarrow \simeq W$ is very easy, when using the placed below matrix form (7.3) of the algorithm. That is why it is left to the reader as an exercise.

$$
\begin{array}{r|ccc|ccc|c|c|ccc|l}
 & & \boldsymbol{a} & & & \boldsymbol{a^*} & & \boldsymbol{A} & \boldsymbol{B} & & \boldsymbol{b} & & \\
\hline
 & & E & & & & & & & & E & & \multirow{2}{*}{$\Big\}\, E_0$} \\
 & & & E & & & & & & & iE & & \\
\hline
 & & & & * & * & * & * & & & & H & \multirow{2}{*}{$\Big\}\, W_0$} \\
U_b^- \{ & * & & & * & * & * & * & * & E & & & \\
\hline
 & a & & & a^* & \multicolumn{2}{c}{\underbrace{\qquad}_{q}} & & & & & &
\end{array}
\qquad (7.3)
$$

(here the matrix $H$ is nonsingular in columns).

The considerations above lead to the following:

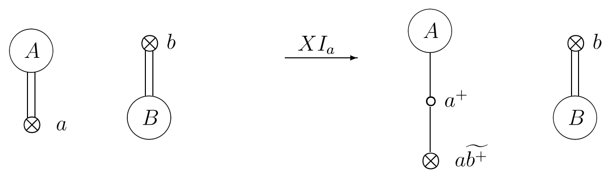
**Theorem 7.1.** *In the case of Differentiation* X *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\{\operatorname{Ind}\mathscr{P} \mid a^+ \subset b^+\}\backslash G_2(b,a) \rightleftarrows \operatorname{Ind}\mathscr{P}'_{(a,b)}.$$

## 8. Differentiation XI

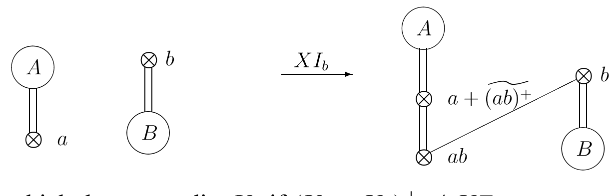
A pair of double incomparable points $(a, b)$ of an equipped poset with primitive involution $\mathscr{P}$ will be called XI-*suitable* if $\mathscr{P} = a^\nabla + b_\triangle$. First of all consider some preliminary steps simplifying the algorithm. Set $A = a^\nabla \backslash a$, $B = b_\triangle \backslash b$. Let $U$ be a representation of $\mathscr{P}$.

(a) Applying Differentiation VII with respect to the pair $(a, \xi)$, where $\xi > b$ is a new added small point with $U_\xi = U_b^+$ (deleted after Differentiation), we obtain the algorithm $\mathrm{XI_a}$ of the form




which decreases $\dim U_0$ if $U_a^+ \not\subset U_b^+$.

(b) Applying Differentiation IX* (i.e. the dual one to IX) with respect to the pair $(ab, b)$, where $ab \prec \{a, b\}$ is a new added double point with $U_{ab} = U_a \cap U_b$ (here $p = a$), we obtain the algorithm $\mathrm{XI_b}$ of the form




which decreases $\dim U_0$ if $(U_a \cap U_b)^+ \not\subset U_b^-$.

(b*) The dual Differentiation $\mathrm{XI}_{\mathrm{b}}^{*}$ decreases $\dim U_0$ in the case $U_a^+ \not\subset (U_a + U_b)^-$.

(c) Now take the matrix $M$ of $U$ and choose the horizontal stripe corresponding to the subspace $U_b^-$

$$\begin{array}{c}
\begin{array}{cccccc} \quad \boldsymbol{a} & & \boldsymbol{A} & \boldsymbol{B} & \quad \boldsymbol{b} & \end{array} \\
\begin{array}{|cc|c|c|cc|}
\hline
X & & * & & & Y \\
\hline
* & * & * & * & E & \\
\hline
\end{array}
\begin{array}{l} \}\, L \\ \}\, U_b^- \end{array}
\end{array},$$

where the block $X$ is supposed to be nonsingular in columns over $\mathbb{C}$. Consider a pair of matrices $N = (X, Y)$ as a representation of the critical set $M_1 = \{a, b\}$ and decompose it into a direct sum of indecomposables $N = \bigoplus_j N_j$, where $N_j = (X_j, Y_j)$. Taking into account (a), (b), (b*) and the selection of $U_b^-$, note that each summand $N_j$ must satisfy the relations $a^+ \subset b^+$; $(ab)^+ \subset b^-$; $a^+ \subset (a+b)^-$ and $b^- = 0$. Moreover, it is easily to see that each summand $N_j$ with the conditions $a^- = 0$ and $a^+ = 1$ (with the sense $1 = U_0$) is a direct summand of the whole $U$.

Therefore, it is enough in the main version $\mathrm{XI}_{\mathrm{c}}$ of Differentiation XI to restrict the considerations by those summands $N_j$ which satisfy the following conditions:

$$\begin{array}{lll}
(1)\ a^+ \subset b^+, & (2)\ b^- = 0, & (3)\ ab = 0, \\
(4)\ a^+ \subset (a+b)^-, & (5)\ a^- \neq 0 \text{ or } a^+ \neq 1. &
\end{array} \tag{8.1}$$

It follows immediately from the matrix classification of indecomposables of the critical set $M_1 = \{a, b\}$ (see Appendix A–C), obtained in [26] with use of the results in [6], that the indecomposables of type $M_1 - 4$ only satisfy all the conditions (8.1).

Now put at the place of the summands $N_j = (X_j, Y_j)$ the corresponding blocks of representations $M_1 - 4$ taken in the form

$$X_j = \boxed{\begin{array}{c} iE \\ J(0) \end{array}}, \quad Y_j = \boxed{\begin{array}{c} E \\ iE \end{array}}$$

(this form is equivalent to the form shown in Appendix A–C). Then group together all the blocks of equal dimension and, using additions of rows $L \xrightarrow{\mathbb{R}} U_b^-$, make the maximal possible number of zero vertical stripes in the matrix block $X \cap U_b^-$. Deleting all imaginary parts of this block, we receive the following form of the matrix $M$ (the blocks $X_j, Y_j$ with numbers of rows $\leqslant 6$ only are shown):

$$
\begin{array}{c|ccccccc|c|c|ccccccc|l}
 & & & & \boldsymbol{a} & & & & \boldsymbol{A} & \boldsymbol{B} & & & & \boldsymbol{b} & & & & \\
\hline
- & iE & & & & & & & & & & E & & & & & & \rule{0pt}{0pt} \\
 & & & & & & & & * & & & iE & & & & & & \}\ N_1 \\
- & & iE & & & & & & & & & & E & & & & & \\
- & & & iE & & & & & & & & & & E & & & & \\
- & & & E & & & & & & & & & iE & & & & & \}\ N_2 \\
 & & & & & & & & * & & & & & iE & & & & \\
- & & & & iE & & & & & & & & & & E & & & \\
- & & & & & iE & & & & & & & & & & E & & \\
- & & & & & & iE & & & & & & & & & & E & \}\ N_3 \\
- & & & & & E & & & & & & & & & iE & & & \\
- & & & & & & E & & & & & & & & & iE & & \\
 & & & & & & & & * & & & & & & & & iE & \\
 & & & & & & & & * & & & & & & & & & \\
\hline
 & \mathbb{R} & \mathbb{R} & & \mathbb{R} & & & * & * & * & E & & & & & & & \}\ U_b^- \\
\hline
 & a_1 & a_2 & - & a_3 & - & - & q & & & \xi & b_1 & - & b_2 & - & - & b_3 & \\
 & \longleftarrow & & & \longleftarrow & & & \longleftarrow & & & \longrightarrow & \longrightarrow & & \longrightarrow & & & &
\end{array}
\tag{8.2}
$$

where the blocks, marked by $\mathbb{R}$, are real.

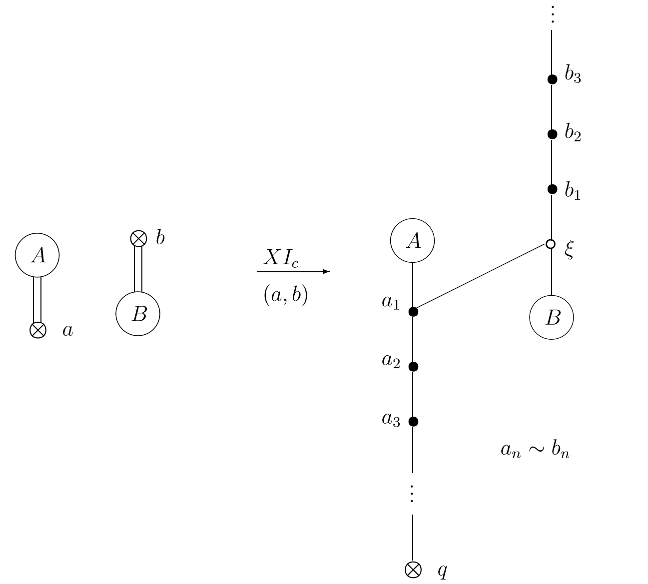
It is allowed to make arbitrary admissible transformations of this matrix not changing the "deleted parts" situated at crossing of all but the last rows of each horizontal stripe $N_j$ and all but the first (the last) columns of each vertical stripe $X_j(Y_j)$ marked in (8.2) by the sign "−". Analyzing the appeared problem, we obtain the main version $\mathrm{XI_c}$ (compare with Differentiation III in [23]).

The *derivative poset* (of type (c)) of the set $\mathscr{P}$ with respect to XI-suitable pair $(a, b)$ is an equipped poset with primitive involution $\mathscr{P}' = \mathscr{P}'_{(a,b)}$, obtained from $\mathscr{P}$ as follows:

(1) the point $a$ is replaced by the infinite decreasing chain $a_1 > a_2 > \cdots > a_n > \cdots > q$, and the point $b$ is replaced by the infinite increasing chain $\xi < b_1 < b_2 < \cdots < b_n < \cdots$, where $a_n \sim b_n$ $(n \geqslant 1)$ are big, $\xi$ is small and $q$ is double; [10]

(2) the relations $a_1 < A,\ a_1 < \xi$ and $B < \xi$ (and the induced ones) are added.

[10] One may get rid of the point $\xi(q)$ using the relation $B^+ = b^-$ (the relations $a^- \subset b^-$ and $a^- = (a_\triangle \backslash a)^+$).




Denote by $\mathscr{R}$ the full subcategory of the category $\mathscr{P}$-*sp* formed by all those representations, the corresponding blocks $N_j = (X_j, Y_j)$ of which (considered as representations of $M_1 = \{a, b\}$) satisfy the restrictions (8.1). Let $\mathscr{R}'$ be the full subcategory of the category $\mathscr{P}'$-*sp* formed by all representations $W$, which satisfy the relations $a_n^- = a_{n+1}^+$ for all $n \geqslant 1$ (i.e. in the matrix realization $T$ of $W$ with the condition $\underline{\dim}\, W = \underline{\dim}\, T$ all stripes $T_{b_n}$ are nonsingular in columns).

Let $U$ be a representation of $\mathscr{P}$. The *coupling of degree n* of $\mathbb{R}$ -subspaces $U_a, U_b \subset \widetilde{U}_0$ ($n \geqslant 1$) is an $\mathbb{R}$- subspace $[U_a - U_b]^{[n]}$ in $\widetilde{U}_0$ of the form

$$[U_a - U_b]^{[n]} = \left\{(t_0, t_{2n}) \mid (t_{2k}, t_{2k+1}) \in U_a;\ (t_{2k+1}, t_{2k+2}) \in U_b;\ k \in \overline{0, n-1}\right\}$$

(in particular, $[U_a - U_b] = \{(t_0, t_2) \mid (t_0, t_1) \in U_a;\ \ (t_1, t_2) \in U_b\}$, $[U_a - U_b]^{[2]} = \{(t_0, t_4) \mid (t_0, t_1), (t_2, t_3) \in U_a; (t_1, t_2), (t_3, t_4) \in U_b\}$).

The *differentiation functor* $'\colon \mathscr{R} \longrightarrow \mathscr{R}'$ is given by the formulas

$$\begin{aligned} &U_0' = U_0,\\ &U_K' = U_K \quad \text{for } K \neq \{a_n, b_n\}, \xi, q,\\ &U'_{(a_n,b_n)} = [U_a - U_b]^{[n]} \cap (U_b^-, U_0) + (0, U_a^+),\\ &U_\xi' = U_b^-, \quad U_q' = U_a \cap \widetilde{U}_b^-. \end{aligned} \tag{8.3}$$

for a representation $U$ and by the equality $\varphi' = \varphi$ for a morphism $\varphi : U_0 \longrightarrow V_0$. Note that $((M_1 - 4)_{2n})' = P(b_n) \oplus P^{2n-1}(A, b_1)$, where the lower index $2n$ denotes the number of rows of the representation $M_1 - 4$.

For $U \in \operatorname{Ob} \mathscr{R}$ denote by $U^{\downarrow}$ such a direct summand of the object $U'$, for which $U' = U^{\downarrow} \oplus P^m(A, b_1)$, where $m = \dim(U_a^+ + U_b^-)/U_b^-$. Taking, as usually, the complementing pair of subspaces $(E_0, W_0)$ in $U_0$ with respect to the pair $(U_a^+, U_b^-)$, we have $U^{\downarrow} = W$, where $W_K = U'_K \cap \{W_0^K\}$.

Clearly, $\dim W_0 < \dim U_0$ if $U_a^+ \not\subset U_b^-$.

The *primitive* representation $W^{\uparrow}$, for which $(W^{\uparrow})' \simeq W \oplus P^m(A, b_1)$, is restored for $W \in \mathrm{Ob}\,\mathscr{R}'$ as follows. Set $W_{(a_n,b_n)} = \underline{W}_{(a_n,b_n)} \oplus F_n$ for some complement $F_n$ of dimension $p_n$ with a base $(f_{01}^n, f_{n1}^n), \dots, (f_{0p_n}^n, f_{np_n}^n)$. For each $n \geqslant 1$ consider a new $\mathbb{R}$-space $E_n$ of dimension $(2n-1)p_n$ with the base $\{e_{ij}^n\} \cup \{f_{kj}^n\}$, where $i \in \overline{1, n}$, $k \in \overline{1, n-1}$, $j \in \overline{1, p_n}$, and for $E_0 = \bigoplus_n E_n$ set $W^{\uparrow} = U$, where

$$
\begin{aligned}
&U_0 = W_0 E_0,\\
&\dot{U}_K = W_K \quad \text{for } K \neq a, b,\\
&\dot{U}_a = W_q + \bigoplus_n \bigoplus_{i=1}^n \{(f_{i-1,1}^n, e_{i1}^n), \dots, (f_{i-1,p_n}^n, e_{ip_n}^n)\},\\
&\dot{U}_b = \widetilde{W}_\xi + \bigoplus_n \bigoplus_{i=1}^n \{(e_{i1}^n, f_{i1}^n), \dots, (e_{ip_n}^n, f_{ip_n}^n)\}.
\end{aligned}
\tag{8.4}
$$

Comparing the constructions $\downarrow$ and $\uparrow$ (and using the matrix interpretation (8.2)), we obtain the following fact.

**Theorem 8.1.** *In the case of Differentiation* $\mathrm{XI_c}$ *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\mathrm{Ind}\,\mathscr{R} \rightleftarrows \mathrm{Ind}\,\mathscr{R}'.$$

In the following sections sometimes instead of “Differentiation I”, “Differentiation II”, ... we write briefly “D-I”, “D-II”, ...

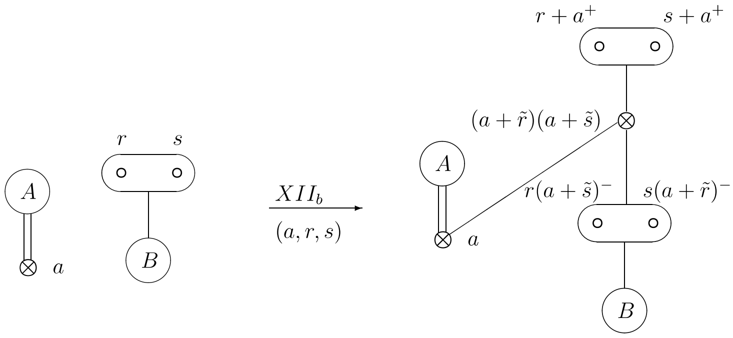
## 9. Differentiation XII

A triad $(a, r, s)$ of an equipped poset with primitive involution $\mathscr{P}$, where $a$ is double and $r, s$ are small, is called XII-*suitable* if $\mathscr{P} = a^{\triangledown} + \{r, s\}_{\vartriangle}$. Set $A = a^{\triangledown} \backslash a$, $B = \{r, s\}_{\vartriangle} \backslash \{r, s\}$. As in the case of Differentiation XI, first consider some preliminary reductions simplifying the algorithm. Let $U$ be a representation of $\mathscr{P}$.

(a) One can use the simplest variant of D-VII with respect to the pair $(a, \xi)$, where $\xi$ is a new added small point, $\{r, s\} < \xi$ and $U_\xi = U_r + U_s$ (with the subsequent deleting of $\xi$). It decreases $\dim U_0$ if $U_a^+ \not\subset U_r + U_s$ or, shortly, $a^+ \not\subset r + s$ (see the convention on notations in Section 2).
(b) Making the following sequence of transformations in the corresponding lattice of subspaces:
   - adding the sum $a + \tilde{r}$ with the relations $a \prec a + \tilde{r}$ and $r \lhd a + \tilde{r}$ and the condition $U_{a+\tilde{r}} = U_a + \widetilde{U}_r$,

- applying D-VIII with respect to the pair $(a, a + \tilde r)$ with the subsequent deleting of the point-sum $a + \tilde r$,
- applying D-VIII with respect to the pair $(a, (a + \tilde r)(a + \tilde s))$,

we obtain the algorithm $\mathrm{XII_b}$ in the form

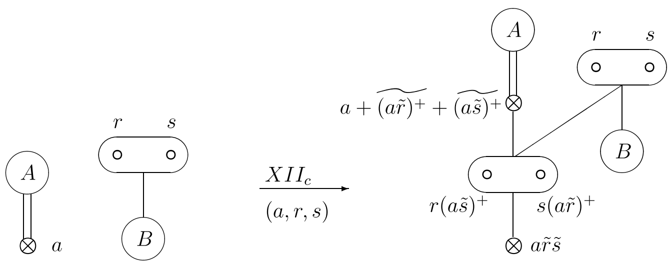



which decreases $\dim U_0$ in the case $U_a^+ \not\subset (U_a + \widetilde U_r)^-(U_a + \widetilde U_s)^-$ or, shortly, $a^+ \not\subset (a + \tilde r)^-(a + \tilde s)^-$, i.e. $\widetilde{a^+} \not\subset (a + \tilde r)(a + \tilde s)$.

(c) Making another sequence of transformations, namely:

- adding the intersection $a\tilde r$, where $a\tilde r \prec a$ and $a\tilde r \lhd r$, with the condition $U_{a\tilde r} = U_a \cap \widetilde U_r$,
- applying D-VII with respect to the pair $(a\tilde r, s)$ and deleting the appeared point-intersection $a\tilde r\tilde s$,
- completing with respect to a special pair $((a\tilde r)^+, s)$ by replacing the point $(a\tilde r)^+$ by $s(a\tilde r)^+$ (surely, $s(a\tilde r)^+ < s$),
- applying D-VII with respect to the pair $(a\tilde s, r)$, deleting the point-intersection $s(a\tilde r)^+ = s(\widetilde{(a\tilde r)^+})^-$ and completing with respect to a special pair $((a\tilde s)^+, r)$ by means of replacing the point $(a\tilde s)^+$ by the point $r(a\tilde s)^+ < r$,
- applying D-VII with respect to the pair $(\widetilde{(a\tilde r)^+}, s)$, completing with respect to a special pair $((a\tilde r)^+, s)$ by means of replacing the point $(a\tilde r)^+$ by the point $s(a\tilde r)^+ < s$, and deleting two new point-intersections $\tilde s(a + \widetilde{(a\tilde r)^+} + \widetilde{(a\tilde s)^+})$ and $\tilde s\widetilde{(a\tilde r)^+}$,

we obtain as a result the following algorithm$\mathrm{XII_c}$




It decreases $\dim U_0$ if $(U_a \cap \widetilde U_r)^+ \not\subset U_s$.

(d) Now take the matrix $M$ of $U$ and select in its lower part the horizontal stripe corresponding to the subspace $U_r \cap U_s$

| $\boldsymbol{a}$ | $\boldsymbol{A}$ | $\boldsymbol{B}$ | $\boldsymbol{r}$ | | $\boldsymbol{s}$ | | |
|---|---|---|---|---|---|---|---|
| $X$ | $*$ | | | $Y$ | | $Z$ | |
| $*$ | $*$ | $*$ | $E$ | | $E$ | | $\}\ U_r \cap U_s$ |

Consider the triple $(X, Y, Z)$ as a representation of the critical set $L_1 = \{a, r, s\}$ and decompose it into a direct sum of indecomposables. Obviously, each summand $V$, satisfying the conditions $V_a^- = 0$ and $V_a^+ = V_0$, is a direct summand of the whole $U$. Hence, taking in account the situations (a)–(c), we may restrict ourselves by only those summands which satisfy the following conditions:

$$\begin{array}{lll} (1)\ rs = 0, & (2)\ a^+ \subset r + s, & (3)\ a \neq 0 \text{ or } a^+ \neq 1, \\ (4)\ \widetilde{a^+} \subset (a + \tilde{r})(a + \tilde{s}), & (5)\ a\tilde{r} \subset \tilde{s} \text{ and } a\tilde{s} \subset \tilde{r}. & \end{array} \tag{9.1}$$

It is easy to verify (see Appendix A–C, where the matrix classification of the critical set $L_1 = \{a, r, s\}$, obtained in [26], is presented) that all this restrictions are valid for the representations of type $L_1 - 9$ and also for three trivial: $[\delta_0]$, $[\delta_0 + \delta_r]$ and $[\delta_0 + \delta_s]$.

Writing at the place of the summands $K_j = (X_j, Y_j, Z_j)$ of type $L_1 - 9$ the corresponding blocks in the form

| $\boldsymbol{a}$ | $\boldsymbol{r}$ | $\boldsymbol{s}$ |
|---|---|---|
| $iE$ | $E$ | $0$ |
| $J(0)$ | $E$ | $E$ |

(see Appendix A–C) and also taking into account the blocks corresponding to trivial summands $[\delta_0 + \delta_r]$ and $[\delta_0 + \delta_s]$, we obtain the following matrix (in which the blocks of type $L_1 - 9$ with at most four rows are shown):

| | $\boldsymbol{a}$ | | | | $\boldsymbol{A}$ | $\boldsymbol{B}$ | $\boldsymbol{r}$ | | | | | $\boldsymbol{s}$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| – | $iE$ | | | | | | | $E$ | | | | | | | | | $\}\ N_1$ |
| | | | | | $*$ | | | $E$ | | | | | $E$ | | | | |
| – | | $iE$ | | | | | | | $E$ | | | | | | | | $\}\ N_2$ |
| – | | | $iE$ | | | | | | | $E$ | | | | | | | |
| – | | | $E$ | | | | | | $E$ | | | | | $E$ | | | |
| | | | | | $*$ | | | | | $E$ | | | | | $E$ | | |
| | | | | | $*$ | | | | | | $E$ | | | | | | |
| | | | | | $*$ | | | | | | | | | | | $E$ | |
| | | | | | $*$ | | | | | | | | | | | | |
| | $\mathbb{R}$ | $\mathbb{R}$ | | $*$ | $*$ | $*$ | $E$ | | | | | $E$ | | | | | $\}\ U_r \cap U_s$ |
| | $a_1$ | $a_2$ | $-$ | $q$ | | | $\xi$ | $b_1$ | - | $b_2$ | $r^+$ | $\xi$ | $b_1$ | $-$ | $b_2$ | $s^+$ | |
| | $\longleftarrow \cdots$ | | $\longleftarrow$ | | | | $\longrightarrow$ | | $\cdots$ | | $\longrightarrow$ | $\longrightarrow$ | | $\cdots$ | | $\longrightarrow$ | |

(9.2)

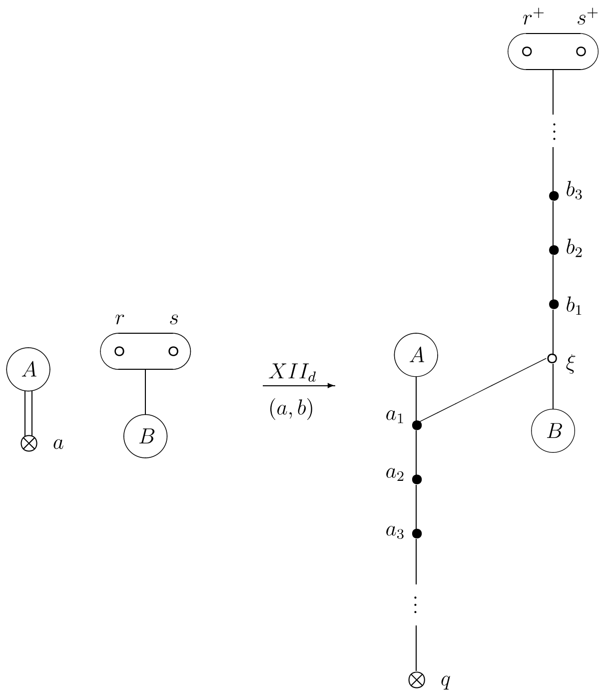
with the same notations as in the case of D-XI.

Remark that modulo “deleted rows” the columns, corresponding to the points $\xi$ and $b_n$ of the vertical stripes $r$ and $s$, must be equal and, consequently, are identified.

Examining the admissible transformations which do not change the “deleted part” (the rows and columns of which are marked by the dash “$-$”) we come to the following constructions.

The *derivative poset* of the set $\mathscr{P}$ with respect to XII-suitable pair of points $(a, b)$ is an equipped poset with primitive involution $\mathscr{P}' = \mathscr{P}'_{(a,b)}$ obtained from $\mathscr{P}$ in the following way:

(1) the point $a$ is replaced by the infinite decreasing chain $a_1 > a_2 > \cdots > a_n > \cdots > q$, and the dyad $\{r, s\}$ is replaced by the infinite increasing semichain $\xi < b_1 < b_2 < \cdots < b_n < \cdots < \{r^+, s^+\}$, where the points $a_n \sim b_n$ are big ($n \geqslant 1$), $q$ is double , $\xi$ is small and $\{r^+, s^+\}$ is a small dyad;
(2) the relation $a_1 < \xi$ is added (plus all the induced relations)




Let $\mathscr{R}$ be the full subcategory of the category $\mathscr{P}$-*sp* formed by all those objects, for which the corresponding representation $(X, Y, Z)$ of the set $L_1 = \{a, r, s\}$ is decomposable into a direct sum of representations of type $L_1 - 9$ and $[\delta_0]$, $[\delta_0 + \delta_r]$, $[\delta_0 + \delta_s]$ only. Let $\mathscr{R}'$ be the full subcategory in $\mathscr{P}'$-*sp* formed by all those representations $W$, which satisfy the relations $a_n^- = a_{n+1}^+$ for $n \geqslant 1$ and $rs = \sum_n b_n^+$.

For a representation $U$ of $\mathscr{P}$ the *coupling of degree* $n$ $(n \geqslant 1)$ of $\mathbb{R}$-subspaces $U_a$ and $(U_r, U_s)$ in $\widetilde{U}_0$ is an $\mathbb{R}$-subspace $[U_a - (U_r, U_s)]^{[n]}$ in $\widetilde{U}_0$ of the form $[U_a - (U_r, U_s)]^{[n]} = \{(t_0, t_{2n}) \mid (t_{2k}, t_{2k+1}) \in U_a;\ (t_{2k+1} + t_{2k+2}, t_{2k+2}) \in (U_r, U_s)\}$ where $k = 0, 1, \ldots, n-1$ (in particular, $[U_a - (U_r, U_s)] = \{(t_0, t_2) \mid (t_0, t_1) \in U_a;\ (t_1 + t_2, t_2) \in (U_r, U_s)\}$, $[U_a - (U_r, U_s)]^{[2]} = \{(t_0, t_4) \mid (t_0, t_1), (t_2, t_3) \in U_a; (t_1 + t_2, t_2), (t_3 + t_4, t_4) \in (U_r, U_s)\}$).

The *differentiation functor* $'$ : $\mathscr{R} \longrightarrow \mathscr{R}'$ is determined by the conditions

$$\begin{aligned}
&U_0' = U_0,\\
&U_K' = U_K \quad \text{for } K \neq \{a_n, b_n\}, \xi, q, r^+, s^+,\\
&U(a_n, b_n)' = [U_a - (U_r, U_s)]^{[n]} \cap (U_r \cap U_s, U_0) + (0, U_a^+),\\
&U_\xi' = U_r \cap U_s, \quad U_q' = U_a \cap (\widetilde{U_r \cap U_s}),\\
&U_{r^+}' = U_r + U_a^+, \quad U_{s^+}' = U_s + U_a^+
\end{aligned} \tag{9.3}$$

for a representation $U$ and by the equality $\varphi' = \varphi$ for a morphism $\varphi : U_0 \longrightarrow V_0$.

Note that $((L_1 - 9)_{2n})' = P(b_n) \oplus P^{2n-1}(A, b_1)$, where $2n$ is the number of rows of the representation $L_1 - 9$.

For $U \in \operatorname{Ob}\mathscr{R}$ denote by $U^\downarrow$ such an object of the category $\mathscr{R}'$, for which $U' = U^\downarrow \oplus P^m(A, b_1)$ with $m = \dim(U_a^+ + U_r \cap U_s)/U_r \cap U_s$. Taking the complementing pair $(E_0, W_0)$ in $U_0$ with respect to the pair $(U_a^+, U_r \cap U_s)$, we have $U^\downarrow = W$, where $W_K = U_K' \cap \{W_0^K\}$ for all $K$.

Here $\dim W_0 < \dim U_0$ if $U_a^+ \not\subset U_r \cap U_s$.

The definition of the *primitive* representation $W^\uparrow$, satisfying the condition $(W^\uparrow)' \simeq W \oplus P^m(A, b_1)$ for $W \in \operatorname{Ob}\mathscr{R}'$, is analogous to the case of D-XI. Namely, setting $W_{(a_n, b_n)} = \underline{W}_{(a_n, b_n)} \oplus F_n$ for a complement $F_n$ with the base $(f_{01}^n, f_{n1}^n), \ldots, (f_{0p_n}^n, f_{np_n}^n)$, and also $W_{r^+} = \underline{W}_{r^+} \oplus R$ and $W_{s^+} = \underline{W}_{s^+} \oplus S$ for some complements $R$, $S$, consider for each $n \geqslant 1$ a new $\mathbb{R}$-space $E_n$ of dimension $(2n-1)p_n$ with the base $\{e_{ij}^n\} \cup \{f_{kj}^n\}$, where $i \in \overline{1, n},\ k \in \overline{1, n-1},\ j \in \overline{1, p_n}$ and for $E_0 = \bigoplus_n E_n$ set $W^\uparrow = U$, where

$$\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_K = W_K \quad \text{for } K \neq a, r, s,\\
&\dot{U}_a = W_q + \bigoplus_n \bigoplus_{i=1}^n \{(f_{i-1,1}^n, e_{i1}^n), \ldots, (f_{i-1,p_n}^n, e_{ip_n}^n)\},\\
&\dot{U}_r = W_\xi \oplus R + \bigoplus_n \bigoplus_{i=1}^n \{e_{i1}^n + f_{i1}^n, \ldots, e_{ip_n}^n + f_{ip_n}^n)\},\\
&\dot{U}_s = W_\xi \oplus S + \bigoplus_n \bigoplus_{i=1}^n \{f_{i1}^n, \ldots, f_{ip_n}^n\}.
\end{aligned} \tag{9.4}$$

The discussion above yields:

**Theorem 9.1.** *In the case of Differentiation* $\mathrm{XII_d}$ *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

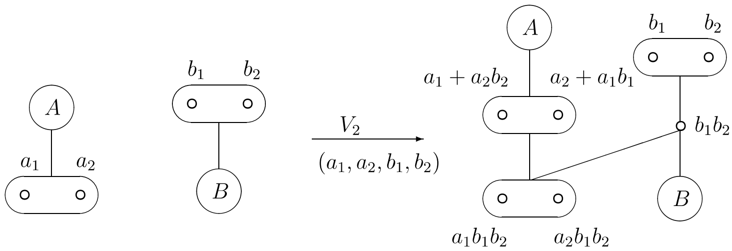
$$\mathrm{Ind}\,\mathscr{R} \rightleftarrows \mathrm{Ind}\,\mathscr{R}'.$$

*Remark on Differentiation* $\mathrm{V}_2$. It appears, the described in [23] Differentiation $\mathrm{V}_2$ can be reduced (analogously to the situations $XII_{b,c}$) to a composition of two Differentiations I.

Namely, let $(a_1, a_2, b_1, b_2)$ be a V-suitable small tetrad of points of an ordinary poset $\mathscr{P}$, i.e. $\mathscr{P} = \{a_1, a_2\}^{\triangledown} + \{b_1, b_2\}_{\vartriangle}$. Set $A = \{a_1, a_2\}^{\triangledown} \backslash \{a_1, a_2\}$, $B = \{b_1, b_2\}_{\vartriangle} \backslash \{b_1, b_2\}$. Then (all considered points are small):

- adding the point-intersections $a_1b_1$ and $b_1b_2$ with the natural order relations,
- applying D-I with respect to the pair $(a_1b_1, b_2)$ with the subsequent deleting of the point-intersection $a_1b_1$,
- applying D-I with respect to the pair $(a_2b_2, b_1)$ with the subsequent adding natural order relations $a_1b_1 < a_2 + a_1b_1$ and $a_2b_2 < a_1 + a_2b_2$,
- completing with respect to arising special pairs $(a_1b_1, b_2)$ and $(a_2b_2, b_1)$ by means of replacing the points $a_1b_1$ and $a_2b_2$ by the points $a_1b_1b_2$ and $a_2b_2b_1$ respectively and adding natural relations $a_ib_1b_2 < b_1b_2$ $(i = 1, 2)$,

we obtain exactly the algorithm $\mathrm{V}_2$ built in [23] (where there was an additional relation $b_1b_2 = B^+$ permitting to omit the point $b_1b_2$):




It decreases $\dim U_0$ if $a_1b_1 \not\subset b_2$ or $a_2b_2 \not\subset b_1$.

## 10. Differentiation XIII

First consider three simple preliminary lemmas.

**Lemma 10.1.** *If a representation V of the critical set* $M_1 = \{a, b\}$ *satisfies the condition* $(V_a + V_b)^- = 0$, *then it is a direct sum of trivial indecomposables of four types*: $[\delta_0]$, $D(a)$, $D(b)$ *and* $D(a, b)$.

**Proof.** From the one hand the statement is a direct consequence of the classification of indecomposables of $M_1$ (see Appendix A–C) due to which the only possible indecomposables of $M_1$ are those of type $1, 3, \tilde{3}, 5$ of minimal possible dimension. From the other hand, it can be proved easily directly. □

**Lemma 10.2.** *If in the matrix problem* (a), *where* $\xi$ *is double*,

$$\begin{array}{rl} & \begin{array}{c} \mathbb{C} \\ \xrightarrow{\qquad} \\ \xi \quad \psi \end{array} \\ \mathbb{R}\uparrow & \begin{array}{l} \left\{ \begin{array}{|c|c|} \hline E & X \\ \hline iE & Y \\ \hline \end{array} \right. \\ \ \ \begin{array}{|c|c|} \hline 0 & \Omega \\ \hline \end{array} \end{array} \\ & \quad \text{(a)} \end{array}$$

*the matrix* $\Omega$ *is* $\mathbb{C}$*-nonsingular in columns*, *then it is possible (using the shown* $\mathbb{C}$*-additions of columns from the left to the right and* $\mathbb{R}$*-additions of rows from the bottom stripe to the top stripes*) *to convert the matrices* $X$, $Y$ *into arbitrary matrices* $X'$, $Y'$ *over* $\mathbb{C}$ *of the same size* (*in particular*, *into zero matrices*).

**Proof.** The rows of $\Omega$ may be added to the rows of $X$, $Y$ with coefficients in $\mathbb{C}$ (as, for instance, adding the row $j$ of $\Omega$ to the row $k$ of $X$ and annihilating then all added elements by admissible $\mathbb{C}$-additions of columns, we add automatically the row $j$ of $\Omega$ to the row $k$ of $Y$ with the coefficient $-i$). □

For any complex matrix $N$ denote by $N^0$ some maximal collection of its linearly independent columns (which can be chosen not uniquely).

**Lemma 10.3.** *Assume that in the matrix problem* (b), *where* $\xi$ *is double*,

$$\begin{array}{c} \mathbb{C} \\ \xrightarrow{\qquad} \\ \xi \quad \Sigma \\ \begin{array}{|c|c|} \hline E & S \\ \hline iE & T \\ \hline 0 & N \\ \hline \end{array} \\ \text{(b)} \end{array}$$

*the placed over* $N^0$ *columns of the matrices* $S$, $T$ *are zero*, *and*, *moreover*, *none of possible* $\mathbb{C}$*-linear combinations of columns of the whole matrix is equal to a nonzero real vector*. *Then* (*using the shown admissible* $\mathbb{C}$*-additions of columns from the left to the right*) *it is possible to annihilate also the rest of the columns of the matrices* $S$, $T$ (*converting them*, *consequently*, *into completely zero matrices*).

**Proof.** Any column of the stripe $\Sigma$ is some $\mathbb{C}$-linear combination of the added (from the left) columns and of those columns from $\Sigma$ which prolong the columns of $N^0$

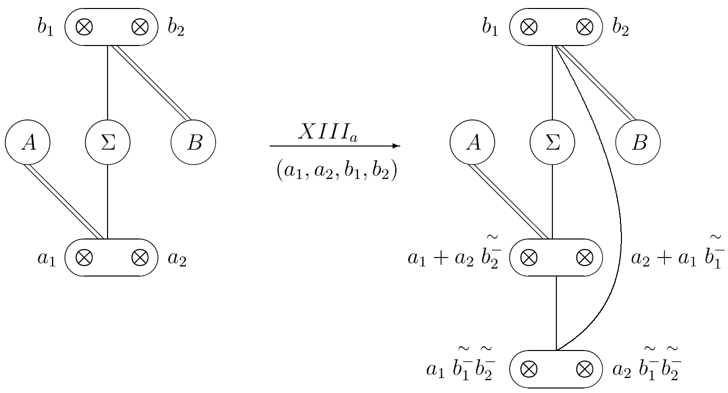
(otherwise a suitable $\mathbb{C}$-linear combination of columns of the whole matrix gives a nonzero real vector, contradiction). $\square$

Let $\mathscr{P}$ be an equipped poset. A pair of double weakly comparable dyads $\{a_1, a_2\} \prec \{b_1, b_2\}$ in $\mathscr{P}$ will be called XIII-*suitable* if $\mathscr{P} = \{a_1, a_2\}^{\nabla} + \{b_1, b_2\}_{\vartriangle} + \Sigma$, where $\Sigma$ is the interior of the convex envelope $[a_1, a_2, b_1, b_2]$. Denote $A = \{a_1, a_2\}^{\nabla} \backslash \{a_1, a_2\}$, $B = \{b_1, b_2\}_{\vartriangle} \backslash \{b_1, b_2\}$. We start considering some special variants of the differentiation simplifying the algorithm. Let $U$ be a representation of $\mathscr{P}$.

(a) Using the following sequence of transformations in the lattice of subspaces:

- adding a double point-intersection $a_1\widetilde{b_1^-}$ with the relations $a_1\widetilde{b_1^-} \prec a_1$ and $a_1\widetilde{b_1^-} \vartriangleleft b_1$,
- applying D-IX with respect to the pair $(a_1\widetilde{b_1^-}, b_2)$ and deleting after that the point-intersection $a_1\widetilde{b_1^-}$,
- applying D-IX with respect to the pair $(a_2\widetilde{b_2^-}, b_1)$ (after which there appears again the point $(a_1\widetilde{b_1^-})$,
- adding natural relation-inclusions $a_i\widetilde{b_i^-} \prec a_j + a_i\widetilde{b_i^-}$ $(i \neq j)$ and then strengthening the special relations $a_i\widetilde{b_i^-} \prec b_j$ by means of replacing each point $a_i\widetilde{b_i^-}$ by the point $a_i\widetilde{b_i^-}\widetilde{b_j^-} \vartriangleleft b_j$,

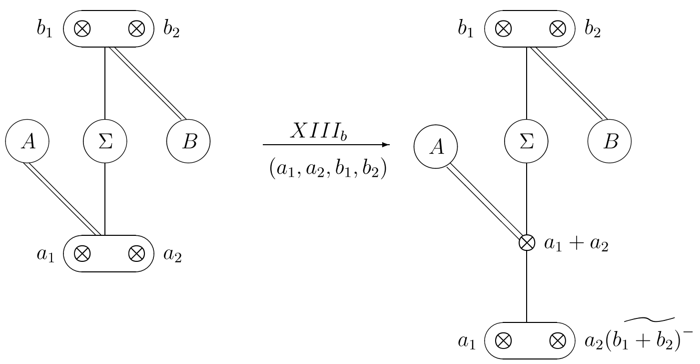
we obtain the algorithm $\mathrm{XIII_a}$ of the form




Due to the construction, it transforms an equipped poset into equipped one and decreases $\dim U_0$ if $a_i\widetilde{b_i^-} \not\subset \widetilde{b_j^-}$ at least for one pair $i \neq j$.

Since one can permute the points $a_1, a_2$, the algorithm may be applied for decreasing $\dim U_0$ in the case $a_k\widetilde{b_i^-} \not\subset \widetilde{b_j^-}$ for some triple $i, j, k$ with the condition $i \neq j$ (the dual algorithm $\mathrm{XIII_a^*}$ really works if $\widetilde{a_j^+} \not\subset \widetilde{a_i^+} + b_k$, where $i \neq j$).

(b) Adding to $\mathscr{P}$ a double point-sum $b_1 + b_2$ with the relations $\{b_1, b_2\} \prec b_1 + b_2$ and applying D-IX with respect to the pair $(a_1, b_1 + b_2)$ (together with the subsequent deleting the added point-sum $b_1 + b_2$), we obtain the algorithm $\mathrm{XIII_b}$ of the form




It is used in the case $a_i^+ \not\subset (b_1 + b_2)^-$ at least for one $i$ (the dual one, $\mathrm{XIII_b^*}$, is used in the case $(a_1a_2)^+ \not\subset b_i^-$); this condition guarantees decreasing $\dim U_0$.

(c) Assume now that $U$ satisfies the relations $a_k\widetilde{b_i^-} \subset \widetilde{b_j^-}$ for all $i, j, k$ with $i \neq j$ (otherwise one can apply D-$\mathrm{XIII}_a$ decreasing $\dim U_0$) and the relation $(a_1 + a_2)^+ \subset (b_1 + b_2)^-$ (otherwise D-$\mathrm{XIII}_b$ is applied). It is also convenient to accept the relation denoted by $\gamma(b_1, b_2)$ which means $U_{b_1} \cap U_{b_2} = U_\Sigma + \widetilde{U_B^+}$ (otherwise one can extend $\Sigma$ adding to it a new maximal double point $\eta$ with $\Sigma \prec \eta \prec \{b_1, b_2\}$ and $U_\eta = U_{b_1} \cap U_{b_2}$).

Denote $\Delta = U^-_{\{b_1,b_2\}} = U^-_{b_1} \cap U^-_{b_2}$ and $\nabla = [(U_{a_1} + \widetilde{\Delta}) \cap (U_{a_2} + \widetilde{\Delta})]^+$ (note that $\Delta \subset \nabla$).

Consider the matrix $M$ of $U$ (of minimal dimension $\underline{\dim}\, M = \underline{\dim}\, U$) and select in its lower part the horizontal stripe $\Delta$, corresponding to the subspace $\Delta \subset U_0$. The pair of matrices $M_{a_1}\backslash\Delta$ and $M_{a_2}\backslash\Delta$ can be considered as a realization of some representation $V$ of the critical subset $M_1 = \{a_1, a_2\}$, and, due to Lemma 10.1, $V$ is a direct sum of indecomposables of the types $[\delta_0]$, $D(a_1)$, $D(a_2)$ and $D(a_1, a_2)$. Hence, the pair of matrices $M_{a_1}\backslash\Delta$ and $M_{a_2}\backslash\Delta$ can be reduced to the form

$$\begin{array}{|c|c|c|c|c|c|l}
\multicolumn{3}{c}{\boldsymbol{a_1}} & \multicolumn{3}{c}{\boldsymbol{a_2}} & \\ \hline
 & E & & & E & & \multirow{2}{*}{$\left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\} E_0$} \\ \cline{1-6}
 & iE & & & iE & & \\ \hline
 & & L_1 & & & L_2 & Q \\ \hline
\end{array} \tag{10.1}$$

with the union-matrix $L_1 \cup L_2$ being a direct sum of the indecomposables $D(a_1)$, $D(a_2)$ and $[\delta_0]$ (note that $L_1 \cup L_2$ is $\mathbb{C}$ -linearly nonsingular in columns).

Denote by $E_0$ and $Q$ the horizontal stripes of the whole matrix $M$ corresponding to just in the same way denoted stripes in (10.1) (see the matrix (10.2) below). It is clear that the stripes $E_0$ and $W_0 = Q \cup \Delta$ of $M$ correspond to some complementing pair of subspaces $(E_0, W_0)$ in $U_0$ with respect to the pair $(\nabla, \Delta)$ (the stripe $\nabla$ is just $E_0 \cup \Delta$).

Obviously, $M_B \cap (E_0 \cup Q) = 0$ and also one may accept $M_A \cap E_0 = 0$ (due to the relation $\{a_1, a_2\} \lhd A$) and $M_{\{b_1,b_2\}} \cap \Delta = 0$ (due to the inclusion $\widetilde{\Delta} \subset U_\Sigma + \widetilde{U_B^+}$ being a consequence of the relation $\gamma(b_1, b_2)$).

Using the admissible transformations, turn into zero the part of the stripe $\Delta$ placed below the block

$$\boxed{\begin{array}{c} E \\ iE \end{array}}$$

of the matrix $M_{a_1} \cap E_0$. Moreover, using the additions of columns $M_{a_1} \xrightarrow{\mathbb{C}} M_{\{b_1,b_2\}}$, turn into zero the lower semistripe of the horizontal stripe $M_{\{b_1,b_2\}} \cap E_0$.

Set $N = M_\Sigma \cap Q$. Recall that $N^0$ denotes some maximal number of $\mathbb{C}$-linearly independent columns in $N$.

Consider the matrices $Z_1 = M_{b_1} \cap Q$ and $Z_2 = M_{b_2} \cap Q$ as a realization of some representation $S$ of the critical set $\{b_1, b_2\}$ and select inside them the equal vertical stripes $G$ and $G$ corresponding to some $\mathbb{C}$-base of the intersection $S_{b_1} \cap S_{b_2}$, i.e. present them in the form

$$Z_1 = \boxed{\begin{array}{c|c|c} 0 & G & H_1 \end{array}} \quad \text{and} \quad Z_2 = \boxed{\begin{array}{c|c|c} 0 & G & H_2 \end{array}}.$$

Since the matrix $\Omega = L_1 \cup L_2 \cup N^0 \cup G \cup H_1 \cup H_2$ (with $G$ belonging to $Z_2$) is $\mathbb{C}$-nonsingular in columns, we can turn into zero, using Lemma 10.2, the part of the stripe $E_0$ placed above $\Omega$. Then, applying Lemma 10.3, turn into zero the rest of the elements of the block $M_\Sigma \cap E_0$. As a result, $M$ takes the form

$$\begin{array}{ccccccccccccccccl}
 & \boldsymbol{a_1} & & & \boldsymbol{a_2} & & \boldsymbol{A} & \boldsymbol{\Sigma} & \boldsymbol{B} & & \boldsymbol{b_1} & & & \boldsymbol{b_2} & & & \\
\hline
\multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{E} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{E} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{X_1} & \multicolumn{1}{c|}{Y} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{X_2} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \\
\multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{iE} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{iE} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \left.\right\} E_0 \\
\hline
\multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{L_1} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{L_2} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{N} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{G} & \multicolumn{1}{c|}{H_1} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{G} & \multicolumn{1}{c|}{H_2} & Q \\
\hline
\multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \Delta \; \left.\right\} W_0 \\
\hline
\end{array} \tag{10.2}$$

Here, in fact, one can apply arbitrary $\mathbb{C}$-elementary transformations of rows of the matrices $X_1, X_2, Y$, due to the possible permutation of two horizontal semistripes in $E_0$ with the subsequent additions of columns $M_{a_1} \xrightarrow{\mathbb{C}} M_{\{b_1,b_2\}}$ and the obvious restoring the form of the matrix $M_{\{a_1,a_2\}} \cap E_0$. Moreover, the columns of $X_1$ and, in fact, of $X_2$ can be added over $\mathbb{C}$ to the columns of $Y$. For $X_2$ it is made indirectly by additions of its columns to the columns of the right neighboring vertical stripe with the subsequent restoring of the damaged zeroes by row additions $Q \xrightarrow{\mathbb{R}} E_0$ and of the other damaged parts by applying Lemmas 10.2 and 10.3.

Therefore, the triple $T = (X_1, X_2, Y)$ is nothing else as an ordinary matrix representation in the sense of [14] over $\mathbb{C}$ of the trivial poset $\{x_1 < y > x_2\}$ having indecomposables in one-dimensional space only. In addition, the matrix $Y$ must be nonsingular in columns (otherwise the relation $\gamma(b_1, b_2)$ isn't satisfied) and the union of matrices $X_1 \cup Y \cup X_2$ must be nonsingular in rows (otherwise the relation $(a_1 + a_2)^+ \subset (b_1 + b_2)^-$ is not true).

And since $T$ contains none of the direct summands $[\delta_0 + \delta_{x_1} + \delta_{x_2}]$ (leading to extension of $\Delta$) and none of the summands $[\delta_0 + \delta_{x_i}]$ (destroying the relations $\widetilde{a_k b_i^-} \subset \widetilde{b_j^-}$ with $i \neq j$), it is decomposed into a direct sum of indecomposables $[\delta_0 + \delta_y]$ only. Hence, one can take $Y = E$ and write the placed above $G$ part of the matrix $M_{b_1} \cap E_0$ in the form

$$\boxed{\begin{matrix} E \\ 0 \end{matrix}}\,.$$

But for a convenience of the future considerations we convert it into the form

$$\boxed{\begin{matrix} iE \\ E \end{matrix}}\,,$$

obtained with help of the column additions $M_{a_1} \xrightarrow{\mathbb{C}} M_{b_1}$. So, $M$ is reduced to the form

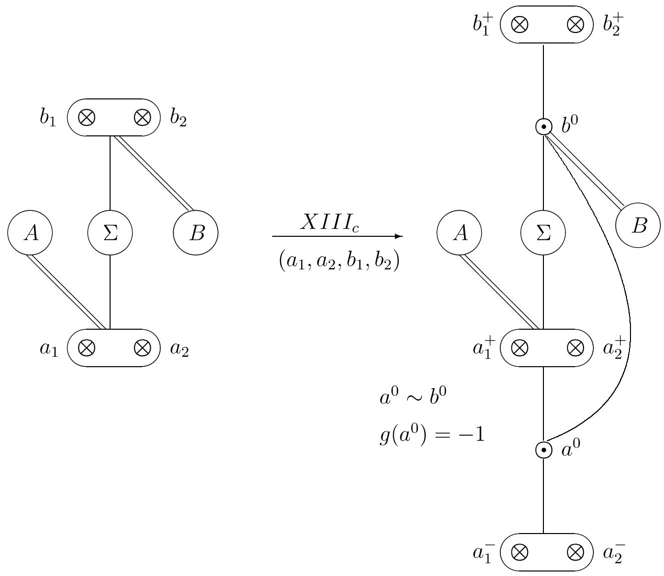
$$M = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|cl}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{a_1}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{a_2}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{A}} & \multicolumn{1}{c}{\boldsymbol{\Sigma}} & \multicolumn{1}{c}{\boldsymbol{B}} & \multicolumn{1}{c}{\boldsymbol{b_1}} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{\boldsymbol{b_2}} & & \\
\hline
 & E & & & E & & & & & iE & & & & & \multirow{2}{*}{$\}\ E_0$} \\
\hline
 & iE & & & iE & & & & & E & & & & & \\
\hline
 & & L_1 & & & L_2 & * & N & & G & H_1 & G & H_2 & Q & \multirow{2}{*}{$\}\ W_0$} \\
\hline
* & & * & * & * & * & * & * & * & & & & & \Delta & \\
\hline
\multicolumn{1}{c}{\otimes} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\otimes} & \multicolumn{1}{c}{\otimes} & \multicolumn{1}{c}{\odot} & \multicolumn{1}{c}{\otimes} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\odot} & \multicolumn{1}{c}{\otimes} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\otimes} & & \\
\multicolumn{1}{c}{a_1^-} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a_1^+} & \multicolumn{1}{c}{a_2^-} & \multicolumn{1}{c}{a^0} & \multicolumn{1}{c}{a_2^+} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{b^0} & \multicolumn{1}{c}{b_1^+} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{b_2^+} & &
\end{array} \tag{10.3}$$

A complementing pair $(E_0, W_0)$, corresponding to the form (10.3), will be called *admissible* (certainly, this pair is not defined uniquely).

Investigating the arising in the stripe $W_0$ matrix problem and determining all those admissible transformations, which do not change the stripe $E_0$ (and save the equality of the blocks $G$), we come to the next definitions.

The *derivative poset of type* (c) of the set $\mathscr{P}$ with respect to XIII-suitable tetrad of points $(a_1, a_2, b_1, b_2)$ is an equipped poset with involution $\mathscr{P}' = \mathscr{P}'_{(a_1,a_2,b_1,b_2)}$ obtained from $\mathscr{P}$ in the following way:

(1) the points $a_i, b_i$ receive new notations $a_i^-, b_i^+$;
(2) a simple double dyad $\{a_1^+, a_2^+\}$ and a pair of bidouble points $a^0 \sim b^0$ of genus $-1$ are added with the relations $\{a_1^-, a_2^-\} \prec a^0 \prec \{a_1^+, a_2^+\} \prec \Sigma \prec b^0 \prec \{b_1^+, b_2^+\}$;
(3) strong relations $a^0 \lhd b^0$; $\{a_1^+, a_2^+\} \lhd A$; $B \lhd b^0$ (plus all induced relations) are added.




Set

$$\mathscr{R} = \left\{ \mathscr{P}\text{-}sp \mid a_k\widetilde{b_i^-} \subset \widetilde{b_j^-} (i \neq j);\ (a_1 + a_2)^+ \subset (b_1 + b_2)^-;\ \gamma(b_1, b_2) \right\}.$$

Let $(E_0, W_0)$ be some fixed admissible complementing pair of subspaces in $U_0$ with respect to the pair $(\nabla, \Delta)$.

In contrast to the previous, relatively more simple algorithms, beginning this case, we shall define the object $U^\downarrow$ immediately, omitting a description of the differentiation functor.[11]

If $U$ is an object of the category $\mathscr{R}$, the *reduced derivative* object $U^\downarrow = W$ of the category $\mathscr{P}'$-$sp$ is given by the following relations:

[11] For the nearest goals (proof of the tameness criteria, classification of indecomposables) it is enough to deal with the objects $U^\downarrow$ only, not considering morphisms in details.

$W_0$ is taken from the admissible complementing pair,

$$\begin{aligned}
&W_{a_i^-} = U_{a_i} \cap \widetilde{\Delta} \quad (i = 1, 2),\\
&W_{(a^0,b^0)} = \left\{ (y - x, t - \widehat{x}) \,\middle|\, \begin{array}{l} x \in U_{a_1} \cap \widetilde{\widetilde{\nabla}} \\ y \in U_{a_2} \cap \widetilde{\nabla} \\ t \in U_{b_1} \cap (U_{b_2} + \widetilde{\widetilde{\nabla}}) \end{array} \right\} \cap \widetilde{W}_0^2,\\
&W_{a_i^+} = (U_{a_i} + U_{a_j} \cap \widetilde{\widetilde{\nabla}}) \cap \widetilde{W}_0 \quad (i \neq j),\\
&W_{b_i^+} = (U_{b_i} + \widetilde{\widetilde{\nabla}}) \cap \widetilde{W}_0 \quad (i = 1, 2),\\
&W_K = U_K \cap \{W_0^K\} \quad \text{for the remaining } K \in \Theta'.
\end{aligned} \tag{10.4}$$

The object $U^{\downarrow}$ does not depend, up to isomorphism, on the choice of the admissible complementing pair $(E_0, W_0)$, as it can be easily verified with help of the form (10.3). Remark that $U^{\downarrow} = 0$ for an indecomposable $U$ iff $U \simeq H(a_1, a_2, b_1)$, where $H(x, y, z)$ denotes a sincere on the subset $\{x \prec z \succ y\}$ representation of the form

| $x$ | $y$ | $z$ |
|---|---|---|
| 1 | 1 | $i$ |
| $i$ | $i$ | 1 |

.

Denote by $\mathscr{R}'$ the full subcategory in $\mathscr{P}'$-$sp$ formed by all objects $U^{\downarrow}$ for $U \in \operatorname{Ob}\mathscr{R}$.

In order to restore for any object $W$ in $\mathscr{R}'$ the *primitive* object $W^{\uparrow} = U$ in $\mathscr{R}$ with the condition $U^{\downarrow} \simeq W$, take the following way.

Present the $\overline{\Lambda}_2$-space $W_{(a^0,b^0)}$ in the form $W_{(a^0,b^0)} = \underline{W}_{(a^0,b^0)} \oplus V$, where $V$ is some complement with the $\overline{\Lambda}_2$-base $(f_1, g_1), \ldots, (f_m, g_m)$, and the $\mathbb{C}$-space $W_{b_i^+}$ in the form $W_{b_i^+} = \underline{W}_{b_i^+} \oplus \mathscr{H}_i$ with some complements $\mathscr{H}_i$ $(i = 1, 2)$. Consider a new $\mathbb{R}$-space $E_0$ of dimension $2m$ with the base $e_1, e_1', \ldots, e_m, e_m'$ and a new $\mathbb{C}$-space $C_0$ of dimension $m$ with the base $r_1 = e_1 + \mathrm{i}e_1', \ldots, r_m = e_m + \mathrm{i}e_m'$ and set $W^{\uparrow} = U$, where

$$\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_K = W_K \quad \text{for } K \neq a_1, a_2, b_1, b_2,\\
&\dot{U}_{a_1} = W_{a_1^+} + C_0,\\
&\dot{U}_{a_2} = W_{a_2^+} + \{r_1 + f_1, \ldots, r_m + f_m\},\\
&\dot{U}_{b_1} = \mathscr{H}_1 + \{\widehat{r}_1 + f_1, \ldots, \widehat{r}_m + f_m\},\\
&\dot{U}_{b_2} = \mathscr{H}_2 + \{g_1, \ldots, g_m\}.
\end{aligned} \tag{10.5}$$

Comparing the operations of differentiation (10.4) and integration (10.5), we establish, with help of the matrix presentation (10.3), the main property of Differentiation $\mathrm{XIII_c}$.

**Theorem 10.4.** *In the case of Differentiation* $\mathrm{XIII_c}$ *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{R}\backslash H(a_1, a_2, b_1) \rightleftarrows \operatorname{Ind}\mathscr{R}',$$

*where*

$$\mathscr{R} = \left\{\mathscr{P} - sp \mid a_k\widetilde{b_i^-} \subset \widetilde{b_j^-}\ (i \neq j);\ (a_1 + a_2)^+ \subset (b_1 + b_2)^-;\ \gamma(b_1, b_2)\right\}$$

*and* $\mathscr{R}'$ *is the image of the category* $\mathscr{R}$ *with respect to the action of the functor* $'$.

It is clear that D-XIII$_\mathrm{c}$ decreases $\dim U_0$ if $\Delta \neq \nabla$, i.e. if $(U_{a_1} + \widetilde{\Delta}) \cap (U_{a_2} \cap \widetilde{\Delta}) \neq \widetilde{\Delta}$.

**Remark 10.5.** It is possible to omit the relations of the category $\mathscr{R}$differentiating any representation of $\mathscr{P}$. Then the derivative poset $\mathscr{P}'$ will be obtained from $\mathscr{P}$ by replacing the subset $\{a_1, a_2\} \prec \Sigma \prec \{b_1, b_2\}$ by the subset $\{a_1^-, a_2^-\} \prec \{\pi_1, \pi_2\} \prec a^0 \prec \xi \prec \{a_1^+, a_2^+\} \prec \Sigma \prec \eta \prec b^0 \prec \{b_1^+, b_2^+\}$, where $a^0 \sim b^0$ are bidouble points of genus $-1$, all other points are simple double and $\{a_1^+, a_2^+\} \lhd A;\ B \lhd \eta;\ \xi \lhd \eta$. Of course, here the matrix form will be more bulky with respect to (10.3).

(d) At last, consider a representation $U$ of $\mathscr{P}$ satisfying both the relations of the category $\mathscr{R}$ and the dual ones

$$\begin{aligned} &a_k\widetilde{b_i^-} \subset \widetilde{b_j^-}\ \ (i \neq j), \quad (a_1 + a_2)^+ \subset (b_1 + b_2)^-, \quad \gamma(b_1, b_2), \\ &\widetilde{a_i^+} \subset b_k + \widetilde{a_j^+}\ \ (i \neq j), \quad (a_1a_2)^+ \subset (b_1b_2)^-, \quad \gamma^*(a_1, a_2), \end{aligned} \tag{10.6}$$

and also the relation $\Delta = \nabla$ and the dual one in the form

$$\begin{aligned} &(a_1 + \widetilde{\Delta})(a_2 + \widetilde{\Delta}) = \widetilde{\Delta}, \\ &b_1\widetilde{\nabla}_1 + b_2\widetilde{\nabla}_1 = \widetilde{\nabla}_1, \end{aligned} \tag{10.7}$$

where $\nabla_1 = U^+_{\{a_1,a_2\}} = U^+_{a_1} + U^+_{a_2}$ (as it was explained before, if one of them is not satisfied, then one of the described earlier algorithms can be applied, including D-XIII$_\mathrm{c}$ or D-XIII$^*_\mathrm{c}$).

In this case let $(E_0, W_0)$ be some complementing pair of subspaces in $U_0$ with respect to the pair $(\nabla_1, \Delta)$. Due to (10.7), the horizontal stripe of $M$, denoted in (10.1) by $E_0$, is absent. Now the corresponding role plays a part of the stripe denoted in (10.1) by $Q$. This part corresponds to some maximal collection of linearly independent rows of the matrix $L_1 \cup L_2$. Reducing the matrices $L_1$ and $L_2$, converting into zero the blocks $*$ of the stripe $\Delta$ under them and arguing then analogously to the case c), present $M$ in the form (with new $E_0, W_0, Q, N, G, H_i, X_i, Y$):

$$M = \begin{array}{cc} \begin{array}{ccccccccccccc} \multicolumn{2}{c}{\boldsymbol{a_1}} & \multicolumn{2}{c}{\boldsymbol{a_2}} & \boldsymbol{A} & \boldsymbol{\Sigma} & \boldsymbol{B} & \multicolumn{3}{c}{\boldsymbol{b_1}} & \multicolumn{3}{c}{\boldsymbol{b_2}} \end{array} & \\ \left|\begin{array}{cc|cc|c|c|c|ccc|ccc} & E & & & & & & X_1 & Y & & X_2 & & \\ & iE & & & & & & & & & & & \\ & & & E & & & & X_1' & Y' & & X_2' & & \\ & & & iE & & & & & & & & & \\ \hline & & & & * & N & & & G & H_1 & & G & H_2 \\ \hline * & & * & & * & * & * & & & & & & \end{array}\right| & \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} E_0 \\ \left.\begin{array}{c} Q \\ \Delta \end{array}\right\} W_0 \end{array} \end{array} \tag{10.8}$$

Obviously, $\mathscr{G} \cap \widetilde{\nabla}_1 = 0$, where $\mathscr{G}$ is the $\mathbb{C}$-space generated by the columns of the matrix $M_{b_1}$ prolonging the columns of $G$. Hence, due to the second equality of (10.7), $V_1 + V_2 = \widetilde{\nabla}_1$, where $V_i$ is the $\mathbb{C}$-space in $\widetilde{U}_0$ generated by the columns of $M_{b_i}$, prolonging the columns of $X_i$, and by the columns of the matrices $M_{a_1}$ and $M_{a_2}$.

So, the columns of the matrix

$$\begin{array}{|c|} \hline Y \\ Y' \\ \hline \end{array}$$

are $\mathbb{C}$-linear combinations of columns of the matrix

$$\begin{array}{|c|c|} \hline X_1 & X_2 \\ \hline X_1' & X_2' \\ \hline \end{array} \tag{10.9}$$

and can be turned into zero. Taking $Y = Y' = 0$, we automatically extract, as a direct summand of $U$, the matrix representation located in the stripe $E_0$.

Clearly, the matrix problem for the block (10.9) consists of independent $\mathbb{C}$-elementary transformations of each horizontal or vertical stripe. This problem was solved completely in [11] (over an arbitrary field $k$). It is one-parameter and (up to permutations of the stripes) has the indecomposables of 7 types shown in Appendix A–C.

For the proof of the main results it is enough for us to know that the problem (10.9) is one-parameter. Nevertheless, we are able to give more detailed information concerning the splittable summand. Taking into account the mentioned classification (see Appendix A–C), the choice of $\Delta = U^-_{\{b_1,b_2\}}$ and the relations (10.6) and (10.7), we conclude that the block (10.9) of the matrix (10.8) has the form VII from Appendix A–C. Consequently we get:

**Proposition 10.6.** *An indecomposable representation $U$ of an equipped poset $\mathscr{P}$ with* XIII*-suitable tetrad of points $(a_1, a_2, b_1, b_2)$ satisfies the conditions* (10.6) *and* (10.7) *and the condition $U^+_{\{a_1,a_2\}} \not\subset U^-_{\{b_1,b_2\}}$ if and only if it belongs to a complex series of the form*

$$\begin{array}{cccc} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{b_1} & \boldsymbol{b_2} \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{E} & \multicolumn{1}{c|}{E} \\ \hline \multicolumn{1}{|c|}{iE} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{E} & \multicolumn{1}{c|}{E} & \multicolumn{1}{c|}{X} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{iE} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \end{array} \tag{10.10}$$

*where $X$ is any Jordan block over $\mathbb{C}$ with the eigenvalue $\lambda \in \mathbb{C}\backslash\{0, 1\}$.*

## 11. Differentiation XIV

Let $\mathscr{P}$ be an equipped poset with involution with the set $\Theta$ of the equivalence classes of its points. Let $U_0$ be some $\mathbb{R}$-space. Take some additional notations.

For a representation $U$ of $\mathscr{P}$ and some bidouble point $a \in \mathscr{P}$ denote $U_a = \{x \in \widetilde{U}_0 \mid (x, 0) \in U_{(a,a^*)}\}$, $\widehat{U}_a = \{x \in \widetilde{U}_0 \mid (x, y) \in U_{(a,a^*)}$ for some $y \in \widetilde{U}_0\}$. It holds $U_a \subset \widehat{U}_a \subset \widetilde{U}_0$. For a simple double point $a$ set $\widehat{U}_a = U_a$.

Also set $U_a^- = U_a \cap U_0$ and $U_a^+ = \operatorname{Re} \widehat{U}_a$. Obviously, $U_a^- \subset U_a^+ \subset U_0$ (compare with the definition of the subspaces $U_x^- \subset U_x^+ \subset U_0$ given in Section 2 for equipped posets).

It will be used subsequently also a subspace $U_a^\# = \widehat{U}_a \cap U_0$ satisfying the inclusion $U_a^- \subset U_a^\# \subset U_a^+$.

Now pass over the algorithm. A pair of points $(a, b)$ of the set $\mathscr{P}$, where $a$ is bidouble, $b$ is simple double and $a \prec b$, is called XIV-*suitable* if $\mathscr{P} = a^\nabla + b_\vartriangle + \Sigma$, where $\Sigma$ is the interior of the interval $[a, b]$. Set $A = a^\nabla \backslash a,\ B = b_\vartriangle \backslash b$.

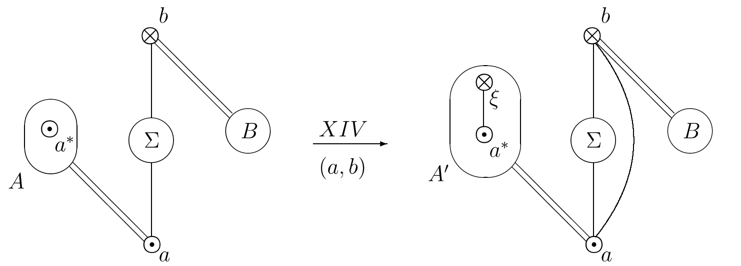
For arbitrary representation $U$ of $\mathscr{P}$ denote $\Delta = U_b^-$, $\nabla = U_a^+$ and choose some complementing pair $(E_0, W_0)$ in $U_0$ with respect to the pair $(\nabla, \Delta)$. Considering the matrix $M$ of $U$, select in it, as usually, the lower horizontal stripe $\Delta$, the upper one $E_0$ and the middle one $Q$. Then reduce $M$ with help of the standard, already well developed technics (using Lemmas 10.2 and 10.3) to the following form (possibly, with superfluous columns in the stripe $M_b$ and changed because of the reduction subspaces $E_0$ and $W_0$)

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|l}
\multicolumn{2}{c}{\boldsymbol{a}} & \multicolumn{2}{c}{\boldsymbol{a^*}} & \multicolumn{1}{c}{\boldsymbol{A}} & \multicolumn{1}{c}{\boldsymbol{\Sigma}} & \multicolumn{1}{c}{\boldsymbol{B}} & \multicolumn{2}{c}{\boldsymbol{b}} & \\
\hline
 & E & & & & & & & & \multirow{2}{*}{$\Big\} E_0 \quad (\text{here } a^* \in A)$} \\
\hline
 & iE & & & & & & & & \\
\hline
 & & * & * & * & N & & & H & Q \quad \Big\} W_0 \\
\hline
* & & * & * & * & * & * & E & & \Delta \\
\hline
\multicolumn{1}{c}{\odot} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\odot} & \multicolumn{1}{c}{\otimes} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{\substack{\otimes \\ b}}} & \\
\multicolumn{1}{c}{a} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a^*} & \multicolumn{1}{c}{\xi} & \multicolumn{5}{c}{} &
\end{array}
\tag{11.1}
$$

where the columns of $H$ are $\mathbb{C}$-linearly independent modulo columns of $N$. The corresponding to this form complementing pair $(E_0, W_0)$ is called *admissible*. Analyzing the corresponding admissible transformations of the stripe $W_0$ (which do not change $E_0$), come to the following construction.

The *derivative* set of the set $\mathscr{P}$ with respect to XIV-suitable pair $(a, b)$ is an equipped poset with involution $\mathscr{P}' = \mathscr{P}'_{(a,b)}$ obtained from $\mathscr{P}$ as follows:

(1) the point $a^*$ is replaced by the chain $a^* \prec \xi$, where $\xi$ is a new simple double point (and, as before, $a \sim a^*$);
(2) the relation $a \vartriangleleft b$ is added.




(possibly, $a^* \in B$ or $a^* \in \Sigma$).

The *reduced derivative* representation $U^{\downarrow} = W$ of $\mathscr{P}'$ is determined by the relations

$$\begin{aligned} &W_0 \text{ is taken from the admissible complementing pair,} \\ &W_{(a,a^*)} = U_{(a,a^*)} \cap \widetilde{W}_0^2, \\ &W_\xi = \widehat{U}_{a^*} \cap \widetilde{W}_0, \\ &W_K = U_K \cap \{W_0^K\} \quad \text{for the remaining classes } K \in \Theta'. \end{aligned} \tag{11.2}$$

Here $\dim U_0^{\downarrow} < \dim U_0$ $for$ $\nabla \not\subset \Delta$, and $U^{\downarrow} = 0$ for indecomposable $U$ iff

$$U \simeq D(a) = \begin{array}{cc} a & a^* \\ \end{array} \begin{array}{|c|c|} \hline 1 & 0 \\ i & 0 \\ \hline \end{array}.$$

If $W$ is any representation of $\mathscr{P}'$, then the *primitive* representation $W^{\uparrow}$ of $\mathscr{P}$ with the property $(W^{\uparrow})^{\downarrow} \simeq W$ is restored in the following way. Present $\mathbb{C}$-space $W_\xi$ in the form $W_\xi = \underline{W}_\xi \oplus F$, where $F$ is a complement with the base $f_1, \dots, f_m$, take new $\mathbb{R}$-space $E_0$ of dimension $2m$ with the base $e_1, e'_1, \dots, e_m, e'_m$ and $\mathbb{C}$-space $C_0$ of dimension $m$ with the base $r_1 = e_1 + \mathrm{i}e'_1, \dots, r_m = e_m + \mathrm{i}e'_m$ and set

$$\begin{aligned} &U_0 = W_0 \oplus E_0, \\ &\dot{U}_K = W_K \quad \text{for } K \neq \{a, a^*\}, \\ &\dot{U}_{(a,a^*)} = W_{(a,a^*)} + \{(r_1, f_1), \dots, (r_m, f_m)\}. \end{aligned} \tag{11.3}$$

The discussion above yields

**Theorem 11.1.** *In the case of Differentiation* XIV *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\operatorname{Ind} \mathscr{P} \backslash D(a) \rightleftarrows \operatorname{Ind} \mathscr{P}'.$$

## 12. Differentiation XV

Let $\mathscr{P}$ be an equipped poset with involution. A triple of points $(a, b_1, b_2)$ in $\mathscr{P}$, where $a$ is bidouble, $\{b_1, b_2\}$ is a simple double dyad and $a \prec \{b_1, b_2\}$, will be called XV-*suitable* if $\mathscr{P} = a^\nabla + \{b_1, b_2\}_\vartriangle + \Sigma$, where $\Sigma$ is the interior of the convex envelope $[a, b_1, b_2]$. Set $A = a^\nabla \backslash a,\ B = \{b_1, b_2\}_\vartriangle \backslash \{b_1, b_2\}$.

We may restrict our considerations by dealing with only those representations $U$ of $\mathscr{P}$, which satisfy the relation $a^+ \subset (b_1 + b_2)^-$ (otherwise it works D-XIV with respect to the pair $(a, \eta)$, where $\eta$ is a new added simple double point with the conditions $\{b_1, b_2\} \prec \eta$ and $U_\eta = U_{b_1} + U_{b_2}$) and the relation $\gamma(b_1, b_2)$ (see D-XIII$_\mathrm{c}$).

Denote $\Delta = U^-_{\{b_1,b_2\}}$, $\nabla = U^+_a$, choose some complementing pair $(E_0, W_0)$ with respect to the pair $(\nabla, \Delta)$ and consider the matrix $M$ of $U$ with $\underline{\dim}\, M = \underline{\dim}\, U$. Arguing completely analogously to the case D-XIII$_\mathrm{c}$ (with a new $\nabla$ and with several simplifications: under omitted point $a_2$, under $L_1 = \varnothing$ and under the existing equivalence $a^* \sim a$ with $a = a_1$) until to the moment (10.2) inclusively, we can reduce $M$ to the following form being analogous to the form (10.2) (for $a^* \in A$):

$$\begin{array}{c|cc|cc|c|c|c|ccc|ccc|l}
 & \multicolumn{2}{c|}{\boldsymbol{a}} & \multicolumn{2}{c|}{\boldsymbol{a^*}} & \boldsymbol{A} & \boldsymbol{\Sigma} & \boldsymbol{B} & \multicolumn{3}{c|}{\boldsymbol{b_1}} & \multicolumn{3}{c|}{\boldsymbol{b_2}} & \\
\hline
 & & E & & & & & & X_1 & Y & & X_2 & & & \multirow{2}{*}{$\Big\}\, E_0$} \\
 & & iE & & & & & & & & & & & & \\
\hline
 & & & * & * & * & N & & & G & H_1 & & G & H_2 & Q \ \Big\}\, W_0 \\
\hline
 & * & & * & * & * & * & * & & & & & & & \Delta
\end{array} \qquad (12.1)$$

where the matrix $N^0 \cup G \cup H_1 \cup H_2$ is $\mathbb{C}$-nonsingular in columns.

As in the case D-XIII$_\mathrm{c}$, for the matrices $X_1, Y, X_2$ we obtain a trivial problem on ordinary representations in the sense of [14] over $\mathbb{C}$ of the poset $\{x_1 < y > x_2\}$. The corresponding admissible indecomposable summands are now representations not only of type $[\delta_0 + \delta_y]$, but also $[\delta_0 + \delta_{x_1}]$ and $[\delta_0 + \delta_{x_2}]$, since in the considered situation there are no analogs of the relations $a_k \widetilde{b_i^-} \subset \widetilde{b_j^-}$ from D-XIII$_c$. Therefore, the matrix $M$ can be reduced to the form (being a generalized analog of the form (10.3))

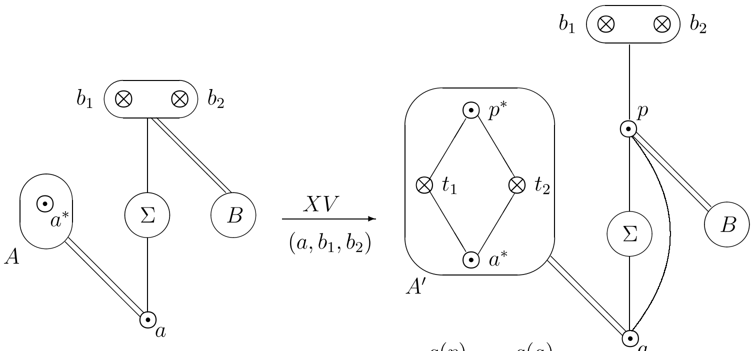
$$M = \begin{array}{cccc|cccc|c|c|c|ccc|ccc|l}
\multicolumn{4}{c|}{\boldsymbol{a}} & \multicolumn{4}{c|}{\boldsymbol{a^*}} & \boldsymbol{A} & \boldsymbol{\Sigma} & \boldsymbol{B} & \multicolumn{3}{c|}{\boldsymbol{b_1}} & \multicolumn{3}{c|}{\boldsymbol{b_2}} & \\
\hline
 & E & & & & & & & & & & iE & & & & & & \\
 & iE & & & & & & & & & & E & & & & & & \\
 & & E & & & & & & & & & & & & iE & & & \\
 & & iE & & & & & & & & & & & & E & & & E_0 \\
 & & & E & & & & & & & & & iE & & & & & \\
 & & & iE & & & & & & & & & E & & & & & \\
\hline
 & & & & * & * & * & * & * & N & & & G & H_1 & & G & H_2 & Q \ \Big\}\, W_0 \\
\hline
* & & & & * & * & * & * & * & * & * & & & & & & & \Delta \\
\hline
\odot & & & & \odot & \otimes & \otimes & \odot & & & & & \odot & \otimes & & & \otimes & \\
a & & & & a^* & t_1 & t_2 & p^* & & & & & p & b_1 & & & b_2 &
\end{array} \qquad (12.2)$$

The complementing pair $(E_0, W_0)$ in $U_0$ with respect to the pair $(\nabla, \Delta)$ is called *admissible*.

Carrying out the standard investigations of possible admissible transformations of the stripe $W_0$ (not changing $E_0$), we come to the main definitions of the algorithm.

The *derivative set* of the set $\mathscr{P}$ with respect to XV-suitable triple $(a, b_1, b_2)$ is an equipped poset with involution $\mathscr{P}' = \mathscr{P}'_{(a,b_1,b_2)}$ obtained from $\mathscr{P}$ in the following way:

(1) the point $a^*$ is replaced by the subset $a^* \prec \{t_1, t_2\} \prec p^*$, where $\{t_1, t_2\}$ is a simple double dyad and $p^*$ is bidouble;
(2) it is added to the subset $a \prec \Sigma \prec \{b_1, b_2\}$ a bidouble point $p$ with the relations $a \prec \Sigma \prec p \prec \{b_1, b_2\}$, where $g(p) = -g(a)$,
(3) the relations $a \lhd p$ and $B \lhd p$ are added.



(possibly, $a^* \in B \cup \Sigma$). $g(p) = -g(a)$

Set $\mathscr{R} = \{\mathscr{P}\text{-}sp \mid a^+ \subset (b_1 + b_2)^-;\ \gamma(b_1, b_2)\}$. For some fixed admissible complementing pair $(E_0, W_0)$ define the *reduced derivative* representation $U^\downarrow = W$ of $\mathscr{P}'$ by the relations:

$$
\begin{aligned}
&W_0 \text{ is taken from the admissible complementing pair,}\\
&W_{(a,a^*)} = U_{(a,a^*)} \cap \widetilde{W}_0^2,\\
&W_{b_i} = (U_{b_i} + \widetilde{\nabla}) \cap \widetilde{W}_0 \quad (i = 1, 2),\\
&W_{(p^*,p)} = \left\{ (y, t - \widehat{x}) \,\middle|\, \begin{matrix} (x, y) \in U_{(a,a^*)} \\ t \in U_{b_1} \cap (U_{b_2} + \widetilde{\nabla}) \end{matrix} \right\} \cap \widetilde{W}_0^2,\\
&W_{t_i} = \left\{ y \,\middle|\, \begin{matrix} (x, y) \in U_{(a,a^*)} \\ x \in U_{b_i} \end{matrix} \right\} \cap \widetilde{W}_0 \quad (i = 1, 2),\\
&W_K = U_K \cap \{W_0^K\} \quad \text{for the remaining } K \in \Theta'.
\end{aligned}
\tag{12.3}
$$

Here $\dim U_0^\downarrow < \dim U_0$ for $\nabla \neq \Delta$, and $U^\downarrow = 0$ for an indecomposable $U$ iff $U \simeq H(a, b_i)$, where $i \in \{1, 2\}$ and

$$
H(a, b_i) = \begin{array}{c} \begin{array}{ccc} a & a^* & b_i \end{array} \\ \begin{array}{|c|c|c|} \hline 1 & 0 & i \\ i & 0 & 1 \\ \hline \end{array} \end{array}.
$$

If $\mathscr{R}'$ is the full subcategory in $\mathscr{P}'$-*sp* formed by the objects $U^{\downarrow}$ for $U \in \mathrm{Ob}\,\mathscr{R}$, then for a restoring of the *primitive* object $W^{\uparrow}$ with the standard property $(W^{\uparrow})^{\downarrow} \simeq W$ consider the corresponding direct sums:

(a) $W_{(p^*,p)} = \underline{W}_{(p^*,p)} \oplus V$, where $V$ is a complement with the base $(f_1, g_1), \ldots, (f_{m_0}, g_{m_0})$;
(b) $W_{t_j} = \underline{W}_{t_j} \oplus \Omega_j$, where $\Omega_j$ is a complement with the base $w_{j1}, \ldots, w_{jm_j}$, $j = 1, 2$;
(c) $W_{b_j} = \underline{W}_{b_j} \oplus \mathscr{H}_j$ $(j = 1, 2)$.

Now take new $\mathbb{R}$-space $E_0$ of dimension $2m = 2(m_0 + m_1 + m_2)$ with the base $e_1, e_1', \ldots, e_{m_0}, e_{m_0}', t_{j1}, t_{j1}', \ldots, t_{jm_j}, t_{jm_j}'(j = 1, 2)$ and $\mathbb{C}$-space $C_0$ of dimension $m$ with the base $r_1 = e_1 + \mathrm{i}e_1', \ldots, r_{m_0} = e_{m_0} + \mathrm{i}e_{m_0}'$, $s_{j1} = t_{j1} + \mathrm{i}t_{j1}', \ldots, s_{jm_j} = t_{jm_j} + \mathrm{i}t_{jm_j}'$ $(j = 1, 2)$ and set $W^{\uparrow} = U$, where

$$\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_K = W_K \quad \text{for } K \neq \{a, a^*\}, b_1, b_2,\\
&\dot{U}_{(a,a^*)} = W_{(a,a^*)} + \big\{(e_1, f_1), \ldots, (e_{m_0}, f_{m_0})\big\}\\
&\qquad\qquad + \big\{(s_{j1}, w_{j1}), \ldots, (s_{jm_j}, w_{jm_j}) \mid j = 1, 2\big\},\\
&\dot{U}_{b_1} = \mathscr{H}_1 + \big\{\widehat{s}_{11}, \ldots, \widehat{s}_{1m_1}\big\} + \big\{\widehat{r}_1 + g_1, \ldots, \widehat{r}_{m_0} + g_{m_0}\big\},\\
&\dot{U}_{b_2} = \mathscr{H}_2 + \big\{\widehat{s}_{21}, \ldots, \widehat{s}_{2m_2}\big\} + \big\{g_1, \ldots, g_{m_0}\big\}.
\end{aligned} \tag{12.4}$$

Comparing and analyzing (12.2)–(12.4), we obtain:

**Theorem 12.1.** *In the case of Differentiation* XV *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\mathrm{Ind}\,\mathscr{R}\backslash\big\{H(a, b_i) \mid i = 1, 2\big\} \rightleftarrows \mathrm{Ind}\,\mathscr{R}',$$

*where* $\mathscr{R} = \{\mathscr{P}\text{-}sp \mid a^+ \subset (b_1 + b_2)^-; \gamma(b_1, b_2)\}$ *and* $\mathscr{R}'$ *is the full subcategory of the category* $\mathscr{P}'$*-sp formed by the objects of the form* $U^{\downarrow}$ *with* $U \in \mathrm{Ob}\,\mathscr{R}$.

## 13. Differentiation XVI

A pair of bidouble weakly comparable but not equivalent points $a \prec b$ of an equipped poset with involution $\mathscr{P}$ is called XVI-*suitable* if $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma$, where $\Sigma$ is the interior of the interval $[a, b]$. Set $A = a^{\triangledown}\backslash a$, $B = b_{\vartriangle}\backslash b$.

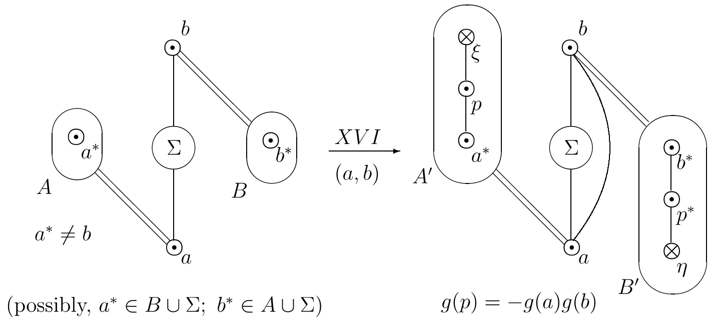
One can impose on representations $U$ of $\mathscr{P}$ the relation $\gamma(b)$ denoting $\widetilde{U_\Sigma + U_B^+} = U_b$ (otherwise the subset $\Sigma$ is extended by adding a simple double point $\xi$ with the relations $\Sigma \prec \xi \prec b$ and $B \lhd \xi$ and the condition $U_\xi = U_b$). As a consequence, we have the inclusions $U_b^- \subset U_\Sigma + \widetilde{U_B^+}$.

Denote $\Delta = U_b^-$, $\nabla = U_a^+$. Consider the matrix $M$ of $U$ and, using standard arguing (analogous, for instance, to the case D-XIII$_\mathrm{c}$ but being much more simple) and also Lemmas 10.2 and 10.3, reduce it to the form (for $a^* \in A$, $b^* \in B$):

$$
M = \begin{array}{c|ccc|ccc|c|c|c|ccc|ccc|l}
 & & \boldsymbol{a} & & & \boldsymbol{a^*} & & \boldsymbol{A} & \boldsymbol{\Sigma} & \boldsymbol{B} & & \boldsymbol{b^*} & & & \boldsymbol{b} & & \\
\hline
 & & & E & & & & & & & & & & & & & \\
 & & & iE & & & & & & & & & & & & & E_0 \\
 & & E & & & & & & & & & & & & iE & & \\
 & & iE & & & & & & & & & & & & E & & \\
\hline
 & & & & * & * & * & * & N & & & & & & & H & Q \quad W_0 \\
\hline
 & * & & & * & * & * & * & * & * & * & * & * & & & & \Delta \\
\hline
 & \odot & & & \odot & \odot & \otimes & & & & \otimes & \odot & \odot & & & \odot & \\
 & a & & & a^* & p & \xi & & & & \eta & p^* & b^* & & & b &
\end{array}
\tag{13.1}
$$

where the columns of $H$ are $\mathbb{C}$-linearly independent modulo columns of $N$, and the horizontal stripes $E_0$ and $W_0$ correspond to the *admissible* complementing pair of subspaces $(E_0, W_0)$ in $U_0$ with respect to $(\nabla, \Delta)$. The obtained form (13.1) is a base for the construction of Differentiation XVI.

The *derivative* equipped poset with involution $\mathscr{P}' = \mathscr{P}'_{(a,b)}$ with respect to the pair $(a, b)$ is obtained from $\mathscr{P}$ by means of replacing the point $a^*(b^*)$ by the weak three-point chain $a^* \prec p \prec \xi$ $(\eta \prec p^* \prec b^*)$, where $p \sim p^*$ are bidouble points, and $\xi, \eta$ are simple double ones, and also by adding the relation $a \lhd b$. Moreover, $g(p) = -g(a)g(b)$.




**Remark.** If we put on representations of $\mathscr{P}$ the additional relations $\alpha(a)$ and $\beta(b)$ (defined in the following section), then the derivative poset $\mathscr{P}'$ will not contain the points $\xi$ and $\eta$.

Let $U$ be a representation of $\mathscr{P}$. In order to define the reduced derivative representation $U^\downarrow$ of the set $\mathscr{P}'$, we have to introduce the notion of a crossed coupling of a family of subspaces in $\widetilde{U}_0^2$.

Assume $(a_1, a_1^*), \dots, (a_n, a_n^*)$ is any sequence of ordered pairs of equivalent bidouble points of the set $\mathscr{P}$ (repetitions are allowed). The *crossed coupling of degree* $n$ ($n \geqslant 1$) of the ordered family of subspaces $U_{(a_1,a_1^*)}, \dots, U_{(a_n,a_n^*)}$ is a subspace in $\widetilde{U}_0^2$ of the form

$$\begin{aligned}\textstyle\bowtie_{i=1}^n U_{(a_i,a_i^*)} &= U_{(a_1,a_1^*)} \bowtie U_{(a_2,a_2^*)} \bowtie \cdots \bowtie U_{(a_n,a_n^*)}\\ &= \left\{(x_1, y_n) \mid (x_i, y_i) \in U_{(a_i,a_i^*)} \text{ and } y_i = \widehat{x}_{i+1} \text{ for all possible } i\right\}.\end{aligned}$$

The crossed coupling of $n$ copies of one and the same space $U_{(a,a^*)}$ will be denoted by $U_{(a,a^*)}^{[n,\bowtie]}$.

**Remark.** If to assume that $\bowtie_{i=1}^n U_{(a_i,a_i^*)} = U_{(t,t^*)}$ for some new pair of bidouble points $t \sim t^*$, then (as it can be verified easily) the genus of the point $t$ is bound up with the genera of the points $a_i$ in the following way:

$$g(t) = (-1)^{n+1} g(a_1) \cdots g(a_n).$$

In particular, for $U_{(t,t^*)} = U_{(a,a^*)}^{[n,\bowtie]}$, we have

$$g(t) = (-1)^{n+1}(g(a))^n = \begin{cases} g(a) & \text{for } n = 2k+1,\\ -1 & \text{for } n = 2k.\end{cases}$$

The *reduced derivative* representation $U^\downarrow = W$ of $\mathscr{P}'$ is determined by the relations

$$\begin{aligned}
&W_0 \text{ is taken from the admissible complementing pair,}\\
&W_{(a,a^*)} = U_{(a,a^*)} \cap \widetilde{W}_0^2, \quad W_{(b,b^*)} = [U_{(b,b^*)} + (\widetilde{\nabla}, 0)] \cap \widetilde{W}_0^2,\\
&W_{(p,p^*)} = [U_{(a^*,a)} \bowtie U_{(b,b^*)}] \cap \widetilde{W}_0^2,\\
&W_\xi = \widehat{U}_a \cap \widetilde{W}_0, \quad W_\eta = U_{b^*} \cap \widetilde{W}_0,\\
&W_K = U_K \cap \{W_0^K\} \quad \text{for the remaining } K \in \Theta'.
\end{aligned} \tag{13.2}$$

It holds $\dim U_0^\downarrow < \dim U_0$ for $\nabla \neq \Delta$, and $U^\downarrow = 0$ for an indecomposable $U$ exactly in the case $U \simeq D(a)$ or $U \simeq H(a, b)$.

Denote $\mathscr{R} = \{\mathscr{P}\text{-}sp \mid \gamma(b)\}$. Let $\mathscr{R}'$ be the full subcategory in $\mathscr{P}'\text{-}sp$ formed by the objects $U^\downarrow$ for all $U \in \operatorname{Ob}\mathscr{R}$.

Take $W \in \operatorname{Ob}\mathscr{R}'$. In order to restore the *primitive* object $W^\uparrow$ with the condition $(W^\uparrow)^\downarrow \simeq W$, fix a direct sum $W_\xi = \underline{W}_\xi \oplus F$, where $F$ is a complement with the base $f_1, \dots, f_{m_0}$, and also a direct sum $W_{(p,p^*)} = \underline{W}_{(p,p^*)} \oplus G$, where $G$ is a complement with the base $(g_1, h_1), \dots, (g_{m_1}, h_{m_1})$. Considering a new $\mathbb{R}$-space $E_0$ of dimension $2m = 2(m_0 + m_1)$ with the base $e_1, e'_1, \dots, e_{m_0}, e'_{m_0}, t_1, t'_1, \dots, t_{m_1}, t'_{m_1}$ and a new $\mathbb{C}$-space $C_0$ of dimension $m$ with the base $r_1 = e_1 + \mathrm{i}e'_1, \dots, r_{m_0} = e_{m_0} + \mathrm{i}e'_{m_0}, s_1 = t_1 + it'_1, \dots, s_{m_1} = t_{m_1} + it'_{m_1}$, set $W^{\uparrow} = U$, where

$$
\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_K = W_K \quad \text{for } K \neq \{a, a^*\}, \{b, b^*\},\\
&\dot{U}_{(a,a^*)} = W_{(a,a^*)} + \big\{(r_1, f_1), \dots, (r_{m_0}, f_{m_0})\big\}\\
&\qquad\qquad + \big\{(s_1, g_1), \dots, (s_{m_1}, g_{m_1})\big\},\\
&\dot{U}_{(b,b^*)} = W_{(b,b^*)} + \big\{(\widehat{s}_1, h_1), \dots, (\widehat{s}_{m_1}, h_{m_1})\big\}.
\end{aligned}
\tag{13.3}
$$

Comparing (13.1)–(13.3) and analyzing the whole described construction, we get:

**Theorem 13.1.** *In the case of Differentiation* XVI *the operations* $\downarrow$ *and* $\uparrow$ *induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{R}\backslash\{D(a), H(a,b)\} \rightleftarrows \operatorname{Ind}\mathscr{R}',$$

*where* $\mathscr{R} = \{\mathscr{P}\text{-}sp \mid \gamma(b)\}$ *and* $\mathscr{R}'$ *is the full subcategory in* $\mathscr{P}'$*-sp formed by the objects* $U^{\downarrow}$ *for* $U \in \operatorname{Ob}\mathscr{R}$.

## 14. Differentiation XVII

A pair $a \prec a^*$ of bidouble weakly comparable points of an equipped poset with involution $\mathscr{P}$ will be called XVII-*suitable* if $\mathscr{P} = a^{\triangledown} + a^*_{\vartriangle} + \Sigma$, where $\Sigma$ is the interior of the interval $[a, a^*]$. Set $A = a^{\triangledown}\backslash a,\ B = a^*_{\vartriangle}\backslash a^*$.

Let $U$ be a representation of $\mathscr{P}$ and $M$ be the matrix realization of $U$ satisfying the natural condition $\underline{\dim}\, M = \underline{\dim}\, U$. We say that the representation $U$ satisfies the relation $\beta(x)$, where $x$ is some bidouble point, if the stripe $M_x$ is nonsingular in columns over $\mathbb{C}$.

From now on we may assume that $U$ satisfies the relation $\beta(a^*)$. Really, otherwise consider a new equipped poset with involution $\mathscr{P}'$ obtained from $\mathscr{P}$ by replacing the point $a$ by a weak chain $\eta \prec a$, where $\eta$ is a new simple double point . Then pass from $U$ to the representation $U'$ of the set $\mathscr{P}'$ with $U'_0 = U_0,\ U'_\eta = U_a$ and $U'_K = U_K$ for the remaining classes $K$. After such a transition $U'$ will satisfy the relation $\beta(a^*)$ and, moreover, the coordinate $d_{(a,a^*)}$ of the vector $d = \underline{\dim}\, U$ will be decreased (when transforming into the vector $d' = \underline{\dim}\, U'$) on the value $d'_\eta$.

Also impose the relation $U^+_a \subset U^\#_{a^*} = \widehat{U}_{a^*} \cap U_0$, which will be denoted by $\alpha(a)$. If it is not satisfied, one can verify easily (using Lemmas 10.2 and 10.3) that the matrix $M$ is reduced to the form

$$M = \begin{array}{c|c|c|c|c|c|c|c|l}
\multicolumn{1}{c}{} & \multicolumn{2}{c}{\boldsymbol{a}} & \multicolumn{1}{c}{\boldsymbol{A}} & \multicolumn{1}{c}{\boldsymbol{\Sigma}} & \multicolumn{1}{c}{\boldsymbol{B}} & \multicolumn{2}{c}{\boldsymbol{a^*}} & \\
\cline{2-8}
 & & E & & & & & & \multirow{2}{*}{$\Big\}\, S$} \\
\cline{2-8}
 & & iE & & & & & & \\
\cline{2-8}
 & * & & * & * & * & * & * & T \\
\cline{2-8}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\odot} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\odot} & \multicolumn{1}{c}{\otimes} & \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a^*} & \multicolumn{1}{c}{\xi} &
\end{array} \tag{14.1}$$

where the horizontal stripes $S$ and $T$ correspond to some complementing pair of subspaces $(S, T)$ with respect to the pair $(U_a^+, U_{a^*}^{\#})$. Hence, in the stripe $T$ there appears naturally the problem on representations of an equipped poset with involution $\mathscr{P}_1$ obtained from $\mathscr{P}$ by replacing the point $a^*$ by a weak chain $a^* \prec \xi$, where $\xi$ is a new simple double point (here the coordinate $d_0$ of the vector $d = \underline{\dim}\, U$ is decreased on the value $\dim_{\mathbb{R}} S$).

**Remark.** In fact (14.1) in nothing else as a matrix scheme of some simple additional differentiation with respect to the bidouble point $a$ (which can be applied in other similar situations).

Moreover, impose the third relation $\gamma(a^*)$ which was introduced earlier (see D-XVI) is tantamount to the equality $U_\Sigma + \widetilde{U}_B^+ = U_{a^*}$ and implies the inclusion $U_{a^*}^- \subset U_\Sigma + \widetilde{U}_B^+$.

Set $\Delta = U_{a^*}^-$, $\nabla = U_a^+$, consider some complementing pair of subspaces $(E_0, W_0)$ in $U_0$ with respect to the pair $(\nabla, \Delta)$ and reduce the matrix $M$ in accordance with the following scheme:

(1) Select in the lower part of the matrix $M$ the horizontal stripe corresponding to the subspace $\Delta$, and in the upper part the stripe corresponding to $E_0$. The remaining middle part denote by $Q$.
(2) One may assume $M_a \cap Q = 0$, $M_A \cap E_0 = 0$ and (due to the relation $\gamma(a^*)$) $M_{a^*} \cap \Delta = 0$. Obviously, also $M_B \cap (E_0 + Q) = 0$.
(3) Reduce the block $M_{a^*} \cap Q$ to the form $\boxed{0 \mid H}$, where the matrix $H$ is $\mathbb{C}$-nonsingular in columns. Denote $M_\Sigma \cap Q = N$.
(4) Reduce the block $M_a \cap E_0$ (partitioned respectively into two vertical stripes) as a matrix representation of a two-point weak chain. Taking in account the choice of the subspace $\Delta = U_{a^*}^-$, we obtain the indecomposables of type $\boxed{\begin{matrix} E \\ iE \end{matrix}}$ only plus some zero vertical columns. As a result, the block $M_{a^*} \cap E_0$ will be partitioned into four vertical stripes, in particular, the matrix $H$ will take the form $H = \boxed{H_0 \mid H_\infty}$.
(5) Allowing for Lemmas 10.2 and 10.3, we can admit $M_\Sigma \cap E_0 = 0$ and $X \cap E_0 = 0$, where $X$ is the located under $H$ part of the block $M_{a^*} \cap E_0$.
(6) Using natural additions of columns or rows, turn into zero those horizontal stripes of the matrix $M_{a^*} \cap E_0$, which correspond to the cells $iE$ of the block

$M_a \cap E_0$. Also turn into zero those parts of the block $M_a \cap \Delta$, which are located just under the same cells $iE$. As a result, the matrix $M$ takes the form

$$M = \begin{array}{ccccccccccc|l}
 & & \boldsymbol{a} & & \boldsymbol{A} & \boldsymbol{\Sigma} & \boldsymbol{B} & & \boldsymbol{a}^* & & & \\
\hline
 & & & E & & & & X & Y & & & \\
 & & & iE & & & & & & & & E_0 \\
 & E & & & & & & Z & \Lambda & & & \\
 & iE & & & & & & & & & & \\
\hline
 & & & & * & N & & & & H_0 & H_\infty & Q \quad \\
* & & * & & * & * & * & & & & & \Delta \quad W_0
\end{array} \tag{14.2}$$

where the blocks $X, Y, Z, \Lambda$ form a matrix problem of the following type (this problem arises by using only those admissible transformations of the matrix $M$, which do not change the blocks $E$, $iE$ and zero blocks).

It is given a rectangular matrix over $\mathbb{C}$ partitioned into four blocks $X, Y, Z, \Lambda$

$$\begin{array}{c|c|c|c}
 & \mathbb{C} \longrightarrow & \mathbb{C} & \\
\hline
\mathbb{C} & X & Y & \\
\downarrow & & & \\
\mathbb{C} & Z & \Lambda & \sim \\
\hline
 & & \sim &
\end{array} \tag{14.3}$$

in such a way that the block $\Lambda$ is square. One can apply to it the next transformations:

(1) independent $\mathbb{C}$-elementary transformations of the first stripe rows and of the first stripe columns; [12]
(2) independent additions of the first stripe rows to the second stripe rows and of the first stripe columns to the second stripe columns;
(3) applying to the second stripe rows any $\mathbb{C}$-elementary transformation $S$, we must apply to the second stripe columns:
(a) the conjugate-inverse transformation $\overline{S}^{-1}$ in the case $g(a) = 1$,
(b) the inverse transformation $S^{-1}$ in the case $g(a) = -1$
(this correlation between the transformations of the second stripe rows and second stripe columns is marked in (14.3) by the symbol $\sim$).

Therefore, the square matrix $\Lambda$ in (14.3) is transformed (besides possible additions of rows or columns of the blocks $Y, Z$) by the *consimilarity transformation* $S\Lambda\overline{S}^{-1}$

[12] Naturally, by the first stripe rows (first stripe columns) we mean the rows (columns) of the first horizontal (vertical) stripe.

(see [9]) in the case (a) or by the ordinary *similarity transformation* $S\Lambda S^{-1}$ in the case (b). Respectively, we will call the problem (14.3) *having the type* (a) or (b).

Now solve the problem (14.3). If $X \neq 0$ or $Y \neq 0$, reduce the first horizontal stripe by admissible transformations to the form

$$
\begin{array}{c}
\longrightarrow \\
\begin{array}{|c|c|c|c|}
\hline
E & & & \\
\hline
 & & E & \\
\hline
 & & & \\
\hline
\end{array}
\end{array}
$$

(this is a trivial problem on representations over $\mathbb{C}$ in the sense of [14] of an ordinary poset of two comparable points). Then turn into zero the columns of the second horizontal stripe located under the blocks $E$. It turns out that in the second stripe there arises again the problem of type (14.3), but of smaller size. The analogous reduction can be made by dealing with the first vertical stripe (in the case $X \neq 0$ or $Z \neq 0$). Continuing this procedure, in the long run we obtain the problem

$$
\begin{array}{|c|c|}
\hline
X_1 & Y_1 \\
\hline
Z_1 & \Lambda_1 \\
\hline
\end{array}
$$

of type (14.3) with zero or empty blocks $X_1$, $Y_1$ and $Z_1$, i.e. in fact we isolate the square cell $\Lambda_1$ as a direct summand of the problem.

The summand $\Lambda_1$ may be presented in the canonical form under consimilarity transformations in the situation (a) (see [9, Theorem 3.1]) or under similarity transformations in the situation (b) (the usual Jordan normal form). Analyzing the described reduction procedure, we can find easily the canonical form of the remaining "discrete" indecomposable matrices of type (14.3). Namely, it takes place

**Proposition 14.1.** *The canonical indecomposable forms for the matrix problem* (14.3) *are exhausted by the following five forms*:

$$
\begin{array}{cc}
 & 1 \\
1 & \begin{array}{|c|}\hline 1 \\ \hline\end{array}
\end{array}
\qquad
\begin{array}{cc}
 & 2 \\
\downarrow & \begin{array}{|c|}\hline 0\ldots01 \\ \hline J_n^-(0) \\ \hline\end{array}
\end{array}
\qquad
\begin{array}{cc}
 & \longrightarrow \\
2 & \begin{array}{|c|c|}\hline \begin{matrix}1\\0\\\vdots\\0\end{matrix} & J_n^-(0) \\ \hline\end{array}
\end{array}
\qquad
\begin{array}{cc}
 & \longrightarrow \\
\downarrow & \begin{array}{|c|c|}\hline 0 & 0\ldots01 \\ \hline \begin{matrix}1\\0\\\vdots\\0\end{matrix} & J_n^-(0) \\ \hline\end{array}
\end{array}
$$

(1) (2) (3) (4)

$$
\begin{array}{cc}
 & 2 \\
2 & \begin{array}{|c|}\hline \mathcal{F} \\ \hline\end{array}
\end{array}
\ \text{(for the type (a))}
\qquad
\begin{array}{cc}
 & 2 \\
2 & \begin{array}{|c|}\hline J_n(\lambda) \\ \hline\end{array}
\end{array}
,\ \lambda \in \mathbb{C}\ \text{(for the type (b))}
$$

(5a) (5b) $(n \geq 1)$

*where the canonical form* $\mathscr{F}$ (*of size* $n \times n$) *under consimilarity transformations* $S\Lambda\overline{S}^{-1}$ *is presented in* [9, *Theorem* 3.1], *the conventions on the notations* $J_n(\lambda)$, $J_n^-(0)$ *are given in Notations for Appendix* A–C *to the present article*, *and the numbers near the blocks denote the index numbers of the horizontal or vertical stripes* (*where it is necessary*).

Obviously, if the subproblem (14.3) has direct summands of type (5a) or (5b), then the whole problem (14.2) has direct summands of the form

$$T(\Lambda') = \begin{array}{cc} a & a^* \\ \end{array}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\begin{array}{|c|c|} \hline E & \Lambda' \\ \hline iE & 0 \\ \hline \end{array}, \quad \text{where } \Lambda' = \begin{cases} \mathscr{F} & \text{for } g(a) = 1, \\ J_n(\lambda) & \text{for } g(a) = -1, \end{cases} \tag{14.4}$$

and $\det \Lambda' \neq 0$ due to the relation $\beta(a^*)$. Therefore, when constructing Differentiation, one can exclude from considerations the direct summands of type (5a) and (5b) of the subproblem (14.3). Moreover, this subproblem has no direct summands of type (2) and (3), as they contradict the relations $\alpha(a)$ and $\beta(a^*)$ respectively.

So, we may assume that the subproblem (14.3) admits direct summands of types (1) and (4) only. Now put these summands into the form (14.2), permute in a suitable way the columns of the stripes $M_a$ and $M_{a^*}$ and the rows of the horizontal stripe $E_0$, and also substitute the fragment

$$\begin{array}{cc} a & a^* \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c|}{iE} \\ \hline \multicolumn{1}{|c|}{iE} & \multicolumn{1}{c|}{E} \\ \hline \end{array}$$

for each fragment

$$\begin{array}{cc} a & a^* \\ \hline \multicolumn{1}{|c|}{E} & \multicolumn{1}{c|}{E} \\ \hline \multicolumn{1}{|c|}{iE} & \multicolumn{1}{c|}{0} \\ \hline \end{array}$$

(this can be done using suitable admissible transformations). As a result, the matrix (14.2) takes the form (below the summands of type (4) with $n \leqslant 2$ only are shown):

$$M = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|l} \multicolumn{3}{c}{\boldsymbol{a}} & & & & \multicolumn{1}{c}{\boldsymbol{A}} & \multicolumn{1}{c}{\Sigma} & \multicolumn{1}{c}{\boldsymbol{B}} & & & & \multicolumn{1}{c}{\boldsymbol{a}^*} & & & \\ \hline & & E & & & & & & & & iE & & & & & \\ \hline & & iE & & & & & & & & E & & & & & \\ \hline & & & & E & & & & & & & & iE & & & E_0 \\ \hline & & & & & E & & & & & & & & iE & & \\ \hline & & & & iE & & & & & & & & E & & & \\ \hline & & & & & iE & & & & & & & & E & & \\ \hline & & & & & & * & N & & H_0 & & H_1 & & & H_2 & Q \quad W_0 \\ \hline * & * & & * & & & * & * & * & & & & & & & \Delta \\ \hline \multicolumn{1}{c}{\odot} & \multicolumn{1}{c}{\odot} & & \multicolumn{1}{c}{\odot} & & & & & & \multicolumn{1}{c}{\odot} & & \multicolumn{1}{c}{\odot} & & & \multicolumn{1}{c}{\odot} & \\ \multicolumn{1}{c}{a_0} & \multicolumn{1}{c}{a_1} & & \multicolumn{1}{c}{a_2} & & & & & & \multicolumn{1}{c}{a_0^*} & & \multicolumn{1}{c}{a_1^*} & & & \multicolumn{1}{c}{a_2^*} & \\ \multicolumn{2}{c}{\longleftarrow} & \multicolumn{2}{c}{\longleftarrow} & & & & & & \multicolumn{2}{c}{\longrightarrow} & & \multicolumn{2}{c}{\longrightarrow} & & \end{array} \tag{14.5}$$

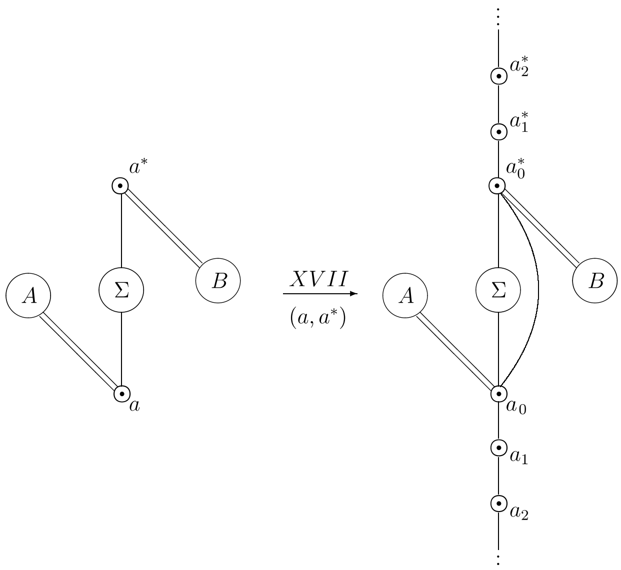
Here $\bigcup_{i=1}^{\infty} H_i = H_\infty$ (certainly, the number of the nonempty blocks $H_i$ is finite).

The corresponding to the form (14.5) complementing pair $(E_0, W_0)$ in $U_0$ with respect to the pair $(\nabla, \Delta)$ is called *admissible*.

Analyzing the matrix problem arising in the stripe $W_0$ (under the invariable part $E_0$) we come to the construction of the XVIIth algorithm.

The *derivative set* of the set $\mathscr{P}$ with respect to the XVII-suitable pair of bidouble points $a \prec a^*$ is an equipped poset with involution $\mathscr{P}' = \mathscr{P}'_{(a,a^*)}$ obtained from $\mathscr{P}$ as follows:

(1) the point $a$ is replaced by the infinite decreasing completely weak chain of bidouble points $a_0 \succ a_1 \succ a_2 \succ \cdots$, and the point $a^*$ is replaced by the infinite increasing completely weak chain $a_0^* \prec a_1^* \prec a_2^* \prec \cdots$, where $g(a_{2k}) = g(a)$ and $g(a_{2k+1}) = -1$ $(k \geqslant 0)$;
(2) the relations $a_0 \lhd A$; $B \lhd a_0^*$ and $a_0 \lhd a_0^*$ are added (plus all the induced ones).




$$g(a_n) = \begin{cases} g(a) & \text{for } n = 2k, \\ -1 & \text{for } n = 2k+1. \end{cases}$$

The *reduced* derivative representation $U^{\downarrow} = W$ of the set $\mathscr{P}'$ is determined by the formulas

$$\begin{aligned} &W_0 \text{ is taken from the complementing admissible pair } (E_0, W_0), \\ &W_{(a_n,a_n^*)} = \left[U_{(a,a^*)}^{[n+1,\bowtie]} + (0, \widetilde{\nabla})\right] \cap \widetilde{W}_0^2, \qquad (14.6) \\ &W_K = U_K \cap \left\{W_0^K\right\} \quad \text{for the remaining } K \in \Theta'. \end{aligned}$$

Obviously, $\dim W_0 < \dim U_0$ if $\nabla \not\subset \Delta$. And $U^{\downarrow} = 0$ (in the case of indecomposable U) iff $U \simeq H(a, a^*)$ or $U \simeq T(\Lambda')$. Hence, under Differentiation XVII the operation $\downarrow$ annihilates infinitely many nonisomorphic indecomposables of the set $\mathscr{P}$.

Set $\mathscr{R} = \{\mathscr{P}\text{-}sp \mid \alpha(a), \beta(a^*), \gamma(a^*)\}$. Denote by $\mathscr{R}'$ the full subcategory in $\mathscr{P}$-$sp$ formed by the objects $U^{\downarrow}$ with $U \in \operatorname{Ob}\mathscr{R}$.

The *primitive* representation $W^{\uparrow}$ of $\mathscr{P}$, where $W \in \operatorname{Ob}\mathscr{R}'$, with the property $(W^{\uparrow})^{\downarrow} \simeq W$ is built in the following way. Set $W_{(a_n, a_n^*)} = \underline{W}_{(a_n, a_n^*)} \oplus G_n$, where $G_n$ is some complement with the base $(r^n_{01}, \widehat{r^n_{n+1,1}}), \ldots, (r^n_{0p_n}, \widehat{r^n_{n+1,p_n}})$, $n \geqslant 0$. Choose a new $\mathbb{R}$-space $E_0$ of dimension $2m = 2\sum_{n=1}^{\infty} np_n$ with the base $e^n_{jk}, f^n_{jk}$ for $n \geqslant 1$, $j \in \overline{1, n}$, $k \in \overline{1, p_n}$ and also choose a $\mathbb{C}$-space $C_0$ of dimension $m$ with the base $r^n_{jk} = e^n_{jk} + \mathrm{i} f^n_{jk}$ for the same $n, j, k$ and set $W^{\uparrow} = U$, where

$$
\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_K = W_K \quad \text{for } K \neq \{a, a^*\},\\
&\dot{U}_{(a,a^*)} = G_0 + \bigoplus_{n=1}^{\infty} \bigoplus_{j=0}^{n} \{(r^n_{j1}, \widehat{r^n_{j+1,1}}), \ldots, (r^n_{jp_n}, \widehat{r^n_{j+1,p_n}})\}.
\end{aligned}
\tag{14.7}
$$

Comparing and analyzing (14.5)–(14.7), we establish the main property of the considered algorithm.

**Theorem 14.2.** *In the case of Differentiation* XVII *the operations $\downarrow$ and $\uparrow$ induce mutually inverse bijections*

$$\operatorname{Ind}\mathscr{R}\backslash\{H(a, a^*), T(\Lambda')\} \rightleftarrows \operatorname{Ind}\mathscr{R}'.$$

## 15. Some properties of the algorithms

Immediately from the described in Sections 3−14 constructions we receive some natural consequences.

**Corollary 15.1.** *Let $\mathscr{P}$ be an equipped poset with involution containing a group of points suitable for one of Differentiations* VII–XVII *or dual to them. Then each indecomposable representation $U$ of $\mathscr{P}$ satisfies the following conditions*:

(1) $\dim U_0^{\downarrow} \leqslant \dim U_0$;

(2) *if* $\dim U_0^{\downarrow} = \dim U_0$, *then $U$ is also a representation of the set $\overline{\mathscr{P}}$ obtained from $\mathscr{P}$ by adding some strong order relations between its points*.

**Remark.** Certainly, the corollary is also valid for the operation of completion (described in Section 4) if we assume in the corresponding situation $U^{\downarrow} = U$ for $U \not\geq D(a)$ and $D^{\downarrow}(a) = 0$.

Defining in a standard way the operations of differentiation and integration of $(\mathbb{R}[t], \mathbb{C}[t])$-representations for equipped posets with involution by means of transition to rational envelopes over the quotient fields $\mathbb{R}(t)$ and $\mathbb{C}(t)$, analogously to the cases of ordinary posets [25] and ordinary posets with involution [23, p. 318] (see also [18, Section 15.5]), we get the following important result for the real series.

**Corollary 15.2.** *For any of the algorithms* VII–XVII (*as well as for the operation of completion*), *a real series of indecomposables is transformed by differentiation or integration into a real series.* [13]

As for the complex series, in general it is not quite clear, how they are differentiated, because the variable $t$ runs through matrices over $\mathbb{C}$ while the admissible transformations of rows are over $\mathbb{R}$. At the same time, the integration procedure is transparent enough (due to its construction) in the case when all new double points, arising by Differentiation, have double "parents" (this is a fact for all the considered algorithms, except of VIII, X and XII).

**Corollary 15.3.** *For any of the algorithms* VII–XVII, *except of* VIII, X *and* XII, (*as well as for the operation of completion*) *each complex series of indecomposables of the derivative set* $\mathscr{P}'$ *is transformed by integration into a complex series of the initial set* $\mathscr{P}$.

It appears, in tame case the analogous statement takes place in fact for all the constructed algorithms, including VIII, X and XII. This is due to the proved in Section 17 (see Theorem 17.1) fact that each of the corresponding complex series has some special support called a *special tower*. Now we present a preliminary Corollary 15.4 used in the proof of Theorem 17.1.

Recall that the *support* of a representation $U$ of a set $\mathscr{P}$ is a subset in $\mathscr{P}$ of the form $\operatorname{Supp} U = \operatorname{Supp} d = \{x \in \mathscr{P} \mid d_x > 0\}$, where $d = \underline{\dim}\, U$.

We call an equipped poset with involution $\mathscr{T}$ a *special tower* if it is an ordinal sum $\mathscr{T} = \{\mathscr{L}_1 < \mathscr{L}_2 < \cdots < \mathscr{L}_n\}$ satisfying three conditions:

(1) $n \geqslant 2$;
(2) each *link* $\mathscr{L}_i$ is either a bidouble point or a simple double dyad;
(3) if a link $\mathscr{L}_i$ is a simple double dyad, then at least one of the relations $\mathscr{L}_{i-1} < \mathscr{L}_i$ and $\mathscr{L}_i < \mathscr{L}_{i+1}$ is week.

[13] In the sense that almost all indecomposables of given dimension, generated over $\mathbb{R}$ by one $(\mathbb{R}[t], \mathbb{C}[t])$-representation, pass again into indecomposables generated over $\mathbb{R}$ by one $(\mathbb{R}[t], \mathbb{C}[t])$-representation.

Let $U$ be a representation of a special tower $\mathscr{T} = \{\mathscr{L}_1 < \cdots < \mathscr{L}_n\}$. We call $U$ *strict* if two next conditions are satisfied:

(1) $\bigcap_{x\in\mathscr{L}_1} U_x = 0,\ \sum_{x\in\mathscr{L}_n} \widehat{U}_x = \widetilde{U}_0$ and also $\sum_{x\in\mathscr{L}_i} \widehat{U}_x = \bigcap_{x\in\mathscr{L}_{i+1}} U_x$ for all $i \in \overline{2, n-1}$;

(2) $U_x^- = 0$ for $x \in \mathscr{L}_1$, $U_x^+ = U_0$ for $x \in \mathscr{L}_n$ and also $U_x^+ = U_y^-$ in the case $x \in \mathscr{L}_i,\ y \in \mathscr{L}_{i+1}$ and $x \lhd y$ for some $i \in \overline{2, n-1}$.

**Corollary 15.4.** *Let $\mathscr{P}$ be an equipped poset with involution satisfying the conditions* (a) *and* (b) *of Theorem* C, *and $\mathscr{P}'$ be the derivative set obtained from $\mathscr{P}$ by applying one of Differentiations* I–V, $\overline{\alpha}$, VII–XVII (*or by the operation of completion*). *Assume $U = W^{\uparrow}$ is a sincere indecomposable representation of $\mathscr{P}$, where $W$ is an indecomposable representation of $\mathscr{P}'$ such that $\mathscr{T} =$* Supp $W$ *is a special tower and the restriction $W \mid \mathscr{T}$ is strict. Then $\mathscr{P}$ is a special tower too and $U$ is its strict representation.*

Note that the verification of the last statement is a rather routine procedure which is being done for each algorithm separately (including several auxiliary operations preceding some of the main differentiation algorithms). These calculations show that only the algorithms XIII−XVII play an essential role in the description of complex series. As for the rest, they are in fact superfluous (since their action is in fact trivial in the corresponding "complex" situation).

The next statement follows immediately from the constructions presented in Sections 3−14.

**Corollary 15.5.** *If an infinite series of indecomposables of equal dimension is annihilated by some of Differentiations* VII–XVII, *then it is included* (*up to isomorphism and up to finite number of indecomposables*) *into one of the following four series*:

(a) *the standard real series $M_1 - 6$ of the critical set $M_1 = \{a, b\}$ in the case of Differentiation* XI;

(b) *the standard real series $L_1 - 11$ of the critical set $L_1 = \{a, r, s\}$ in the case of Differentiation* XII;

(c) *the complex series* (10.10) *of the set $\Omega = \{\{a_1, a_2\} \prec \{b_1, b_2\}\}$, where $a_i, b_i$ are simple double, in the case of Differentiation* XIII;

(d) *the complex series* (14.5) (*for the type* (b)) *of the set $\Sigma = \{a \prec a^*\}$ with $g(a) = -1$, in the case of Differentiation* XVII.

Clearly, all algorithms built for a smaller class may be also used in a larger class if there exists the corresponding suitable group of points. More precisely, for each given Differentiation the definition of a suitable group of points in a larger class is quite natural and actually does not differ from the smaller situation (sufficiently to have all wider structures “hidden” inside the “passive” cones $A$ and $B$ occurred in each Differentiation). Taking this in account, we use below in the proof of the main Theorem C the full spectrum of the algorithms I$-$V, $\overline{\alpha}$ from [23], the completion and the algorithms VII$-$XVII presented in Sections 3–14 above.

Moreover, it is used one more simple operation of restoring the reducibility condition (R), which (just as for ordinary posets with involution [2]) can be broken by Differentiation. The condition (R) is restored easily by applying (the maximal possible number of times) the trivial operation $\mathscr{P} \longrightarrow \mathscr{P}'_a$ described briefly in [2] and mentioned already briefly in Section 1. Recall that if $a \in \mathscr{P}$ is a big point and $N(a) = \{c_1 < \cdots < c_n\}$ is a small chain, $n \geqslant 0$, then $\mathscr{P} = \{X < a < Y\} + N(a)$, where $X = a_\Delta \backslash a,\ Y = a^\nabla \backslash a$, and the set $\mathscr{P}'_a$ is obtained from $\mathscr{P}$ by deleting the point $a$ and replacing the equivalent point $a^*$ by a small chain of $n+2$ points $d^- < d_1 < \cdots < d_n < d^+$. The matrix interpretation of this simple reducing procedure may be given in the form

$$
M = \begin{array}{cccc|c|cccc|c|cccc|l}
 & \boldsymbol{a^*} & & & \boldsymbol{X} & & \boldsymbol{a} & & & \boldsymbol{Y} & \boldsymbol{c_1} & \longrightarrow & \boldsymbol{c_n} & & \\
\hline
 & & & & & & & & E & & & & & & \\
 & & & & & & & E & & & & & & E & \Big\} S \\
 & & & & & & E & & & & & E & & & \\
\hline
 & & & & & & & & & * & * & & * & & \\
* & * & * & * & * & & & & & & * & & * & & U_X^+ \ \Big\} T \\
\hline
d^- & d_1 & d_n & d^+ & & & & & & & c_1 & & c_n & &
\end{array}
$$

obtained from the initial matrix representation $M$ of $\mathscr{P}$ by the following sequence of steps (assume $a^* \in X$):

(1) place at the bottom of $M$ the horizontal stripe corresponding to the subspace $U_X^+$;
(2) consider above the stripe $U_X^+$ a subproblem on representations over $\mathbb{R}$ of a trivial ordinary poset $\{a; c_1 < \cdots < c_n\}$ and write in the upper horizontal stripe $S$ its direct summands of type $P(a), P(a, c_1), \ldots, P(a, c_n)$;
(3) make obvious zeroes under the summands $P(a), P(a, c_i)$ in the stripe $U_X^+$ and obtain in the stripe $T$ the problem on representations of the set $\mathscr{P}'_a$.

Therefore, it holds:

**Lemma 15.6.** *Let $\mathscr{P}$ be an equipped poset with involution containing such a big point $a$ that the set $N(a)$ of all points, incomparable with $a$, is a small chain. Then*:

(a) *there exists a natural one-to-one correspondence between indecomposables of the sets $\mathscr{P}$ and $\mathscr{P}'_x$ which induces a one-to-one correspondence between the corresponding series of representations*;
(b) *the equipped poset $\mathscr{P}$ is tame* (*wild*) *if and only if the equipped poset $\mathscr{P}'_a$ is tame* (*wild*).

## 16. Combinatorics

In this purely combinatorial Section under a tame (ordinary) poset we mean a poset not containing the subsets $N_1, \dots, N_6$.

We say that a subset $\Gamma$ of an equipped poset with involution is a *garland* if: (1) $\Gamma$ is a semichain; (2) $\Gamma$ contains no bidouble points; (3) all dyads in $\Gamma$ are small; (4) all order relations in $\Gamma$ are strong. [14]

**Lemma 16.1** [2, Lemma 1]. *Let $S$ be a tame ordinary poset with at least two minimal elements. Then $S$ contains* I*-suitable pair of points $(a_1, b)$ or* V*-suitable quadruple of points $(a_1, a_2, b_1, b_2)$, where $a_1, a_2 \in \min S$.*

The following lemma generalizes to the equipped situation the corresponding statement for ordinary posets with involution (see [2, Lemma 2]).

**Lemma 16.2.** *Let $Q$ be an equipped poset with primitive involution which contains at least two minimal or two maximal elements, has tame evolvent $\overset{\vee}{Q}$ and does not contain weak order relations between points. Then there exists a collection of points in $\mathscr{P}$ suitable to one of Differentiations* I–V, $\overline{\alpha}$, VII *and* X–XII *considered up to duality.*

**Proof.** Assume $|\min Q| \geqslant 2$. The points $x', x'' \in \overset{\vee}{Q}$, corresponding to a big or double *paternal* point $x \in Q$, will be called *new*. All other points in $\overset{\vee}{Q}$ (which coincide with small points in $Q$) are called *old*.

(a) If there exists some big or double point $a \in \min Q$, then $\Gamma = Q \backslash a^{\triangledown}$ is a garland, hence, for $\max \Gamma = \{b\}$ the pair $(a, b)$ is $\overline{\alpha}$, II, III, X, VII (with $C = \varnothing$), X*— or XI-suitable, and for $\max \Gamma = \{b_1, b_2\}$ the triple $(a, b_1, b_2)$ is IV-suitable or XII-suitable.

(b) Let $\min Q$ be a small set. Due to Lemma 16.1 the evolvent $\overset{\vee}{Q}$ contains I-suitable pair $(a_1, b)$ or V-suitable quadruple $(a_1, a_2, b_1, b_2)$, where $a_i \in \min \overset{\vee}{Q}$ and also $a_i \in \min Q$.

Assume that $(a_1, b)$ is I-suitable pair of points in $\overset{\vee}{Q}$. If the point $b$ is old, then the pair $(a_1, b)$ is I-suitable in $Q$, and if $b$ is new, with the paternal point $\overline{b}$, then $(a_1, \overline{b})$ is $\overline{\alpha}*$-suitable or VII*-suitable pair in $Q$.

Assume that $(a_1, a_2, b_1, b_2)$ is V-suitable quadruple in $\overset{\vee}{Q}$. If the points $b_1, b_2$ are old, then this quadruple is V-suitable in $Q$. But if the point $b_1$ is new, with the paternal

[14] In other words, a garland is any "strong" ordinal sum of small points, of big points, of simple double points and of small dyads.

point $\overline{b}_1$, then automatically $\overline{b}_1$ is the paternal point for $b_2$ and the triple $(a_1, a_2, \overline{b}_1)$ is IV*-suitable or XII*-suitable in $Q$. This finishes the proof. $\square$

For any subset $A$ of an equipped poset with involution $\mathscr{P}$ the sets $A_{\text{prim}}$ and $\overset{\vee}{A}_{\text{prim}}$ are defined naturally (completely analogously to the case $A = \mathscr{P}$). And the subset $A$ will be called *admissible* if it satisfies two conditions:

(a) each bidouble point of $A$ is comparable with all other points in $A$;
(b) the evolvent $\overset{\vee}{A}_{\text{prim}}$ of the set $A_{\text{prim}}$ is tame.

**Remark.** If $A$ itself is closed with respect to action of the involution (in particular, if $A = \mathscr{P}$), we obtain the notion of an admissible equipped poset with involution.

**Lemma 16.3.** *Let $\mathscr{P}$ be an admissible equipped poset with involution containing at least one pair of weakly comparable points. Then $\mathscr{P}$ contains a special pair of points or a collection of points suitable for one of Differentiations* I, VII–IX *and* XIII–XVII *considered up to duality.*

**Proof.** Consider a weak relation $a \prec b$ in $\mathscr{P}$ of the greatest possible length (i.e. with the maximal $n$ for which there exists a chain $a = x_0 \prec x_1 \prec \cdots \prec x_n = b$). Denote $X = \{x \mid x \prec b \text{ and } x \notin [a, b]\}$, $Y = \{y \mid a \prec y \text{ and } y \notin [a, b]\}$, $C = N(a, b)$. As the relation $a \prec b$ is of maximal length and $\mathscr{P}$ is admissible, it holds $X \subset N(a)$, $Y \subset N(b)$, $|X| \leqslant 1$, $|Y| \leqslant 1$, $C$ is a small chain, $|C| \leqslant 2$ and also $X \lhd C \lhd Y$ for $C \neq \varnothing$ and $X \lhd Y$ or $X \prec Y$ for $C = \varnothing$. Moreover, it is easy to see that $\mathscr{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma + C + X + Y$, where $\Sigma$ is the interior of the interval $[a, b]$, and $X \prec \Sigma \prec Y$. Set $A = a^{\triangledown} \backslash a$, $B = b_{\vartriangle} \backslash b$. Consider possible cases (up to duality).

(1) Both points $a, b$ are simple double.
  (a) Assume $C = \varnothing$. Then $X = x$ and $Y = y$ are one-point sets, otherwise the pair $(a, b)$ is special, IX-suitable or IX*-suitable. Hence, $\mathscr{P} = [a, x, y, b] + A + B$ and also $A \subset \{y, b\}^{\text{up}}$; $B \subset \{a, x\}_{\text{down}}$ and $\{a, x\} \lhd A$; $B \lhd \{y, b\}$. Thus for $x \lhd y$ we have IX-suitable pair $(x, b)$ and for $x \prec y$ we have XIII-suitable quadruple $(a, x, y, b)$.
  (b) Assume $C = \{c\}$. If $X = Y = \varnothing$, it appears VII-suitable pair of points $(a, b)$. If $X = x$ and $Y = \varnothing$, then $B \subset \{a, x\}_{\text{down}}$ and $\Sigma$ is a simple double chain. And since $a_{\wedge} \backslash c_{\vartriangle}$ is a garland (incomparable with the point $x$), then for $\xi \in \min(\mathscr{P} \backslash c_{\vartriangle})$ the pair $(\xi, c)$ is I-suitable (VII-suitable) for ordinary (simple double) point $\xi$. If, at last, $X = x$, $Y = y$, then, as in a), the pair $(x, b)$ is IX-suitable.
  (c) Assume $C = \{d < c\}$. Here $\Sigma = \varnothing$ and for $|X| = |Y| = 1$ the pair $(x, b)$ is, as before, IX-suitable. Suppose $Y = \varnothing$ and $X$ is arbitrary and $\xi \in \min(\mathscr{P} \backslash c_{\vartriangle})$. Since $a, \xi \in N(c, d)$, then $\xi \leqslant a$ and, as it is easy to verify, again the pair $(\xi, c)$ is I-suitable (VII-suitable) for ordinary (double) point $\xi$.

(2) The point $a$ is bidouble and the point $b$ is simple double. Here $N(a) = X = C = \varnothing$ and $B < a$, hence, for $Y = \varnothing$ (for $Y = y$) we have XIV-suitable pair of points $(a, b)$ (XV-suitable triple $(a, y, b)$).
(3) Both points $a, b$ are bidouble. Then $N(a) = N(b) = X = Y = C = \varnothing$ and the pair of points $(a, b)$ is XVI-suitable or XVII-suitable. This finishes the proof of the lemma. □

A subset of an equipped poset with involution will be called a *tower* if it is an arbitrary ordinal sum of any double points and of simple double diads (with arbitrary order relations, both weak and strong). Obviously, each tower is an admissible subset.

We call a tower $\mathscr{T}$ *closed* if every of the subsets $\min \mathscr{T}$, $\max \mathscr{T}$ (independently of each other) coincides with either a bidouble point or a simple double dyad. We say that a tower $\mathscr{T}'$ *condenses* a tower $\mathscr{T}$ if $\mathscr{T} \subset \mathscr{T}'$ and $\min \mathscr{T} = \min \mathscr{T}'$, $\max \mathscr{T} = \max \mathscr{T}'$.

The following lemma is rather obvious.

**Lemma 16.4.** *Let $\mathscr{P}$ is an admissible equipped poset with involution, and $\mathscr{P}_1$ be an equipped poset with involution obtained from $\mathscr{P}$ by applying any of the following operations*:

(a) *some big point is replaced by a garland*;
(b) *between some small dyad $\{p, q\}$ with the condition $p^{\triangledown} \backslash p = q^{\triangledown} \backslash q = A$ $(p_{\vartriangle} \backslash p = q_{\vartriangle} \backslash q = A)$ and the subset $A$ a garland is inserted*; [15]
(c) *between some double point $a$ with the condition $a^{\vee} \backslash a = a^{\triangledown} \backslash a = A$ $(a_{\wedge} \backslash a = a_{\vartriangle} \backslash a = A)$ and the subset $A$ a garland is inserted*;
(d) *some closed tower is condensed*;
(e) *some bidouble point is replaced by a tower*.

*Then the set $\mathscr{P}_1$ is admissible too*.

From the previous lemmas we derive the following important fact.

**Lemma 16.5.** *If $\mathscr{P}$ is an admissible equipped poset with involution, then under each of Differentiations* I–V, $\overline{\alpha}$ *and* VII–XVII *considered up to duality* (*and also under the reducing procedure*) *the derivative equipped poset with involution $\mathscr{P}'$ is admissible too*.

**Proof.** Actually, for all Differentiations, except I, VII, VIII and IX, (and for the reducing procedure) the statement follows immediately from the geometric construction of the algorithms (see the corresponding diagrams in the text) and from

[15] That is, it is assumed that $\{p, q\} \lhd \Gamma \lhd A$ $(A \lhd \Gamma \lhd \{p, q\})$, where $\Gamma$ is the inserted garland.

Lemma 16.4. In any of this cases the derivative set $\mathscr{P}'$ is obtained from $\mathscr{P}$ by applying one or several listed in Lemma 16.4 operations plus, possibly, additions of some order relations.

As for the algorithms I and VII−IX, the evolvent $\overset{\vee}{\mathscr{P}}'_{\text{prim}}$ is obtained from the evolvent $\overset{\vee}{\mathscr{P}}_{\text{prim}}$ by a single or double application of Differentiation I for ordinary poset (see Remarks 3.1, 5.1 and 6.1). But it is well known that if an ordinary poset contains none of the subsets $N_1, \ldots, N_6$, then the derivative one neither does (remind that this fact was proved by purely combinatorial means in [24], Proposition 2.1, and by using of the Tits quadratic form in [21], Section 2, item 4). Therefore, the condition (b) of the definition of an admissible set is preserved for Differentiations I and VII–IX as well. Since the preservation of the condition (a) is rather obvious, this completes the proof of the lemma. $\square$

## 17. Proof of Theorem C

We have to prove both the tameness and wildness statements of Theorem C.

*Tameness.* Here we establish the following more general result.

**Theorem 17.1.** *If a reduced equipped poset with involution $\mathscr{P}$ satisfies the conditions* (a) *and* (b) *of Theorem* C, *then $\mathscr{P}$ is tame. Moreover, if such $\mathscr{P}$ has a sincere complex series of indecomposables, then $\mathscr{P}$ is a special tower and the mentioned series is strict.* [16]

**Proof.** We have to show, in particular, that $\mu(d) < \infty$ for the dimension vector $d = \underline{\dim}\, U$ of any sincere indecomposable representation $U$ of $\mathscr{P}$. This will be proved (as well as the second part of the theorem) by induction on the number $d_0 = \dim_{\mathbb{R}} U_0$.

The base of induction is the case of the nondifferentiable sets or nondifferentiable sincere complex series. As it follows from the previous considerations, except of the trivial one-point sets, the only nondifferentiable sincere one is the set $\Psi = \{a \lhd a^*\}$ with $a \sim a^*$ being bidouble (of any genus), which is one-parameter according to Lemma 2.1. And the only nondifferentiable sincere complex series are strict series of special towers of the form (10.10) and (14.5).

So, one may assume $|\mathscr{P}| \geqslant 2$ and $\mathscr{P} \neq \Psi$ and $U$ is not a strict representation of the form (10.10) or (14.5). Then, moreover, if $\mathscr{P}$ contains no weak relations, it contains no bidouble points, i.e. $\mathscr{P} = \mathscr{P}_{\text{prim}}$ (otherwise, since $U$ is sincere and indecomposable, due to the condition (a) of Theorem C it must be $\mathscr{P} = \Psi$).

[16] That is, each representation, belonging to this series, is strict.

It is also clear, due to the conditions (a) and (b) of the theorem, that for each big or simple double point $x \in \mathscr{P}$ the subset $N(x)$ of all points incomparable with $x$ is a garland. In addition, one can assume that for any two points $x \neq y$ the following natural condition takes place

$$U_x^+ \subset U_y^- \;\Rightarrow\; x \lhd y. \tag{17.1}$$

Otherwise we may complete $\mathscr{P}$ by the relation $x \lhd y$ and consider $U$ as a representation of the completed poset. For the same reason (under a nontrivial $U$) the set $\mathscr{P}$ may be considered as not containing special pairs of simple double points (see Lemma 4.1).

Now, if $\mathscr{P}$ contains no weak relations, then (as it was mentioned) $\mathscr{P} = \mathscr{P}_{\text{prim}}$ and due to the indecomposability of $U$ and reducibility of $\mathscr{P}$ it holds $|\min \mathscr{P}| \geqslant 2$. Hence, according to Lemma 16.2 there exists a collection of points in $\mathscr{P}$ suitable for one of Differentiations I–V, $\overline{\alpha}$, VII (with $C = \varnothing$) and X–XII or for one of the dual algorithms.

If $\mathscr{P}$ contains weak relations, then by Lemma 16.3 it contains a collection of points suitable for one of the algorithms I, VII–IX and XIII–XVII or for one of the dual algorithms.

Therefore, in each case the set $\mathscr{P}$ contains a collection of points suitable for one of Differentiations I–V, $\overline{\alpha}$ and VII–XVII (considered up to duality). Applying the chosen algorithm (together with the subsequent reducing procedure) to all indecomposables of a given dimension $d$, we obtain, in general, several (but a finite number) derivative vectors $d^{\downarrow}$ (being the dimensions of the reduced derivative representations $U^{\downarrow}$) satisfying, due to the condition (17.1) and Corollary 15.1, the inequality $d_0^{\downarrow} < d_0$. In accordance with Lemma 16.5 these vectors $d^{\downarrow}$ correspond to equipped posets with involution with the restrictions (a) and (b) of Theorem C. Now from the inductive hypothesis we have $\mu(d^{\downarrow}) < \infty$ for all these $d^{\downarrow}$. Hence, in accordance with Corollaries 15.1, 15.2 and 15.4 of the present paper and also Corollaries 1 and 2 from [23], it holds $\mu(d) < \infty$. Moreover, due to the induction and Corollary 15.4, the second part of Theorem 17.1 takes place too. This finishes the proof both of Theorem 17.1 and of the tameness statement of Theorem C.

*Wildness.* Let $\mathscr{P}$ be a reduced equipped poset with involution not satisfying one of the conditions (a) and (b) of Theorem C. We have to verify that $\mathscr{P}$ possesses some wild generator (which will be written below in a matrix form).

Assume the condition (a) is not satisfied. Then $\mathscr{P}$ contains a dyad $\{a, b\}$, where $a$ is bidouble and $b$ is arbitrary.

If $b = a^*$, take a matrix representation $M$ of $\mathscr{P}$ with $M_{a^*} = E$, $M_x = \varnothing$ for $x \neq a, a^*$ and $M_a = Z$ for arbitrary matrix $Z$ over $\mathbb{C}$ of the corresponding size. It leads immediately to the classic wild matrix problem on the pair of matrices $(X, Y)$ over $\mathbb{R}$, where $X + \mathrm{i}Y = Z$.

If $b \neq a^*$, then in the worst (for the catching of wildness) case one can consider $b$ as a small point and also suppose, for example, that $a^* \lhd \{a, b\}$. Then, setting again $M_{a^*} = E,\ M_x = \varnothing$ for $x \neq a, a^*, b$ and using strong column additions $M_{a^*} \longrightarrow M_a, M_b$, we receive in the stripes $M_a$ and $M_b$ (considered over $\mathbb{C}$ and $\mathbb{R}$ respectively) the well-known wild problem on representations of a triad $\Upsilon = \{x, x^*, b\}$, where $x \sim x^*$ is a pencil. It has, for instance, in the dimension $d = (d_0; d_{(x,x^*)}, d_b) = (2; 2, 1)$ a wild generator of the form

$$\begin{array}{cc|cc|c} \multicolumn{2}{c}{x} & \multicolumn{2}{c}{x^*} & b \\ \hline X & Y & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

where $X, Y$ are the variables of the corresponding free algebras $\mathbb{R}\langle X, Y\rangle$ and $\mathbb{C}\langle X, Y\rangle$.

Assume the condition (b) is not satisfied, i.e. the evolvent $\overset{\vee}{Q}$ of the set $Q = \mathscr{P}_{\text{prim}}$ contains one of the subsets $N_1, \dots, N_6$. One can show easily in this case $Q$ itself contains one of the ordinary posets $N_1, \dots, N_6$ or one of the equipped posets $W_1, \dots, W_9$ listed in Section 1, or one of the following subsets:

$$W_{10} = \{\otimes\ \bullet\ \circ\},\quad W_{11} = \{\otimes\ \bullet\ \bullet\},\quad W_{12} = \{\otimes \otimes \bullet\},\quad W_{13} = \left\{\begin{array}{c}\otimes\\ |\\ \otimes\end{array}\ \bullet\right\},$$

$$W_{14} = \{\bullet\ \circ\ \circ\,\circ\},\quad W_{15} = \{\bullet\ \bullet\ \circ\},\quad W_{16} = \{\bullet\ \bullet\ \bullet\},\quad W_{17} = \left\{\bullet\ \circ\ \begin{array}{c}\circ\\ |\\ \circ\end{array}\right\}.$$

Let us investigate the possible situations.

(a) The case of the subsets $N_1, \dots, N_6$ needs no considerations as it was solved in [12] (note that here everything is reduced to the set $\Upsilon$, since in the sequence $\Upsilon, N_1, \dots, N_6$ each previous set is contained in the derivative of the next one under a suitable differentiation).

We also will take in account that, due to [2], each ordinary reduced poset with involution $S$, containing one of the subsets $W_{14}, \dots, W_{17}$ is wild. Such $S$ is called below a *counterset*.

(b) In the case of the equipped posets $W_1, \dots, W_9$, one may pay not attention to the last three sets since in the sequence $W_4, W_7, W_8, W_9$ each previous set is contained in the derivative of the next one under a suitable variant of Differentiation VII. For the rest sets $W_1, \dots, W_6$ the minimal wild generators are as follows:

$$
\begin{array}{c} W_1 \\ \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 0 \\ Z & 1 & 0 & 1 \\ \hline \end{array} \\ \begin{array}{cccc} \otimes & & & \end{array} \end{array}
\quad
\begin{array}{c} W_2, W_3 \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ Z & i & 0 \\ \hline \end{array} \\ \begin{array}{ccc} \otimes & \otimes & x \end{array} \end{array}
\quad
\begin{array}{c} W_4 \\ \begin{array}{|cc|cc|c|c|} \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & Z & 0 & 0 & 1 & 0 \\ i & 1 & 0 & 0 & 0 & 1 \\ \hline \end{array} \\ \begin{array}{cccc} \otimes & & & \longrightarrow \end{array} \end{array}
\quad
\begin{array}{c} W_5 \\ \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 0 \\ 1 & Z & 0 & 1 \\ i & 0 & 0 & 0 \\ \hline \end{array} \\ \begin{array}{cccc} \otimes & \otimes & & \end{array} \\ \longrightarrow \end{array}
\quad
\begin{array}{c} W_6 \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ 1 & Z & i \\ i & 0 & 0 \\ \hline \end{array} \\ \begin{array}{ccc} \otimes & \otimes & \otimes \end{array} \\ \longrightarrow \end{array},
$$

with $Z = X + \mathrm{i}Y$ where $X, Y$ are the corresponding variables.

(c) Obviously, the set $W_{12}$ is not simpler than the set $W_2$, and the sets $W_{10}$ and $W_{14}$ are transformed into $W_4$ and $N_2$ respectively when reducing the stripe $M_{x^*}$, where $x$ is the unique shown big point (one can assume $N(x^*) = \varnothing$ in the worst for wildness case).

(d) Assume $Q \supset W_{11} = \{a, x, y\}$, where $a$ is double. If $x^* = y$, then $W_{11}$ is not simpler than $\Upsilon$. If $x^* \neq y$, we can reduce the stripes of the points $x^*, y^*$ (being comparable in the worst case with all points of $W_{11}$) and to obtain an equipped subset of type $(\widetilde{1}, 2, 2) \supset W_4$.

(e) Assume $Q \supset W_{16} = \{x, y, z\}$. If $x^* = y$, the set $\{x, x^*, z\}$ is not simpler than $\Upsilon$. That is why consider the points $x, y, z$ as mutually nonequivalent.

If $x^* \in N(y^*)$, we have a simpler counterset $\{x, y, x^*, y^*, \xi\}$, where $\xi$ is a small point replacing $z$.

If $x^* \in N(x)$, then, reducing the stripes of the points $y^*, z^*$, we obtain as a subset a counterset $\{x; x^*; p; q_1 < q_2\}$, where $p, q$ are small.

So, the points $x^*, y^*, z^*$ are mutually comparable and, moreover, each of them is comparable with each of the points $x, y, z$. But then one of them is not comparable with some other point , say $\xi$, which (in the worst case) is small. Therefore, reducing the stripes of the points $x^*, y^*, z^*, \xi$, we obtain a subset $N_3$.

(f) Assume $Q \supset W_{13} = \{a \prec b; x\}$, where $x$ is big. If $x^* \in N(x)$, then (in the worst case) $Q$ contains the subset $\{x^* < a \prec b; x\}$ which is wild. Really, applying D-III in [23] with respect to the pair $(x^*, x)$ together with the subsequent reduction of all stripes $M_{x_i^*}$, we obtain the set of type $(\widetilde{2}, \infty) \supset (\widetilde{2}, 3) = W_7$.

If $x^* \in N(a)$, then in the worst case (and up to duality) $Q$ contains the subset $\{x > x^* < b \succ a\}$ and the application of D-X with respect to the pair $(x^*, a)$ leads to an equipped poset $\{q; a \prec b\} = W_6$.

If $x^*$ is comparable with $x, a, b$, then it is not comparable with some other point $\xi$ (which is small in the worst case) and after reduction of the stripes, corresponding to the points $x^*, \xi$, we obtain an equipped subset of the form $(\widetilde{2}, 3) = W_7$.

(g) Assume $Q \supset W = W_{17} = \{x_0; p; q_1 < q_2\}$, where $x_0$ is big. Then in each sequence $X = \{x_0, x_0^*, x_1, x_1^*, \ldots, x_n, x_n^*\}$, $n \geqslant 0$, with the condition $x_i \in N(x_{i-1}^*)$ all points are different (otherwise $W \cup X$ is a counterset) and the subset $N(x_n^*)$ contains no small dyads (otherwise $W \cup X \cup D$ is a counterset, where $D$ is the mentioned dyad). Hence, there exists such a sequence $X$, for which the subset $N(x_n^*)$

contains a double point $a$. Now it is enough to use the placed below fact (#) describing the operation of "substitution of a double point ". This operation is nothing else as some modified equipped analog of the explained in Section 15 operation $\mathscr{P} \longrightarrow \mathscr{P}'_x$ (see Lemma 15.6). It also admits a simple matrix explanation (in the spirit of D-X) which is left for the reader as an exercise.

(#) Let $S$ be an equipped poset with primitive involution and without weak relations containing some big point $x$ incomparable with at least one double point. Denote by $S^{\#}_x$ the set obtained from $S$ by deleting the point $x$ and replacing the equivalent point $x^*$ by a new double point. Then there exists a natural one-to-one correspondence between all indecomposables of the set $S^{\#}_x$ (considered up to finite number of trivial ones) and some *part* of indecomposables of the initial set $S$, which induces the one-to-one correspondence between their series.

Setting now $S = W \cup X \cup \{a\}$ and applying step by step the operation $S^{\#}_x$ for $x = x^*_n, x^*_{n-1}, \dots, x^*_0$, we obtain an equipped poset containing the subset $W_4$ (at the place of $W$).

(h) At last, assume $Q \supset W = W_{15} = \{x_0, y_0, p\}$, where $x_0, y_0$ are big. Then $x^*_0 \neq y_0$, otherwise $W = \Upsilon$. Arguing completely analogously to the previous case, we build two sequences $X = \{x_0, x^*_0, \dots, x_n, x^*_n\}$ and $Y = \{y_0, y^*_0, \dots, y_m, y^*_m\}$, where $n, m \geqslant 0$, with the conditions $x_i \in N(x^*_{i-1})$ and $y_j \in N(y^*_{j-1})$ such that the subsets $N(x^*_n)$ and $N(y^*_m)$ contain double points $a$ and $b$ respectively (possibly, $a = b$). Moreover, all points in $X$ and $Y$ together are different. Setting now $S = W \cup X \cup Y \cup \{a, b\}$ and applying the operation $S^{\#}_t$ step by step for $t = x^*_n, x^*_{n-1}, \dots, x^*_0$ and $t = y^*_m, y^*_{m-1}, \dots, y^*_0$, we obtain, as a result, an equipped poset containing the subset $W_2$ (at the place of $W$). This finishes the proof of Theorem C. $\square$

Theorems A and B are special cases of Theorem C, and Theorem D is its simple consequence (deduced with help of the reducing procedure based on Lemma 15.6).

The equivalence of the conditions (a)–(c) in Lemma 1.1 is left as an easy exercise for the reader. One has to take in account, in particular, that if $d$ is a vector, corresponding to an equipped poset $\mathscr{P}$, and $\overset{\vee}{d}$ is its naturally defined vector-evolvent, corresponding to the set $\overset{\vee}{\mathscr{P}}$ (namely, $\overset{\vee}{d}_0 = d_0$, $\overset{\vee}{d}_x = d_x$ for $x \in \mathscr{P}^0$ and $\overset{\vee}{d}_{x'} = \overset{\vee}{d}_{x''} = d_x$ for $x \in \mathscr{P} \backslash \mathscr{P}^0$), then $f_{\mathscr{P}}(d) = f_{\overset{\vee}{\mathscr{P}}}(\overset{\vee}{d})$.

## Notations for Appendix A–C

Below in Appendix A–C by $E_n$ we denote the $n \times n$ identity matrix. If some matrix is equipped with an arrow of the form $\leftarrow, \rightarrow, \uparrow, \downarrow$, it means that a zero column or row must be added to that matrix respectively from the left, right, above, below (in the case of several arrows one has to add several zero columns or rows).

For a square matrix $X$ set $P_X(\lambda) = \det(\lambda E - X)$.

Denote by $J_n(0)$, $J_n(1)$ the $n \times n$ Jordan blocks with the eigenvalues 0 and 1 (in which the adjacent to the main diagonal identities can be written in arbitrary way: both under and above the diagonal). By $J_n^+(0)$ ($J_n^-(0)$) we denote the Jordan block with the identities written just above (below) the diagonal.

$$\text{For } F = \boxed{\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}}\,,\ \Delta = \boxed{\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}} \text{ and } T_{2n} = \underset{2n \times 2n}{\boxed{\begin{array}{cccccc} F & \Delta & & & & \\ & F & \Delta & & \mathrm{O} & \\ & & F & & & \\ & & & \ddots & & \\ & \mathrm{O} & & & F & \Delta \\ & & & & & F \end{array}}}$$

set $P_{2n} = T_{2n} + \mathrm{i}E_{2n}$,

$$C_n = \underset{(n+1) \times n}{\boxed{\begin{array}{c} iE_n \\ \hline \begin{array}{cccc} 0 & \dots & 0 & 1 \end{array} \end{array}}}\,, \qquad Q_{2n+1} = \underset{(2n+2) \times (2n+1)}{\boxed{\begin{array}{c|c} 1 & 0 \ \ 0 \ \ \dots \ 0 \\ i & 1 \ \ 0 \ \ \dots \ 0 \\ \hline 0 & \\ \vdots & P_{2n} \\ 0 & \end{array}}}$$

The number $h$ denotes the "step" of the corresponding dimension $d = \underline{\dim}\, U$; $f = f(d)$ is the value of the quadratic form $f$ and $\partial = \partial(U) = \partial(d) = (\mu, d)$ is the *defect* of a representation , where ( , ) is the corresponding to an equipped poset $\mathscr{P}$ nonsymmetric bilinear form given by the formula

$$(d, h) = d_0 h_0 + \sum_{x \in \mathscr{P}} f_x d_x h_x + \sum_{x > y} p_{xy} f_x f_y d_x h_y - h_0 \sum_{x \in \mathscr{P}} f_x d_x.$$

In the part B (C) of Appendix A–C the type $\widetilde{m}$ of a representation $U$ is obtained from the type $m$ by the permutation $a \leftrightarrow b$ ($r \leftrightarrow s$).

The matrix classification of the part A was obtained in [11], and of the parts B and C in [26] (with the use of the results from [6] for the part B).

## Appendix A. Representations of "2×2-quadruple" $\begin{array}{|c|c|}\hline * & * \\ \hline * & * \\ \hline\end{array}$ over any field k

With independent $k$-elementary transformations of each horizontal or vertical stripe

(up to permutations of the horizontal or vertical stripes)

| I | II | III | IV | V | VI | VII |
|---|---|---|---|---|---|---|
| $\begin{array}{\|c\|c\|}\hline E_n^{\uparrow} & E_n^{\downarrow} \\ \hline E_n & E_n \\ \hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline E_{n+1} & E_n^{\uparrow} \\ \hline E_{n+1} & E_n^{\downarrow} \\ \hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline E_n^{\leftarrow} & E_n^{\rightarrow} \\ \hline E_{n+1} & E_{n+1} \\ \hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline E_n & E_n^{\leftarrow} \\ \hline E_n & E_n^{\rightarrow} \\ \hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline E_{n+1} & E_n^{\uparrow} \\ \hline E_n^{\rightarrow} & E_n \\ \hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline E_n & E_n \\ \hline J_n(0) & E_n \\ \hline\end{array}$ | $\begin{array}{\|c\|c\|}\hline E_n & E_n \\ \hline X & E_n \\ \hline\end{array}$ |

$X$ is a Frobenius block over $k$ with $P_X(\lambda) \neq \lambda^n$

## Appendix B. Representations of the critical set $M_1 = \{\overset{a}{\otimes}\ \overset{b}{\otimes}\}$ over $(\mathbb{R},\mathbb{C})$

(up to permutation of the points $a \leftrightarrow b$)

$\begin{array}{cc} a & b \\ \end{array}$
$\begin{array}{|c|c|}\hline A & B \\ A' & B' \\ \hline\end{array}$ – representation $U$ in the matrix form

$k_1 = dim_{\mathbb{R}} U_a^-$ $\quad l_1 = codim_{\mathbb{R}} U_a^+$

$k_2 = dim_{\mathbb{R}} U_b^-$ $\quad l_2 = codim_{\mathbb{R}} U_b^+$

$k_3 = dim_{\mathbb{C}} U_a \cap U_b$ $\quad l_3 = codim_{\mathbb{C}}(U_a + U_b)$

$\mu = \begin{smallmatrix} 1 & 1 \\ & 2 \end{smallmatrix}$

| Type | $\underline{min}$ $\underline{dim}$ | $h$ | $f$ | $\partial$ | $k_1$ | $k_2$ | $k_3$ | $l_1$ | $l_2$ | $l_3$ | $A$ | $B$ | $A'$ | $B'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\begin{smallmatrix}00\\1\end{smallmatrix}$ | $\mu$ | 1 | $-2$ | 0 | 0 | 0 | 1 | 1 | 1 | $E_n^{\uparrow}$ | $E_n^{\downarrow}$ | $iE_n$ | $iE_n$ |
| $1^*$ | $\begin{smallmatrix}11\\1\end{smallmatrix}$ | $\mu$ | 1 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | $E_n^{\rightarrow}$ | $E_n^{\leftarrow}$ | $iE_{n+1}$ | $iE_{n+1}$ |
| $2 = \widetilde{2}^*$ | $\begin{smallmatrix}10\\1\end{smallmatrix}$ | $\mu$ | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | $E_{n+1}$ | $E_n^{\uparrow}$ | $iE_n^{\rightarrow}$ | $iE_n$ |
| 3 | $\begin{smallmatrix}10\\2\end{smallmatrix}$ | $\mu$ | 2 | $-2$ | 0 | 0 | 0 | 0 | 2 | 1 | $E_{n+1}$ | $E_n^{\uparrow}$ | $iE_{n+1}$ | $iE_n^{\downarrow}$ |
| $3^*$ | $\begin{smallmatrix}12\\2\end{smallmatrix}$ | $\mu$ | 2 | 2 | 0 | 2 | 1 | 0 | 0 | 0 | $E_n$ | $E_n^{\rightarrow}$ | $iE_n$ | $iE_n^{\leftarrow}$ |
| $4 = 4^*$ | $\mu$ | $\mu$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | $E_n$ | $E_n$ | $iJ_n(0)$ | $iE_n$ |
| $5 = 5^*$ | $\mu$ | $\mu$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | $E_n$ | $E_n$ | $iJ_n(1)$ | $iE_n$ |
| $6 = 6^*$ | $\begin{smallmatrix}\mu\\2\mu\end{smallmatrix}$ | $\begin{smallmatrix}\mu\\2\mu\end{smallmatrix}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $E_n$ | $E_n$ | $iX$ | $iE_n$ |

$X$ is a Frobenius block over $\mathbb{R}$ with $P_X(\lambda) \neq \lambda^n, (\lambda - 1)^n$ and $|det\, X| \leq 1$

## C. Representations of the critical set $L_1 = \{\overset{a}{\otimes}\ \overset{r}{\circ}\ \overset{s}{\circ}\}$ over $(\mathbb{R}, \mathbb{C})$

(up to permutation of the points $r \leftrightarrow s$)

$$U \leftrightarrow \begin{array}{c} \begin{array}{ccc} a & r & s \end{array} \\ \boxed{\begin{array}{c|c|c} A & R & S \\ A' & R' & S' \end{array}} \end{array} \qquad \mu = \begin{array}{ccc} 1 & 1 & 1 \\ & 2 & \end{array}$$

$k_1 = dim_{\mathbb{R}} U_a^-$ $\quad l_1 = codim_{\mathbb{R}} U_a^+$

$k_2 = dim_{\mathbb{R}} U_r \cap \underline{U}_s$ $\quad l_2 = codim_{\mathbb{R}}(U_r + \underline{U}_s)$

$k_3 = dim_{\mathbb{C}} U_a \cap \widetilde{U}_r$ $\quad l_3 = codim_{\mathbb{C}}(U_a + \widetilde{U}_r)$

$k_4 = dim_{\mathbb{C}}(U_a \cap \widetilde{U}_s)$ $\quad l_4 = codim_{\mathbb{C}}(U_a + \widetilde{U}_s)$

$X$ is any Frobenius block over $\mathbb{R}$

| Type | $min$ $\underline{dim}$ | $h$ | $f$ | $\partial$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $A$ | $R$ | $S$ | $A'$ | $R'$ | $S'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 000 1 | $\mu$ | 1 | $-2$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | $J_n^-(0)$ | $E_n$ | $E_n$ | $C_n$ | $O$ | $E_n^{\uparrow}$ |
| $1^*$ | 111 1 | $\mu$ | 1 | 2 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | $E_n^{\rightarrow}$ | $O$ | $E_n^{\leftarrow}$ | $C_n^{\leftarrow}$ | $E_{n+1}$ | $E_{n+1}$ |
| 2 | 010 1 | $\mu$ | 1 | $-1$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | $E_n$ | $O$ | $E_n$ | $iE_n^{\downarrow}$ | $E_{n+1}$ | $E_n^{\uparrow}$ |
| $2^*$ | 101 1 | $\mu$ | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | $E_n^{\rightarrow}$ | $E_n$ | $O$ | $C_n^{\leftarrow}$ | $E_n^{\downarrow}$ | $E_{n+1}$ |
| 3 | 110 2 | $\mu$ | 1 | $-1$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | $E_{n+1}$ | $O$ | $E_n^{\downarrow}$ | $iE_{n+1}$ | $E_{n+1}$ | $E_n^{\uparrow}$ |
| $3^*$ | 112 2 | $\mu$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | $E_n^{\rightarrow}$ | $E_n^{\leftarrow}$ | $O$ | $C_{n+1}$ | $E_{n+1}^{\downarrow}$ | $E_{n+2}$ |
| 4 | 100 1 | $\mu$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | $E_{n+1}$ | $O$ | $E_n^{\downarrow}$ | $iE_n^{\leftarrow}$ | $E_n$ | $E_n$ |
| $4^*$ | 011 1 | $\mu$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | $E_n$ | $O$ | $E_n^{\leftarrow}$ | $iE_n^{\downarrow}$ | $E_{n+1}$ | $E_{n+1}$ |
| 5 | 122 4 | $2\mu$ | 2 | $-2$ | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | $E_{2n}^{\uparrow\uparrow\rightarrow}$ | $E_{2n+2}$ | $E_{2n+2}$ | $C_{2n+1}$ | $O$ | $E_{2n+2}$ |
| $5^*$ | 322 4 | $2\mu$ | 2 | 2 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | $E_{2n+2}^{\rightarrow}$ | $O$ | $E_{2n+2}$ | $C_{2n+1}^{\leftarrow\leftarrow}$ | $E_{2n+2}$ | $E_{2n+2}$ |
| 6 | 100 2 | $2\mu$ | 2 | $-2$ | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | $E_{2n}^{\rightarrow}$ | $E_{2n}$ | $E_{2n}$ | $C_{2n+1}$ | $O$ | $E_{2n}^{\uparrow\uparrow}$ |
| $6^*$ | 122 2 | $2\mu$ | 2 | 2 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | $E_{2n}^{\rightarrow}$ | $O$ | $E_{2n}^{\leftarrow\leftarrow}$ | $C_{2n+1}$ | $E_{2n+2}$ | $E_{2n+2}$ |
| $7 = \widetilde{7}^*$ | 120 2 | $2\mu$ | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | $E_{2n}^{\leftarrow}$ | $O$ | $E_{2n}$ | $Q_{2n+1}$ | $E_{2n+2}$ | $O$ |
| $8 = 8^*$ | $2\mu$ | $2\mu$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | $E_{2n}$ | $E_{2n}$ | $O$ | $P_{2n}$ | $O$ | $E_{2n}$ |
| $9 = 9^*$ | $\mu$ | $\mu$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | $E_n$ | $O$ | $E_n$ | $iJ_n^+(0)$ | $E_n$ | $E_n$ |
| $10 = 10^*$ | $\mu$ | $\mu$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | $E_n$ | $O$ | $J_n^+(0)$ | $iE_n$ | $E_n$ | $E_n$ |
| $11 = 11^*$ | $\mu$ $2\mu$ | $\mu$ $2\mu$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $E$ | $O$ | $E$ | $iE$ | $E$ | $X$ |

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