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Identity excluding groups

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Abstract

We consider identity excluding groups. We first show that motion groups of totally disconnected nilpotent groups are identity excluding. We prove that certain class of p-adic algebraic groups which includes algebraic groups whose solvable radical is type R have identity excluding property. We also prove the convergence of averages of representations for some solvable groups which are not necessarily identity excluding.

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1. Introduction

Let G be a locally compact, σ-compact metrizable group with a right invariant Haar measure m. Let $\mathcal{P}(G)$ be the space of regular Borel probability measures on G. The convolution of any two measures μ and λ in $\mathcal{P}(G)$ is defined by $\mu * \lambda(f) = \iint f(xy)\,d\mu(x)\,d\lambda(y)$ for all continuous bounded functions on G. Let $L^p(G)$, $1 \leqslant p < \infty$, be the space of all measurable functions f with $\int |f|^p < \infty$ and $L^\infty(G)$ be the space of all (essentially) bounded measurable functions on G.

For $\mu, \lambda \in \mathcal{P}(G)$ and $x \in G$, $\mu\lambda$, $x\mu$ and μx denote $\mu * \lambda$, $\delta_x * \mu$ and $\mu * \delta_x$ and μ^n denotes the nth convolution power of μ.

For $\mu \in \mathcal{P}(G)$ and $f \in L^1(G)$, we define the convolution operator $\mu * f$ by $\mu * f(x) = \int f(xy)\,d\mu(y)$ for all $x \in G$.

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We now recall that a measure μ in $\mathcal{P}(G)$ is called *ergodic* if $\frac{1}{n}\sum \mu^k * f \to 0$ for all $f \in L^1(G)$ with $\int f = 0$ and μ is called *weak mixing* if $\frac{1}{n}\sum |\langle \mu^k * f, g\rangle| \to 0$ for all $g \in L^\infty(G)$ and all $f \in L^1(G)$ with $\int f = 0$.

It is known that any weak mixing measure is ergodic and also strictly aperiodic. The converse of this is known as *weak mixing problem*. Lin and Wittmann show that strong convergence of μ^n-averages of unitary representation gives affirmative answer to weak mixing problem (Theorem 3.1 of [7]). Thus, the relationship between ergodic and weak mixing is closely related to the representation theoretic property, called identity excluding. Here we investigate the class of identity excluding groups. We first observe that for groups whose connected component of identity is compact, trivial representation is not weakly contained in any non-trivial irreducible representation implies identity excluding. Using this observation we prove that (i) motion groups of totally disconnected nilpotent groups are identity excluding and (ii) a class of p-adic algebraic groups which includes groups whose solvable radical is type R and the general affine group has identity excluding property. We also show the convergence of averages of unitary representation for splitable solvable Zariski-connected p-adic algebraic groups.

2. Preliminaries

We introduce notions and prove lemmas that are needed to prove the main results.

Definition 2.1. A unitary representation of a locally compact group G in a Hilbert space $\mathcal{H}$ is a homomorphism $T : G \to \mathcal{B}_u(\mathcal{H})$, the space of unitary operators on $\mathcal{H}$ such that for each $v \in V$, the map $g \mapsto T(g)v$ is continuous. For any subgroup H of G and any representation T of H, $T|_H$ denotes the representation of H obtained by restricting T to H. Let I be denote the trivial irreducible representation.

Throughout this article by a group we mean a locally compact, σ-compact metric group and by a representation we mean an unitary representation.

Definition 2.2. We say that a representation T of a group G *weakly contains the trivial representation* or $I \prec T$ if there exists a sequence (v_n) of unit vectors in $\mathcal{H}$ such that $\|T(g)v_n - v_n\| \to 0$ for all $g \in G$: such sequence (v_n) of unit vectors is called *approximate fixed point*.

Remark 2.1. The standard definition of weak containment of trivial representation requires the convergence $\|T(g)v_n - v_n\| \to 0$ to be uniform on compact subsets of G. It may be seen as follows that our definition is equivalent to the standard definition: if there exists a sequence (v_n) such that $\|T(g)v_n - v_n\| \to 0$ for all $g \in G$, then for any $f \in L^1(G)$ with $f \geqslant 0$ and $\int f = 1$, we have $\|T(f)\| = 1$ and hence by 1.3, Chapter 3 of [12], T weakly contains the trivial representation according to the standard definition.

Definition 2.3. We say that a group G has *identity excluding property* if there is no non-trivial irreducible representation T for which there is a dense subgroup D of G such that $I_D \prec T|_D$.

Definition 2.4. For any $\mu \in \mathcal{P}(G)$ and any representation T of G, the μ-average T_μ is defined as $T_\mu(v) = \int T(g)v\, d\mu(g)$ for any vector v. It is easy to see that $||T_\mu|| \leqslant 1$.

We now recall two non-degeneracy conditions for measures on groups which are necessary for weak mixing.

Definition 2.5. A $\mu \in \mathcal{P}(G)$ is called *adapted* if the closed subgroup generated by the support of μ is G and μ is called *strictly aperiodic* if there is no proper closed normal subgroup a coset of which contains the support of μ.

Identity excluding groups was introduced in [5] and also considered in [7] and [15]. It is shown in [7] that nilpotent groups are identity excluding. Also, Lin and Witmann proved that for adapted, strictly aperiodic measure μ on a group G with identity excluding property,

(1) $\|T_\mu^n\| \to 0$ for any non-trivial irreducible unitary representation T,
(2) (T_μ^n) converges strongly for any unitary representation T, and
(3) in addition if μ is ergodic, then μ is weak mixing.

We now prove some lemmas which are needed in the sequel.

Lemma 2.1. *Let G be a group and H be a closed abelian normal subgroup of G such that $G/C(H)$ is finite where $C(H)$ is the centralizer of H. Suppose T is an irreducible representation of G such that $I \prec T$. Then T is trivial on H.*

Proof. Let T be an irreducible representation of G in a Hilbert space $\mathcal{H}$. Suppose $I \prec T$. For any character χ of H, define $V_\chi = \{v \in \mathcal{H} \mid T(h)v = \chi(h)v \text{ for all } h \in H\}$. Then it is easy to see that V_χ is a $C(H)$-invariant closed subspace of $\mathcal{H}$. Let χ be such that V_χ is a non-trivial subspace. Let $g_0C(H), g_1C(H), \ldots, g_kC(H)$ be a system of coset representative of $C(H)$ in G with $g_0 = e$. Now for each $0 \leqslant i \leqslant k$, define $\chi_i(h) = \chi(g_i h g_i^{-1})$ for all $h \in H$. Then each χ_i is a character of H and $T(g_i)V_\chi = V_{\chi_i} = V_i$, say. Then the orthogonal sum $\bigoplus V_i$ is a G-invariant closed subspace of $\mathcal{H}$. Thus, $\mathcal{H} = \bigoplus V_i$. Since $I \prec T$, there exists a $i \geqslant 0$ such that $I \prec T_i$ where T_i is the representation of H in V_i given by χ_i. This shows that χ_i is trivial and hence χ is trivial. Thus, $T(H)$ is trivial. □

Lemma 2.2. *Let G be a group whose connected component of identity is compact and T be an unitary representation of G in a Hilbert space $\mathcal{H}$. Suppose there is a dense subgroup H of G and a sequence (v_n) of unit vectors in $\mathcal{H}$ such that $\|T(g)v_n - v_n\| \to 0$ for all $g \in H$. Then $\|T(g)v_n - v_n\| \to 0$ for all $g \in G$. In particular, $I \not\prec T$ for any non-trivial irreducible representation of G implies G is identity excluding.*

Proof. Let G^0 be the connected component of identity in G. Then G/G^0 is totally disconnected and has compact open subgroups, see Theorem II.7.7 of [3]. Since G^0 is compact, G itself has compact open subgroups. Let K be a compact open subgroup of G. Let $\mathcal{H}_K = \{v \in \mathcal{H} \mid T(K)v = v\}$. Now for each $n \geqslant 1$, there exists a $a_n \in \mathcal{H}_K$ and a $b_n \in \mathcal{H}_K^\perp$ such that $v_n = a_n + b_n$. It can be easily seen that $\mathcal{H}_K^\perp$ is also $T(K)$-invariant. This implies that for $g \in H \cap K$, $\|T(g)v_n - v_n\| = \|T(g)b_n - b_n\| \to 0$. Since H is dense in G, $H \cap K$ is dense in K. We may assume that H is a Borel subgroup of G. Thus, $H \cap K$ is a Borel subgroup of K. Since K is metrizable, we can choose a countable dense subset E of K such that $E \subset H \cap K$. Let $\mu = \sum_{x \in E} r_x \delta_x$ where $r_x > 0$ for all $x \in E$ and $\sum r_x = 1$. Then μ is an adapted, strictly aperiodic probability measure on K such that $\mu(H \cap K) = 1$. Let ρ be the representation of K in $\mathcal{H}_K^\perp$ such that $\rho(g)$ is the restriction of $T(g)$ to $\mathcal{H}_K^\perp$ for any $g \in K$. Then $\|(\rho_\mu^k - I)b_n\| \to 0$ as $n \to \infty$. Suppose $\rho_\mu^k - I$ is not invertible for any $k \geqslant 1$. Then $\|\rho_\mu^k\| = 1$ for all $k \geqslant 1$. Since K is a SIN-group by Theorem 2.11 of [7], $I \prec \rho$. Since K has property (T), $\mathcal{H}_K^\perp$ has a non-trivial $T(K)$-fixed vector. This is a contradiction. This shows that $(\rho_\mu^k - I)$ is invertible for some $k \geqslant 1$. Thus, $b_n \to 0$. Now for any $g \in K$, $\|T(g)v_n - v_n\| = \|T(g)b_n - b_n\| \leqslant 2\|b_n\| \to 0$. Thus, $\{g \in G \mid \|T(g)v_n - v_n\| \to 0\}$ is a dense subgroup containing an open subgroup K and hence $\|T(g)v_n - v_n\| \to 0$ for all $g \in G$. □

3. Motion groups

We first prove the identity excluding for motion groups of totally disconnected nilpotent groups.

Theorem 3.1. *Let G be a split compact extension of a totally disconnected nilpotent group. Then G has identity excluding property.*

Proof. Let N be a totally disconnected nilpotent normal subgroup of G such that G/N is compact. Since G is a split compact extension of N, it is easy to see that the connected component of G is compact. In view of Lemma 2.2, it is enough to show that $I \not\prec T$ for any non-trivial irreducible representation.

Let T be an irreducible representation of G such that $I \prec T$. Let Z be the center of N. We now prove that $T(Z)$ is trivial. Let K be a compact open subgroup of Z and define $L = \bigcap_{g \in G} gKg^{-1}$. Then L is a compact normal of subgroup of G. Since $I \prec T$, $T(L)$ has a non-trivial fixed point. Since L is a normal subgroup of G, space of fixed points for $T(L)$ is a non-trivial $T(G)$-invariant subspace. Thus, $T(L)$ is trivial. Since K is a open subgroup of Z, $N(K)$, the normalizer of K is an open subgroup of G containing N. Since G/N is compact, $N(K)$ is a open subgroup of finite index in G. Thus, L is a finite intersection of conjugates of K. Since Z is a normal subgroup of G, L is a open subgroup of Z. Now by replacing G by G/L, we may assume that Z is discrete. Let $x \in Z$. There exists a finitely generated normal subgroup F of G such that $x \in F \subset Z$ (we may choose F to be the group generated by the finite set $\{gxg^{-1} \mid g \in G\}$). It is easy to see that $\mathrm{Aut}(F)$ is a discrete group. Since $H \subset Z$, the center of N and G/N is compact, the centralizer of F is finite. By Lemma 2.1, $T(F)$ is trivial. In particular, $T(x)$ is trivial. Since $x \in Z$ is arbitrary, $T(Z)$ is trivial.

Since N is nilpotent, by using the central series of N, we conclude that $T(N)$ is trivial. Since any compact group is identity excluding, $T(G)$ itself is trivial. Thus, G is identity excluding. □

We next prove that finite extension of nilpotent groups are identity excluding.

Theorem 3.2. *Let G be a finite extension of a nilpotent group. Then G is identity excluding.*

Proof. Let N be a nilpotent subgroup of G such that G/N is finite. Then replacing N by a subgroup of N, we may assume that N is a normal subgroup of G (the subgroup may be chosen to be $\bigcap xNx^{-1}$).

Let T be an irreducible representation of G in a Hilbert space $\mathcal{H}$. Suppose there exists a dense set D of G such that $I \prec T|_D$. Since G/N is finite, $D \cap N$ is dense in N. Let $N_0 = N$ and $N_i = [N, N_{i-1}]$ for all $i > 0$. Then N_k is contained in the center of N for some $k \geqslant 0$. Also, $D \cap N_k$ is dense in N_k.

We now claim that T is trivial. For any character χ of N_k, we define $\mathcal{H}_\chi = \{v \in \mathcal{H} \mid T(g)v = \chi(g)v$ for all $g \in N_k\}$. Let C be the centralizer of N_k in G. Then C is a normal subgroup of finite index in G. Also, $\mathcal{H}_\chi$ is C-invariant closed subspace of $\mathcal{H}$. Let χ be such that $\mathcal{H}_\chi$ is a non-trivial subspace. Let $g_0C, g_1, \ldots, g_kC$ be a system of coset representative of C in G with $g_0 = e$. Now for each $0 \leqslant i \leqslant k$, define $\chi_i(x) = \chi(g_i x g_i^{-1})$ for all $x \in N_k$. Then each χ_i is a character of N_k and $g_i\mathcal{H}_\chi = \mathcal{H}_{\chi_i} = \mathcal{H}_i$, say. Then the orthogonal sum $\bigoplus \mathcal{H}_i$ is a $T(G)$-invariant closed subspace of $\mathcal{H}$. Thus, $\mathcal{H} = \bigoplus \mathcal{H}_i$. Since $I \prec T|_D$, there exists a $i \geqslant 0$ such that $I \prec T_i|_{D \cap N_k}$ where T_i is the representation of N_k in $\mathcal{H}_i$ given by χ_i. This shows that χ_i is trivial and hence χ is trivial. Thus, $T(N_k)$ is trivial. Using the central series of N, we may prove that $T(N)$ is trivial. Since finite groups are identity excluding, T itself is trivial. □

We now deduce the following for groups of polynomial growth. We recall that a compactly generated group G (with Haar measure m) is said to be of *polynomial growth* if there exists an integer $l > 0$ and a constant $c > 0$ such that $m(U^n) \leqslant cn^l$ for all $n \geqslant 1$ where U is a compact neighborhood of identity generating G: see [2,9] for results on polynomial growth.

Corollary 3.1. *Let G be a compactly generated group of polynomial growth. Suppose the connected component of identity is compact. Then G has identity excluding property.*

Proof. Let G be a compactly generated group of polynomial growth. By Theorem 2 of [9], there exists a compact normal subgroup K of G such that G/K is a Lie group of polynomial growth. Suppose the connected component of identity in G is compact. Then G has a compact open subgroup and hence G/K also has a compact open subgroup. Since G/K is a Lie group, this implies that the connected component of identity is a compact open normal subgroup. Thus, G has a compact open normal subgroup, let it be L. So for any representation T, the space of $T(L)$-fixed points are invariant under G and hence it is enough to show that G/L has identity excluding property. Since G/L is a discrete group of polynomial growth, by Gromov's theorem in [2], we get that G/L is a finite extension of a nilpotent group. Now the result follows from Theorem 3.2. □

4. p-adic algebraic groups

We first make the following observation:

Proposition 4.1. *Let T be an unitary representation of a group G. Suppose for some $f \in L^1(G)$ with $f \geqslant 0$, $T(f)$ is a compact operator. Then $I \prec T$ implies T has a non-trivial fixed point.*

Proof. Let $f \in L^1(G)$ such that $f \geqslant 0$ and $T(f)$ is a compact operator. Then we may assume that $\int f = 1$. Suppose $I \prec T$. Then there exists a sequence (v_n) of unit vectors such that $\|T(g)v_n - v_n\| \to 0$ for all $g \in G$. This implies that $\|v_n - T(f)v_n\| \to 0$. Since $T(f)$ is a compact operator, $(T(f)v_n)$ has a convergent subsequence. By passing to a subsequence we may assume that $T(f)v_n \to v$. Since $\|T(f)v_n - v_n\| \to 0$, $v_n \to v$. This implies that $T(G)v = v$ and $\|v\| = 1$. Thus, T has a non-trivial fixed point. □

Thus, the above result shows that totally disconnected CCR-groups have identity excluding property (see [1] for details on CCR-groups). It is known that only algebraic groups that are CCR are the direct products of a semisimple group and a group of type R ([8] and [13]). Here we prove that p-adic algebraic groups which are semidirect product of semisimple groups and groups of type R have identity excluding property, that is any p-adic algebraic group has identity excluding property if the solvable radical is type R. By an algebraic group over a local field $\mathbb{K}$ of characteristic zero, we mean the group of $\mathbb{K}$-points of an algebraic group defined over $\mathbb{K}$.

Definition 4.1. We say that a finite-dimensional vector space V over a local field of characteristic zero is of *type* R_Γ where Γ is a group of automorphisms of V, if the eigenvalues of each element of Γ are of absolute value one. We say that a Lie group G is of *type* R_Γ where Γ is a group of Lie automorphisms of G, if the Lie algebra of G is of type R_Γ and a Lie group G is said to be of *type* R if G is of type $R_{\mathrm{Ad}(G)}$ where Ad is the adjoint representation of G: see [4] for results on type R real Lie groups and [14] for results on p-adic Lie groups of type R.

Theorem 4.1. *Let G be any p-adic algebraic group. Let U be the unipotent radical of G. Let $U_0 = U$, $U_i = [U, U_{i-1}]$ for $i > 0$. For any $i > 0$ and for any G-invariant subspace W of U_i/U_{i+1} define $\phi_{i,W} : G \to GL(W)$ by $\phi_{i,W}(g)xU_{i+1} = gxg^{-1}U_{i+1}$ for all $xU_{i+1} \in W \subset U_i/U_{i+1}$. Suppose for each $i > 0$ and W as above, either $\phi_{i,W}(G)$ is non-amenable or W is of type $R_{\phi_{i,W}(G)}$. Then G has identity excluding property.*

Remark 4.1. Suppose G is an algebraic subgroup of $GL(W)$, for some finite-dimensional vector space W. Suppose G is reductive. Then G is amenable implies that G has no non-trivial non-compact simple factors. Thus, for amenable G, W is of type R_G if and only if the center of G has no split torus, that is anisotropic.

Proof. Let S be a reductive Levy subgroup of G and U be the unipotent radical of G. Then G is the semidirect product of S and U. Let Z be the center of U. Then Z contains U_k where $k \geqslant 0$ is such that $U_k \neq (e)$ but $U_{k+1} = (e)$. Let $V = U_k$.

We prove the result by induction on dimension of U. It may be noted that G/W satisfies the hypothesis for any irreducible G-invariant subspace W of V. Suppose $\dim(U) = 0$. Then G is a reductive group. Since G is a CCR-group, G has identity excluding property.

If $\dim(U) > 0$. Then V is a non-trivial normal subgroup of G. Suppose V contains a irreducible G-subspace W such that the image of G in $GL(W)$ is non-amenable. It is easy to see that the action of G on $\widehat{W}$ is also irreducible. By 4.15 and 5.15 of [16], we get that (G, W) has strong relative property (T) (see [16] for details). Let T be a irreducible representation of G such that $I \prec T$. Define a representation T_1 of the semidirect product of G and W by $T_1((g, w)) = T(gw)$. Then $I \prec T_1|_G$ and hence $T_1(W)$ has a non-trivial fixed point. So, $T(W)$ has a non-trivial fixed point. Since W is a normal subgroup, $T(W)$ is trivial. Now by induction hypothesis $T(G/W)$ is trivial and hence $T(G)$ is trivial.

Suppose, for any irreducible G-subspace W of V, the image of G in $GL(W)$ is amenable. Then by assumption W is of type $R_{\phi_{i,W}(G)}$. This implies that any split torus of G, centralizes W and since U centralizes W, $\phi_{i,W}(G)$ is compact. Thus, $\{gxg^{-1} \mid g \in G\}$ is compact in V for any $x \in V$.

Suppose G is a p-adic algebraic group, then V is a increasing union of compact normal subgroups of G. Let (M_i) be a sequence of compact normal subgroups of G such that $V = \bigcup M_i$ and $M_i \subset M_{i+1}$ for all $i > 0$. Let T be an irreducible representation of G such that $I \prec T$. Then for each $i > 0$, $I \prec T|_{M_i}$. For each $i > 0$, since M_i has property (T), $T(M_i)$ has a non-trivial fixed point. For each $i > 0$, since M_i is normal in G, the space of fixed points of $T(M_i)$ is a non-trivial $T(G)$-invariant closed subspace. Thus, $T(M_i)$ is trivial for all $i > 0$. This shows that $T(V)$ is trivial. By applying induction hypothesis to G/V, we get that T is trivial. Thus, by Lemma 2.2, G has identity excluding property. □

Corollary 4.1. *Suppose G is a p-adic algebraic group whose solvable radical is of type R. Then G satisfies the hypothesis of Theorem* 4.1 *and G is identity excluding.*

Proof. Suppose for some $i > 0$, there is an irreducible normal subgroup W of U_i/U_{i+1} such that the image of G in $GL(W)$ is amenable. Then any semisimple Levy subgroup of G has only compact orbits in W. Since the solvable radical is of type R, Gx is compact for any $x \in W$. Thus, G satisfies the hypothesis of Theorem 4.1. □

The following is easy to verify.

Corollary 4.2. *The semidirect product of $GL_n(\mathbb{Q}_p)$ and $\mathbb{Q}_p^n$ (for $n > 1$) verifies the hypothesis of Theorem* 4.1 *but its solvable radical is not of type R.*

We now prove a partial converse to Theorem 4.1.

Theorem 4.2. *Let G be a p-adic algebraic group and U be the unipotent radical of G. Suppose the solvable radical of G is not of type R and U is of type R_L where L is a semisimple Levy subgroup of G. Then G is not identity excluding.*

Proof. It is enough to prove the result for a quotient group of G. Let S be the solvable radical of G. Since S is not of type R, let $U' = [U, U]$, then S/U' is also not of type R. Thus, replacing G by G/U', we may assume that unipotent radical is abelian. Since S is not of type R, there exists an element $x \in S$ such that the subspace $\{v \in U \mid x^n v x^{-n} \to e\}$ is non-empty and it is G-invariant. Thus, again replacing G by a quotient of G, we may assume that the split torus A_s of S is one-dimensional and for any $x \in A_s \setminus (e)$, x or x^{-1} contracts U.

Now, let χ be a non-trivial character of U. Let H be the stabilizer of χ in the reductive Levy part of G. Define ρ on HU by $\rho(hu) = \chi(u)$. Then ρ defines an irreducible unitary representation of HU. Now by Mackey's normal subgroup analysis (see [10] and [11]), the induced representation T of G from ρ is irreducible. We now claim that $I \prec T$. Let $G_1 = A_s HU$. By assumption, the non-compact simple factors of L centralizes U. This implies that G/G_1 is compact. Let T_1 be the induced representation of G_1 from ρ. Then T is the induced representation from T_1, by Chapter III, 1.11 of [12], it is enough to show that $I \prec T_1$. As in example of [5], we can prove that $I \prec T_1$. □

5. Convergence of representation averages for some solvable algebraic groups

Lin and Witmann show that the convergence of (T_μ^n) in the strong operator topology implies that any strictly aperiodic ergodic measure is weak mixing. We now consider convergence of (T_μ^n) for measures on any solvable groups. We have proved that solvable p-adic algebraic groups are identity excluding if and only if it is of type R, here we prove that μ^n-averages of representations on split solvable p-adic algebraic groups (which are not necessarily identity excluding), are strongly convergent. We recall that a solvable p-adic algebraic group is called *split* if maximal torus is splitting.

Theorem 5.1. *Let G be a split solvable Zariski-connected p-adic algebraic group which is not of type R. Let μ be an adapted and strictly aperiodic probability measure on G. Let T be a representation of G. Then (T_μ^n) converges in the strong topology.*

Proof. In view of Theorem 2.2 of [7], we may assume that T is irreducible. Let U be the unipotent radical of G. Let A be a maximal of torus of G. Since G is split, A is a split torus. Since G is not of type R, both U and A are non-trivial. Then the center of U is non-trivial. Let Z be the center of U. Replacing G by $G/\ker(T)$, we may assume that T is a faithful representation. We also assume that T has an approximate fixed point, otherwise T_μ^n converges. Since T is faithful, G has no non-trivial center.

Let $\widehat{Z}$ be the dual of Z. We now claim that the stabilizer of a non-trivial point in $\widehat{Z}$ is a proper subgroup of G. Suppose there is a non-trivial point χ in $\widehat{Z}$ whose stabilizer is the whole group. Then there exists a one-dimensional subspace W of Z, which is a normal subgroup of G. Then W is either type R_G or W is contracted by an element of A. Suppose

W is type R_G. Then since G is a split solvable group, G action on W is trivial. Since center of G is trivial, this is a contradiction. So, we may assume that there is an element g of A contracting W. Then $\chi(x) = g^{-n} \cdot \chi(x) = \chi(g^n \cdot x) \to 1$ for any $x \in Z$. This is a contradiction. Thus, the stabilizer of any non-trivial point of $\widehat{Z}$ is a proper subgroup of G.

By Mackey's theorem (see 13.3, Theorem 1 of [6]), there exists a proper subgroup H containing U and an irreducible representation ρ of H in a Hilbert space E such that the induced representation of G from ρ is T (up to equivalence).

We now claim that G is a semidirect product of an abelian group and H. Let $A_0 = A \cap H$. Then A_0 is subtorus of A. Since A splits in G, there exists a subtorus A_1 of A such that A is the direct product $A_0 \times A_1$. Since A_0 is a proper split torus, A_1 is a non-trivial split torus.

Hence, we identify G/H with A_1. Let m be Haar measure on A_1. Then under the identification, m is a G-invariant measure on G/H.

Let $L^2(G, H, \rho)$ be the space of all measurable functions $f : G \to E$ such that

(i) $f(xh) = \rho(h) f(x)$ for all $x \in G$ and $h \in H$,
(ii) $\int_{A_1} \|f(x)\|^2 \, dm(x) < \infty$.

Since the induced representation from ρ is T, T is defined on $L^2(G, H, \rho)$ by $T(g) f(x) = f(gx)$ for all $f \in L^2(G, H, \rho)$ and all $x, g \in G$ (see Chapter I, 5.2 of [12]).

Now for any $g \in G$, there exists unique $s \in A_1$ and unique $h \in H$ such that $g = sh$. Thus, for $f \in L^2(G, H, \rho)$, we have

$$
\begin{aligned}
\|T_\mu f\|^2 &= \left| \int_{A_1} \langle T_\mu f(x), T_\mu f(x) \rangle \, dm(x) \right| \\
&\leqslant \int_{A_1} \int_G \int_G \left| \langle f(g_1 x), f(g_2 x) \rangle \right| d\mu(g_1) \, d\mu(g_2) \, dm(x) \\
&\leqslant \int_{A_1} \int_G \int_G \left| \langle \rho(x^{-1} h_1 x) f(s_1 x), \rho(x^{-1} h_2 x) f(s_2 x) \rangle \right| d\mu \, d\mu \, dm \\
&\qquad \text{where} \quad g_1 = s_1 h_1 \quad \text{and} \quad g_2 = s_2 h_2 \\
&\leqslant \int_{A_1} \int_G \int_G \| f(s_1 x) \| \, \| f(s_2 x) \| \, d\mu \, d\mu \, dm.
\end{aligned}
$$

Let λ be the image of μ in A_1 under the canonical map $g \mapsto s$ and R be the regular representation of A_1 in $L^2(A_1)$. Then from the above calculations we get that $\|T_\mu^n f\| \leqslant \|R_\lambda^n F\|$ where $F(x) = \|f(x)\|$ for all $x \in A_1$. Since nilpotent groups are identity excluding and A_1 is non-compact, we get that $\|T_\mu^n f\| \to 0$. □

We now consider any split solvable group.

Theorem 5.2. *Let G be any split solvable Zariski-connected p-adic algebraic group and μ be an adapted and strictly aperiodic probability measures on G. Suppose T is a representation of G. Then (T_μ^n) converges strongly.*

Proof. Suppose G is of type R. Then G is a nilpotent group and hence it is identity excluding. Suppose G is not type R, then strong convergence of (T_μ^n) follows from Theorem 5.1. □

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