Quotient Rings and Localization for Noetherian Rings

R. B. Warfield, Jr.*

University of Washington, Seattle, Washington 98195

Communicated by A. W. Goldie

Received July 23, 1980

Criteria are obtained for the localizability of a Noetherian ring at a semiprime ideal $S$, particularly when $S$ is the nil radical of $R$. The results are applied to the existence of quotient rings. A decomposition theorem is obtained for certain Noetherian rings which admit Artinian quotient rings.

In order to prove that a Noetherian ring $R$ with nil radical $N$ has an Artinian quotient ring, it is necessary to show that the set $\mathcal{O}(N)$ of elements regular modulo $N$ forms an Ore set. In Section 2 we obtain several criteria for this (i.e., for the localizability of the semiprime ideal $N$), one of which is parallel to Small's theorem on quotient rings [18], and another of which states that $\mathcal{O}(N)$ is Ore if and a certain list of prime ideals consists entirely of minimal primes. It is easily seen that $R$ has an Artinian quotient ring if and only if $\mathcal{O}(N)$ is Ore and, in addition, $\text{Ass}(R_{\alpha})$ consists entirely of minimal primes, so we obtain an analogue of the standard commutative result on quotient rings.

Ginn and Moss [3] show that if $R$ is Noetherian and has an Artinian quotient ring, then $R$ is the product of an Artinian ring and a ring with zero socle. The analogous result for higher Krull dimension is easily seen to be false, even for semiprime rings. However, in Section 3 we show that if for a particular ring $R$ with an Artinian quotient ring, the ring $R/N$ has a decomposition into a factor of Krull dimension less than $\alpha$ and a factor with no nonzero right ideals of Krull dimension less than $\alpha$, then (given some standard ideal invariance and symmetry conditions involving Krull dimension) $R$ has a similar decomposition.

In Section 4 we obtain a partial generalization of the results of Section 2, replacing the ideal $N$ by an arbitrary semiprime ideal $S$ such that $S/N$ is localizable in $R/N$. The condition for $S$ to be localizable involves the behavior of certain bimodules, and can be expressed as saying that a certain

* This research was supported in part by a grant from the National Science Foundation.
family of primes in $R/S$ consists entirely of minimal primes. As a corollary, we obtain a condition for $R$ to have a full quotient ring.

The criteria in this paper for the localizability of the nil radical $N$, and later for an arbitrary semiprime ideal $S$, can be expressed as the requirement that a certain set of primes arising as associated primes of bimodules all be minimal primes (in $R$ or in $R/S$). How useful such a criterion is depends on how easy it is in practice to verify that one has actually checked all the primes one needs to check. We have therefore attempted to make clear in each case that the criteria we use are finite, in that there are a finite number of bimodules whose associated primes must be checked, and we also indicate how such a list of bimodules may be found (Theorems 6 and 16).

The work reported in this paper was initially undertaken while the author was visiting at the University of Leeds in early 1977, when the work in Sections 1 and 2 (except for some terminological changes) was done. The work in Sections 3 and 4 was done in 1979. While it has long been clear that bimodules could be used to provide an obstruction to localizability (e.g., [8]), only recently have situations been found when they can also be used to provide sufficient conditions for localizability, the best results being Müller's in [17]. The author's work on localizability reported here overlaps with recent unpublished work of A. Jategaonkar, as is indicated in more detail in remarks in connection with Theorems 6 and 16.

Throughout this paper, $R$ will be a ring with identity, and except in a few lemmas it will be Noetherian (on both sides). Since we are concerned throughout with classical localizations and quotient rings, we close this introduction by summarizing the basic facts about inverting the elements in a subset of a ring.

We recall that if $R$ is a ring and $C$ a subset, $C$ is right Ore if, for all $c \in C$ and $r \in R$, there are a $c' \in C$ and an $r' \in R$ such that $cr' = rc'$. Equivalently, we call an $R$-module $A$ $C$-torsion if for all $a \in A$, there is a $c \in C$ with $ac = 0$, and $C$ is right Ore if for all $c \in C$, $R/cR$ is $C$-torsion.

A set $C$ is right reversible if for $r \in R$, the existence of an element $c \in C$ with $cr = 0$ implies there is a $c' \in C$ with $rc' = 0$.

**Lemma 1.** Let $R$ be a ring and $C$ a multiplicatively closed subset which is right Ore. Then

(i) For any right $R$-module $A$, the subset $t_C(A) = \{a \in A: ac = 0 \text{ for some } c \in C\}$ is a submodule of $A$.

(ii) If $R$ is right Noetherian, the right Ore set $C$ is right reversible.

(iii) If $C$ is right reversible, then there is a localization $RC^{-1}$ and a natural map $\phi: R \to RC^{-1}$ whose kernel is $t_C(R)$ and whose cokernel is $C$-torsion as a right $R$-module. For all $c \in C$, $\phi(c)$ is invertible in $RC^{-1}$, and every element of $RC^{-1}$ is of the form $\phi(r) \phi(c)^{-1}$ for some $r \in R$, and $c \in C$. 


Furthermore, the map $\phi$ is universal with respect to homomorphisms of $R$ into rings which take the elements of $C$ to invertible elements. $RC^{-1}$ is flat as a left $R$-module and the kernel of the localization map $A \to A \otimes RC^{-1}$ (taking $a$ to $a \otimes 1$) is $t_{C}(A)$.

Proof. The first point is obvious, the second is a remark in [12, p. 1075] and the third is in [2, p. 415].

If $C$ is a right and left Ore set in $R$, we call the ring $RC^{-1}$ a quotient ring for $R$ if the natural map $R \to RC^{-1}$ is injective, (i.e., if $t_{C}(R_{C}) = 0,$) and a full quotient ring if, in addition $C = Q(0)$, the set of all regular elements of $R$.

1. NOETHERIAN BIMODULES

If $R$ and $S$ are rings and $R S$ is an $R-$S bimodule, then $B$ is a Noetherian bimodule if it is Noetherian on both sides (i.e., as a right $S$-module and as a left $R$-module). In our applications, we will start with a Noetherian ring $R$, and consider bimodules of the form $I/J$, where $I$ and $J$ are ideals of $R$ and $J \leq I$. These bimodules are clearly Noetherian and we refer to them as bimodules which are (bimodule) subfactors of $R$. It will sometimes be desirable to regard such a subfactor as a bimodule over other rings, for example, if $B$ is a subfactor of $R$ and $NB = BN = 0$, we may want to regard $B$ as an $R/N-R/N$ bimodule.

We make frequent use in what follows of Goldie's notion of reduced rank [1]. If $R$ is a Noetherian ring, we let $N$ be the nil radical of $R$ and $Q$ the right quotient ring of $R/N$. If $A$ is any right $R$-module, then, for some integer $n$, there is a sequence of submodules $0 = A_{0} < A_{1} < \cdots < A_{n} = A$ such that for all $i$, $A_{i+1}N \leq A_{i}$. We let $\rho(A)$ be the sum of the uniform ranks of the right $Q$-modules $(A_{i+1}/A_{i}) \otimes Q$. It is an easy consequence of the flatness of $Q$ over $R/N$ and the Schreier refinement theorem that this is well defined. It is also clear that $\rho(A_{R}) = 0$ if and only if $A$ is $\mathcal{S}(N)$-torsion. For a bimodule $R S$, where $R$ and $S$ are Noetherian, there are two reduced ranks, $\rho(B_{S})$ and $\rho(R B)$.

Lemma 2. Let $R$ and $S$ be rings and $R B_S$ a Noetherian bimodule. Then

(i) If $r(B)$ and $l(B)$ are the right and left annihilators of $B$ (in $S$ and $R$, respectively), then $R/l(B)$ is a left Noetherian ring and $S/r(B)$ is a right Noetherian ring.

(ii) If $r(B) = P$, where $P$ is prime, then either $B_{S/P}$ is nonsingular or there is a sub-bimodule $D$ such that $r(D)$ is a prime ideal properly containing $P$.

(iii) If $R/l(B)$ is semiprime, $B$ is left nonsingular over $R/l(B)$, and $D$ is a sub-bimodule of $B$ which is essential on the left, then $\rho(B/D)_S = 0$. 

Proof: Let \( X = \{x_1, \ldots, x_n\} \) be a finite subset of \( B \) that generates \( B \) as a left module and as a right module. Since \( X \) generates \( B \) on the left, it follows that \( r(B) = \cap_{i=1}^n r(x_i) \). Hence, \( S/r(B) \) is isomorphic to a right submodule of a direct sum of \( n \) copies of \( B \), and hence \( S/r(B) \) is right Noetherian. This proves (i). For (ii), we suppose that \( B_{S/P} \) is not nonsingular, and let \( T \) be its right singular submodule. By the same argument as above, \( S/r(T) \) is isomorphic to a right submodule of a finite direct sum of copies of \( T \), and hence is singular over \( S/P \). Now let \( D \) be a sub-bimodule of \( T \) whose right annihilator is maximal (among the right annihilators of nonzero submodules of \( T \)). It is easy to see that \( r(D) \) is prime and properly contains \( P \). Finally if the hypotheses of (iii) hold and \( D \) is a submodule essential on the left, and \( X \) is a subset which generates \( B \) on both sides, then there is a regular element \( c \) in \( R/(\ell(B)) \) such that \( cX \leq D \). It follows that \( cB \leq D \). Since \( B \) is left nonsingular, \( cB \) is right isomorphic to \( B \), so \( \rho(cB_S) = \rho(B_S) \). It follows that \( \rho(B/cB)_S = \rho(B/D)_S = 0 \).

**Definition (Jategaonkar).** If \( _RB_S \) is a Noetherian bimodule, \( B \) is a cell if (i) \( r(B) \) and \( \ell(B) \) are prime ideals, and (ii) every nonzero sub-bimodule is essential on both sides.

Lemma 2 implies that if \( B \) is a cell and the right and left annihilators are \( P \) and \( Q \), then \( B \) is nonsingular as a left \( R/P \) and as a right \( S/Q \)-module, and also that given condition (i), condition (ii) holds on both sides if it holds on one.

If \( B \) is a Noetherian bimodule, it is clear from Lemma 2(ii) that we can bind a series of sub-bimodules \( 0 = B_0 < B_1 < \cdots < B_n = B \) such that \( B_{i+1}/B_i \) is a cell (\( 0 \leq i < n \)). Such a series of sub-bimodules is called a cellular series. The resulting factors \( B_{i+1}/B_i \) are by no means unique. However, the Schreier refinement theorem easily shows that the set of cells, whose left annihilators are minimal primes, are unique up to order and subisomorphism, and the same holds on the right. We will call these the left maximal cells of \( B \) and the right maximal cells of \( B \), respectively. It is clear that there are no left maximal cells if and only if \( \rho(\ell(B)) = 0 \).

We reiterate the important parts of these remarks as a lemma for emphasis.

**Lemma 3.** Let \( _RB_S \) be a Noetherian bimodule and \( 0 = B_0 < B_1 < \cdots < B_n = B \) and \( 0 = B'_0 < B'_1 < \cdots < B'_n = B \) two cellular series of sub-bimodules. Let \( \{C_1, \ldots, C_m\} \) and \( \{C'_1, \ldots, C'_k\} \) be the factors of the first and second series, respectively, which are left maximal cells. Then \( m = k \), and after suitable renumbering we have \( r(C_i) = r(C'_i) \), \( \ell(C_i) = \ell(C'_i) \), and there are sub-bimodules \( D_i \leq C_i \), \( D'_i \leq C'_i \) which are essential on each side, and such that \( D_i \cong D'_i \), \( 1 \leq i \leq m \). In particular, the primes \( r(C_i) \) (\( 1 \leq i \leq m \)) are uniquely determined independently of the choice of cellular series.
To require that the primes \( r(C_i) \) be minimal primes, in the above lemma, is the same as requiring that the left maximal cells of \( B \) be right maximal. Most of the localization criteria of this paper can be expressed as the condition that, for certain bimodules, the cells which are maximal on one side be maximal on the other. The remaining two lemmas of this section are technical results giving conditions on a bimodule which imply or are equivalent to this condition (that left maximal cells are right maximal.)

**Lemma 4.** Let \( _RB_S \) be a Noetherian bimodule and suppose that \( R \) is left Noetherian and semiprime, and that \( B \) is left nonsingular (or, equivalently, \( \text{Ass}(R) \) consists exclusively of minimal primes.) Then there is a cellular series \( 0 = B_0 < B_1 < \cdots < B_n = B \) such that for all \( i \) (\( 0 \leq i < n \)), either \( t(B_{i+1}/B_i) \) is a minimal prime, or \( r(B_{i+1}/B_i) \) is not a minimal prime. (Equivalently, all of the right maximal cells of \( B \) are left maximal.)

**Proof.** We choose a series of sub-bimodules \( 0 = D_0 < D_1 < \cdots < D_m = B \) such that \( B_i/D_i \) is left nonsingular, and every sub-bimodule of \( D_{i+1}/D_i \) is left essential (\( 0 < i < n \)). The bimodules \( D_{i+1}/D_i \) are not necessarily cells, because we do not know enough about their structure on the right. However, Lemma 2(iii) implies that if we take a cellular series for \( D_{i+1}/D_i \), the first cell (starting from the bottom) will have a minimal prime as its left annihilator, and the rest will have zero right reduced rank—that is, their right annihilators will be nonminimal primes. This proves Lemma 4.

**Lemma 5.** Let \( _RB_S \) be a Noetherian bimodule, where \( R \) and \( S \) are left and right Noetherian, respectively, but \( B \) is not assumed to be faithful on either side, and let \( N \) and \( M \) be the nil radicals of \( R \) and \( S \). Then the following four conditions on \( B \) are equivalent.

(i) All the right maximal cells of \( B \) are left maximal.

(ii) If \( B' \) is a bimodule subfactor of \( B \) and \( t(B') = P \) and \( r(B') = Q \) are prime ideals in \( R \) and \( S \), and \( B' \) is nonsingular as a left \( R/P \) and as a right \( S/Q \)-module, and \( P \) is minimal, then so is \( Q \).

(iii) Every \( R/N-S/N \) bimodule which is a (bimodule) subfactor of \( B \) and which is left singular (over \( R/N \)) is also right singular.

(iv) There is a series of sub-bimodules \( 0 = B_0 < B_1 < \cdots < B_n = B \) satisfying (a) \( NB_{i+1} \leq B_i \) (\( 0 \leq i < n \)) and (b) for each \( i \), \( B_{i+1}/B_i \) as a left \( R/N \)-module is either singular or nonsingular, such that for each \( i \) such that \( B_{i+1}/B_i \) is singular on the left, \( p(B_{i+1}/B_i)_S = 0 \).

**Proof.** Assuming (i), we prove (ii) by noticing that if we take a cellular series for \( B' \), at least one right maximal cell must arise, but no left maximal cells could arise if \( P \) were not minimal. Of course, (i) is trivially a special
case of (ii). To prove (iii), note that such a bimodule has no left maximal cells, and so, if (i) holds, must have no right maximal cells, so it has zero right reduced rank, and hence is right singular as an $S/M$-module. If (iii) holds, then again it is trivial that (i) holds, so we now need to show that (i) and (iv) are equivalent.

We suppose that (i) holds and that we have a series of sub-bimodules of $B$ satisfying conditions (a) and (b) of (iv). (These always exist.) Suppose that $B_{i+1}/B_i$ is a factor which is singular as a left $k/N$-module. Then in a cellular series for $B_{i+1}/B_i$, none of the cells can be left maximal, and hence by (i), none will be right maximal. Hence, $p(B_{i+1}/B_i)_s = 0$. Conversely, if there is a particular series of sub-bimodules of $B$ satisfying the conclusions of (iv), then we must show that (i) holds. Refining the given series into a cellular series, we see that no left maximal cells arise from factors $B_{i+1}/B_i$ which are left singular (over $R/N$), and Lemma 4 implies that the left maximal cells arising from the other factors are right maximal. This proves that (iv) implies (i) and completes the proof of Lemma 5.

2. LOCALIZING AT $N$ AND ARTINIAN QUOTIENT RINGS

A commutative Noetherian ring has an Artinian quotient ring if and only if all of its associated primes are minimal. This study began with the attempt to find a noncommutative theorem which was appropriately parallel to this one, and which reduced to this in the commutative case. In 1977, the author found a criterion concerning associated primes which was equivalent to the existence of a two-sided artinian quotient ring for two-sided Noetherian rings (Theorem 9 below). The result was extremely one-sided, and obviously not the right result, and so was not published; but it was the first criterion which reduced the quotient ring problem to the minimality of a specific computable finite set of primes. (An earlier result relating quotient rings and the minimality of certain primes appears in [20].) In this section we give a more refined analysis of the same question. We let $R$ be a Noetherian ring and $N$ its nil radical. We let $\mathscr{P}(N)$ be the set of elements of $R$ which are regular modulo $N$. We first seek conditions guaranteeing that $\mathscr{P}(N)$ is a left or right Ore set. This is equivalent to looking for conditions to guarantee that the well-known quotient ring for $R/N$ extends to a localization for $R$ itself. We obtain necessary and sufficient conditions for this. Appropriately, the conditions are vacuously verified in the commutative case, and one version can be expressed as the requirement that a certain (finite) set of primes be minimal. We then see that the existence of an Artinian quotient ring is equivalent to the requirement that $\mathscr{P}(N)$ be right Ore (trivial in the commutative case) plus the condition that the associated primes be minimal.
We recall that for any right module $M$, an ideal $I$ is \textit{associated} to $M$ if there is a submodule $N \leq M, N \neq 0$, such that for all nonzero submodules $N'$ of $N, I = \text{Ann}(N')$. It is easy to see that $I$ is a prime ideal. The set of associated primes is denoted $\text{Ass}(M)$. If $B$ is an $R-S$ bimodule, we have two sets of associated primes, $\text{Ass}_R(B)$ and $\text{Ass}(B_S)$.

**Definition.** In any Noetherian ring $R$, let
\[ g^R(O) = \{ c \in R : \rho(r(c)) = 0 \} \]
and
\[ g^R(O) = \{ c \in R : \rho(\ell(c)) = 0 \}. \]

Note that by an easy reduced rank argument, both of the above sets are contained in $g^R(N)$.

**Theorem 6.** The following properties of a Noetherian ring $R$ are equivalent.

(i) $g^R(N)$ is right Ore.

(ii) $g^R(N) = g^R_*(0)$.

(iii) All of the right maximal cells of $R$ are left maximal.

**Remark.** The equivalence of (i) and (ii) is the analogue in this situation of Small's theorem [19, 11. Lemma 5 contains three other conditions equivalent to (iii). Note that (iii) is a finite criterion in that it gives a specific finite set of primes to check.

**Proof.** We first remark that there is an analogue of the pseudo-Ore condition (as in [4, 2.5]): if $r \in R$ and $c \in g^R_*(0)$, there is a $c' \in g^R(N)$ with $ac' \in cR$. This is an immediate consequence of the fact that $\rho(R/cR) = 0$. This shows immediately that (ii) implies (i).

We next recall that if $g^R(N)$ is right Ore, then so is its image in any factor ring $R/I$, and by Lemma 1, (ii), the image of $g^R(N)$ in $R/I$ is also right reversible. Hence, if $B$ is an $R-R$ bimodule which is a subfactor of $R$, and if $B$ is right $g^R(N)$-torsion-free then $B$ is also left $g^R(N)$-torsion-free. This immediately implies condition (iii).

We must now show that (iii) implies (ii). We suppose that (ii) is false, so that there is a $c \in g^R(N), c \in g^R_*(0)$, so that $\rho(r(c)) > 0$. We use the chain of ideals $r(N^k)$ and find an integer $k$ such that $r(c) \cap r(N^{k+1})$ has larger reduced rank (on the right) than $r(c) \cap r(N^k)$. (We admit the possibility that $k = 0$, where $r(N^0) = 0$.) Since $r(N^{k+1})/r(N^k)$ is a left $R/N$-module, it has a (left) singular submodule $T$ which we regard as an $R-R$ bimodule. Clearly $\rho(\ell T) = 0$, but since $T$ contains a right submodule isomorphic to
we see that \( \rho(T_n) > 0 \). If we take a cellular series for the bimodule \( T \), then none of the resulting cells will be left maximal, but one at least will have positive reduced-rank on the right, and hence will be right maximal. Hence (iii) fails if (ii) fails. This completes the proof of the theorem.

**Corollary 7.** If \( R \) is Noetherian and satisfies any of the conditions of the above theorem, then \( R \) has an Artinian quotient ring if and only if \( \text{Ass } R \) consists of minimal primes.

**Proof:** The condition is clearly necessary. If it were to hold without \( \mathcal{E}(N) \) being left Ore, then by Theorem 6, \( \mathcal{E}(N) \neq \mathcal{E}_*(0) \), so there would be a \( c \in \mathcal{E}(N) \) with \( \ell(c) \neq 0 \). This is impossible, since the condition on \( \text{Ass } R \) implies that \( \tau_{\mathcal{E}(N)}(R_\mathcal{R}) = 0 \). Hence we get a two-sided localization, and its kernel is zero, again because of the condition on \( \text{Ass } R_\mathcal{R} \).

**Remark.** Jategaonkar has obtained independently a result similar to Corollary 7—that if \( R \) is Noetherian, \( \text{Ass}(R_\mathcal{R}) \) consists of minimal primes, and \( R \) satisfies condition (iii) of Theorem 6 and the corresponding condition with left and right reversed, then \( R \) has an Artinian quotient ring.

**Corollary 8.** A Noetherian semiprime ring \( S \) has the property that for all Noetherian rings \( R \) with prime radical \( N \) and \( R/N \cong S \), \( N \) is a localizable semiprime ideal, if and only if there is no \( S-S \) bimodule, finitely generated on each side, which is singular on one side and nonsingular on the other.

**Proof.** Theorem 6 and Lemma 5.

**Corollary 9.** A Noetherian ring \( R \) with nil radical \( N \) has an Artinian classical quotient ring if and only if the elements of \( \text{Ass}(R_\mathcal{R}) \) and \( \text{Ass} R/r(N^k) \) (where \( k \geq 0 \)) are all minimal primes.

**Remark.** This result was proved by the author in 1977, and an alternative proof has been given in [5].

**Proof.** The condition is clearly necessary. Conversely, using the sequence of ideals \( r(N^k) \), the condition shows that the successive factors are all nonsingular on the left, so Lemma 5 and Theorem 6 show that \( \mathcal{E}(N) \) is right Ore. The result now follows from Corollary 7.

There is some redundancy in this result, in that to show \( \mathcal{E}(N) \) is right Ore, it is not in fact necessary that the primes in \( \text{Ass}(R/r(N^k)) \) should all be minimal. The result might suggest that in general the primes in the list arising from Theorem 6(iii) (the left annihilators of right maximal cells) might all appear among those of \( \text{Ass}(R/r(N^k)) \) (for various \( k \)) but this also is false, as the following example shows. Let \( A = (\mathbb{Z} \to \mathbb{Z}) \) and \( I = M_2(\mathbb{Z}) \), and
Let $R = \left( \begin{array}{c} A' \\ \frac{A}{I} \end{array} \right)$. Let $S$ be the socle of $R$, which is also a minimal prime with $R/S \cong A$. Let $M$ be the other minimal prime and $P$ and $Q$ the two primes corresponding to the primes in $A$ minimal over $I$. Then
\[
 \text{Ass}_R(r(N)) = \{P, S\},
\]
\[
 \text{Ass}_R(r(N^2)/r(N)) = \{M\},
\]
while the set of left annihilators of the right maximal cells is $\{P, Q, S, M\}$.

3. DECOMPOSITIONS OF RINGS WITH ARTINIAN QUOTIENT RINGS

In this section we give conditions for a Noetherian ring with an Artinian quotient ring to be a product of rings which are homogeneous with respect to Krull dimension, and we remark how the ideas of this paper can be used to give yet another proof of the converse result—that a Noetherian ring which is homogeneous has an Artinian quotient ring. As is usual when working with Krull dimension, we find it necessary to introduce some additional hypotheses. These hypotheses, as usual, have the frustrating feature that they are rather diverse, and yet there are no known examples in which any of them fail to hold.

We use the notion of (noncommutative) Krull dimension, as in [7], and if $R_B$ is a Noetherian bimodule, we use $|B|$ to denote the Krull dimension of $B$ as a right $S$-module, and $|B|_l$ to denote the Krull dimension of $B$ as a left $R$-module. We will generally use the following hypotheses on a ring $R$:

- $K_1(\alpha)$: If $P$ is a prime ideal, then $|R/P|_l < \alpha$ if and only if $|R/P|_r < \alpha$.
- $K_2(\alpha)$: If $P$ is a prime ideal and $|R/P|_r < \alpha$ and $T$ is an ideal, then $|T/PT|_l < \alpha$.
- $K_2(\alpha)'$: If $P$ is a prime ideal and $|R/P|_l < \alpha$ and $T$ is an ideal, then $|T/TP|_l < \alpha$.

The above three conditions all together are equivalent (easily) to the following condition, sometimes called Krull symmetry:

- $B(\alpha)$: For any $R$-bimodule $B$ which is a subfactor of $R$, $|B|_l < \alpha$ if and only if $|B|_r < \alpha$.

The condition that $K_3(\alpha)$ should hold for all $\alpha$ was introduced by the author in unpublished 1977 notes, and has been used in [9] (where it is called "right prime ideal invariance") and [21] (where it is called "condition ($)"). No Noetherian rings are known which fail to satisfy these conditions for all $\alpha$, but the only value of $\alpha$ for which they are known to be true is $\alpha = 1$. 
(Condition $K_1(1)$ follows from Wedderburn's theorem and $K_2(1)$ is an important result of Lenagan [13].)

For any ordinal $\alpha$ there is an ideal $S_\alpha(R)$ such that $|S_\alpha(R)|_r < \alpha$ and every nonzero right ideal of $R/S_\alpha(R)$ has Krull dimension $\geq \alpha$. We want conditions to guarantee that there is a central idempotent $e$ of $R$ such that $S_\alpha(R) = eR$, or, equivalently, for $R$ to be a product $R_1 \times R_2$, where $|R_1|_r < \alpha$ and $S_\alpha(R_2) = 0$.

**Theorem 10.** Let $R$ be a Noetherian ring with nil radical $N$ and $\alpha$ an ordinal such that $R$ satisfies hypothesis $B(\alpha)$, and suppose that $R$ has an Artinian quotient ring. Then $R$ has a product decomposition $R = R_1 \times R_2$ with $|R_1|_r < \alpha$ and $S_\alpha(R_2) = 0$ if and only if the same conclusion holds for $R/N$.

**Remark.** Since $B(1)$ holds for all Noetherian rings, and every semiprime right Goldie ring is the product of a right Artinian ring and a ring with trivial right socle, this result contains the theorem of Ginn and Moss [3] that a Noetherian ring with an Artinian quotient ring is a product of an Artinian ring and a ring with a trivial socle.

**Lemma 11.** Let $R$ be a Noetherian ring and $\alpha$ an ordinal. Then (i) $S_\alpha(R) = R$ if and only if $S_\alpha(R/N) = R/N$. (ii) If $R$ satisfies $K_2(\alpha)$ and $S_\alpha(R) = 0$ then $S_\alpha(R/N) = 0$, and (iii) If $R$ has an Artinian quotient ring and $S_\alpha(R/N) = 0$, then $S_\alpha(R) = 0$.

**Proof.** The first statement follows from the fact that a homomorphic image of a module of Krull dimension less than $\alpha$ has Krull dimension less than $\alpha$, and the fact that as a right $R$-module, $R$ is a finite extension of finitely generated modules annihilated by $N$. Statement (ii) was proved by Lenagan for fully bounded rings in [14], and by the author, using a similar argument, in general in his unpublished 1977 notes. However, Stafford's approach to similar results in [21] seems more natural, so we refer to [21, 3.2], for the proof that $|R/P|_r \geq \alpha$ for all minimal primes $P$, noting that for a given $\alpha$, $K_2(\alpha)$ can be used instead of what he calls (2), and that every minimal prime is affiliated. Finally, if $R$ has a classical quotient ring and $S_\alpha(R) \neq 0$, then there is a $P \in \text{Ass}(R_R)$ such that $|R/P|_r < \alpha$. (This follows from Lemma 2 and the fact that a finitely generated nonzero, nonsingular $R/P$-module has the same Krull dimension as $R/P$.) Since $R$ has a classical quotient ring, $P$ must be minimal, so $P \in \text{Ass}(R/N)_R$, and $S_\alpha(R/N) \neq 0$.

**Proof of Theorem 10.** We first suppose that $R = R_1 \times R_2$, where $S_\alpha(R_1) = R_1$ and $S_\alpha(R_2) = 0$. Then we get an induced decomposition of $R/N$, and parts (i) and (ii) of Lemma 11 show that this decomposition of $R/N$ is of the desired type.
We then suppose that \( e \) is an idempotent of \( R \) such that its image \( \bar{e} \) in \( R/N \) is central, and \(|\bar{e}(R/N)|, < \alpha \) while \( S_\alpha((1 - \bar{e})R/N) = 0 \). We will show by induction on \( \rho(N_\alpha) \) that \( e \) is central and that \( eR = S_\alpha(R) \). If either \( e \) or \( 1 - e \) is 0, then parts (i) and (iii) of Lemma 11 show that this is correct. We assume then that \( e \neq 0 \), and part (ii) of Lemma 11 shows that \( S_\alpha(R) \neq 0 \). If \( S_\alpha(R) \cap N = 0 \), then the result is trivial, so we assume not. Let \( P \in \text{Ass}(S_\alpha(R) \cap N) \). We claim that \( \ell(P) \) is nonsingular as a right \( R/P \)-module. If not, Lemma 2 implies that \( \ell(P) \) contains a right ideal \( I \) with \( r(I) \) properly containing \( P \), which violates the condition that \( \text{Ass}(R) \) consists entirely of minimal primes (since \( R \) has an Artinian quotient ring). Let \[
abla = \ell(P) \cap N \cap r(N).
\]
Clearly, \( \nabla \neq 0 \). We claim that \( R/\nabla \) also has an Artinian quotient ring, so that by induction (on \( \rho(N) \)) we may assume the theorem is true for \( R/\nabla \). To see this, one checks that if \( Q \) is the quotient ring for \( R \), then \( \ell_Q(P) \) (the left annihilator of \( P \) in \( Q \)) and \( r_\alpha(N) \) are both ideals of \( Q \), and that \( \ell_Q(P) \cap R = \ell(P), \ r_\alpha(N) \cap R = r(N), \) and \( \text{nil}(Q) \cap R = N \). It follows that if \[
abla' = r_\alpha(N) \cap \ell_Q(P) \cap \text{nil}(Q),
\]
Then \( \nabla' \cap R = \nabla \) and \( Q/\nabla' \) is an Artinian quotient ring for \( R/\nabla \).

We may assume by induction that \(|(eR + B)/B|, < \alpha \) and \( S_\alpha([(1 - e)R + B]/B) = 0 \). Hence, \( S_\alpha(R) = eR + B \) and to show \( S_\alpha(R) = eR \), we need only show that \( \nabla \cap [(1 - e)R] = 0 \), since \( \nabla \) is an ideal. If \( r \in \nabla \cap [(1 - e)R] \), then \( er = 0 \) and \( Nr = 0 \), so right multiplication by \( r \) induces a homomorphism of left modules \((1 - e)R/N \to B \). If \( I \) is the left submodule of \( B \) generated by the images of all such homomorphisms, then \( I \) is an \( R-\nabla \) bimodule whose right annihilator is a minimal prime, so \( \text{Ass}(eR) \) must also consist of minimal primes, since \( R \) has an Artinian quotient ring. (This does not use the full force of Theorem 6, but only the fact that \( G(N) \) is right reversible, and hence \( I \) must be left \( G(N) \)-torsion-free.) However, since \(|B|, < \alpha \) and \( S_\alpha([(1 - \bar{e})R/N] = 0 \), hypothesis \( B(\alpha) \) implies that the same statements hold on the left, so \( I \) is singular as a left \( R/N \)-module. This contradiction shows that \( \nabla \cap [(1 - e)R] = 0 \), and, therefore, that \( S_\alpha(R) = eR \).

We have shown that \( eR \) is an ideal, so to complete the proof, we need only show that \((1 - e)R \) is a left ideal, where we know by induction that \((1 - e)R + B \) is an ideal. To show this we must show that there are no right homomorphisms \((1 - e)(R/N) \to B \). Just as before, this follows immediately from the fact that such a homomorphism would yield a nonzero homomorphism of right modules \((1 - \bar{e})(R/N) \to B \).
DEFINITION. A ring \( R \) is said to be (right) \( K \)-homogeneous if it has right Krull dimension and for every nonzero right ideal \( I, |I| = |R|_r \).

COROLLARY 12. If \( R \) is a Noetherian ring satisfying condition \( B(\alpha) \) for all \( \alpha \), such that \( R \) has an Artinian quotient ring and \( R/N \) is a product of \( K \)-homogeneous rings, then \( R \) is a product of \( K \)-homogeneous rings.

Remark. Conversely, if \( R \) is Noetherian and satisfies \( K_2(\alpha) \) for all \( \alpha \), and is \( K \)-homogeneous, then \( R \) has an Artinian quotient ring, as was shown for fully bounded rings by Gordon [6], right fully bounded rings by Lenagan [14], and general by the author (unpublished 1977 notes), Krause [9] (a short proof), and in a more general setting by Krause et al. [11]. (Still other proofs appear in [17] and [10].) Theorem 13 and Corollary 14 below show how this is related to the results of this paper.

THEOREM 13. If \( R \) is a Noetherian ring with nil radical \( N \) such that \( |R|_r = \alpha \) and \( R/N \) is \( K \)-homogeneous, and \( R \) satisfies condition \( K_2(\alpha) \), then \( \mathcal{O}(N) \) is a right Ore set.

Proof. According to Theorem 6, it suffices to consider a bimodule \( B \) which is a subfactor of \( R \), such that \( \ell(B) = P \) and \( r(B) = Q \) are both primes, and \( P \) not minimal, and show \( Q \) is not minimal. Condition \( K_2(\alpha) \) shows \( |B|_r < \alpha \), and since \( B_{K,Q} \) is nonsingular, \( |B|_r = |R/Q|_r \). Lemma 11, part (ii), implies that \( Q \) is not minimal.

COROLLARY 14. If \( R \) is a Noetherian ring which is \( K \)-homogeneous and satisfies \( K_2(\alpha) \), where \( \alpha = |R|_r \), then \( R \) has an Artinian quotient ring.

Proof. Theorem 13 and Corollary 7.

It follows, of course, that if a Noetherian ring satisfies \( K_2(\alpha) \) for all \( \alpha \) and is a product of \( K \)-homogeneous rings, then it has an Artinian quotient ring. In [5], Goldie and Krause find a condition which can be thought of as a weakening of this condition and which (in the presence of hypothesis \( B(\alpha) \) for all \( \alpha \)) is equivalent to the existence of an Artinian quotient ring.

4. SEMIPRIME IDEALS LOCALIZABLE MODULO THE RADICAL

We recall (from [4]) that for any ideal \( I \) of \( R \), we let

\[ \mathcal{O}(I) = \{ c \in R : c + I \text{ is regular in } R/I \} \]

A semiprime ideal \( S \) of the right Noetherian ring \( R \) is localizable if \( \mathcal{O}(S) \) is a right and left Ore set, and right localizable if \( \mathcal{O}(S) \) is a right Ore set. If \( S \) is a right localizable semiprime ideal, then the localization is usually denoted
$R_S$ and is a semilocal ring with Jacobson radical $SR_S$, and $R_S/SR_S$ can be identified with the (Goldie) right quotient ring of $R/S$.

In this section we consider a Noetherian ring $R$ with nil radical $N$ and a semiprime ideal $S$ such that $S/N$ is localizable in $R/N$ and seek conditions for $S$ to be localizable or right localizable. Since $N$ itself has this property, this problem is a generalization of the problem considered in Section 2.

Whether or not it is reasonable to use localizability modulo $N$ as a condition on a semiprime ideal $S$ depends on what localization problem one is considering. Our interest in this condition stems from examples which arise in the study of full quotient rings. In [21, 4.1] Stafford shows that for a Noetherian ring $R$ satisfying some (possibly redundant) ideal invariance conditions, there is a finite set $X$ of primes such that $\mathcal{O}(0) = \bigcap_{P \in X} \mathcal{O}(P)$. For these rings, therefore, the problem of the existence of a full quotient ring is contained in the general problem of localizing at a finite set of primes. As is shown, for example, in [18, Lemma 3], if $X$ is a finite set of primes and $S$ is the intersection of those primes maximal in $X$, then the set $\bigcap_{P \in X} \mathcal{O}(P)$ is a (right) Ore set if and only if $S$ is (right) localizable, in which case $\mathcal{O}(S) = \bigcap_{P \in X} \mathcal{O}(P)$. For these rings, therefore, the existence of a full quotient ring is equivalent to the localizability of a particular semiprime ideal. For the examples which arise naturally in this connection, it is frequently clear that $S/N$ is localizable.

**Lemma 15.** Let $R$ be a ring, $P$ a prime ideal of $R$ such that $R/P$ is right Goldie, and $S$ a semiprime ideal of $R$ such that $R/S$ is right Goldie, and $\mathcal{O}(S)$ is a right Ore set. Then $R/P$ is $\mathcal{O}(S)$-torsion-free if $P$ is contained in any of the primes minimal over $S$, and otherwise $R/P$ is $\mathcal{O}(S)$-torsion.

**Proof.** If $P$ is contained in a prime $Q$ which is minimal over $S$, then $R/Q$ is $\mathcal{O}(S)$-torsion-free, so since $R/Q$ is a homomorphic image of $R/P$, $R/P$ cannot be $\mathcal{O}(S)$-torsion. If $R/P$ were not $\mathcal{O}(S)$-torsion-free, then $\mathcal{O}(I(S)) = \bigcap_{P \in X} \mathcal{O}(P)$ would be an ideal of $R/P$, hence essential, and hence would contain an isomorphic copy of $R/P$. Since this would make $R/P$ $\mathcal{O}(S)$-torsion, which it is not, we conclude that $R/P$ is $\mathcal{O}(S)$-torsion-free.

Conversely, if $P$ is not contained in any prime minimal over $S$, then $(P + S)/S$ is essential in $R/S$ and hence contains a regular element of $R/S$. It follows that $P$ contains an element of $\mathcal{O}(S)$, so $R/P$ is $\mathcal{O}(S)$-torsion.

**Theorem 16.** Let $R$ be a Noetherian ring with nil radical $N$ and $S$ a semiprime ideal of $R$ such that $S/N$ is right localizable in $R/N$. Then the following are equivalent:

(i) $\mathcal{O}(S)$ is right Ore.

(ii) For every cell $B$ which is a subfactor of $R$ such that $r(B)$ is a prime minimal over $S$, $\ell(B)$ is contained in a prime minimal over $S$. 

There is a sequence of ideals, \( 0 = D_0 < D_1 < \cdots < D_n = R \), with the properties that (a) if \( L_i = D_{i+1}/D_i \), then \( NL_i = L_iN = 0 \) \((0 < i < n)\), and (b) if \( A_i = L_i/L_iS \), regarded as an \( R-R/S \) bimodule, then every right maximal cell of \( A_i \) is left \( \mathcal{Q}(S) \)-torsion-free.

**Proof.** Assuming (i), we recall from Lemma 1 that \( \mathcal{Q}(S) \) is right reversible. If \( B = I/J \), where \( I \) and \( J \) are ideals of \( R \), we apply the reversibility in the ring \( R/J \) to conclude that since \( B \) is right \( \mathcal{Q}(S) \)-torsion-free, it must be left \( \mathcal{Q}(S) \)-torsion-free. Statement (ii) now follows from Lemma 15. (In detail, if \( P = \ell(B) \), then as a left module, \( R/P \) is a submodule of a finite direct sum of copies of \( B \), so if \( B \) is left \( \mathcal{Q}(S) \)-torsion-free, the elements of \( \mathcal{Q}(S) \) are left regular modulo \( P \). Since in \( R/P \), left regular elements are right regular, \( R/P \) is right \( \mathcal{Q}(S) \)-torsion-free, and Lemma 15 shows that (ii) holds.)

To show that (ii) implies (iii), we note that sequences of ideals satisfying condition (a) always exist, and Lemma 15 (applied to the semiprime ideal \( S/N \) in \( R/N \)) and condition (ii) imply that any such sequence satisfies (iii).

To prove (i), we need to show that if \( c \in \mathcal{Q}(S) \), then \( R/cR \) is \( \mathcal{Q}(S) \)-torsion. It suffices to consider the sequence of ideals given in (iii), \( 0 < D_1 < \cdots < D_n = R \), and letting \( L_i = D_{i+1}/D_i \), show that \( L_i/cL_i \) is (right) \( \mathcal{Q}(S) \)-torsion. Since \( S/N \) is a right localizable semiprime ideal in \( R/N \), there is a right localization \( (R/N)_S = (R/N)(\mathcal{Q}(S/N))^{-1} \) (in the notation of Lemma 1), and a routine computation shows that \((R/N)_S \) is a semilocal ring with Jacobson radical \( S(R/N)_S \), such that \((R/N)_S/S(R/N)_S \) can be identified with the (Goldie) right quotient ring of \( R/S \). Since \( L_iN = 0 \), we may apply this localization on the right, and to show that \( M_i = L_i/cL_i \) is \( \mathcal{Q}(S) \)-torsion, it suffices to show that \( M_i \otimes (R/N)_S = 0 \). By Nakayama's lemma applied to \( M_i \otimes (R/N)_S \), it suffices to show that \( M_i/M_iS \) is \( \mathcal{Q}(S) \)-torsion. This is a homomorphic image of \( L_i/[cL_i + L_iS] \cong A_i/cA_i \), where \( A_i = L_i/L_iS \). Therefore, we have reduced our problem to showing that \( A_i/cA_i \) is (right) \( \mathcal{Q}(S) \)-torsion. We choose a cellular series for \( A_i \) and let \( B \) be a cell arising from this series. It will suffice to show that \( B/cB \) is \( \mathcal{Q}(S) \)-torsion. Since \( B \) is a right \( R/S \)-module, it is clear that \( B \) itself is already right \( \mathcal{Q}(S) \)-torsion unless its right annihilator is a prime minimal over \( S \). It therefore suffices to consider a right maximal cell of the bimodule \( _R(A_i)_{R/S} \), that is, one whose right annihilator is minimal over \( S \). We are at last ready to use hypothesis (iii). We infer from (iii) that \( _RB \) is \( \mathcal{Q}(S) \)-torsion-free. This means that left multiplication by \( c \) is injective. Since \( _RB_{R/S} \) is finitely generated and nonsingular, the fact that \( cB \cong B \) implies that \( B/cB \) is singular as an \( R/S \)-module. This is precisely what we needed to prove, so we have shown that (iii) implies (i).

**Remark.** I have been advised by A. Jategaonkar that in unpublished work he has independently proved a theorem very similar to the equivalence of conditions (i) and (ii) in Theorem 16. (Technical differences are that...
Jategaonkar's result is for right Noetherian rings, and the nilpotent ideal \(N\) is replaced by an ideal \(I\) satisfying the Artin–Rees condition.

The point of the rather complicated condition (iii) in Theorem 16 is to reduce the verification of the condition to a finite number of steps, so that it is computationally feasible in examples.

**Theorem 17.** Let \(R\) be a Noetherian ring with nil radical \(N\) and \(S\) a semiprime ideal of \(R\) such that \(S/N\) is localizable in \(R/N\). Then \(S\) is localizable if and only if for every cell \(B\) which is a subfactor of \(R\), \(r(B)\) is a prime minimal over \(S\) if and only if \(t(B)\) is a prime minimal over \(S\).

**Proof.** That the second condition implies that \(\mathcal{S}(S)\) is right and left Ore follows from Theorem 16. Conversely, if \(\mathcal{S}(S)\) is Ore, then part (ii) of Theorem 16 shows that if \(B\) is a cell which is a subfactor of \(R\), and if \(r(B)\) is minimal over \(S\), then \(t(B)\) is contained in a prime \(Q\) which is minimal over \(S\). If \(Q \neq t(B)\), then \(B/SB\) is a proper factor of \(B\) and from the definition of a cell it is clear that \(SB\) is right essential in \(B\), so \(B/SB\) is singular as a right \(R/S\)-module. However, \(B/SB\) is not left \(\mathcal{S}(S)\)-torsion, since \(B\) is not, and this contradicts the left reversibility of \(\mathcal{S}(S)\) (which follows from the fact that \(\mathcal{S}(S)\) is left Ore, by Lemma 1).

**Remark.** In the special case of \(FBN\) rings, Theorem 17 follows from results of Müller ([15, 16, Corollary 17].)

It is clear that Theorems 16 and 17 are finite criteria, in that if one uses the method of Theorem 16(iii), one obtains a finite set of primes to check, and in many examples this is not computationally difficult. On the other hand, one does not get a canonical series of cells to look at, as one does for the problem of localizing at \(N\) in Theorem 6. It would be interesting to know whether there was such a series.

**Corollary 18.** Let \(R\) be a Noetherian ring with nil radical \(N\) such that \(\mathcal{S}(0) = \bigcap_{P \in X} \mathcal{S}(P)\), where \(X\) is a finite set of primes. Let \(Y\) be the subset of \(X\) consisting of those primes in \(X\) maximal in \(X\), and let \(S = \bigcap_{P \in Y} P\). Then \(R\) has a full quotient ring if and only if \(S/N\) is a localizable ideal in \(R/N\) and for every cell \(B\) which is a subfactor of \(R\), \(r(B) \in Y\) if and only if \(t(B) \in Y\).

This follows from Theorem 17 and [18, Lemma 3]. We call that in [21] it is shown that for many Noetherian rings, \(\mathcal{S}(0)\) has this form.

**Example.** Theorem 17 cannot be improved to say that \(\mathcal{S}(S)\) is localizable if \(S/N\) is localizable in \(R/N\) and there is a cellular series for \(R\) such that every cell arising from this series is \(\mathcal{S}(S)\)-torsion-free on the left if and only if it is \(\mathcal{S}(S)\)-torsion-free on the right. (This would be parallel to
Theorem 6.) In this example, $\mathcal{E}(S) = \mathcal{E}(0)$, so it is also an example, for the quotient ring problem, and all of the cells that arise are $\mathcal{E}(S)$-torsion-free on each side. (This ring was shown to me by J. T. Stafford, in another connection.) Let $T = k[x, y]/(xy, x^2)$ (where $k$ is any field), let $\overline{T} = T/\text{nil}(T) \cong k[y]$, and let $R = (\begin{smallmatrix} T \\ 0 \end{smallmatrix} T)$. Let $P'$ be the prime ideal of $T$ defined by $P' = (x, y)/(xy, x^2)$ and let $P$ and $Q$ be the prime ideals of $R$ defined by $P = (\begin{smallmatrix} T \\ 0 \end{smallmatrix} T)$, $Q = (\begin{smallmatrix} 0 \\ T \end{smallmatrix} T)$. Then if $S = P \cap Q$, we have $\mathcal{E}(0) = \mathcal{E}(S)$. There is an obvious cellular series of length four, in which all of the factors are $\mathcal{E}(S)$-torsion-free on each side. If we let $B = (\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} T)$, then the bimodule $B/BS$ demonstrates that $\mathcal{E}(S)$ is not right Ore (using Theorem 16) since $B/BS$ is right $\mathcal{E}(S)$-torsion-free but left $\mathcal{E}(S)$-torsion.

Given a semiprime Noetherian ring $T$ and a finite set $X$ of prime ideals in $T$, it is easy to construct a Noetherian ring $R$ with $R/N \cong T$ such that this set $X$ corresponds to the primes in $R$ which we need to localize at in order to get a full quotient ring. From this it is easy to give examples in which there is no full quotient ring. It is also easy to construct examples in which $\mathcal{E}(0) = \bigcap_{P \in X} \mathcal{E}(P)$ and in which even though this set is not Ore, there is a larger finite set $Y$ of primes such that $\bigcap_{P \in Y} \mathcal{E}(P)$ is Ore, so that $R$ has a semilocal ring of quotients, even though it is not a full quotient ring. The following example shows that even this is not always possible.

**Example.** A noetherian ring which is not semilocal in which if $G$ is an Ore set satisfying $\mathcal{E}(0) \supseteq G \supseteq U$, where $U$ is the set of units of $R$, then $G = U$.

This example is based on an idea of Müller [15, p. 609]. We begin by considering the automorphism $\sigma$ of the complex polynomial ring $\mathbb{C}[x]$ which takes $x$ to $x + 1$. The powers of $\sigma$ take the ideal $(x)$ to the ideals $(x + n)$, $n \in \mathbb{Z}$. We invert all other primes in $\mathbb{C}[x]$, thus obtaining a principal ideal domain $D$ with an automorphism $\sigma$ which transitively permutes the primes of $D$. We now let $A = D \oplus D$, where the ring multiplication is defined by $(a, b)(a', b') = (aa', ab' + \sigma(a') b)$. If $M = (0) \oplus D$ is the nil radical of $A$, then $M/(x + n)M$ is a bimodule, left isomorphic to $D/(x + n)D$ and right isomorphic to $D/(x + n - 1)D$. An easy argument using the right and left reversibility of an Ore set shows that although $\mathcal{E}(M)$ is an Ore set in $A$, there is no Ore set properly contained between $\mathcal{E}(M)$ and the group of units of $A$. Finally, we let $R = (\begin{smallmatrix} D \\ D \end{smallmatrix} \mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers, and $A$ acts on $D/\mathbb{x}D$ by way of the natural map $A \rightarrow A/M \cong D$. If $P'$ is the ideal of $A$ which is the inverse image of $xD$ under this map. $P = (\begin{smallmatrix} P' \\ \mathbb{C} \end{smallmatrix} D)$, and $Q = (\begin{smallmatrix} \mathbb{C} \\ 0 \end{smallmatrix} D)$, then $\mathcal{E}(0) = \mathcal{E}(P) \cap \mathcal{E}(Q)$. Not only is this not an Ore set, but $\mathcal{E}(0)$ contains no Ore set properly containing the group of units.

REFERENCES