On the Rational Spectra of Graphs with Abelian Singer Groups

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ABSTRACT

Let G be a finite abelian group. We investigate those graphs $\mathcal{G}$ admitting G as a sharply 1-transitive automorphism group and all of whose eigenvalues are rational. The study is made via the rational algebra $\mathcal{Q}(G)$ of rational matrices with rational eigenvalues commuting with the regular matrix representation of G. In comparing the spectra obtainable for graphs in $\mathcal{Q}(G)$ for various G's, we relate subschemes of a related association scheme, subalgebras of $\mathcal{Q}(G)$, and the lattice of subgroups of G. One conclusion is that if the order of G is fifth-power-free, any graph with rational eigenvalues admitting G has a cospectral mate admitting the abelian group of the same order with prime-order elementary divisors.

1. INTRODUCTION

Let G be a finite abelian group of order n represented by $n \times n$ permutation matrices A_g [$A_g(h, k) = 1$ if and only if $gh = k$]. The complex algebra $\mathbb{C}[G]$ generated by these matrices is self-centralizing, and hence a graph $\mathcal{G}$ on n vertices admits G as a regular (sharply 1-transitive) automorphism group if and only if its adjacency matrix A is in $\mathbb{C}[G]$. We are concerned here with such graphs all of whose eigenvalues are rational, and to that end we consider the algebra $\mathcal{Q}(G)$ *over the rational field* of those matrices in $\mathbb{C}[G]$ with rational entries *and* rational eigenvalues. Notice the matrices in $\mathbb{C}[G]$ may be simultaneously diagonalized, so this is indeed an algebra. We are primarily interested in this study in comparing the spectra realizable by graphs in $\mathcal{Q}(G)$ for various groups G. In particular we concentrate on the question of when

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every graph in $\mathcal{A}(G)$ has a cospectral "mate" in $\mathcal{A}(H)$. When this happens we say H spectrally dominates G. We prove for example:

THEOREM 1.1. *If n is fifth-power-free, the abelian group with prime-order elementary divisors spectrally dominates every abelian group of order n.*

The general question of spectral domination appears intractible with the present techniques, and our results come by considering the embeddability of the algebra $\mathcal{A}(H)$ into $\mathcal{A}(G)$. These algebras are, in fact, association algebras (*over the rational field*), and the issue relates to the question of subschemes of the corresponding association scheme for $\mathcal{A}(G)$ as well as to the issue of embedding the lattice of subgroups of H in that of G. All of these in turn relate to appropriate column tactical decompositions of the eigenmatrix for the scheme underlying $\mathcal{A}(G)$.

The applications to graphs with rational eigenvalues reveal some special consequences of this spectral restriction. For example, we can show the following.

THEOREM 1.2. *If a graph $\mathcal{G}$ with rational eigenvalues admits the cyclic group of order n as a regular automorphism group, it admits any abelian group of order n.*

2. BACKGROUND

In this section we review briefly the techniques developed in [1,2] for the study of $\mathcal{A}(G)$. Throughout, G denotes an abelian group of order n written additively. By a *symmetric table* for G we mean a function $\lambda: G\times G \to Z_m$ (the cyclic group whose order is the *exponent* of G) with (1) $\lambda(g,h)=\lambda(h,g)$, (2) $\lambda(g,h_1+h_2)=\lambda(g,h_1)+\lambda(g,h_2)$, (3) $\lambda(g,h)=0$ for all h only for $g=0$. Normally λ will be the obvious function corresponding to the "natural" isomorphism of G with its character group. If ζ is a complex mth root of unity, the unitary matrix U with $U(g,h)=(1/\sqrt{n})\zeta^{\lambda(g,h)}$ is a diagonalizing matrix for $\mathbf{C}[G]$. One then observes the (Fourier) transform $\tilde{}$ defined by for $A=\Sigma_g a_g A_g$,

$$\tilde{A}=\sum_{h\in G}\left(\sum_{g\in G} a_g\zeta^{\lambda(g,h)}\right)A_h, \tag{2.1}$$

is a linear isomorphism. Moreover $\tilde{\tilde{A}}=nA^t$ (the transpose), so $\tilde{}$ interchanges entries and eigenvalues. Denoting the Hadamard (componentwise) product of two matrices A and B by $A*B$, we have

$$\widetilde{AB}=\tilde{A}*\tilde{B},$$

$$\widetilde{A*B}=\frac{1}{n}\tilde{A}\tilde{B}. \tag{2.2}$$

This shows that $\mathcal{Q}(G)$ is an algebra with respect to $*$ also, a so called association algebra [3]. Let $\tau(G)$ be the number of cyclic subgroups of G, and let $d(G,H)$ be the maximum d such that both G and H have elements of order d. Let us further put $\bar{g}$ for the subgroup generated by g, and use $\otimes$ for the tensor product. The following is established in [2]:

THEOREM 2.1. *The rational algebra $\mathcal{Q}(G)$ is the association algebra of dimension $\tau(G)$ for the association scheme with relation matrices*

$$R_{\bar{g}}\equiv\sum_{\bar{h}=\bar{g}}A_h.$$

If $d(G,H)\leqslant 2$, then $\mathcal{Q}(G\times H)=\mathcal{Q}(G)\otimes\mathcal{Q}(H)$ and the eigenmatrices satisfy $E(G\times H)=E(G)\otimes E(H)$.

It will be useful to keep in mind that the algebra $\mathcal{Q}(G)$ is the linear span not only of the relation matrices $R_{\bar{g}}$ but also of a corresponding family of orthogonal idempotents $F_{\bar{g}}=(1/n)\tilde{R}_{\bar{g}}$, summing to I [the common Frobenius covariants of $\mathcal{Q}(G)$] and that the $\tau(G)\times\tau(G)$ eigenmatrix $E(G)$ expresses these bases in terms of each other. That is,

$$R_{\bar{g}}=\sum_{\bar{h}}e_{\bar{g},\bar{h}}F_{\bar{h}}.$$

From another point of view, since $R_{\bar{g}}=\tilde{F}_{\bar{g}}$, E is the matrix of the transform $\tilde{}$ on $\mathcal{Q}$; and since $\tilde{\tilde{A}}=nA$ ($A=A^t$ in $\mathcal{Q}$), we have $E^2=nI$ for example. The rows of E give the spectra of the $R_{\bar{g}}$ [multiplicity in the $\bar{h}$th column being the rank of $F_{\bar{h}}$, $\phi(|\bar{h}|)$ (Euler's function)].

Dually, the rows of $(1/n)E$ reveal the entries of $F_{\bar{g}}$, the $\bar{h}$th column entry being the coefficient of $R_{\bar{h}}$.

The graphs in $\mathcal{Q}(G)$ are essentially the $(0,1)$ matrices here, and these are sums of relation matrices with spectrum given by the corresponding sum of the rows of E.

We require one more result from [2]. Let G be a p-group. The eigenmatrix $E(G)$ is given by

$$E(\bar{g},\bar{h})=\begin{cases}\phi(|\bar{g}|) & \text{if } \lambda(g,h)=0,\\ \dfrac{-\phi(|\bar{g}|)}{p-1} & \text{if } \lambda(g,h)\neq 0,\quad \lambda(g,ph)=0,\\ 0 & \text{otherwise.}\end{cases}$$

Finally, some additional notation. $\mathcal{C}(G)$ will denote the poset of cyclic subgroups of G and $\mathcal{S}(G)$ the lattice of all subgroups. For $K\in\mathcal{S}(G)$ we use $K^{\perp}=\{g|\lambda(g,k)=0$ for every $k\in K\}$. If G is a p-group, we use $\bar{h}^*$ for the predecessor $\overline{ph}$ of $\bar{h}$.

3. SUBSCHEMES AND SUBALGEBRAS

A sub-association-algebra of $\mathcal{Q}(G)$ corresponds to a sub-association-scheme, which essentially is an appropriate partition of the relation matrices $R_{\bar{g}}$. In general let $\pi=\{X_1,\dots,X_m\}$ be a partition of $\mathcal{C}(G)$, and let $\mathcal{R}(\pi)=\{\Sigma_{\bar{g}\in X_i}R_{\bar{g}}|X_i\in\pi\}$ and $\mathcal{F}(\pi)=\{\Sigma_{\bar{g}\in X_i}F_{\bar{g}}|X_i\in\pi\}$. The linear span of $\mathcal{R}(\pi)$ is a Hadamard product algebra, and the span of $\mathcal{F}(\pi)$ is an ordinary product algebra. The orthogonal idempotents for the (ordinary) algebra generated by $\mathcal{R}(\pi)$ are sums of the idempotents $F_{\bar{g}}$ over the cells of an appropriate *induced partition*, π^+, defined as follows. The cyclic subgroups $\bar{g}$ and $\bar{k}$ are in the same cell X_j^+ of π^+ if and only if

$$\sum_{\bar{h}\in X_i} e_{\bar{h},\bar{g}}=\sum_{\bar{h}\in X_i} e_{\bar{h},\bar{k}}$$

for all $X_i\in\pi$. We then put

$$\lambda^+_{Xi,Xj}=\sum_{\bar{h}\in X_i} e_{\bar{h},\bar{g}}\qquad(\text{any}\quad \bar{g}\in X_j^+). \tag{3.1}$$

If π^+ has t cells, the $m\times t$ matrix $\Lambda=(\lambda^+_{Xi,Xj})$ records the column sums of the blocks in the block decomposition of E given by the partition π on the rows and π^+ on the columns. If indeed the $\mathcal{R}(\pi)$ define a subscheme, Λ will be its eigenmatrix and necessarily $m=t$ (Λ is square). Surprisingly this is also sufficient.

THEOREM 3.1. *Let $\pi = \{X_1, \ldots, X_m\}$ be a partition of $\mathcal{C}(G)$ with corresponding induced partition $\pi^+ = \{X_1^+, \ldots, X_t^+\}$ as above. Then the following are equivalent:*

(i) $t = m$,
(ii) $\pi = \pi^{++}$,
(iii) $\mathcal{R}(\pi)$ *defines a subscheme (with association algebra spanned by* $\mathcal{F}(\pi^+)$).

Proof. Consider the vector space $V(\pi)$ of $\tau(G)$-tuples [indexed by $\mathcal{C}(G)$] which are constant on the cells of π. Then E maps $V(\pi)$ into $V(\pi^+)$, dimension$V(\pi) = m$, dimension$V(\pi^+) = t$, and thus $m \leqslant t$. Equality holds if and only if $V(\pi)$ is mapped onto $V(\pi^+)$, which is then mapped again by E onto $V(\pi^{++}) = V(\pi)$, since $E^2 = nI$. (Note that in general here π^{++} is a refinement of π.) As to (iii), observe

$$\sum_{\bar{g} \in X_i} R_{\bar{g}} = \sum_{X_j^+ \in \pi^+} \lambda^+_{X_i, X_j} \left(\sum_{\bar{g} \in X_j^+} F_{\bar{g}} \right),$$

whence the span of $\mathcal{R}(\pi)$ is contained in the span of $\mathcal{F}(\pi^+)$, and $t = m$ if and only if they are equal and so an association algebra. ■

As mentioned in the introduction, we wish to investigate spectral domination from the point of view of algebra (and scheme) embeddings. Hence we proceed to investigate when the algebra corresponding to $\mathcal{R}(\pi)$ is, in fact, $\mathcal{Q}(H)$ for some abelian group H.

4. EMBEDDING AND COLLAPSING

We begin with some remarks concerning embeddings of the lattice $\mathcal{S}(H)$ of subgroups of the abelian group H into $\mathcal{S}(G)$. These lattices are well known [4] to be products of corresponding lattices for the Sylow subgroups. Further, if $T: \mathcal{S}(H) \to \mathcal{S}(G)$ is a lattice embedding where G and H are of the same order, then the condition $|T(K)| = |K|$ for all $K \in \mathcal{S}(H)$ (i.e., T is size preserving) is equivalent to T sending Sylow p-subgroups to Sylow p-subgroups. There are lattice embeddings not of this sort, but we shall be concerned only with size preserving maps. Before proceeding to clarify the relevance of this, we establish a lemma. Note that for p-groups of the same order any lattice embedding is size-preserving.

LEMMA 4.1. *Let G and H be finite abelian p-groups of the same order, and $T: \mathcal{S}(H) \to \mathcal{S}(G)$ be a lattice embedding. For any $g \in G$ there is a unique $\bar{h} \in \mathcal{C}(H)$ such that $\bar{g} \leq T(\bar{h})$ and $\bar{g} \not\leq T(\bar{h}^*)$.*

Proof. If $U(\bar{h}) = \{\bar{g} \mid \bar{g} \leq T(\bar{h})\}$, then $\bigvee_{\bar{g} \in U(\bar{h})} \bar{g} = T(\bar{h})$ and $\sum_{\bar{g} \in U(\bar{h})} \phi(|\bar{g}|) = |T(\bar{h})| = |\bar{h}|$. Now if $P(\bar{h}) = U(h) - U(\bar{h}^*)$, the lemma claims $\{P(\bar{h}) \mid \bar{h} \in \mathcal{C}(G)\}$ is a partition of $\mathcal{C}(G)$. Disjointness is clear. Now

$$\sum_{\bar{g} \in \bigcup P(\bar{h})} \phi(|\bar{g}|) = \sum_{\bar{h} \in \mathcal{C}(H)} \sum_{\bar{g} \in P(\bar{h})} \phi(|\bar{g}|) = \sum_{\bar{h} \in \mathcal{C}(H)} (|\bar{h}| - |\bar{h}^*|)$$

$$= \sum_{\bar{h} \in \mathcal{C}(H)} \phi(|\bar{h}|) = |H| = |G| = \sum_{\bar{g} \in \mathcal{C}(G)} \phi(|\bar{g}|).$$

Hence $\bigcup P(\bar{h}) = \mathcal{C}(G)$. ■

Now for abelian G and H of the same order we say that G *collapses to* H if and only if there is a partition π of $\mathcal{C}(G)$ such that $\pi = \pi^{++}$ and the collapsed matrix $\Lambda(3.1)$ is $E(H)$. We refer to such a partition π as a collapsing partition.

THEOREM 4.2. *Let G and H be finite abelian groups of the same order. Then there is a one-to-one correspondence between any two of the following sets:*

(i) *the association-algebra embeddings of $\mathcal{A}(H)$ into $\mathcal{A}(G)$,*
(ii) *the size-preserving lattice embeddings of $\mathcal{S}(H)$ into $\mathcal{S}(G)$,*
(iii) *the collapsing partitions of G to H.*

Proof. If $\tau: \mathcal{A}(H) \to \mathcal{A}(G)$ is an algebra embedding, we see that for any $K \leq H$, if $A_K = \sum_{g \in K} A_g$, then $\tau(A_K)$ is a $(0,1)$ matrix and $A_K^2 = |K| A_K$, so $\tau(A_K) = A_{T(K)}$ for some subgroup $T(K) \leq G$ with $|T(K)| = |K|$. The mapping $T: \mathcal{S}(H) \to \mathcal{S}(G)$ thus defined is clearly 1-1 and size-preserving. From $A_{K_1} * A_{K_2} = A_{K_1 \cap K_2}$ we have $A_{T(K_1)} * A_{T(K_2)} = A_{T(K_1) \cap T(K_2)} = A_{T(K_1 \cap K_2)}$. Hence T is meet-preserving. That $A_{K_1} A_{K_2} = |K_1 \cap K_2| A_{K_1 K_2}$ gives the corresponding result for joins, and T is a lattice embedding. Conversely, let $T: \mathcal{S}(H) \to \mathcal{S}(G)$ be a size-preserving lattice embedding. We assume here that G and H are p-groups, which is sufficient from our earlier remarks. We define a linear transformation $\tau: \mathcal{A}(H) \to \mathcal{A}(G)$ by $\tau(A_{\bar{h}}) = A_{T(\bar{h})}$. Now for any $K \leq H$ we claim $\tau(A_K) = A_{T(K)}$. Note that in fact $R_{\bar{h}} = A_{\bar{h}} - A_{\bar{h}^*}$, so $A_K = \sum_{\bar{h} \leq K} R_{\bar{h}}$, and thus $\tau(A_K) = \sum_{\bar{h} \leq K} \tau(R_{\bar{h}}) = \sum_{\bar{h} \leq K} (A_{T(\bar{h}^*)}) = \sum_{\bar{g}} a_{\bar{g}} R_{\bar{g}}$ for suitable $a_{\bar{g}} \in \mathbb{Q}$. If $a_{\bar{g}} \neq 0$ then $R_{\bar{g}} \leq A_{T(\bar{h})} - A_{T(\bar{h}^*)}$, so $g \in T(\bar{h}) \leq T(K)$ and

$a_{\bar{g}}=1$. If $\bar{g}\leqslant T(K)$, then by Lemma 4.1, $a_{\bar{g}}=1$. This establishes the claim, and the remaining details are straightforward. The correspondence between collapsing partitions π is covered by Theorem 3.1. ■

COROLLARY 4.3. *Any abelian group of order n collapses to the cyclic group Z_n. In particular Z_n is spectrally dominated by any abelian group of order n.*

This corollary is clear, since $\mathcal{S}(Z_p r)$ is a chain. Actually more than spectral domination occurs here. For any graph in $\mathcal{C}(Z_n)$ there is an isomorphic graph in $\mathcal{C}(G)$. That is,

THEOREM 4.4. *If a graph $\mathcal{G}$ with rational eigenvalues admits the cyclic group Z_n as a regular Singer group, then $\mathcal{G}$ admits every abelian group G of order n.*

Proof. What actually happens here is that the embedding of $\mathcal{C}(Z_n)$ in $\mathcal{C}(G)$ can be accomplished with a fixed permutation similarity $X\to P^tXP$. The cyclic scheme underlying $\mathcal{C}(Z_n)$ associates (relates) g and h according to the order of $g-h$. Hence it suffices to prove that for any abelian group G of order n there is a bijection, $f\colon Z_n\to G$, such that $f(x)-f(y)$ determines the order of $x-y$. [Then for any graph in $\mathcal{C}(Z_n)$ one obtains an isomorphic graph in $\mathcal{C}(G)$ by joining $f(x)$ and $f(y)$ if and only if x is joined to y.] Furthermore, from Theorem 2.1 we may restrict ourselves to p-groups. For $n=p^r$ we proceed by induction on r, $r=1$ being clear. For $n=P^{k+1}$, let f be the bijection from the subgroup $\bar{p}$ in Z_n onto a subgroup H of G of index p. The elements of A_n have a unique representation in the form $k+lp$, where $0\leqslant k\leqslant p-1$ and $0\leqslant l\leqslant P^k-1$. If $g+H$ generates the quotient G/H, the elements of G have a unique representation in the form $kg+h$ where $0\leqslant k\leqslant p-1$ and $h\in H$. Put $\hat{f}\colon Z_n\to G$ by

$$\hat{f}(k+lp)=kg+f(lp).$$

This is easily a bijection, and if $\hat{f}(x)-\hat{f}(y)=lg+h$, either $l\neq 0$, whence $x-y$ is a generator of Z_n, or $l=0$ and $\hat{f}(x)-\hat{f}(y)=f(x')-f(y')$ for $x',y'\in\bar{p}$ with $x-y'=x'-y'$, and thus by induction the order of $x-y$ is determined. ■

In order to obtain our main result on collapsing (and consequent spectral domination) in the next section, we need the following technical equivalent of collapsing. As we need only one direction of the equivalence, we state and prove only that.

THEOREM 4.5. *Let G and H be p-groups of the same order. Let π and Φ be partitions of $\mathcal{C}(G)$, each with $\tau(H)$ classes: $\pi=\{X_{\bar{h}}\mid \bar{h}\in\mathcal{C}(H)\}$, $\Phi=\{Y_{\bar{h}}\mid \bar{h}\in\mathcal{C}(H)\}$. Suppose the following hold:*

(i) *For each $\bar{h}\in\mathcal{C}(H)$, $\Sigma_{\bar{g}\in X_{\bar{h}}}\phi(\bar{g})=\phi(\bar{h})$.*

(ii) *If $\bar{h}$, $\bar{k}\in\mathcal{C}(H)$ and $\lambda(\bar{h},\bar{k})=0$, then $\lambda(\bar{g},\bar{f})=0$ for any $\bar{g}\in X_{\bar{h}}$ and $\bar{f}\in Y_{\bar{k}}$.*

(iii) *If $\bar{h}$, $\bar{k}\in\mathcal{C}(H)$ and there is some $\bar{f}\in Y_{\bar{k}}$ with $\lambda(\bar{f},\bar{g})=0$ for all $\bar{g}\in X_{\bar{h}}$, then $\lambda(\bar{h},\bar{k})=0$.*

Then G collapses to H (with π as a collapsing partition and $\Phi=\pi^{+}$).

Proof. We construct an embedding of $\mathcal{S}(H)$ into $\mathcal{S}(G)$ in stages. Start by defining $T_0(\bar{h})=\vee_{\bar{g}\in X_{\bar{h}}}\bar{g}$ for $\bar{h}\in\mathcal{C}(H)$. We show $T_0(\bar{h}^*)\leqslant T_0(\bar{h})$. Suppose not; then there exists $\bar{g}\in X_{\bar{h}^*}$ such that $\bar{g}\not\leqslant T_0(\bar{h})$. Then there exists $\bar{f}$ such that $\bar{f}\leqslant T_0(\bar{h})^{\perp}$ but $\lambda(\bar{f},\bar{g})\neq 0$. But then $\lambda(\bar{f},\bar{l})=0$ for every $\bar{l}\in X_{\bar{h}}$, so if $\bar{f}\in Y_{\bar{k}}$, by (iii), $\lambda(\bar{h},\bar{k})=0$, so $\lambda(\bar{h}^*,\bar{k})=0$ so by (ii), $\lambda(\bar{f},\bar{g})=0$, a contradiction.

Next, for $A\in\mathcal{S}(H)$, define $T(A)=\vee_{\bar{h}\leqslant A}T_0(\bar{h})$. We have shown above that $T(\bar{h})=T_0(\bar{h})$. Note that by (ii) and (iii) if $\bar{f}\in Y_{\bar{k}}$, then $T(\bar{h})\leqslant\bar{f}^{\perp}$ if and only if $\lambda(\bar{h},\bar{k})=0$.

We proceed to show T is the desired embedding. Clearly if $\bar{h}\leqslant A$ then $T(\bar{h})\leqslant T(A)$. Conversely, suppose $T(\bar{h})\leqslant T(A)$. Then for any $\bar{k}\leqslant A^{\perp}$, $\bar{f}\in Y_{\bar{k}}$, we have $T(A)\leqslant\bar{f}^{\perp}$, so $T(\bar{h})\leqslant\bar{f}^{\perp}$, so $\lambda(\bar{h},\bar{k})=0$, so $A^{\perp}\leqslant\bar{h}^{\perp}$, so $\bar{h}\leqslant A$. Then easily, for A, $B\in\mathcal{S}(H)$, $A\leqslant B$ if and only if $T(A)\leqslant T(B)$. In particular $|A|=|T(A)|$. Next we show that $\bar{g}\leqslant T(A)$ if and only if $\bar{g}\in X_{\bar{h}}$ for some $\bar{h}\leqslant A$. The sufficiency is clear. Assume $\bar{g}\leqslant T(A)$. Then by (i),

$$\sum_{\substack{\bar{g}\in X_{\bar{h}}\\ \bar{h}\leqslant A}}\phi(\bar{g})=\sum_{\bar{h}\leqslant A}\phi(\bar{h})=|A|=|T(A)|=\sum_{\bar{g}\leqslant T(A)}\phi(\bar{g}),$$

so the two sets must be equal. That T preserves intersections follows easily. Also clear is that $T(A\vee B)\geqslant T(A)\vee T(B)$. Let $\bar{g}\leqslant T(A\vee B)$. Then $\bar{g}\in X_{\bar{h}}$ for some $\bar{h}\leqslant A\vee B$, so $\bar{h}\leqslant\bar{h}_1\vee\bar{h}_2$ for some $\bar{h}_1\leqslant A$, $\bar{h}_2\leqslant B$. Let $\bar{f}\leqslant[T(\bar{h}_1)\vee T(\bar{h}_2)]^{\perp}$. Then if $\bar{f}\in Y_{\bar{k}}$, we have $\lambda(\bar{h}_1,\bar{k})=\lambda(\bar{h}_2,\bar{k})=0$, so $\bar{h}_1\vee\bar{h}_2\leqslant\bar{k}^{+}$, so $\lambda(\bar{h},\bar{k})=0$, so $\lambda(\bar{g},\bar{f})=0$, so $\bar{g}\leqslant T(\bar{h}_1)\vee T(\bar{h}_2)\leqslant T(A)\vee T(B)$, and we are finished. ∎

5. APPLICATIONS AND EXAMPLES

In this section we apply the previous development and in particular Theorem 4.5 to establish the theorem mentioned in the introduction.

THEOREM 5.1. *If n is fifth-power-free, the abelian group with prime-order elementary divisors spectrally dominates every abelian group of order n.*

The theorem follows from the following three collapsing results (together with Corollary 4.3), where we use Theorem 4.5 with $\pi = \Phi$. We prove only the first one, the proofs of the other two being similar.

THEOREM 5.2. *Let p be a prime.*

(i) *For $n \geqslant 1$, $\mathbb{Z}_p^{n+1}$ collapses to $\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{n-1}$.*
(ii) *$\mathbb{Z}_p^4$ collapses to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.*
(iii) *$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ collapses to $\mathbb{Z}_{p^3} \times \mathbb{Z}_p$.*

Proof of (i). The nonzero cyclic subgroups of $\mathbb{Z}_p^{n+1}$ are given by the $n+1$-tuples $(a_0, a_1, \dots, a_n)$ with $a_i \in \mathbb{Z}_p$ and leading nonzero term 1. The nonzero cyclic subgroup of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{n-1}$ are given by $(1, a_2, \dots, a_n)$, $(p, a_2, \dots, a_n)$, where $a_2, \dots, a_n \in \mathbb{Z}_p$, and $(0, a_2, \dots, a_n)$ with $a_i \in \mathbb{Z}_p$ and leading nonzero term 1. For $\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{n-1}$ let λ be standard: $\lambda((x, a_2, \dots, a_n), (y, b_2, \dots, b_n)) = xy + p(a, b_2 + \cdots + a_n b_n) \bmod p^2$, and for $\mathbb{Z}_p^{n+1}$,

$$\lambda((a_0, a_1, \dots, a_n), (b_0, b_1, \dots, b_n)) = a_0 b_1 + a_1 b_0 + a_2 b_2 + \dots + a_n b_n \bmod p.$$

We shall use $[h]$ for $X_{\bar{h}}$. Let

$$[(1, a_2, \dots, a_n)] = \{(1, a_1, a_2, \dots, a_n) | a_1 \in \mathbb{Z}_p\},$$

$$[(p, a_2, \dots, a_n)] = \{(0, 1, a_2, \dots, a_n)\},$$

$$[(0, a_2, \dots, a_n)] = \{(0, 0, a_2, \dots, a_n)\}.$$

The verification of conditions (i), (ii), and (iii) is straightforward. ■

We continue with some negative results.

REMARK 5.3. Let p be a prime. Then $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ never collapses to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$. The nature of the difficulty is the cyclic subgroup $(\overline{p, 0, 0})$, which by Lemma 4.1 has to be in the image of any embedding: it is covered by too many elements, and no comparable element exists in $\mathcal{C}(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2})$.

REMARK 5.4. Although $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ does not collapse to $\mathbb{Z}_4 \times \mathbb{Z}_4$, one can nevertheless prove, by exhaustion of cases, that the former group does spectrally dominate the latter—which in turn does not dominate $\mathbb{Z}_8 \times \mathbb{Z}_2$, since the following bipartite graph with spectrum $5, -5, 3, -3, 1^{(6)}, -1^{(6)}$ is in

$\mathcal{C}(\mathbb{Z}_8 \times \mathbb{Z}_2)$ but has no cospectral mate in $\mathcal{C}(\mathbb{Z}_4 \times \mathbb{Z}_4)$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

More surprising is the fact that the elementary p-group does not in general collapse to any other group of the same order.

Remark 5.5. $\mathbb{Z}_2^6$ does not collapse to $\mathbb{Z}_4^3$. One indeed proves that there is no lattice embedding, but the proof is rather technical, so we omit it.

Remark 5.6. As the case of $n = 16$ seems to indicate, it may be conjectured that for two groups of order p^r, G, and H, if the elementary divisors of G are a finer partition than the elementary divisors of H, then G dominates H. This would of course imply that the elementary group dominates any other group.

Finally, a conjecture of a more general sort is

Conjecture. Let π be a partition of $\mathcal{C}(G)$; then $\pi^{+++} = \pi^{+}$.

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