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# Morphisms of Extensions of $C^*$-Algebras: Pushing Forward the Busby Invariant

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We study completions of diagrams of extensions of $C^*$-algebras of the form

$$\begin{tikzcd} 0 \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & 0 \end{tikzcd}$$

in which all three $C^*$-algebras in one of the rows and either the ideal or the quotient in the other are given, along with the three morphisms between them. We find universal solutions to all four of these problems under restrictions of varying severity, on the given vertical maps and describe the solutions in terms of push-outs and pull-backs of certain diagrams. Our characterization of the universal solution to one of the diagrams yields a concrete description of various amalgamated free products. This leads to new results about the $K$-theory of amalgamated free products, verifying the Cuntz conjecture in certain cases. We also obtain new results about extensions of *matricial field* $C^*$-algebras, verifying partially a conjecture of Blackadar and Kirchberg. Finally, we show that almost commuting unitary matrices can be uniformly approximated by commuting unitaries when an index obstruction vanishes. © 1999 Academic Press



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## 1. INTRODUCTION

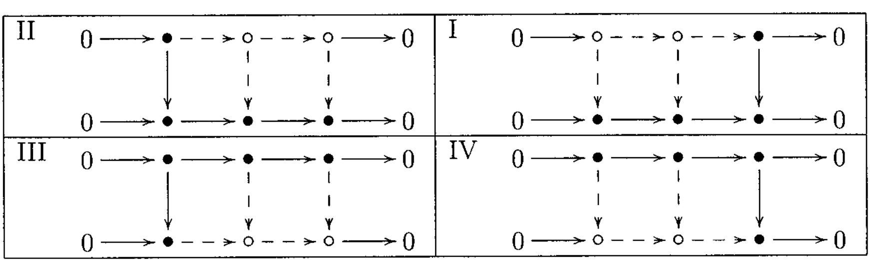
It is well-known that in the categories of (not necessarily unital) algebras and $C^*$-algebras, there exist (necessarily unique) universal solutions to the *pull-back* and *push-out* problems. One may formulate these problems diagrammatically as

Such diagrams are to be read as follows: Given objects and morphisms symbolized by black dots and solid arrows, does there exist a *universal* choice of objects and morphisms symbolized by white dots and dashed arrows, making the diagram commute? The first question translates to the defining diagram for the *amalgamated free product*

$$\begin{tikzcd} & B \arrow[dr] \arrow[drr, bend left] & & \\ A \arrow[ur] \arrow[dr] & & B *_A C \arrow[r, dotted] & D \\ & C \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

and the existence of this object gives the affirmative answer to our first problem. The *pull-back* construction gives the affirmative answer to the second.

The study of extensions is central to the theory of $C^*$-algebras and has widespread applications to most of its subfields. One might mention the classification problem, *KK*-theory, exact $C^*$-algebras, and quasidiagonality as four different examples of research areas in which extensions play a pivotal role. Continuing the train of thought above, we are lead to consider the four problems



where we now, besides commutativity, require horizontal exactness everywhere.

The purpose of this article is to settle the question of the existence of non-trivial solutions to these problems, with universal solutions identified whenever possible. Along the way it will be made clear that the four problems, despite their apparent similarities, have very different solutions, requiring quite different methods. In particular, the advantage of working in the category of $C^*$-algebras instead of just algebras over $\mathbb{C}$ increases dramatically with the number of the diagram.

In fact, Diagram I can always be filled out universally in both of these categories, for exactly the same reason. Indeed, if a system

$$\begin{tikzcd} & & & B_1 \arrow[d, "\beta"] & \\ 0 \arrow[r] & A \arrow[r, "\iota"'] & X \arrow[r, "\pi"'] & B \arrow[r] & 0 \end{tikzcd}$$

is given, the universal solution is

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota_1"] \arrow[d, equal] & X_1 \arrow[r, "\pi_1"] \arrow[d, "\chi"] & B_1 \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\iota"'] & X \arrow[r, "\pi"'] & B \arrow[r] & 0 \end{tikzcd}$$

where $X_1$ is the pull-back over $X$ and $B_1$, $\pi_1$ and $\chi$ the universal maps, and $\iota_1$ the map induced by

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d, "\iota"] & B_1 \arrow[d, "\beta"] \\ X \arrow[r, "\pi"'] & B \end{tikzcd}$$

We leave the details to the reader.

To explain how we are going to deal with the other three problems, we recall that the study of ($C^*$-)algebraic extensions is based on two tools—the pull-back construction, which we have already mentioned, and the *multiplier algebra*, which is the universal unital ($C^*$-)algebra containing the given algebra as an essential ideal. When an extension

$$0 \to A \xrightarrow{\iota} X \xrightarrow{\pi} B \to 0$$

is given, there is a morphism

$$\theta\colon X \to M(A)$$

given by $\theta(x)a = \iota^{-1}(x\iota(a))$, and a morphism

$$\eta\colon B \to M(A)/A$$

will be induced. This is an invariant of the isomorphism class of the extension, the *Hochschild* or *Busby* map, depending on whether one works in the category of algebras or the category of $C^*$-algebras. It has been known since [13] that if an algebra $A$ (over $\mathbb{C}$, *without unit*) is an essential ideal in itself, then this correspondence between (isomorphism classes of) extensions of $A$ by another algebra $B$ with $\mathrm{Hom}(B, M(A)/A)$ is a bijection. The extension algebra $X$ associated to $\eta\colon B \to M(A)/A$ will be the universal solution of a diagram of type I

$$\begin{tikzcd} & & & B \arrow[d, "\eta"] & \\ 0 \arrow[r] & A \arrow[r] & M(A) \arrow[r] & M(A)/A \arrow[r] & 0 \end{tikzcd}$$

where the bottom line is the maximal essential extension of $A$. We shall often refer to this situation by

$$0 \to A \to X \to B \to 0 \qquad [\eta].$$

The situation working in the category of $C^*$-algebras is similar; all one must do is replace the Hochschild theory by the corresponding theory due to Busby ([3]). As every $C^*$-algebra is an essential ideal in itself, we do not need to make this assumption explicitly.

With these tools at hand, e.g. in the $C^*$-algebra case, we may describe the universal solution to Diagram I in a different manner: If the original extension has Busby map $\eta$, the completed extension will have Busby map $\eta\beta$. Again, we leave the details to the reader.

When we pass to the diagrams II and III, we encounter a problem which rarely has a solution in the category of algebras, but very often does in the category of $C^*$-algebras. The difficulty stems from the multiplier construction not being functorial. When trying to perform Busby theory on the given data in diagrams II and III, we need to induce from the given vertical map $\alpha\colon A \to A_1$ a map $\tilde{\alpha}\colon M(A)/A \to M(A_1)/A_1$. This is only possible for a very narrow class of algebra morphisms, whereas it is possible for a large and natural class of $C^*$-algebra morphisms. With the functoriality problem out of the way, we are able to solve problems II and III fully. We shall develop a tool, Theorem 2.2 below, which contains the full answer. Furthermore, we shall describe the universal objects as pull-backs and push-outs for certain diagrams.

While a universal solution to diagram IV exists, for rather trivial reasons, when the given vertical map is surjective, it is the case of injective vertical maps which is important. We predict that universal solutions will only very rarely exist in this case. Furthermore, even to achieve non-trivial solutions requires heavy use of the analytical side of $C^*$-algebra theory, in particular of corona algebra methods, as described in [23] and [27]. In order to get just a single nontrivial completion of the diagram, the given vertical map must be *corona extendible* ([22]), a notion which turns up in the study of *projective* and *semiprojective* $C^*$-algebras. Surprisingly many morphisms satisfy this strong condition, and we present a new class of corona extendible maps in this paper.

Diagrams III and IV are of special interest to applications in $C^*$-algebra theory. Taken together they give a way to analyze $\mathrm{Hom}(X, -)$ in terms of $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(B, -)$ when $X$ is an extension of $A$ by $B$. For an example of why this may be desirable, consider the extension

$$0 \to S\mathbf{M}_n \to \mathbb{I}_n \to \mathbb{C} \to 0$$

involving two of the most basic $C^*$-algebras and the nonunital dimension drop algebra. The latter $C^*$-algebra has, justifiably, been getting much press of late, especially with regard to torsion coefficient $K$-theory and classification problems, cf. [5], [7], [6], [4]. The study of maps out of $\mathbb{I}_n$ (in [17], for example) was essentially based on the isomorphism

$$\mathbb{I}_n = S\mathbf{M}_n *_{C_0(]0,1[)} C_0(]0,1]).$$

This a special case of our Proposition 4.2, which identifies the universal completion to diagram III. Many approximate results about $\mathrm{Hom}(\mathbb{I}_n, -)$ (those in [17]) are turned into exact results by using the corona extendibility concept linked to Diagram IV.

The paper is organized as follows. In the first two sections we develop two tools—one algebraic, the other analytical in nature—which are fundamental to most of our further results. We are optimistic that the applications of those tools are not limited to what is presented here. We then turn our attention towards Diagrams II, III and IV, solve the question of universal solutions under certain restrictions on the vertical map, and describe why corona extendibility is important for Diagram IV. Expanding the results in [22], we then devise a large new class of corona extendible maps.

Turning to applications, we study instances of *matricial* corona extendibility, in the sense that only maps into corona algebras over $\bigoplus \mathbf{M}_{n_k}$ must be extendible. Combining several results in the paper, we find a class of matricially corona extendible maps, and this leads to new results about the class of *matricial field* algebras defined and studied by Blackadar and

Kirchberg. Our results also combine with results by Loring, based on Friis and Rørdam's proof of Lin's theorem regarding almost commuting self-adjoint matrices, to show that certain maps defined on the two-torus have a property related to matricial corona extendibility. Out of this we derive the result that almost commuting unitary matrices may be approximated by commuting unitaries when a natural index obstruction vanishes.

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## 2. MORPHISMS OUT OF EXTENSIONS

The main result of this section allows a full description of morphisms out of an extension $X$ of $A$ by $B$, in terms of morphisms out of $A$ and $B$. We prove the result using a reformulation concerning maps between extensions, and this basic result is what we shall be using throughout the paper.

The results and their proofs are essentially algebraic in nature, but, as mentioned above, require what appears to be unrealistic restrictions in categories without topology. By contrast, the result holds true for $C^*$-algebras when a certain morphism falls in a large and natural class, the *proper* morphisms.

### 2.1. *Proper Maps*

To find a subcategory of algebras and morphisms for which the multiplier construction is functorial, we must in addition to the already mentioned hypothesis that the objects should be essential ideals in themselves, add the condition

$$A_1 = \alpha(A)A_1 = A_1\alpha(A)$$

on the morphisms from $A$ to $A_1$. In this case, we may functorially define $\bar{\alpha}\colon M(A) \to M(A_1)$ by the formula

$$\bar{\alpha}(m)[\alpha(a)a_1] = \alpha(ma)a_1.$$

to obtain an extension of $\alpha$. As $\bar{\alpha}$ extends $\alpha$, we also get a morphism $\tilde{\alpha}$ between the *corona algebras* $C(A) = M(A)/A$ and $C(A_1) = M(A_1)/A_1$.

In the class of $C^*$-algebras, we only need to require one of the equivalent properties:

(i) $A_1 = \overline{\alpha(A)A_1} = \overline{A_1\alpha(A)}$.

(ii) $\alpha(A)$ is is contained in no proper, hereditary $C^*$-subalgebra of $A_1$.

(iii) The image under $\alpha$ of an approximate unit for $A$ is an approximate unit for $A_1$.

We say that $\alpha$ is *proper* when this is the case, and use the last property to define $\bar{\alpha}$. Let an approximate unit $(u_\lambda)$ for $A$ be given, and define

$$\bar{\alpha}(m)\, a = \lim_{\Lambda} \alpha(mu_\lambda)\, a$$

which is indeed a $*$-homomorphism from $M(A)$ to $M(A_1)$ extending $\alpha$. Clearly $\bar{\alpha}$ is unique, so the map does not depend upon the choice of $(u_\lambda)$ and the construction is functorial. We again define $\tilde{\alpha}\colon C(A) \to C(A_1)$ in the obvious way.

To explain the terminology and demonstrate that this is a natural class, we return to commutative $C^*$-algebras. In a careless moment a $C^*$-algebraist might be quoted for saying that there is a covariant functor between the categories of commutative C*-algebras with morphisms and the category of locally compact Hausdorff spaces with continuous maps. This, however, is wrong! Some morphisms (e.g. the zero morphism) do not correspond to continuous maps and some maps (non-proper ones) do not correspond to morphisms. The correct categories are:

| Compact spaces | | Unital commutative $C^*$-algebras |
|---|---|---|
| Continuous maps | $\leftrightarrow$ | Unital morphisms |

and

| Locally compact spaces | | Commutative $C^*$-algebras |
|---|---|---|
| Proper continuous maps | $\leftrightarrow$ | Proper morphisms |

The bridge between these two categories are "pointed spaces", i.e. locally compact spaces $X$ with an additional point "$\infty$", written $X^+$, where $X^+$ is the one-point compactification if $X$ is non-compact, and otherwise $\infty$ is an isolated point adjoined to $X$, with continuous maps $g\colon X^+ \to Y^+$ with $g(\infty) = \infty$. This corresponds to commutative $C^*$-algebras with arbitrary unital morphisms via forced unitization. Note how, when both $X$ and $Y$ are noncompact, precisely the maps $g$ such that $g^{-1}(\{\infty\}) = \{\infty\}$ will reduce to proper continuous maps of $X$ into $Y$.

This analogy is made sharper by the noncommutative Tietze extension theorem. A surjection $A \to A_1$ is certainly proper, and by [26, 10] the induced morphism $M(A) \to M(A_1)$ is surjective. Consider the case $A = C_0(X)$ and $A_1 = C(Z)$ where $Z$ is closed in $X$. The fact that any element in $M(A_1)_{sa} = C_b(Z, \mathbb{R})$ lifts to an element in $M(A)_{sa} = C_b(X, \mathbb{R})$ is the reason for calling this an extension theorem.

## 2.2. *Existence of Morphisms Out of Extensions*

LEMMA 2.1. *Given a commutative diagram*

$$\begin{tikzcd} A_1 \arrow[r, "\iota_1"] \arrow[d, "\alpha"] & X_1 \arrow[d, "\chi"] \\ A \arrow[r, "\iota"] & X \end{tikzcd}$$

*in which the horizontal maps are embeddings of ideals, with $\alpha$ proper, and letting*

$$\theta\colon X \to M(A), \qquad \theta_1 : X_1 \to M(A_1), \quad \bar{\alpha}\colon M(A_1) \to M(A)$$

*denote the morphisms induced by $\iota$, $\iota_1$, and $\alpha$, we have*

$$\theta\chi = \bar{\alpha}\theta_1.$$

*Proof.* Recall that

$$\theta(x)a = xa, \qquad \theta_1(y)\, a_1 = ya_1.$$

A dense set of elements in $A$ are of the form $\alpha(a_1)\, a$, and we have

$$\begin{aligned} \chi(x_1)\iota(\alpha(a_1)a) &= \chi(x_1\iota_1(a_1))\iota(a) \\ &= \chi(\iota_1(x_1 a))\iota(a) \\ &= \iota\alpha(x_1 a_1)\iota(a) \\ &= \iota(\alpha(x_1 a_1)a), \end{aligned}$$

proving that $\theta\chi = \bar{\alpha}\theta_1$. ∎

THEOREM 2.2. *Let Busby maps $\eta_1 : B_1 \to C(A_1)$ and $\eta\colon B \to C(A)$ be given, as well as morphisms $\alpha\colon A_1 \to A$ and $\beta\colon B_1 \to B$, and assume that $\alpha$ is proper, so that we have an induced morphism $\tilde{\alpha}\colon C(A) \to C(A_1)$. Consider the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "\iota_1"] \arrow[d, "\alpha"] & X_1 \arrow[r, "\pi_1"] \arrow[d, dashed] & B_1 \arrow[r] \arrow[d, "\beta"] & 0 & {[\eta_1]} \\
0 \arrow[r] & A \arrow[r, "\iota"] & X \arrow[r, "\pi"] & B \arrow[r] & 0 & {[\eta].}
\end{tikzcd}$$

*Then there exists a morphism $\chi\colon X_1 \to X$, necessarily unique, making the diagram above commute, exactly if*

$$\tilde{\alpha}\eta_1 = \eta\beta.$$

*Proof.* We shall prove the result using the 3D diagram

$$\begin{tikzcd}
& 0 \arrow[r] & A_1 \arrow[r] \arrow[dd, "\alpha"] & M(A_1) \arrow[r] \arrow[dd, "\overline{\alpha}"] & C(A_1) \arrow[r] \arrow[dd, "\tilde{\alpha}"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "\iota_1"'] \arrow[ur, equal] \arrow[dd] & X_1 \arrow[r, "\pi_1"'] \arrow[ur, "\theta_1"] \arrow[dd, dashed, "\chi"] & B_1 \arrow[r] \arrow[ur, "\eta_1"] \arrow[dd, "\beta"] & 0 & \\
& 0 \arrow[r] & A \arrow[r] & M(A) \arrow[r] & C(A) \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"'] \arrow[ur, equal] & X \arrow[r, "\pi"'] \arrow[ur, "\theta"] & B \arrow[r] \arrow[ur, "\eta"] & 0 &
\end{tikzcd}$$

Note that by construction, the leftmost square and the top, bottom, and back faces are commutative.

Suppose a $\chi$ exists so that the front faces commute. By Lemma 2.1 both the front face and the center square are commutative, whence so is the rightmost square.

Conversely, suppose that the rightmost square is commutative. We conclude that the two morphisms from $X_1$ to $C(A)$ going over $M(A)$ and $B$, respectively, agree, and by the universal property of $X$, we can fill out the diagram with a unique morphism $\chi$ for which the center and right front squares commute. A diagram chase shows that also the left front square commutes. ▮

*Remark* 2.3. Uniqueness of the map $\chi$ may fail in the absence of properness. Consider for instance the diagrams of trivial extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "0"] & \mathbb{C}\oplus\mathbb{C} \arrow[r] \arrow[d, "\chi_i"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}\oplus\mathbb{C} \arrow[r] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

with morphisms $\chi_1(a, b) = (0, b)$ and $\chi_2(a, b) = (b, b)$.

We shall use Theorem 2.2 to give a complete description of the set of morphisms out of an extension $C^*$-algebra. Define the idealizer of $\varphi(A)$ in $Y$ as

$$I(\varphi(A) : Y) = \{ y \in Y \mid y\varphi(A) + \varphi(A)y \subseteq \varphi(A) \}.$$

As quotient we obtain the *eigenalgebra*

$$E(\varphi(A) : Y) = I(\varphi(A) : Y)/\varphi(A),$$

cf. [24]. Thus working only with the $C^*$-subalgebra $\varphi(A)$ of $Y$ we have created an extension

$$0 \to \varphi(A) \xrightarrow{\iota} I(\varphi(A) : Y) \xrightarrow{\pi} E(\varphi(A) : Y) \to 0$$

giving rise to a Busby map

$$\zeta: E(\varphi(A) : Y) \to C(\varphi(A)).$$

THEOREM 2.4. *Given an extension of $C^*$-algebras*

$$0 \to A \to X \to B \to 0 \qquad [\eta]$$

*and a $C^*$-algebra $Y$, there is a bijective correspondence between elements $\bar{\varphi}$ in* $\mathrm{Hom}(X, Y)$ *and pairs* $(\varphi, \psi)$, *where* $\varphi \in \mathrm{Hom}(A, Y)$ *and* $\psi \in \mathrm{Hom}(B, E(\varphi(A) : Y))$, *such that*

$$\tilde{\varphi}\zeta = \eta\psi.$$

*Here $\zeta$ is the Busby map for the extension*

$$0 \to \varphi(A) \to I(\varphi(A) : Y) \to E(\varphi(A) : Y) \to 0$$

*and $\tilde{\varphi}$ is the map induced by the proper morphism $\varphi: A \to \varphi(A)$.*

*Proof.* Let $\bar{\varphi}$ be given and denote by $\varphi$ its restriction to $A$. Since $A$ is an ideal in $X$, $\varphi(A)$ is an ideal in $\bar{\varphi}(X)$, and $\bar{\varphi}$ in fact maps into $I(\varphi(A) : Y)$. We get a diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"'] & X \arrow[r] \arrow[d, "\bar{\varphi}"'] & B \arrow[r] \arrow[d, "\psi"'] & 0 \\
0 \arrow[r] & \varphi(A) \arrow[r] & I(\varphi(A): Y) \arrow[r] & E(\varphi(A): Y) \arrow[r] & 0
\end{tikzcd}$$

with $\psi$ induced by the other two vertical morphisms. Since $\tilde{\varphi}\zeta = \eta\psi$ by Theorem 2.2, we have established a map out of $\mathrm{Hom}(X, Y)$ with the described properties. The map is a bijection also as a consequence of this result.

## 3. MULTIPLIER REALIZATIONS

In contrast to the tools developed in the previous section, our next result is highly analytical in nature. Its proof is based on the fact that the multiplier algebra of a $C^*$-algebra $A$ embeds naturally as the idealizer of $A$ in the enveloping von Neumann algebra $A^{**}$.

### 3.1. *A Technical Theorem*

THEOREM 3.1. *Let $C(E)$ denote the corona of a $\sigma$-unital $C^*$-algebra $E$, and let $D$ and $N$ be separable $C^*$-subalgebras of $C(E)$. For every morphism*

$$\varphi\colon A \to C(E) \cap D' \cap N^{\perp},$$

*where $A$ is a $\sigma$-unital $C^*$-algebra, and every element $m$ in $M(A)$, there is a $z$ in $C(E) \cap D' \cap N^{\perp}$ such that $\varphi(ma) = z\varphi(a)$ and $\varphi(am) = \varphi(a)z$ for each $a$ in $A$. If $0 \leqslant m \leqslant 1$ we can choose $0 \leqslant z \leqslant 1$.*

*Proof.* By linearity it suffices to consider the case $0 \leqslant m \leqslant 1$. If $(u_n)$ is a countable approximate unit for $A$, set

$$x_n = m^{1/2} u_n m^{1/2}, \qquad y_n = (1-m)^{1/2}\, u_n (1-m)^{1/2}.$$

Then we obtain monotone increasing sequences $(x_n)$ and $(y_n)$ in $A_+$, such that $x_n \nearrow m$ and $1 - y_n \searrow m$ in $A^{**}$. Equivalently,

$$\lim \|(m - x_n)a\| = \lim \|(1 - y_n - m)a\| = 0$$

for every $a$ in $A$. (Strict convergence, cf. [25, 3.12.9].)

By Kasparov's technical theorem, see [23, 3.5] or [27, 8.3], there is an element $e$ in $C(E) \cap D' \cap N^{\perp}$ such that $0 \leqslant e \leqslant 1$ and $\varphi(A)(1-e) = 0$. Now consider the sequences $(\varphi(x_n))$ and $(e - \varphi(y_n))$ in $C(E)_+$, and note that one is increasing, the other decreasing, with $\varphi(x_n) \leqslant e - \varphi(y_m)$ for all $n, m$ (because $e$ is a unit for $\varphi(A)$). Since $C(E)$ has the asymptotically Abelian, countable Riesz separation property (AA–CRISP) by [23, 3.4] or [27, 6.7], there is an element $z$ in $C(A) \cap D'$ such that

$$\varphi(x_n) \leqslant z \leqslant e - \varphi(y_n)$$

for all $n$. In particular, $0 \leqslant z \leqslant e \leqslant 1$, so $z \in N^\perp$. For each $a$ in $A$ we have

$$\begin{aligned}
&\|(z-\varphi(x_n))^{1/2}\,\varphi(a)\|^2 \\
&\quad = \|\varphi(a^*)(z-\varphi(x_n))\varphi(a)\| \\
&\quad \leqslant \|\varphi(a^*)(e-\varphi(y_n)-\varphi(x_n))\varphi(a)\| \\
&\quad \leqslant \|a^*(1-y_n-x_n)a\| \\
&\quad = \|a^*((1-m)^{1/2}\,(1-u_n)(1-m)^{1/2}+m^{1/2}(1-u_n)\,m^{1/2})a\| \\
&\quad \leqslant \|(1-u_n)^{1/2}\,(1-m)^{1/2}\,a\|^2+\|(1-u_n)^{1/2}\,m^{1/2}a\|^2 \\
&\quad \to 0,
\end{aligned}$$

whence $(z-\varphi(x_n))\varphi(a) \to 0$. Since also $(m-x_n)\,a \to 0$ by our choice of $x_n$, it follows that $z\varphi(a)=\varphi(ma)$ for every $a$ in $A$, as desired. ∎

COROLLARY 3.2. *If the morphism $\varphi\colon A \to C(E) \cap D' \cap N^\perp$ is injective, and if $I(A)$ and $A^\perp$ denote the idealizer and the annihilator of $\varphi(A)$ in $C(E)$, there is an extension*

$$0 \to A^\perp \cap D' \cap N^\perp \to I(A) \cap D' \cap N^\perp \to M(A) \to 0$$

*Proof.* As we have seen above, there is always a natural morphism

$$\theta\colon I(A) \to M(A)$$

and it is clear from the definition that $\ker\theta = A^\perp$. In our case we conclude from Theorem 3.1 that this morphism, even restricted to $I(A) \cap D' \cap N^\perp$, is surjective. ∎

*Remark* 3.3. Our proof of Theorem 3.1 uses the Kasparov technical theorem. However, it is possible to give a uniform proof of Theorem 3.1 and the KTT based on lifting properties of $C^*$-algebra elements. See [19].

The potential—which we believe is considerable—of the previous result lies in the possibility of choosing special subalgebras $B$ of $I(A) \cap D' \cap N^\perp$ that have zero intersection with $A^\perp$. Certainly the case where $B$ is simple and defined by a few algebraic relations (like $\mathcal{O}_n$ or the irrational rotation algebras) should be investigated more closely, perhaps in conjunction with projection-creating conditions on $C(E)$, such as having real rank zero.

For immediate consumption we shall consider only the simplest case associated with lifting problems—projective $C^*$-algebras.

THEOREM 3.4. *Let $C(E)$ be the corona algebra of a $\sigma$-unital $C^*$-algebra $E$ and let $D$ and $N$ be separable $C^*$-subalgebras of $C(E)$. If*

*$\varphi\colon A \to C(E) \cap D' \cap N^{\perp}$ is a morphism of a $\sigma$-unital $C^*$-algebra $A$ and $\theta\colon P \to M(A)$ is a representation of a projective $C^*$-algebra $P$ as multipliers of $A$, there is a realization morphism $\psi\colon P \to C(E) \cap D' \cap N^{\perp}$, meaning that*

$$\psi(x)\varphi(a) = \varphi(\theta(x)a), \qquad \forall a \in A, \quad \forall x \in P.$$

*Proof.* Using Corollary 3.2 we obtain the following commutative diagram

$$\begin{tikzcd}
M(A) \arrow[dr, "\bar{\varphi}"] & & P \arrow[ll, "\theta"'] \arrow[dd, dashed, "\psi"] & \\
& M(\varphi(A)) & & \\
A \arrow[uu, "\iota"] \arrow[r, "\varphi"] & \varphi(A) \arrow[u, "\iota"] \arrow[r, "\iota"] & I(\varphi) \arrow[ul, "\pi"] \arrow[r, "\iota"] & C(E) \cap D' \cap N^{\perp}
\end{tikzcd}$$

in which $I(\varphi) = I(\varphi(A) : D' \cap N^{\perp})$. Here, according to the Tietze extension theorem, $\bar{\varphi}$ is a surjective morphism extending $\varphi$ and the morphism $\pi$ from Corollary 3.2 satisfies $\pi(z)y = zy$ for all $z$ in $I(\varphi)$ and $y$ in $\varphi(A)$, and is surjective. Since $P$ is projective there is a morphism $\psi\colon P \to I(\varphi)$ such that $\pi\psi = \bar{\varphi}\theta$. For every $x$ in $P$ and $a$ in $A$ we therefore get

$$\psi(x)\varphi(a) = \pi(\psi(x))\varphi(a) = \bar{\varphi}(\theta(x))\varphi(a) = \varphi(\theta(x)a)$$

as desired. ∎

## 4. UNIVERSAL COMPLETIONS

### 4.1. *Diagram II*

Let a system

$$\begin{tikzcd}
& A_1 \arrow[d, "\alpha"] & & & \\
0 \arrow[r] & A \arrow[r] & X \arrow[r, "\pi"'] & B \arrow[r] & 0
\end{tikzcd} \qquad [\eta]$$

be given, and suppose that $\alpha$ is proper. The universal solution is obtained by two subsequent pull-back constructions. First, one defines $B_1$ as the pull-back

$$\begin{tikzcd} B_1 \arrow[r, "\eta_1"] \arrow[d, "\beta"'] & C(A_1) \arrow[d, "\tilde{\alpha}"] \\ B \arrow[r, "\eta"'] & C(A) \end{tikzcd}$$

and gets the diagram

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\alpha"] & X_1 \arrow[r, "\pi_1"] \arrow[d, "\chi"] & B_1 \arrow[r] \arrow[d, "\beta"] & 0 & {[\eta_1]} \\ 0 \arrow[r] & A \arrow[r] & X \arrow[r, "\pi"'] & B \arrow[r] & 0 & {[\eta]} \end{tikzcd}$$

by the standard Busby construction. Here one observes that Theorem 2.2 applies to give the unique morphism $\chi: X_1 \to X$ making the diagram commute. Furthermore, any other completion

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\alpha"] & X_1' \arrow[r, "\pi_1'"] \arrow[d, "\chi'"] & B_1' \arrow[r] \arrow[d, "\beta'"] & 0 & {[\zeta]} \\ 0 \arrow[r] & A \arrow[r] & X \arrow[r, "\pi"'] & B \arrow[r] & 0 & {[\eta]} \end{tikzcd}$$

of the diagram must factor through this in a unique fashion; for by definition of $B_1$ and Theorem 2.2, there is a unique morphism $\beta'': B_1' \to B_1$ with the properties

$$\eta_1 \beta'' = \zeta \qquad \beta\beta'' = \beta'.$$

By Theorem 2.2 again, there is a unique morphism $\chi'': X_1' \to X_1$ through which the entire completion factors.

We can describe the $C^*$-algebra $X_1$ more explicitly:

PROPOSITION 4.1. *In the universal solution above, the square*

$$\begin{tikzcd} X_1 \arrow[r, "\theta_1"] \arrow[d, "\chi"'] & M(A_1) \arrow[d, "\bar{\alpha}"] \\ X \arrow[r, "\theta"'] & M(A) \end{tikzcd}$$

*is a pull-back.*

*Proof.* There is a commutative diagram

$$\begin{tikzcd}
& B_1 \arrow[rr, "\eta_1"] \arrow[dd, "\beta"] & & C(A_1) \arrow[dd, "\tilde{\alpha}"] \\
X_1 \arrow[ur, "\pi_1"] \arrow[rr, "\theta_1"] \arrow[dd, "\chi"'] & & M(A_1) \arrow[ur, "\kappa_1"] \arrow[dd, "\bar{\alpha}"] & \\
& B \arrow[rr, "\beta"'] & & C(A) \\
X \arrow[ur, "\pi"] \arrow[rr, "\theta"'] & & M(A) \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

A diagram chase shows that if all faces of a cubic diagram commute and if the top, bottom and back faces are pull-backs, so is the front face. ∎

4.2. *Diagram III*

Let a system

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\alpha"] & X \arrow[r, "\pi"] & B \arrow[r] & 0 \\
& A_1 & & &
\end{tikzcd}$$

be given, and assume that $\alpha$ is proper. The universal solution is

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\alpha"] & X \arrow[r, "\pi"] \arrow[d, "\chi"] & B \arrow[r] \arrow[d, equal] & 0 & {[\eta]} \\
0 \arrow[r] & A_1 \arrow[r, "\iota_1"'] & X_1 \arrow[r, "\pi_1"'] & B \arrow[r] & 0 & {[\tilde{\alpha}\eta],}
\end{tikzcd}$$

in which the morphism $\chi$ exists by Theorem 2.2. If another completion is given by the Busby map $\zeta\colon B' \to C(A_1)$, we get a diagram

$$\begin{tikzcd}
& 0 \arrow[r] & A \arrow[r] \arrow[d] \arrow[ddl] & X \arrow[r] \arrow[d] \arrow[ddl] & B \arrow[r] \arrow[d, equal] \arrow[ddl] & 0 & {[\eta]} \\
& 0 \arrow[r] & A_1 \arrow[r] \arrow[dl, equal] & X_1 \arrow[r] \arrow[dl] & B \arrow[r] \arrow[dl] & 0 & {[\tilde{\alpha}\eta]} \\
0 \arrow[r] & A_1 \arrow[r] & X' \arrow[r] & B' \arrow[r] & 0 & {[\zeta]} &
\end{tikzcd}$$

in which the morphisms in the center exist and are unique by Theorem 2.2. The center triangle is commutative as a consequence of uniqueness.

Again, a concrete description of the center $C^*$-algebra in the universal solution is possible.

PROPOSITION 4.2. *In the universal solution above, the square*

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "\alpha"] & X \arrow[d, "\chi"] \\ A_1 \arrow[r, "\iota_1"'] & X_1 \end{tikzcd}$$

*is a push-out.*

*Proof.* We consider the situation

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "\alpha"'] & X \arrow[d, "\chi"'] \arrow[ddr, bend left, "\psi"] & \\ A_1 \arrow[r, "\iota_1"] \arrow[drr, bend right, "\varphi"'] & X_1 \arrow[dr, dotted] & \\ & & Y \end{tikzcd}$$

It is our task to prove that the dotted morphism can be filled in in a unique fashion. To do this, we abbreviate

$$I(\varphi\alpha) = I(\varphi(\alpha(A)) : Y) \qquad I(\varphi) = I(\varphi(A) : Y)$$

$$E(\varphi\alpha) = E(\varphi(\alpha(A)) : Y) \qquad E(\varphi) = I(\varphi(A) : Y)$$

and note that $\psi$ must map into $I(\varphi\alpha)$ since

$$\psi(x)\varphi(\alpha(a)) = \psi(x)\psi(\iota(a)) = \psi(\iota(xa)) = \varphi(\alpha(xa)).$$

We also conclude from $A_1 = \overline{\alpha(A)A_1}$ that $I(\varphi\alpha) \subseteq I(\varphi)$. We get a diagram

$$\begin{tikzcd}[column sep=small] & 0 \arrow[rr] & & \varphi(A_1) \arrow[rr] & & I(\varphi) \arrow[rr] & & E(\varphi) \arrow[rr] & & 0 & {[\zeta_1]} \\ 0 \arrow[rr] & & A_1 \arrow[ur, "\varphi"] \arrow[rr, "\iota_1"'] & & X_1 \arrow[rr, "\pi_1"'] & & B \arrow[ur, "\beta_1"] \arrow[rr] & & 0 & & {[\overline{\alpha}\eta]} \\ & 0 \arrow[rr] & & \varphi(\alpha(A)) \arrow[uu, "\iota_A"'] \arrow[rr] & & I(\varphi\alpha) \arrow[uu, "\iota_X"'] \arrow[rr] & & E(\varphi\alpha) \arrow[uu, "\iota_B"'] \arrow[rr] & & 0 & {[\zeta]} \\ 0 \arrow[rr] & & A \arrow[uu, "\alpha"] \arrow[ur, "\varphi\alpha"] \arrow[rr, "\iota"'] & & X \arrow[uu, "\chi"] \arrow[ur, "\psi"] \arrow[rr, "\pi"'] & & B \arrow[uu, equal] \arrow[ur, "\beta"] \arrow[rr] & & 0 & & {[\eta]} \end{tikzcd}$$

in which all the morphisms on the rightmost square are induced by the morphisms on the other two vertical squares, except $\beta_1$, which is given by

$$\beta_1 = \iota_B \beta.$$

To get a morphism making the top face commute, we need

$$\tilde{\varphi}(\tilde{\alpha}\eta) = \zeta_1 \beta_1$$

according to Theorem 2.2. We also learn from the other implication of this result applied to the bottom and back edges that

$$(\widetilde{\varphi\alpha})\eta = \zeta\beta \qquad \text{and} \qquad \widetilde{\iota_A}\zeta = \zeta_1 \iota_B,$$

whence (1) will follow if we can show that

$$\tilde{\varphi}\tilde{\alpha} = \widetilde{\iota_A}(\widetilde{\varphi\alpha}).$$

This in turn is a direct consequence of functoriality of the extensions of proper morphisms to multiplier algebras, applied to the left square in the 3D diagram above.

Denote the unique induced morphism by $\upsilon$ and consider it as a map into $Y$. Clearly $\upsilon\iota_1 = \varphi$, and $\upsilon\chi = \psi$ by uniqueness of the center morphism making the diagram involving only the lower front and upper back short exact sequences commute. To check uniqueness of the morphism induced, note that if $\upsilon' \colon X_1 \to Y$ is another map with $\upsilon'\iota_1 = \varphi$ and $\upsilon\chi = \psi$, we have $\upsilon'(X_1) \subseteq I(\varphi)$ because $A_1$ is an ideal in $X_1$. Hence $\upsilon'$ restricts to a morphism making the left cube of the 3D diagram commute. We conclude that the upper right square commutes by a diagram chase, whence $\upsilon = \upsilon'$ by uniqueness in Theorem 2.2. ∎

COROLLARY 4.3. *If a diagram of extensions*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\iota} & X & \xrightarrow{\pi} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\chi} & & \| & & \\
0 & \longrightarrow & A_1 & \xrightarrow[\iota_1]{} & X_1 & \xrightarrow[\pi_1]{} & B & \longrightarrow & 0
\end{array}$$

*is given, with $\alpha$ proper, the diagram is universal of type III, and the left square is a pushout.*

*Proof.* If the Busby map for the upper extension is $\eta$, the Busby map for the lower is $\tilde{\alpha}\eta$ by Theorem 2.2 and the diagram is universal of type III. Thus, Proposition 4.2 applies. ∎

Note that if $A$, $A_1$, and $X$ are all commutative in the diagram above, so is $X_1$. We shall apply this result to determine certain amalgamated free products of commutative $C^*$-algebras in Section 6.3 below.

*Remark* 4.4. One should note the duality between Diagrams I and III, and how the solutions come out as pull-back and push-outs, respectively, of the given data. It is important to stress, however, that our proof of existence of a universal solution to Diagram III requires the full force of Busby theory (hence Diagram I) and proper morphisms. Actually, it follows from Remark 2.3 that there is no universal solution to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "0"'] & \mathbb{C}\oplus\mathbb{C} \arrow[r] & \mathbb{C} \arrow[r] & 0 \\ & \mathbb{C} & & & \end{tikzcd}$$

For if there was one, say the trivial extension of $\mathbb{C}$ by $X$, any map out of $\mathbb{C}$ would factor uniquely through $X$, whence $X=\mathbb{C}$, and the solution had to be one of the ones given in Remark 2.3. And then the other solution could not be factored.

Corollary 4.5. *Let morphisms $\iota\colon A\to X$ and $\alpha\colon A\to A_1$ have the properties that $\alpha$ is proper, and that $\iota$ is an inclusion of $A$ an an ideal in $X$. We then have a six term exact sequence*

$$\begin{tikzcd} K_0(A) \arrow[r, "\Phi^0"] & K_0(A_1)\oplus K_0(X) \arrow[r, "\Psi^0"] & K_0(A_1 *_A X) \arrow[d, "\Gamma^0"] \\ K_1(A_1 *_A X) \arrow[u, "\Gamma^1"] & K_1(A_1)\oplus K_1(X) \arrow[l, "\Psi^1"] & K_1(A) \arrow[l, "\Phi^1"] \end{tikzcd}$$

*Proof.* Take $K$-theory on the diagram in 4.3 and apply the Barratt–Whitehead lemma (cf. [11, 17.4]). ∎

As a consequence of our proof the maps can be characterized as follows. Let the morphisms $\iota_1, \pi_1, \chi$ be the ones found in the universal solution above, and $\partial^i$ the connecting maps in the six-term exact sequence corresponding to the extension of $A$ by $B$. Then

$$\Phi^i = \begin{bmatrix} K_i(\alpha) \\ K_i(\iota) \end{bmatrix} \qquad \Psi^i = [-K_i(\iota_1) \quad K_i(\chi)] \quad \Gamma^i = \partial^i K_i(\pi_1)$$

4.3. *Diagram IV*

When a diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & X \arrow[r, "\pi"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ & & & B_1 & \end{tikzcd}$$

is given, with $\beta$ surjective, the universal solution is

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, hook] & X \arrow[r, "\pi"] \arrow[d, equal] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] & X \arrow[r, "\beta\pi"'] & B_1 \arrow[r] & 0 \end{tikzcd}$$

as is easily checked. Here $A_1$ is of course the kernel of $\beta\pi$.

The result above is not completely satisfactory since most of the interesting applications lie in the non-surjective case; in fact embeddings $B \subseteq B_1$ are the most desirable to handle. Here the methods we have used before lead nowhere, and we suspect that universal solutions to the problem will be scarce. This explains our next try at solutions of Diagram IV with forced boundary conditions.

## 5. CORONA EXTENDIBILITY

Working in analogy to the universal solutions to Diagram I and III, we consider the problem

$$\begin{tikzcd} 0 \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] \arrow[d, dashed] & \bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bullet \arrow[r, dashed] & \circ \arrow[r, dashed] & \bullet \arrow[r] & 0. \end{tikzcd}$$

Requiring that the leftmost $C^*$-algebra in the completion is the same as the given one assures that the completion—if it exists—is non-trivial. Since we are not interested in the case where the rightmost morphism is surjective (but willingly injective), constancy on the left side is really optimal. Giving up this demand could result in “solutions” like

$$\begin{tikzcd} 0 \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \bullet \arrow[r, equal] & \bullet \arrow[r] & 0. \end{tikzcd}$$

We know from Theorem 2.2 that a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & X \arrow[r] & B \arrow[r] \arrow[d, "\beta"] & 0 & {[\eta]} \\ & A & & B_1 & & \end{tikzcd}$$

can be completed precisely when there exists a morphism $\eta_1 : B_1 \to C(A)$ with $\eta = \eta_1 \beta$. Solutions to such an extension problem are rarely unique, so we cannot expect universal solutions.

A highly selective (and yet surprisingly large!) class of morphisms $\beta: B \to B_1$ has the property

$$\begin{tikzcd} B \arrow[r, "\eta"] \arrow[d, "\beta"'] & C(E), \\ B_1 \arrow[ur, dashed] & \end{tikzcd}$$

i.e., every morphism $\eta$ into the corona algebra of a *$\sigma$-unital* $C^*$-algebra $E$ may be factored through $B_1$. We shall call such morphisms *corona extendible*, cf. [22, 1.1].

It is clear that every such morphism has a nontrivial completion in the sense described above. The aim of this section is to prove the existence of many corona extendible maps.

### 5.1. *Corona Extendibility and Tensor Products*

LEMMA 5.1. *Let $\theta: A \to X$ be a morphism between $C^*$-algebras and assume that $A$ has a faithful representation on a separable Hilbert space. Then $\theta$ is corona extendible provided only that all injective morphisms $\varphi: A \to C(E)$ extend to $X$.*

*Proof.* Assume that $\theta$ has the weaker property and consider a general morphism $\varphi: A \to C(E)$. Let $\pi: A \to \mathbb{B}(\ell^2)$ be a faithful representation of $A$. Tensoring if necessary $\pi$ with the identity map on $\ell^2$ we may assume that $\pi(A)$ contains no compact operators, i.e. $\pi(A) \cap \mathbb{K} = \{0\}$, and thus consider the injective morphism

$$\pi' : A \to \mathbb{B}(\ell^2)/\mathbb{K} = C(\mathbb{K}).$$

Now $\varphi \oplus \pi' : A \to C(E \oplus \mathbb{K})$ is injective, so by assumption there is an extension $\psi: X \to C(E \oplus \mathbb{K})$ such that $\psi\theta = \varphi \oplus \pi'$. If $\pi_1$ denotes projection on the first summand in $C(E \oplus \mathbb{K}) = C(E) \oplus C(\mathbb{K})$, we can define $\bar{\varphi} = \pi_1 \psi$ to obtain a morphism of $X$ in $C(E)$ such that $\bar{\varphi}\theta = \varphi$. ∎

THEOREM 5.2. *Let $X$ be an extension of $A$ by $P$, where $A$ is $\sigma$-unital and has a faithful representation on a separable Hilbert space and $P$ is projective. Then for every separable, unital, $C^*$-algebra $D$ the embedding map $A \otimes D \to X \otimes D$ is corona extendible, when $\otimes$ denotes the maximal tensor product.*

*Proof.* Consider a morphism $\varphi\colon A \otimes D \to C(E)$, which by Lemma 5.1 we may assume to be injective. Embedding $D$ in $M(A \otimes D)$ as $1 \otimes D$ (where 1 is the unit in $M(A)$) and using Theorem 3.1 we can find a separable $C^*$-subalgebra $D_0$ of $I(\varphi(A \otimes D))$ and a surjective morphism $\sigma\colon D_0 \to D$ such that

$$z\varphi(a \otimes d) = \varphi(a \otimes \sigma(z)d) \tag{2}$$

for all $z$ in $D_0$ and $a \otimes d$ in $A \otimes D$. It follows that $\ker \sigma$ and $\varphi(A \otimes D)$ annihilate each other. Since both are $\sigma$-unital $C^*$-algebras it follows from the SAW* property of $C(E)$ (cf. [26]) that there is an element $e$ in $C(E)$ with $0 \leqslant e \leqslant 1$ such that

$$(\ker \sigma)\, e = (1-e)\, \varphi(A \otimes D) = 0 \tag{3}$$

Consider now the morphism $\varphi'\colon A \to C(E)$ defined by $\varphi'(a) = \varphi(a \otimes 1)$, $a \in A$. Note that $\varphi'(A) \subseteq D_0' \cap \{1-e\}^{\perp}$ by (2) and (3). By Theorem 3.4 there is therefore a morphism

$$\pi'\colon P \to C(E) \cap D_0{}' \cap \{1-e\}^{\perp} \tag{4}$$

such that $\pi'(x)\varphi'(a) = \varphi'(\alpha(x)a)$ for every $x$ in $P$ and $a$ in $A$. Here $\alpha\colon P \to M(A)$ is the morphism that defines the split extension, as described in [21, 2.1].

By definition of the maximal $C^*$-tensor product there is now a morphism

$$\pi' \otimes \iota\colon P \otimes D_0 \to C(E)$$

defined by $(\pi' \otimes \iota)(x \otimes z) = \pi'(x)\, z$. Note here that the ideal $P \otimes \ker \sigma$ of $P \otimes D_0$ is contained in $\ker\, (\pi' \otimes \iota)$, because if $z \in \ker \sigma$ then

$$(\pi' \otimes \iota)(x \otimes z) = \pi'(x)\, z = \pi'(x)\, ez = 0$$

by (3) and (4).

Since we are using the maximal tensor product we have an extension

$$0 \to P \otimes \ker \sigma \to P \otimes D_0 \to P \otimes D \to 0$$

and from the above we see that $\pi' \otimes \iota$ uniquely defines a morphism $\psi\colon P \otimes D \to C(E)$.

We now have the extension

$$0 \to A \otimes D \to X \otimes D \to P \otimes D \to 0$$

which is split exact because the morphism $\alpha \otimes \iota$ provides a right inverse to the quotient map. Now let $(u_n)$ be an approximate unit for $A$ and compute for $z \in D_0$, $d \in D$, $a \in A$, and $x \in P$ that

$$\begin{aligned}
\psi(x \otimes \sigma(z))\varphi(a \otimes d) &= (\pi' \otimes \iota)(x \otimes z)\varphi(a \otimes d) \\
&= \pi'(x) z \varphi(a \otimes d) \\
&= \pi'(x)\varphi(a \otimes \sigma(z) d) \\
&= \lim \pi'(x)\varphi'(a)\varphi(u_n \otimes \sigma(z) d) \\
&= \lim \varphi'(\alpha(x) a)\varphi(u_n \otimes \sigma(z) d) \\
&= \varphi(\alpha(x) a \otimes \sigma(z) d).
\end{aligned}$$

It follows that

$$\psi(y)\, \varphi(z) = \varphi((\alpha \otimes \iota)(y)\, z), \qquad y \in P \otimes D, \quad z \in A \otimes D$$

which by [21, 2.2] means exactly that $(\varphi, \psi)$ defines a morphism $\bar{\varphi}\colon X \otimes D \to C(E)$ that extends $\varphi$. ∎

THEOREM 5.3. *Let $X$ be an extension of a separable $C^*$-algebra $A$ by a projective $C^*$-algebra $P$. Then for every separable $C^*$-algebra $D$ the embedding map $A \otimes D \to X \otimes D$ is corona extendible.*

*Proof.* Consider again an injective morphism $\varphi\colon A \otimes D \to C(E)$. Since we no longer assume that $D$ is unital we have instead an embedding $A \to A \otimes 1 \subseteq M(A \otimes D)$. Using Theorem 3.1 we can find a separable $C^*$-subalgebra $A_0$ of $I(\varphi(A \otimes D))$ and a surjective morphism $\sigma\colon A_0 \to A$ such that

$$z\varphi(a \otimes d) = \varphi(\sigma(z) a \otimes d)$$

for all $z$ in $A_0$ and $a \otimes d$ in $A \otimes D$.

With $\tilde{D}$ the unitization of $D$ we define a morphism $\varphi'\colon A_0 \otimes \tilde{D} \to C(E)$ by

$$\varphi'(z \otimes (d + \lambda 1)) = \varphi(\sigma(z) \otimes d) + \lambda z,$$

for $z$ in $A_0$ and $d+\lambda 1$ in $\tilde{D}$. Moreover, if $\alpha\colon P\to M(A)$ is the Busby map for the extension $X$ we can find a lifting morphism

$$\begin{tikzcd} & M(A_0) \arrow[d, "\bar{\sigma}"] \\ P \arrow[ur, "\beta"] \arrow[r, "\alpha"'] & M(A) \end{tikzcd}$$

because $P$ is projective.

The situation has now been reduced so that we can apply Theorem 5.2 with $A_0, \beta, \tilde{D}$ and $\varphi'$ instead of $A, \alpha, D$ and $\varphi$ to obtain a morphism $\psi\colon P\otimes\tilde{D}\to C(E)$ such that

$$\psi(x\otimes d)\,\varphi'(z\otimes\tilde{e})=\varphi'(\beta(x)z\otimes d\tilde{e})$$

for all $x$ in $P$, $z$ in $A_0$ and $d, \tilde{e}$ in $\tilde{D}$. If we now take $d, e$ in $D$ and for $a$ in $A$ choose $z$ in $A_0$ such that $\sigma(z)=a$, then

$$\begin{aligned}\psi(x\otimes d)\,\varphi(a\otimes e) &= \psi(x\otimes d)\,\varphi(\sigma(z)\otimes e)\\ &= \psi(x\otimes d)\,\varphi'(z\otimes e)\\ &= \varphi'(\beta(x)z\otimes de)\\ &= \varphi(\sigma(\beta(x)z)\otimes de)\\ &= \varphi(\bar{\sigma}(\beta(x))\sigma(z)\otimes de)\\ &= \varphi(\alpha(x)a\otimes de)\\ &= \varphi((\alpha(x)\otimes d)(a\otimes e)).\end{aligned}$$

It follows that the pair $(\psi\mid P\otimes D, \varphi)$ defines an extension $\bar{\varphi}\colon X\otimes D\to C(E)$, as desired. ∎

*Remark* 5.4. To illustrate the possibilities of the preceding result, consider the situation where $x$ is a self-adjoint element in the multiplier algebra $M(A)$ of a separable $C^*$-algebra $A$. Let $\mathrm{sp}_{ess}(x)$ denote the "essential spectrum" of $x$, i.e. the spectrum of the image of $x$ in $C(A)$. If now $\mathrm{sp}_{ess}(x)$ is an interval containing 0 and $X=C^*(A, x)$, then the embedding map $A\otimes D\to X\otimes D$ is corona extendible for every separable $C^*$-algebra $D$. The argument for this is quite simple. If $P=C_0(\mathrm{sp}_{ess}(x)\setminus\{0\})$, then $P$ is projective and the map $f\mapsto f(x)$ given by spectral theory defines a morphism $\alpha\colon P\to M(A)$ corresponding to the (split) extension

$$0\to A\to X\to P\to 0,$$

consequently the conditions in Theorem 5.3 are met.

If the element $x$ above is invertible, the result may fail. To consider the simplest case, let $A$ be a non-unital $C^*$-algebra and with $x=1$, the unit in $M(A)$, put $\tilde{A}=A+\mathbb{C}1$. Examples where the embedding $A\otimes D\to\tilde{A}\otimes D$ is not corona extendible occur already in the commutative case. To be specific, let $A=C_0(]0,1])$ and $D=\mathbb{C}\oplus\mathbb{C}$. Then

$$A\otimes D=A\oplus A=\{f\in C([-1,1])\mid f(0)=0\}$$
$$=C^*\langle x\mid \|x\|\leqslant 1, x=x^*\rangle.$$

On the other hand $\tilde{A}=C([0,1])$, so $\tilde{A}\otimes D=C([0,1])\oplus C([0,1])$. If we now let $\mathbb{R}_+=[0,\infty[$ and set $E=C_0(\mathbb{R}_+)$, then

$$C(E)=C_b(\mathbb{R}_+)/C_0(\mathbb{R}_+)=C(\beta\mathbb{R}_+\backslash\mathbb{R}_+)$$

and this algebra has no non-trivial projections because $\beta\mathbb{R}_+\backslash\mathbb{R}_+$ is connected by [12, 3.5]—it has one end, cf. [1]. Choose any element $f$ in $C(\beta\mathbb{R}_+\backslash\mathbb{R}_+)$ such that $\|f\|=1$, $f=f^*$ and $\pm 1\in\mathrm{sp}(f)$ (e.g. the continuation of $f(x)=\sin x$ to $\beta\mathbb{R}_+$). Then $\mathrm{sp}(f)=[-1,1]$, so $C^*(f)=A\otimes D$, providing an embedding of $A\otimes D$ into $C(E)$. This embedding can not be extended to $\tilde{A}\otimes D$, because the latter algebra contains non-trivial projections.

## 6. FURTHER APPLICATIONS

### 6.1. *Extensions of Codimension One*

In this section we apply our results about Diagram III to find a characterization of extensions of $A$ by $\mathbb{C}$ when $A$ is $\sigma$-unital. In this, the *strict* Urysohn lemma ([28]) is instrumental.

LEMMA 6.1. *If $A$ is $\sigma$-unital and there is an extension*

$$0\to A\to X\xrightarrow{\pi}\mathbb{C}\to 0,$$

*then there exist morphisms $\alpha$, $\bar{\alpha}$ with $\alpha$ proper, so that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & \mathbb{C} & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\bar{\alpha}} & & \| & & \\
0 & \longrightarrow & C_0(]0,1[) & \longrightarrow & C_0(]0,1]) & \longrightarrow & \mathbb{C} & \longrightarrow & 0
\end{array}$$

*commutes.*

*Proof.* By [28] we can find $h$ in $X$ with $0 \leqslant h \leqslant 1$, such that $\pi(h) = 1$ and $h - h^2$ is a strictly positive element in $A$. Defining $\bar{\alpha}$ by sending $t$ to $h$, we note that the ideal generated by $t - t^2$ is $C_0(]0, 1[)$. Hence $\bar{\alpha}$ restricts to a map into $A$ which is proper by assumption. ∎

By Corollary 4.3 the diagram above is universal of type III, and since universal solutions (filled out identically to the right) are equivalent, we conclude:

PROPOSITION 6.2. *Any extension*

$$0 \to A \to X \to \mathbb{C} \to 0$$

*with $A$ $\sigma$-unital is equivalent to the extension*

$$0 \to A \to A *_{C_0(]0,\,1[)} C_0(]0, 1]) \to \mathbb{C} \to 0$$

*arising from a proper morphism $\alpha$: $C_0(]0, 1[) \to A$.*

In examples, it is often easy to find a concrete choice of $h$ (and hence of $\alpha$). For instance, in the case of the extension leading to the nonunital dimension drop algebra mentioned in the introduction, one may take $h = t\mathbf{1}$.

6.2. *Matricial Field Algebras*

Recall that, according to Blackadar and Kirchberg, a separable $C^*$-algebra is a *matricial field* (an *MF-algebra*) if it is an asymptotic inductive limit of finite-dimensional $C^*$-algebras. They show ([2, 3.2.2]) that this happens precisely when $A$ can be embedded in a corona algebra of a special form, viz.

$$A \subseteq \prod \mathbf{M}_{n_k} \Big/ \bigoplus \mathbf{M}_{n_k} = M\left(\bigoplus \mathbf{M}_{n_k}\right) \Big/ \bigoplus \mathbf{M}_{n_k}$$

for some (infinite, possibly repetitive, non-decreasing) sequence $(n_k)$ in $\mathbb{N}$. $C^*$-algebras that are residually finite dimensional (i.e. embeds into $\prod \mathbf{M}_{k_n}$, see [9]) are *MF*-algebras [2, 3.2.2], so all projective $C^*$-algebras are *MF*-algebras by [21, §1].

By Busby theory $A$ is an *MF*-algebra if and only if there exists an essential extension

$$0 \to \bigoplus \mathbf{M}_{n_k} \to X \to A \to 0,$$

and this can be reformulated, [2, 3.2.2], as the condition that there is an essential and quasi-diagonal extension

$$0 \to \mathbb{K} \to X \to A \to 0.$$

It is clear why corona extendibility is relevant for this question. But since we only need to consider a limited sort of corona extendibility involving only $C(E)$ where $E = \bigoplus \mathbf{M}_{n_k}$ we shall use the term *matricially corona extendible* for morphisms with this property. We then get:

LEMMA 6.3. *Given an extension*

$$0 \to A \xrightarrow{\iota} X \xrightarrow{\pi} B \to 0,$$

*where both $A$ and $B$ are MF, and $\iota$ is matricially corona extendible, then $X$ is MF.*

*Proof.* By assumption we have embeddings

$$\varphi : A \to C(E), \qquad \psi : B \to C(F),$$

where $E = \bigoplus \mathbf{M}_{n_k}$, $F = \bigoplus \mathbf{M}_{m_k}$. We may also by assumption extend $\varphi$ to a morphism $\bar{\varphi}: X \to C(E)$. Letting $\bar{\psi} = \psi\pi$ we have a morphism out of $X$ with $\ker \bar{\psi} = A$. Consequently,

$$\ker \bar{\varphi} \cap \ker \bar{\psi} = \{0\},$$

and we have an injective morphism $\bar{\varphi} \oplus \bar{\psi}$ of $X$ into

$$C(E) \oplus C(F) = C\left(\bigoplus \mathbf{M}_{r_k}\right),$$

where $(r_k) = \bigcup \{m_k\} \cup \{n_k\}$, suitably reordered, possibly with repetitions. ∎

Extending a previous result, [22, 5.7], we now have:

THEOREM 6.4. *Let $A$, $D$ and $P$ be separable, nuclear $C^*$-algebras and assume that $A \otimes D$ is MF and $P$ is projective. Then for any extension $X$ of $A$ by $P$ the $C^*$-algebra $X \otimes D$ is MF.*

*Proof.* Since $P$ is projective, it can be embedded in the mapping cone of $P$ (see [21, §2]) and is therefore contractible; i.e. there is a path $\{\pi_t \mid 0 \leqslant t \leqslant 1\}$ in $\mathrm{Hom}(P)$, continuously embedded in $C(P \times [0, 1], P)$, such that $\pi_0 = 0$ and $\pi_1 = \mathrm{id}$. Evidently this implies that also $P \otimes D$ is contractible, and therefore quasi-diagonal by [29]. But every nuclear, quasi-diagonal $C^*$-algebra is *MF* by [2, 3.2.2], so $P \otimes D$ is *MF*. We can therefore apply Proposition 6.3 according to Theorem 5.3. ∎

Lemma 6.5. *The natural morphism*

$$C_0(\,]0,2[\,) \to C_0(\,]0,1]\,) \oplus C_0(\,[1,2[\,)$$

*is matricially corona extendible.*

*Proof.* Let a morphism

$$\varphi\colon C_0(\,]0,2[\,) \to \prod \mathbf{M}_{n_k} \Big/ \bigoplus \mathbf{M}_{n_k}$$

be given and extend it to a map from $C_0(\,]0,2[\,)^{\sim}$ to $\prod \mathbf{M}_{n_k}/\bigoplus \mathbf{M}_{n_k}$ by sending the unit to the unit. It suffices that this has an extension to the unitization of $C_0(\,]0,1]\,) \oplus C_0(\,[1,2[\,)$. This is easily seen to be equivalent to finding a logarithm for a unitary in $\prod \mathbf{M}_{n_k}/\bigoplus \mathbf{M}_{n_k}$. As every contraction in $\mathbf{M}_{n_k}$ has a logarithm bounded in norm by 2, this follows. ∎

Theorem 6.6. *Suppose that the sequence*

$$0 \to A \xrightarrow{\ \iota\ } X \xrightarrow{\ \pi\ } \mathbb{C} \to 0$$

*is exact and $A$ is $\sigma$-unital. Then the morphism $\iota$ is matricially corona extendible.*

*Proof.* By Proposition 6.2 there is a diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & X \arrow[r] & \mathbb{C} \arrow[r] & 0 & {[\eta]} \\
0 \arrow[r] & C_0(\,]0,1[\,) \arrow[r] \arrow[u, "\alpha"] & C_0(\,]0,1]\,) \arrow[r] \arrow[u, "\bar{\alpha}"] & \mathbb{C} \arrow[r] \arrow[u, equal] & 0 & {[\zeta]}
\end{tikzcd}$$

with $\alpha$ proper, so that it is universal of type III. Considering the map $\delta_1\colon C_0(\,[1,2[\,) \to \mathbb{C}$ we can take the universal solution to Diagram I and achieve the vertical faces of the diagram

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[rr] & & A \arrow[rr, "\iota"] & & X \arrow[rr, "\pi"] & & \mathbb{C} \arrow[rr] & & 0 & {[\eta]} \\
0 \arrow[rr] & & A \arrow[rr, "\iota_1"] \arrow[ur, equal] & & Y \arrow[rr, "\pi_1"] \arrow[ur, "\lambda"] & & C_0([1,2[) \arrow[rr] \arrow[ur, "\delta_1"] & & 0 & & {[\eta\delta_1]} \\
 & 0 \arrow[rr] & & C_0(]0,1[) \arrow[rr, hook] \arrow[uu, "\alpha"] & & C_0(]0,1]) \arrow[rr] \arrow[uu, "\bar{\alpha}"] & & \mathbb{C} \arrow[rr] \arrow[uu, equal] & & 0 & {[\zeta]} \\
0 \arrow[rr] & & C_0(]0,1[) \arrow[rr, hook] \arrow[uu, "\alpha"] \arrow[ur, equal] & & C_0(]0,2[) \arrow[rr] \arrow[uu, dotted, "\upsilon"] \arrow[ur, "\gamma"] & & C_0([1,2[) \arrow[rr] \arrow[uu, equal] \arrow[ur, "\delta_1"] & & 0 & & {[\zeta\delta_1]}
\end{tikzcd}$$

Here we have identified the pull-back in the lower horizontal face with $C_0(]0,2[)$, whereafter the map $\gamma$ becomes restriction. By the universal properties of the extension involving $Y$ (or directly by Theorem 2.2) there is a morphism $\upsilon\colon C_0(]0,1[)\to Y$ making the diagram commute.

Note also that since $\eta=\bar{\alpha}\zeta$ by universality of the back face, we have

$$\eta\delta_1=\bar{\alpha}\zeta\delta_1,$$

and also the front face is universal with respect to Diagram III. Applying the isomorphism given by Proposition 4.2, we may read off from the diagram that the morphism $\lambda$ between the two amalgamated free products,

$$Y=A *_{C_0(]0,1[)} C_0(]0,2[)\xrightarrow{\lambda} A *_{C_0(]0,1[)} C_0(]0,1])=X$$

is the one given by restricting functions and leaving $A$ fixed.

Suppose now that $\varphi\colon A\to\prod \mathbf{M}_{n_k}/\bigoplus \mathbf{M}_{n_k}$ is a given morphism. Since $C_0([1,2[)$ is projective $\iota_1$ is corona extendible, and $\varphi$ extends to $\varphi'\colon Y\to\prod \mathbf{M}_{n_k}/\bigoplus \mathbf{M}_{n_k}$. Let $\psi=\varphi'\upsilon$ and note that

$$\psi(f)=\varphi'(\iota_1(\alpha(f)))=\varphi(\alpha(f)),$$

for $f$ in $C_0(]0,1[)$. By Lemma 6.5 there is an extension of $\psi$ to

$$\bar{\psi}\colon C_0(]0,3/2])\oplus C_0([3/2,2[)\to\prod \mathbf{M}_{n_k}\Big/\bigoplus \mathbf{M}_{n_k}.$$

Let $\tilde{\lambda}\colon C_0(]0,1])\to C_0(]0,3/2])\oplus C_0([3/2,2[)$ be defined by

$$\tilde{\lambda}(f)=(\tilde{f},0),$$

where

$$\tilde{f}(t)=\begin{cases} f(t) & 0\leqslant t\leqslant 1\\ f(1) & 1\leqslant t\leqslant 3/2.\end{cases}$$

For $f$ in $C_0(]0,1[)$ we have

$$\bar{\psi}\lambda(f)=\bar{\psi}(\tilde{f}\oplus 0)=\bar{\psi}(f\oplus 0)=\psi(f)=\varphi(\alpha(f)).$$

Therefore $\varphi$ and $\bar{\psi}\tilde{\lambda}$ determine the desired extension to the algebra $A *_{C_0(]0,1[)} C_0(]0,1])=X$. ∎

Corollary 6.7. *Suppose that the sequence*

$$0 \to A \to X \to \mathbb{C} \to 0$$

*is exact and $A$ is $\sigma$-unital. If $A$ is MF then $X$ is MF.*

*Proof.* It is clear that $\mathbb{C}$ is *MF*. Hence we may combine Proposition 6.6 and Lemma 6.3 to get the result. ∎

### 6.3. *Almost Commuting Matrices*

It is possible to rephrase the methods used in the short proof of Lin's theorem ([15]) about almost commuting self-adjoint matrices recently found by Friis and Rørdam ([10]) in terms of matricially corona extendible maps. Combining this with our characterization of certain amalgamated free products leads to new results about almost commuting matrices, which we shall present in this section.

The connection between almost commuting matrices and corona extensibility will be explored in detail in [19]. We present here a sketch of some new results, emphasizing the power of our universal solution to diagram III. The ability to "exchange ideals" allows results about very special $C^*$-algebras to be upgraded to work for a broad class of $C^*$-algebras. The situation is akin to that of working on a *CW* complex cell by cell, where the key results are often results about spheres or disks. In fact, we now use Proposition 4.3 to derive theorems about various *CW* complexes from a result in [20] about the two-sphere.

Consider first the following definition:

Definition 6.8. A unital $C^*$-algebra $A$ is *matricially semiprojective* if for any sequence $n_k$ in $\mathbb{N}$ and any unital morphism

$$\varphi\colon A \to \prod \mathbf{M}_{n_k} \Big/ \bigoplus \mathbf{M}_{n_k},$$

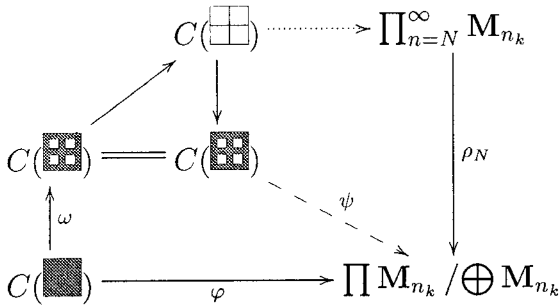
there exists $N$ and a unital morphism

$$\bar{\varphi}\colon A \to \prod_{k=N}^{\infty} \mathbf{M}_{n_k}$$

so that $\rho_N \bar{\varphi} = \varphi$, where $\rho_N$ is defined by

$$\rho_N((b_N, b_{N+1}, \dots)) = (0, \dots, 0, b_N, b_{N+1}, \dots) + \bigoplus \mathbf{M}_{n_k}.$$

To obtain Lin's result it suffices to prove that $C(\mathbb{D})$ is matricially semiprojective. Replacing the disk by a square, we consider the diagram




Here $\omega$ is the inclusion that corresponds to the surjection of spaces arising from identifying the "internal" boundary components to points, and the downward map in the triangle is given via some retraction onto the grid. This triangle will not be commutative, but if the number of holes is large, it can be made to commute up to any positive constant $\varepsilon$ on the image under $\omega$ of some set of generators of $C([0, 1]^2)$. It is implicit in [10] that the morphism $\omega$ is matricially corona extendible, so we can choose $\psi$ to make the lower square commute. Since the $C^*$-algebra over the one-dimensional grid is semiprojective by [18], the dotted morphism making the rightmost square commute exists, so we achieve a lift of $\varphi$ up to $\varepsilon$ on a set of generators. In the finitely generated case, in particular for $C([0, 1]^2)$, the existence of approximate lifts ensures matricial semiprojectivity. See [15] or [20].

We will restate the result from [20] that we need. To do this, we must introduce some terminology regarding "punching holes" in certain $CW$ complexes. Suppose $X$ is a two-dimensional $CW$ complex with one two-cell, that is

$$X = X_1 \cup_{S^1} \mathbb{D}.$$

Here $X_1$ is the one-skeleton and there is a (pushout) diagram

$$\begin{tikzcd}
& X & \\
X_1 \arrow[ur] & & \mathbb{D} \arrow[ul] \\
& S^1 \arrow[ul] \arrow[ur, "\partial"'] &
\end{tikzcd}$$

Let us denote by $X^{[n]}$ the result of replacing a point in the interior of the two-cell by a circle. More specifically, let

$$\mathbb{D}^{[1]} = \{re^{2\pi i\theta} \mid 1 \leqslant r \leqslant 2\}$$

and define $\beta\colon S^1 \to \mathbb{D}^{[1]}$ to send $e^{2\pi i\theta}$ to $2e^{2\pi i\theta}$. Then $X^{[1]}$ is defined by the pushout diagram

$$\begin{tikzcd} & X^{[1]} & \\ X_1 \arrow[ur] & & \mathbb{D}^{[1]} \arrow[ul] \\ & S^1 \arrow[ul] \arrow[ur, "\beta"'] & \end{tikzcd}$$

There is a surjection $\mathbb{D}^{[1]} \to \mathbb{D}$ sending $re^{2\pi i\theta}$ to $(r-1)\,e^{2\pi i\theta}$ and this induces a surjection $X^{[1]} \to X$. Notice that one may give $X^{[1]}$ a cell structure that again has only one two-cell. (Note first that this is true of the annulus $\mathbb{D}^{[1]}$). Therefore we may repeat the procedure, and so define $X^{[n]}$ which surjects onto $X$, and is one-to-one except that $n$ points have inverse image a circle. We shall denote the corresponding unital morphism from $C(X)$ to $C(X^{[n]})$ by $\omega^{[n]}$.

In particular, we have $(S^2)^{[n]}$ which is homeomorphic to the closed disk with $n-1$ holes removed. Let $U$ be any open set, homeomorphic to $\mathbb{R}^2$, that misses the selected $n$ points. Let

$$\iota\colon C_0(\mathbb{R}^2) \to C(S^2)$$

and

$$\iota^{[n]}\colon C_0(\mathbb{R}^2) \to C((S^2)^{[n]})$$

denote the resulting inclusions. Assume further that a morphism $\alpha\colon C_0(\mathbb{R}^2) \to A$ is given. Using $\alpha$, $\iota$, and $\iota^{[n]}$ we can form the amalgamated free products

$$\hat{A} = A *_{C_0(\mathbb{R}^2)} C(S^2)$$
$$\hat{A}^{[n]} = A *_{C_0(\mathbb{R}^2)} C((S^2)^{[n]}).$$

Let $\iota_{C_0(\mathbb{R}^2)}$ denote the canonical inclusion of $C_0(\mathbb{R}^2)$ into $\hat{A}$.

A result in [20] now shows that if $\varphi\colon \hat{A} \to C(\bigoplus \mathbf{M}_{d_k})$ is given, then if the $K$-theory of $\varphi\iota_{C_0(\mathbb{R}^2)}$ is zero (which is a consequence of [16, p. 199] if $K_0(A)$ is pure torsion), then there exists $\bar{\varphi}$ so that

$$\begin{tikzcd} & \widehat{A}^{[n]} \arrow[dr, dotted, "\overline{\varphi}"] & \\ C_0(\mathbb{R}^2) \arrow[r, "\iota_{C_0(\mathbb{R}^2)}"'] & \widehat{A} \arrow[u, "\mathrm{id}_A * \omega^{[n]}"] \arrow[r, "\varphi"'] & C(\bigoplus \mathbf{M}_{d_k}) \end{tikzcd}$$

commutes. We show here how our results regarding Diagram III allow us to identify $\hat{A}$ and $\hat{A}^{[n]}$ coherently for some commutative choices of $A$, proving that the corresponding map $\omega^{[n]}$ has similar properties.

Let $A_d$ denote the following quotient of the half-open annulus:

$$A_d = \{re^{2\pi i\theta} \mid \tfrac{1}{2} < r \leqslant 1\} \Big/ \begin{array}{l}\text{a } d\text{-fold identification} \\ \text{of the boundary } S^1.\end{array}$$

(Notice $A_2 \cup \{\infty\}$ is $\mathbb{RP}^2$), and identify $\mathbb{R}^2$ with

$$\{re^{2\pi i\theta} \mid \tfrac{1}{2} < r \leqslant 1\} \Big/ \begin{array}{l}\text{collapse the boundary} \\ S^1 \text{ to a point.}\end{array}$$

Clearly the identity of the half-open annulus induces a proper map from $A_d$ to $\mathbb{R}^2$, and transposing produces a proper morphism

$$\alpha\colon C_0(\mathbb{R}^2) \to C_0(A_d).$$

We identify

$$S^2 = \mathbb{D} \Big/ \begin{array}{l}\text{collapse the boundary} \\ S^1 \text{ to a point}\end{array}$$

and choose $U \subseteq S^2$ as the canonical embedding given by the identifications above. Finally, fix $n$ points in $S^2 \setminus U$. When $W_d$ denotes the $d$-Moore space (so $W_2 = \mathbb{RP}^2$), we then have

Proposition 6.9. *There are isomorphisms making the diagram*

$$\begin{tikzcd} \widehat{C_0(A_d)} \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_{C_0(A_d)} * \omega^{[n]}"'] & C_0(W_d) \arrow[d, "\omega^{[n]}"] \\ \widehat{C_0(A_d)}^{[n]} \arrow[r, "\cong"'] & C_0((W_d)^{[n]}) \end{tikzcd}$$

*commute.*

*Proof.* There is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(A_d) & \longrightarrow & C_0(W_d) & \longrightarrow & C(\mathbb{D}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & C_0(\mathbb{R}^2) & \longrightarrow & C(S^2) & \longrightarrow & C(\mathbb{D}) & \longrightarrow & 0, \end{array}$$

where we have identified

$$W_d = \mathbb{D} \Big/ \begin{array}{l}\text{a } d\text{-fold identification} \\ \text{of the boundary } S^1.\end{array}$$

In the diagram, the quotient maps arise from the restriction to the closed disks of radius 1/2 at the origin of either $W_d$ or $S^2$. Since $\alpha$ is proper, we may apply Proposition 4.3 to see that the left square is a pushout.

Repeating the last argument, but with identifications

$$(W_d)^{[n]} = \mathbb{D}^{[n]} \Big/ \begin{array}{l}\text{a } d\text{-fold identification} \\ \text{of the “outer” } S^1\end{array}$$

and

$$(S^2)^{[n]} = \mathbb{D}^{[n]} \Big/ \begin{array}{l}\text{collapse the “outer”} \\ S^1 \text{ to a point}\end{array}$$

we find that $\hat{A}^{[n]} \simeq C_0((W_d)^{[n]})$ in a coherent fashion, so that the given isomorphisms lead to a commuting diagram as above. ∎

THEOREM 6.10. *The map $\omega^{[n]}\colon C(W_d) \to C((W_d)^{[n]})$ is corona extendible.*

*Proof.* Since $K_0(C_0(A_d)) = \mathbb{Z}/d\mathbb{Z}$, we may apply [20, Lemma 11(1)] to the morphism $\mathrm{id} *_{C_0(A_d)} \omega^{[n]}$. As any isomorphism is corona extendible, the result follows from Proposition 6.9 above.

In the latter result we may of course place the $n$ holes anywhere on $W_d$, since any isomorphism is corona extendible. It is then straightforward to modify the Friis-Rørdam proof of Lin's theorem to conclude:

COROLLARY 6.11. *$C(W_d)$ is matricially semiprojective.*

Since we may identify $\mathbb{RP}^2$ as

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid z_2^2 = (1 - |z_1|)\, z_1\}$$

we obtain:

COROLLARY 6.12. *For every $\varepsilon > 0$ there is a $\delta > 0$ so that if $d$ is an integer and $x$, $y$ are contractions in $\mathbf{M}_d$ and*

$$\|[x, x^*]\| \leqslant \delta \qquad \|[y, y^*]\| \leqslant \delta \quad \|[x, y]\| \leqslant \delta,$$

*and if moreover*

$$\|y^2 - (1 - |x|)\, x\| \leqslant \delta,$$

*then there exists* commuting *normal contractions $\bar{x}$ and $\bar{y}$ in $\mathbf{M}_d$ such that*

$$\|x - \bar{x}\| \leqslant \varepsilon \qquad \text{and} \qquad \|y - \bar{y}\| \leqslant \varepsilon.$$

If in place of $C_0(A_2)$ one uses $C_0(T)$, where

$$T = \{re^{2\pi i\theta} \mid \tfrac{1}{2} < r \leqslant 1\} \Big/ \begin{matrix} e^{2\pi it} \sim e^{2\pi i(3/4 - t)} \\ e^{2\pi i(1/4 + t)} \sim e^{2\pi i(1 - t)} \end{matrix} \qquad \text{for} \quad t \in [0, 1/4],$$

then $C_0(T)^{\sim} \cong C(\mathbb{T}^2)$. We define a proper map $\alpha$ as above, and get by similar reasoning:

PROPOSITION 6.13. *There are isomorphisms making the diagram below commute.*

$$\begin{tikzcd} \widetilde{C_0(T)} \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_{C_0(T)} * \omega^{[n]}"'] & C_0(\mathbb{T}^2) \arrow[d, "\omega^{[n]}"] \\ \widetilde{C_0(T)}^{[n]} \arrow[r, "\cong"'] & C_0((\mathbb{T}^2)^{[n]}) \end{tikzcd}$$

Applying [20, Lemma 11(2)] we get

THEOREM 6.14. *Consider a diagram*

$$\begin{tikzcd} & C\left((\mathbb{T}^2)^{[n]}\right) \arrow[dr, dotted, "\overline{\varphi}"] & \\ C_0(\mathbb{R}^2) \arrow[r, "\iota"] & C(\mathbb{T}^2) \arrow[u, "\omega^{[n]}"] \arrow[r, "\varphi"] & C(\bigoplus \mathbf{M}_{d_k}) \end{tikzcd}$$

*where the $K$-theory of $\varphi\iota$ is zero. There exists $\bar{\varphi}$ so that the diagram commutes.*

In this case, combining the Friis-Rørdam methods with the fact that the winding number index is just $K$-theory (see [8]) yields the following result:

COROLLARY 6.15. *For every $\varepsilon > 0$ there is a $\delta > 0$ so that when $d$ is an integer and $u$ and $v$ are in $\mathscr{U}(\mathbf{M}_d)$ with*

$$\|uv - vu\| \leqslant \delta,$$

*and moreover*

$$\text{winding}\ \#(\lambda \mapsto \det(\lambda uv + (1-\lambda)\, vu)) = 0,$$

*then there exist $\bar{u}$ and $\bar{v}$ in $\mathscr{U}(\mathbf{M}_d)$ that* commute *and for which*

$$\|u - \bar{u}\| \leqslant \varepsilon \qquad \textit{and} \qquad \|v - \bar{v}\| \leqslant \varepsilon.$$

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